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Local energy integrals for effectively hyperbolic operators I

By

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1. Introduction.

In the paper of V. Ja. Ivrii and V. M. Petkov [5], they have proved that the strongly hyperbolic operator is effectively hyperbolic. Here we mean by the strongly hyperbolic operator that the operator for which the Cauchy problem is C^∞ well posed independent of lower order terms. We say that P is effectively hyperbolic (this terminology is due to L. Hörmander, see [4]) if the fundamental matrix of P has non-zero real eigenvalues at every critical point of P.

In this paper, we shall prove the converse for second order operators (in the microlocal sense). That is the effectively hyperbolic operator is strongly hyperbolic. Recently this fact was also proved by N. Iwasaki [7], [8]. In [8], he has proved that effectively hyperbolic operator of second order has a special expression of symbols by solving nonlinear equations and in [7] for such operators, the C^∞ well posedness was proved by the method of V. Ja. Ivrii [6], which is based on transformations of operator powers of operators.

Here, we shall apply directly the energy integral methods to two standard forms in [13], which are obtained by a homogeneous canonical transformation in the cotangent bundle on the space variables. We shall derive the local energy estimate which indicates the loss of regularity of solutions when they transverse a certain hypersurface in the cotangent bundle. Our arguments follow those of [11], [12] quite far.

Let us consider the following operator.

$$P(x_0, x, D_0, D) = D_0^2 - Q(x_0, x, D) + R(x_0, x, D) D_0,$$

where $Q(x_0, x, D)$, $R(x_0, x, D)$ is a classical pseudodifferential operator of second and zero order depending smoothly on x_0, and

$$x = (x_1, \cdots, x_d),\ \xi = (\xi_1, \cdots, \xi_d) \in \boldsymbol{R}^d,\ D = \left(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_d}\right),\ D_0 = \frac{1}{i}\frac{\partial}{\partial x_0}.$$

We also write $X=(x_0, x)$, $\Xi=(\xi_0, \xi)$, $x^{(p)}=(x_p, \cdots, x_d)$, $\xi^{(p)}=(\xi_p, \cdots, \xi_d)$, for $0\leqq p\leqq d$. Denote by $Q_2(x_0, x, \xi)$ the principal symbol of Q_2 which is assumed to be non-negative, and set

$$P_2(x_0, x, \xi_0, \xi)=\xi_0^2-Q_2(x_0, x, \xi).$$

Assume that $(0, 0, 0, \bar{\xi})$ is a critical point, that is, $\mathrm{grad}_{X,\Xi} P_2$ vanishes at this point. Then the effective hyperbolicity of P_2 means that the fundamental matrix F_{P_2} has non-zero real eigen values at this point, where

$$F_{P_2}(X, \Xi)=\begin{bmatrix} P_{2,X,\Xi} & P_{2,\Xi,\Xi} \\ -P_{2,X,X} & -P_{2,\Xi,X} \end{bmatrix}$$

and $P_{2,X,\Xi}$ is the $(d+1)\times(d+1)$ matrix with elements $(\partial^2 P_2/\partial x_i\partial\xi_j)_{i,j=0}^{d}$.

From [13], there exists a homogeneous canonical transformation independent of (x_0, ξ_0), defined in a conic neighborhood of $(0, \bar{\xi})$, taking $(0, \bar{\xi})$ to $(0, \hat{\xi})$, under which P_2 is transformed to one of the followings.

$$(1)_p \begin{cases} \xi_0^2-\sum_{i=0}^{p-1}(x_i-x_{i+1})^2 q_i(X, \xi)-\sum_{i=1}^{p}\xi_i^2 r_i(X, \xi)-\{(x_p-\phi_p(x^{(p+1)}, \xi^{(p+1)}))^2 \\ +\psi_p(x^{(p+1)}, \xi^{(p+1)})\}q_p(X, \xi), \text{ with } \{\phi_p, \{\phi_p, \psi_p\}\}(0, \hat{\xi}^{(p+1)})=0, 0\leqq p\leqq d-1 \end{cases}$$

$$(2)_p \begin{cases} \xi_0^2-\sum_{i=0}^{p-1}(x_i-x_{i+1})^2 q_i(X, \xi)-\sum_{i=1}^{p}\xi_i^2 r_i(X, \xi)-g_p(x^{(p)}, \xi^{(p+1)})r_{p+1}(X, \xi), \\ \text{with } \sum_{i=1}^{p} r_i(0, \hat{\xi})^{-1}>1, \ (\partial^2 g_p/\partial x_p^2)(0, \hat{\xi}^{(p+1)})=0, \ 1\leqq p\leqq d-1, \end{cases}$$

where q_i, r_i is positive and homogeneous of degree 2, 0 respectively, and ψ_p, g_p, vanishing at $(0, \hat{\xi})$, is homogeneous of degree 0, 2 respectively, non-negative, and ϕ_p is homogeneous of degree 0. We may assume that the above functions are defined in $I\times \boldsymbol{R}^d\times U$, where $I=(-T, T)$ and U is a conic neighborhood of $\hat{\xi}$.

In this paper, we treat the operators of type $(1)_p$ with $\mathrm{grad}\,\phi_p(0, \hat{\xi}^{(p+1)})=0$. Making a linear change of coordinates $x^{(p+1)}$, if necessary, we may suppose that $\hat{\xi}=(0, \cdots, 0, \hat{\xi}_d)$, $\hat{\xi}_d\neq 0$. In the last section, we shall give some remarks on the case $(1)_p$ with $\mathrm{grad}\,\phi_p(0, \hat{\xi}^{(p+1)})\neq 0$. Operators of type $(2)_p$ will be treated in a forthcoming paper.

Set

$$x'=(x_1, \cdots, x_p),\ x''=(x_{p+1}, \cdots, x_d),\ \xi'=(\xi_1, \cdots, \xi_p),\ \xi''=(\xi_{p+1}, \cdots, \xi_d),$$

and make a change of scale of variables introducing a parameter μ, $(0<\mu\leqq 1)$;

$$\eta_0=\mu^{1/2}\xi_0,\ \eta'=\mu^{1/2}\xi',\ \eta''=\xi'',\ y_0=\mu^{-1/2}x_0,\ y'=\mu^{-1/2}x',\ y''=x''.$$

Then we are led to the operator

$\xi_0^2-\mu^2\sum_{i=0}^{p-1}(x_i-x_{i+1})^2\tilde{q}_i(X,\ \xi,\ \mu)-\sum_{i=1}^{p}\xi_i^2\tilde{r}_i(X,\ \xi,\ \mu)-\mu^2(x_p-\tilde{\phi}(x'',\ \xi'',\ \mu))^2\times$
$\tilde{q}_p(X,\ \xi,\ \mu)-\mu\tilde{\psi}(x'',\ \xi'',\ \mu)\tilde{q}_p(X,\ \xi,\ \mu)+\mu\tilde{T}_1(X,\xi,\mu)+\mu^{1/2}\tilde{T}_0(X,\ \xi,\ \mu)\xi_0,$

where $\tilde{\phi}(x'',\ \xi'',\ \mu)=\mu^{-1/2}\phi_p(x'',\ \xi'')$, $\tilde{\psi}(x'',\ \xi'',\ \mu)=\psi_p(x'',\ \xi'')$, and $\tilde{f}(X,\ \xi,\ \mu)$ $=f(\mu^{1/2}x_0,\ \mu^{1/2}x',\ x'',\ \mu^{-1/2}\xi',\ \xi'')$.

Obviously, $\tilde{T}_i(X,\ \xi)$ is homogeneous of degree i. Set

$$V(\mu,\ \hat{\xi})=\{(x,\ \xi);\ |x|<\mu^{1/2},\ |\xi-\hat{\xi}|\xi||<\mu^{1/2}|\xi|,\ |\xi|\geqq\mu^{-2}\},$$

and denote by $\phi(x'',\ \xi,\ \mu)$, $\psi(x'',\ \xi,\ \mu)$, $\Gamma_i(\xi,\ \mu)$, $q_i(X,\ \xi,\ \mu)$, $r_i(X,\ \xi,\ \mu)$, $T_i(X,\ \xi,\ \mu)$ an extension of $\tilde{\phi}$, $\tilde{\psi}$, ξ_i, $\tilde{q}_i$, $\tilde{r}_i$, $\tilde{T}_i$ which coincides with the original one in $V(\delta\mu^{1/2},\ \hat{\xi})$ and belongs to a class of pseudodifferential operators of type (1, 0) (of L. Hörmander) with parameter μ, where δ is a positive constant which will be fixed in later. These extensions will be clarified in the next section.

After these modifications, we consider the following operator.

$$(1.1)\quad P_{(\mu)}=\xi_0^2-\mu^2\sum_{i=0}^{p}Y_i(X,\ \xi,\ \mu)^2q_i(X,\ \xi,\ \mu)-\sum_{i=1}^{p}\Gamma_i(\xi,\ \mu)^2r_i(X,\ \xi,\ \mu)$$
$$-\mu\psi(x'',\ \xi,\ \mu)q_p(X,\ \xi,\ \mu)+\mu T_1(X,\ \xi,\ \mu)+\mu^{1/2}T_0(X,\ \xi,\ \mu)\xi_0,$$

where $Y_i(X,\ \xi,\ \mu)=x_i-x_{i+1}$, $0\leqq i\leqq p-1$, $Y_p(X,\ \xi,\ \mu)=x_p-\phi(x'',\ \xi,\ \mu)$. We denote by $P_{(\mu)}^s(X,\ \Xi)$ the subprincipal symbol of $P_{(\mu)}$. Obviously, $\mu^{-1}P_{(\mu)}^s(0,\ 0,\ 0,\ \hat{\xi})$ does not depend on $\mu>0$, and we denote it by $P^s(0,\ 0,\ 0,\ \hat{\xi})$.

To formulate the energy estimate, we introduce some notations.

$$(1.2)\quad J(X,\ \xi,\ \mu)=\{Y(X,\ \xi,\ \mu)^2+\langle\mu\xi\rangle^{-1}\}^{1/2},\ Y(X,\ \xi,\ \mu)=$$
$$x_0-\phi(x'',\ \xi,\ \mu),\ \langle\xi\rangle^2=1+\sum_{j=1}^{d}\xi_j^2.$$

Take $\chi_0(s)\in C^\infty(\boldsymbol{R})$ such that $\chi_0(s)=1$ for $s\geqq-1/4$, $\chi_0(s)=0$ for $s\leqq-1/2$ and define

$$(1.3)\quad J_\pm(X,\ \xi,\ \mu)=\pm\{2\chi_0(\pm Y(X,\ \xi,\ \mu)\langle\mu\xi\rangle^{1/2})-1\}Y(X,\ \xi,\ \mu)+$$
$$+\langle\mu\xi\rangle^{-1/2}.$$

Next, choose $\chi(s)\in C^\infty(\boldsymbol{R})$ with $\chi(s)+\chi(-s)=1$ for $s\in\boldsymbol{R}$, $\chi(s)=0$ for $s\leqq-1$, $\chi(s)=1$ for $s\geqq1$, and set

$$(1.4)\quad \alpha_n^\pm(X,\ \xi,\ \mu)=\chi(\pm n^{1/2}Y(X,\ \xi,\ \mu)\langle\mu\xi\rangle^{1/2}).$$

Using these notations, we introduce the following semi-norm.

$$|||u|||_{n+k,r}^2=||\langle\mu D\rangle^kI_n(n-r)\alpha_n^-u||^2+||\langle\mu D\rangle^kJ_+(-n-r)\alpha_n^+u||^2,$$

where $I_n(n-r)$, $J_+(-n-r)$, $\alpha_n^\pm$ denotes the pseudodifferential operator with symbol $\langle\mu\xi\rangle^nJ_-(X,\ \xi,\ \mu)^{n-r}$, $J_+(X,\ \xi,\ \mu)^{-n-r}$, $\alpha_n^\pm(X,\ \xi,\ \mu)$ respectively, and

$||u||_s$ denotes the usual Sobolev norm in $H^s(\boldsymbol{R}^d)$. We set $||u||_0=||u||$.

Theorem 1. 1. *For any* $L\in\boldsymbol{N}$, *we have*

$$C(n,\ \mu,\ L)\int e^{-2x_0\theta}||P_{(\mu)}u||_{-2L}^2dx_0+\int e^{-2x_0\theta}|||P_{(\mu)}u|||_{n,0}^2dx_0\geqq c_1n\int e^{-2x_0\theta}|||D_0u|||_{n,1}^2dx_0$$
$$+c_2n\int e^{-2x_0\theta}|||u|||_{n+1,0}^2dx_0+c_3\theta\int e^{-2x_0\theta}|||D_0u|||_{n,1/2}^2dx_0+c_3\theta\int e^{-2x_0\theta}|||u|||_{n+1,-1/2}^2dx_0$$
$$+c_4\theta^3\int e^{-2x_0\theta}||u||_{-L}^2dx_0+c_4\theta^{3/2}\int e^{-2x_0\theta}||D_0u||_{-L}^2dx_0,$$

for $n\geqq C_0C,\ 0<\mu\leqq\mu_0(n),\ \theta\geqq\theta_0(n,\ \mu,\ L),\ u\in C_0^\infty(I\times\boldsymbol{R}^d)$, *where* $C=|P^s(0,\ 0,\ 0,\ \hat{\xi})|+1,\ C_0=C_0(q_i(0,\ \hat{\xi}))$.

Theorem 1. 2. *For any* $s\in\boldsymbol{R}$, we *have*

$$C(n,\ s)\int e^{-2x_0\theta}||P_{(1)}u||_{n+s+1}^2dx_0\geqq\theta^{3/2}\int e^{-2x_0\theta}||D_0u||_s^2dx_0$$
$$+\theta^3\int e^{-2x_0\theta}||u||_s^2dx_0,$$

for $n\geqq C_0C,\quad \theta\geqq\theta_0(n,\ s),\ u\in C_0^\infty(I\times\boldsymbol{R}^d)$.

Remark 1. 1. We note that

$$\langle\mu\xi\rangle^nJ_-(X,\ \xi,\ \mu)^n\alpha_n^-(X,\ \xi,\ \mu)+J_+(X,\ \xi,\ \mu)^{-n}\alpha_n^+(X,\ \xi,\ \mu),$$

is equivalent to $\langle\mu\xi\rangle^n$ when $x_0-\phi(x'',\ \xi,\ \mu)\leqq-c$, to $\langle\mu\xi\rangle^{n/2}$ when $|x_0-\phi(x'',\ \xi,\ \mu)|\leqq c\langle\mu\xi\rangle^{-1/2}$ and to 1 when $x_0-\phi(x'',\ \xi\ \mu)\geqq c$ with arbitrary positive c.

2. Preliminaries.

We shall specify the extensions of $\tilde{q}_i$ and others. First we introduce some notations. Let $a(X,\ \xi,\ \mu)\in C^\infty(I\times\boldsymbol{R}^{2d})$, where μ is a positive parameter. We say that $a(X,\ \xi,\ \mu)$ belongs to $(Es)_{\rho,\delta}^{m,l}(\rho\geqq\delta,\ \delta\geqq0)$ if $a(X,\ \xi,\ \mu)$ satisfies the inequalities with some $\mu_1>0$,

$$|a_{(\beta)}^{(\alpha)}(X,\ \xi,\ \mu)|\leqq C_{\alpha,\beta}\mu^{l-\rho|\alpha|-\delta|\beta|}\langle\xi\rangle^{m-|\alpha|},$$

for all $\alpha\in\boldsymbol{N}^d$, $\beta\in\boldsymbol{N}^{d+1}$, $\mu\in(0,\ \mu_1)$ with $C_{\alpha,\beta}$ independent of μ. We also say that $a(X,\ \xi,\ \mu)$ belongs to $S^{-\infty}(\mu)$, if for any $l\in\boldsymbol{N}$, $a(X,\ \xi,\ \mu)$ satisfies the inequlities,

$$|a_{(\beta)}^{(\alpha)}(X,\ \xi,\ \mu)|\leqq C_{\alpha,\beta,l}(\mu)\langle\xi\rangle^{-l-|\alpha|/2+|\beta|/2},$$

for all α, β, $\mu\in(0,\ l(\mu)]$, where $C_{\alpha,\beta,l}(\mu)$ may depend on μ. It is clear from Calderón-Vaillancourt theorem [3] that

$$||a(X,\ D,\ \mu)u||\leqq C(l,\ \mu)||u||_{-l},\ \mu\in(0,\ l(\mu)],\ \text{for any } l\in\boldsymbol{N}.$$

Here $a(X, D, \mu)u$ is defined by

$$a(X, D, \mu)u=\int e^{ix\xi}a(X, \xi, \mu)\hat{u}(\xi)d\xi,\ \ \hat{u}(\xi)=\int e^{-ix\xi}u(x)dx,\ \ u\in C_0^\infty(\boldsymbol{R}^d).$$

Without special mentions, we denote by $\sigma(a(X, D, \mu))$, $\mathrm{op}(a(X, \xi, \mu))$ the symbol of $a(X, D, \mu)$ and the operator with symbol $a(X, \xi, \mu)$. But sometimes, we do not distinguish operators and their symbols.

Let $a_i(X, \xi, \mu)\in(Es)_{\rho,\delta}^{m_i,l_i}(i=1, 2)$. Then it is not difficult to show by standard methods that

$$(2.1)\qquad \sigma(a_1(X, D, \mu)a_2(X, D, \mu))-\sum_{|\alpha|\leqq N-1}\frac{1}{\alpha!}a_1^{(\alpha)}(X, \xi, \mu)a_{2(\alpha)}(X, \xi, \mu)$$

belongs to $(Es)_{\rho,\delta}^{m_1+m_2-N,l_1+l_2-(\rho+\delta)N}$, mod $S^{-\infty}(\mu)$. This means that one can write

$$a_1(X, D, \mu)a_2(X, D, \mu)-\mathrm{op}\left\{\sum_{|\alpha|\leqq N-1}\frac{1}{\alpha!}a_1^{(\alpha)}(X, \xi, \mu)a_{2(\alpha)}(X, \xi, \mu)\right\}=\mathrm{op}(r_N)+\mathrm{op}(r_{-\infty}),$$

with $r_N(X, \xi, \mu)\in(Es)_{\rho,\delta}^{m_1+m_2-N,l_1+l_2-(\rho+\delta)N}$, $r_{-\infty}(X, \xi, \mu)\in S^{-\infty}(\mu)$. Similarly, for $a(X, \xi, \mu)\in(Es)_{\rho,\delta}^{m,l}$, we see that

$$(2.2)\qquad \sigma(a(X, D, \mu)^*)-\sum_{|\alpha|\leqq N-1}\frac{1}{\alpha!}(-1)^{|\alpha|}\overline{a_{(\alpha)}^{(\alpha)}(X, \xi, \mu)}\in(Es)_{\rho,\delta}^{m-N,l-(\rho+\delta)N},$$

mod $S^{-\infty}(\mu)$.

Take $\eta_1(s)\in C_0^\infty(\boldsymbol{R})$ with $0\leqq\eta_1(s)\leqq1$, $\eta_1(s)=1$ for $|s|\leqq1$, $\eta_1(s)=0$ for $|s|\geqq2$ and define $q_{i,2}(X, \xi, \mu)$ by

$$(2.3)\qquad q_{i,2}(X, \xi, \mu)=\{\tilde{q}_i(X, \xi, \mu)-q_i(0, x'', 0, \xi'')\}\times \times\eta_1(\delta^{-2}\mu^{-1}|\xi'|^2|\xi|^{-2})\eta_1(\delta^{-2}\mu|x'|^2).$$

Obviously, $\theta^2\mu^{-1}|\xi'|^2+|\xi''|^2$ is equivalent to $|\xi|^2$ uniformly in $0\leqq\theta\leqq1$, $0<\mu\leqq1$, $0<\delta\leqq\delta_0$ on the support of $\eta_1(\delta^{-2}\mu^{-1}|\xi'|^2|\xi|^{-2})$. Then, from Taylor expansion of order 1, it is easy to see that

$$(2.4)\qquad |q_{i,2}(X, \xi, \mu)|\leqq C\delta|\xi|^2,\ \ \partial_{x'}^\alpha\partial_{\xi'}^\gamma q_{i,2}(X, \xi, \mu)\in(Es)_{1/2,0}^{2-|\gamma|,(|\alpha|-|\gamma|)/2},$$

(making a modification near $\xi=0$, if necessary), where C does not depend on μ and δ. Similarly we set

$$(2.5)\qquad q_{i,1}(x'', \xi)=\{q_i(0, x'', 0, \xi'')-q_i(0, \hat{\xi})|\xi|^2\}\times \times\eta_2(\delta^{-1}(\xi''|\xi|^{-1}-\hat{\xi}''))\eta_2(\delta^{-1}x'')+q_i(0, \hat{\xi})|\xi|^2,$$

with $\eta_2(z)\in C_0^\infty(\boldsymbol{R}^{d-p})$ satisfying $0\leqq\eta_2(z)\leqq1$, $\eta_2(z)=1$ if $|z|\leqq1$, $\eta_2(z)=0$ if $|z|\geqq2$. Now we define $q_i(X, \xi, \mu)$ by

$$(2.6)\qquad q_i(X, \xi, \mu)=q_{i,1}(x'', \xi)+q_{i,2}(X, \xi, \mu).$$

It is clear that this extended $q_i(X, \xi, \mu)$ coincides with $\tilde{q}_i$ in $V(\delta\mu^{1/2}, \hat{\xi})$ and we have

$$(2.7)\quad q_i(X, \xi, \mu) \geqq \{q_i(0, \hat{\xi}) - C\delta\}\, |\xi|^2,\ q_i \in (Es)^{2,0}_{1/2,0},\ q_{i,1}(x'', \xi) \in (Es)^{2,0}_{0,0}.$$

From the asymptotic expansion formula (2.2) and (2.4), we get

$$(2.8)\quad \mathrm{op}\{\mu^2 Y_i(X)^2 q_i(X, \xi, \mu)\} - \mathrm{op}\{\mu^2 Y_i(X)^2 q_i(X, \xi, \mu)\}^* \in (Es)^{1,3/2}_{1/2,0},\quad \mathrm{mod}\ S^{-\infty}(\mu).$$

By the same way, following the formulas (2.3), (2.5) and (2.6), $\tilde{r}_i$ will be extended to $r_i(X, \xi, \mu) = r_{i,1}(x'', \xi) + r_{i,2}(X, \xi, \mu)$, where

$$(2.9)\quad r_i(X, \xi, \mu) \geqq \{r_i(0, \hat{\xi}) - C\delta\},\ \partial^\alpha_{x'}\partial^\gamma_{\xi'} r_{i,2}(X, \xi, \mu) \in (Es)^{0,(|\alpha|-|\gamma|)/2}_{1/2,0}.$$

Choose $\eta_3(s) \in C_0^\infty(\boldsymbol{R})$ so that $\eta_3(s) = s$ if $|s| \leqq 2$, $\eta_3(s) = 0$ if $|s| \geqq 3$ and set

$$\Gamma_i(\xi, \mu) = \mu^{1/2}\eta_3(\mu^{-1/2}(\xi_i |\xi|^{-1} - \hat{\xi}_i))\, |\xi|,\ Z_i(x, \mu) = \mu^{1/2}\eta_3(\mu^{-1/2}x_i),\ 1 \leqq i \leqq d.$$

We observe $\Gamma_i(\xi, \mu)^2 r_i(X, \xi, \mu)$ which coincides with $\xi_i^2 \tilde{r}_i$ in $V(\delta\mu^{1/2}, \hat{\xi})$. Taking (2.9) into account, it follows that

$$(2.10)\quad \mathrm{op}\{\Gamma_i^2 r_i\} - \mathrm{op}\{\Gamma_i^2 r_i\}^* - L_i\Gamma_i \in (Es)^{0,0}_{1/2,0},\ \mathrm{mod}\ S^{-\infty}(\mu),\quad L_i(X, \xi, \mu) \in (Es)^{0,1/2}_{1/2,0}.$$

Next, we define the extensions of $\psi_p(x'', \xi'')$ and $\mu^{-1/2}\phi_p(x'', \xi'')$. Setting

$$\Gamma(\xi, \mu) = (\Gamma_1(\xi, \mu), \cdots, \Gamma_d(\xi, \mu)),\ Z(x, \mu) = (Z_1(x, \mu), \cdots, Z_d(x, \mu)),$$

we define $\psi(x'', \xi, \mu) = \psi_1(x'', \xi, \mu) + \psi_2(x'', \xi, \mu)$ by

$$(2.11)\quad \psi_1(x'', \xi, \mu) = \sum_{|\alpha+\beta|=2} \frac{1}{\alpha!\beta!} Z(x, \mu)^\beta \Gamma(\xi, \mu)^\alpha \partial_x^\beta \partial_\xi^\alpha \psi_p(0, \hat{\xi}')\, |\xi|^{-|\alpha|},$$
$$\psi_2(x'', \xi, \mu) = \Big\{\psi_p(x'', \xi'') - \sum_{|\alpha+\beta|=2} \frac{1}{\alpha!\beta!} x^\beta (\xi |\xi|^{-1} - \hat{\xi})^\alpha \partial_x^\beta \partial_\xi^\alpha \psi_p(0, \hat{\xi}'')\Big\} \times \eta_2(\mu^{-1/2}(\xi'' |\xi|^{-1} - \hat{\xi}''))\eta_2(\mu^{-1/2}x''),$$

with η_2 being as above. Since $\psi_p(x'', \xi'') \geqq 0$, $\psi_p(0, \hat{\xi}'') = 0$, it follows that $\psi_1(x'', \xi, \mu)$ is non-negative. It is also clear that $\psi_1(x'', \xi, \mu) \in (Es)^{0,1}_{1/2,1/2}$, $\psi_2(x'', \xi, \mu) \in (Es)^{0,3/2}_{1/2,1/2}$.

Proposition 2. 1. *$\psi(x'', \xi, \mu) \geqq 0$, $\psi(x'', \xi, \mu) \in (Es)^{0,1}_{1/2,1/2}$,*

$$\partial_{x_i}\partial_{\xi_i}\psi(x'', \xi, \mu) = C_i(x'', \xi, \mu)\partial_{x_i}\partial_{\xi_i}\psi_p(0, \hat{\xi}'')\, |\xi|^{-1} + \tilde{C}_i(x'', \xi, \mu)\, |\xi|^{-1},$$

with $C_i(x'', \xi, \mu) \in (Es)^{0,0}_{1/2,1/2}$ which is equal to 1 in $V(2\mu^{1/2}, \hat{\xi})$ and $\tilde{C}_i(x'', \xi, \mu) \in (Es)^{0,1/2}_{1/2,1/2}$.

Proof. To prove the second assertion, it suffices to note that

$(\partial^2\psi_p/\partial x_i\partial\xi_d)(0, \hat{\xi}'')=0$ and $\eta_3^{(1)}(\mu^{-1/2}(\xi_j|\xi|^{-1}-\hat{\xi}_j))\xi_j\partial_{\xi_i}|\xi|^{-1}$ belongs to $(Es)^{-1,1/2}_{1/2,1/2}$ if $j\leqq d-1$.

Consider $\mu\psi(x'', \xi, \mu)q_p(x, \xi, \mu)\in(Es)^{2,2}_{1/2,1/2}$. In virtue of proposition 2. 1 and (2. 4), it follows that

$$\partial_{x_i}\partial_{\xi_i}(\mu\psi q_p)-\mu C_i\partial_{x_i}\partial_{\xi_i}\psi_p(0, \hat{\xi}'')|\xi|^{-1}q_p\in(Es)^{1,3/2}_{1/2,1/2}.$$

Hence, we have

$$\text{(2. 12)}\quad \mathrm{op}\{\mu\psi q_p\}-\mathrm{op}\{\mu\psi q_p\}^*-i\sum_{j=1}^{d}\mathrm{op}(\mu C_j\partial_{x_j}\partial_{\xi_j}\psi_p(0, \hat{\xi}'')|\xi|^{-1}q_p)\in(Es)^{1,3/2}_{1/2,1/2}.$$

Similarly, following (2. 11), we extend $\mu^{-1/2}\phi_p(x'', \xi'')$ to $\phi(x'', \xi, \mu)$ which belongs to $(Es)^{0,1/2}_{1/2,1/2}$. Consequently, recalling $Y_p(x, \xi, \mu)=x_p-\phi(x'', \xi, \mu)$, it follows from (2. 3) that

$$\text{(2. 13)}\quad \mathrm{op}\{\mu^2Y_p^2q_p\}-\mathrm{op}\{\mu^2Y_p^2q_p\}^*\in(Es)^{1,3/2}_{1/2,1/2},\ \mathrm{mod}\ S^{-\infty}(\mu),\ \mu^2Y_p^2q_p\in(Es)^{2,2}_{1/2,1/2}.$$

Finally, we extend $\tilde{T}_i(X, \xi, \mu)$. Let $\eta_4(s)\in C_0^\infty(\boldsymbol{R}^d)$ be 1 for $|s|\leqq 1$ and be 0 for $|s|\geqq 2$. We define $T_i(X, \xi, \mu)$ by

$$\text{(2. 14)}\quad \begin{aligned} T_{i,1}(X, \xi, \mu)&=\{\tilde{T}_i(X, \xi, \mu)\\ &\quad -T_i(0, \hat{\xi})|\xi|^i\}\eta_4(\delta^{-1}\mu^{-1/2}(\xi|\xi|^{-1}-\hat{\xi}))\eta_4(\delta^{-1}\mu^{-1/2}x),\\ T_0(X, \xi, \mu)&=T_{0,1}(X, \xi, \mu)+T_0(0, \hat{\xi}),\\ T_1(X, \xi, \mu)&=T_{1,1}(X, \xi, \mu)\\ &\quad +T_1(0, \hat{\xi})|\xi|-\frac{i}{2}\sum_{j=1}^{d}(1-C_j)\partial_{x_j}\partial_{\xi_j}\psi_p(0, \hat{\xi}'')|\xi|^{-1}q_p, \end{aligned}$$

where C_j are the same ones in proposition 2. 1. The following properties are easily verified.

$$\text{(2. 15)}\quad \begin{aligned} &T_i(X, \xi, \mu)\in(Es)^{i,0}_{1/2,1/2},\ |T_{i,1}(X, \xi, \mu)|\leqq C\delta|\xi|^i,\\ &T_i(X, \xi, \mu)=\tilde{T}_i(X, \xi, \mu)\ \text{in}\ V(\delta\mu^{-1/2}, \hat{\xi}). \end{aligned}$$

3. Pseudodifferential operators with parameter μ.

We shall say that $a(X, \xi, \mu)\in C^\infty(I\times\boldsymbol{R}^{2d})$ belongs to $J^rS^{m,l}_{1/2}$ if there is $\mu_1>0$ such that

$$|a^{(\gamma)}_{(\alpha)}(X, \xi, \mu)|\leqq C_{\alpha,\gamma}\langle\xi\rangle^{m-|\gamma|/2+|\alpha|/2}\langle\mu\xi\rangle^l J(X, \xi, \mu)^r,$$

holds for all multi indexes $\alpha\in\boldsymbol{N}^{d+1}$, $\gamma\in\boldsymbol{N}^d$ and for $\mu\in(0, \mu_1]$ with $C_{\alpha,\gamma}$ independent of μ. We donote by $J^rS^{m,l}$ the set of all $a(X, \xi, \mu)\in C^\infty(I\times\boldsymbol{R}^{2d})$ such that, with some $\mu_1>0$, the following inequalities are valid,

$$|a^{(\gamma)}_{(\alpha)}(X, \xi, \mu)|\leqq C_{\alpha,\gamma}\langle\xi\rangle^{m-|\gamma|}\langle\mu\xi\rangle^{l+|\alpha|/2+|\gamma|/2}J(X, \xi, \mu)^r,$$

for all $\alpha\in\boldsymbol{N}^{d+1}$, $\gamma\in\boldsymbol{N}^d$, $\mu\in(0, \mu_1]$ with $C_{\alpha,\gamma}$ independent of μ.

Let $a(X, \xi, \mu)\in(Es)^{m,l}_{\rho,\delta}$ and take $\eta(s)\in C^\infty(\boldsymbol{R})$ with $\eta(s)=0$ for $|s|\leqq 1$, $\eta(s)=1$ for $|s|\geqq 2$. On the support of $\eta(\mu^{1+2(\rho+\delta)}\langle\xi\rangle)$, it is clear that

$\mu^{-(\rho+\delta)}\leqq\langle\mu\xi\rangle^{1/2}$, hence $\mu^{-l}a(X, \xi, \mu)\eta(\mu^{1+2(\rho+\delta)}\langle\xi\rangle)$ belongs to $J^0S^{m,0}$. On the other hand, obviously, $1-\eta(\mu^{1+2(\rho+\delta)}\langle\xi\rangle)$ belongs to $S^{-\infty}(\mu)$ and then we have $\mu^{-l}a(X, \xi, \mu)\in J^0S^{m,0}$ modulo $S^{-\infty}(\mu)$. We write $a(X, \xi, \mu)\in \mu^sJ^rS^{m,l}$(resp. $\mu^sJ^rS^{m,l}_{1/2}$) if $\mu^{-s}a(X, \xi, \mu)\in J^rS^{m,l}$(resp. $J^rS^{m,l}_{1/2}$). Also we set $S^{m,l}_{1/2}=J^0S^{m,l}_{1/2}$.

When deriving the energy estimates in the following sections, the operator belonging to $S^{-\infty}(\mu)$ bring only error terms which will be treated in section 8. Therefore we freely omit the term "modulo $S^{-\infty}(\mu)$" throughout this paper except for section 8. According to this abbreviation, we can write

$$a(X, \xi, \mu)\in\mu^lJ^0S^{m,0} \text{ if } a(X, \xi, \mu)\in(Es)^{m,l}_{\rho,\delta}. \tag{3.1}$$

Let $R\in J^0S^{m,l}_{1/2}$. On the support of $\eta(\mu\langle\xi\rangle)$ we have $\langle\xi\rangle\geqq2\mu^{-1}$ and hence $\langle\mu\xi\rangle^s\langle\xi\rangle^{-s}\leqq C_s\mu^s(s\geqq0)$. Again, according to this abbreviation, we get

$$R\in\mu^sJ^0S^{m+s,l-s}_{1/2}(s\geqq0) \text{ if } R\in J^0S^{m,l}_{1/2}. \tag{3.2}$$

Especially, we have

$$\phi^{(\gamma)}_{(\alpha)}(x'', \xi, \mu)\in J^0S^{-|\gamma|,0} \quad \text{for } |\alpha+\gamma|\leqq1. \tag{3.3}$$

The following propositions are easily verified.

Proposition 3. 1. $Y(X, \xi, \mu)=x_0-\phi(x'', \xi, \mu)\in J^1S^{0,0}$, $\langle\mu\xi\rangle^{-1/2}\in J^1S^{0,0}$, $J^rS^{m,l}\subset J^0S^{m,l+r^-/2}$, *with* $r^-=\max(0, -r)$. *Let* $a_i(X, \xi, \mu)\in J^{r_i}S^{m_i,l_i}$ $(i=1, 2)$. *Then we have*

$$a_1(X, \xi, \mu)a_2(X, \xi, \mu)\in J^{r_1+r_2}S^{m_1+m_2,l_1+l_2}.$$

Note that for any $f(s)\in\mathscr{B}(\boldsymbol{R})$ with $f^{(1)}(s)\in C_0^\infty(\boldsymbol{R})$, we have

$$f(Y(X, \xi, \mu)\langle\mu\xi\rangle^{1/2})\in J^0S^{0,0}.$$

Moreover we have the following estimates,

$$|\partial_x^\alpha\partial_\xi^\gamma f(Y(X, \xi, \mu)\langle\mu\xi\rangle^{1/2})|\leqq C_{\alpha,\gamma}J(X, \xi, \mu)^{-|\alpha+\gamma|}\langle\xi\rangle^{-|\gamma|}. \tag{3.3}$$

Proposition 3. 2. *Assume that* $a(X, \xi, \mu)\in J^rS^{m,l}$ *and* $a(X, \xi, \mu)\geqq cJ(X, \xi, \mu)^r\langle\mu\xi\rangle^l\langle\xi\rangle^m$ *with* $c>0$ *independent of* μ. *Then we have*

$$a(X, \xi, \mu)^n\in J^{rn}S^{mn,ln} \text{ for any } n\in\boldsymbol{R}.$$

Especially it follows that $J(X, \xi, \mu)^n\in J^nS^{0,0}$.

Proposition 3. 3. $c_1J(X, \xi, \mu)\geqq J_\pm(X, \xi, \mu)\geqq c_2J(X, \xi, \mu)$ *with* $c_i>0$ *independent of* μ.

Proof. We shall show the inequality for $J_+(X, \xi, \mu)$. When $Y(X, \xi, \mu)\langle\mu\xi\rangle^{1/2}\geqq0$ or $Y(X, \xi, \mu)\langle\mu\xi\rangle^{1/2}\leqq-1/2$, we have $J_+(X, \xi, \mu)=|Y(X, \xi, \mu)|+\langle\mu\xi\rangle^{-1/2}$ from the definition, then the inequality is immediate.

When $-1/2\leqq Y(X, \xi, \mu)\langle\mu\xi\rangle^{1/2}\leqq 0$, it follows that

$$J_{+}(X, \xi, \mu)\geqq\langle\mu\xi\rangle^{-1/2}-|Y(X, \xi, \mu)|\geqq 4^{-1}\{|Y(X, \xi, \mu)|+\langle\mu\xi\rangle^{-1/2}\}.$$

Thus we obtain the desired inequality.

From this proposition, it follows that $J_{\pm}(X, \xi, \mu)^{n}\in J^{n}S^{0,0}$ for any $n\in\boldsymbol{R}$.

Proposition 3. 4. $\partial_{x}^{\alpha}\partial_{\xi}^{\gamma}J_{\pm}(X, \xi, \mu)^{n}\in J^{n-|\alpha+\gamma|}S^{-|\gamma|,0}$ *for* $|\alpha+\gamma|\leqq 1$.

Proof. It suffices to show that $\partial_{x}^{\alpha}\partial_{\xi}^{\gamma}J_{\pm}(X, \xi, \mu)\in J^{1-|\alpha+\gamma|}S^{-|\gamma|,0}$ for $|\alpha+\gamma|\leqq 1$. If we note that $\langle\mu\xi\rangle^{1/2}Y(X, \xi, \mu)$ is bounded on the support of $\chi_{0}^{(1)}(\pm n^{1/2}Y(X, \xi, \mu)\langle\mu\xi\rangle^{1/2})$, the assertion follows from proposition 3. 1.

Next we consider the composition of pseudodifferential operators.

Proposition 3. 5. *Let* $a_{i}(X, \xi, \mu)\in J^{r_{i}}S^{m_{i},l_{i}}(i=1, 2)$. *Then*

$$\sigma(a_{1}(X, D, \mu)a_{2}(X, D, \mu))-\sum_{|\alpha|\leqq N-1}\frac{1}{\alpha!}a_{1}^{(\alpha)}(X, \xi, \mu)a_{2(\alpha)}(X, \xi, \mu)$$
$$\in J^{r_{1}+r_{2}}S^{m_{1}+m_{2}-N,l_{1}+l_{2}+N}.$$

Let $a(X, \xi, \mu)\in J^{r}S^{m,l}$ *then* $a(X, D, \mu)^{*}\in J^{r}S^{m,l}$ *and*

$$\sigma(a(X, D, \mu)^{*})-\sum_{|\alpha|\leqq N-1}\frac{1}{\alpha!}(-1)^{|\alpha|}\overline{a_{(\alpha)}^{(\alpha)}(X, \xi, \mu)}\in J^{r}S^{m-N,l+N},$$

where $a(X, D, \mu)^{*}$ *denotes the adjoint of* $a(X, D, \mu)$ *with respect to the scalar product in* $L^{2}(\boldsymbol{R}^{d})$.

Proof. (cf. [2], [1]) Set

$$d(y, \xi, \eta, \mu)=1+\langle\mu\xi\rangle|y|^{2}+\langle\mu\xi\rangle^{-1}|\eta|^{2}.$$

Then remarking the following inequalities,

$$J(x_{0}, x+y, \xi, \mu)^{s}\leqq C_{s}J(X, \xi, \mu)^{s}d(y, \xi, \eta, \mu)^{|s|/2},$$
$$J(X, \xi+\theta\eta, \mu)^{s}\leqq C_{s}J(X, \xi, \mu)^{s}d(y, \xi, \eta, \mu)^{|s|/2}, \text{ for } |\xi|\geqq 2(1+|\eta|),$$
$$J(X, \xi+\theta\eta, \mu)^{s}\leqq C_{s}J(X, \xi, \mu)^{s}\langle\eta\rangle^{3|s|/2}, \text{ for } |\xi|\leqq 2(1+|\eta|),$$
$$|\partial_{\xi}^{\gamma}\partial_{y}^{\beta}\partial_{\eta}^{\delta}d(y, \xi, \eta, \mu)^{-K}|\leqq C_{\gamma,\beta,\delta}\langle\mu\xi\rangle^{|\beta|/2-|\delta|/2}\langle\xi\rangle^{-|\gamma|}d(y, \xi, \eta, \mu)^{-K},$$

with C_{s}, $C_{\gamma,\beta,\delta}$ independent of μ, one can proceed in the same way as in [12].

Proposition 3. 6. *Let* $a_{i}(X, \xi, \mu)\in J^{r_{i}}S_{1/2}^{m_{i},l_{i}}(i=1, 2)$. *Then* $a_{1}(X, D, \mu)a_{2}(X, D, \mu)\in J^{r_{1}+r_{2}}S_{1/2}^{m_{1}+m_{2},l_{1}+l_{2}}$. *If* $a(X, D, \mu)\in J^{r}S_{1/2}^{m,l}$, *then* $a(X, D, \mu)^{*}\in J^{r}S_{1/2}^{m,l}$.

Now we study the sharp Gårding type inequality.

Proposition 3. 7. *Let* $a(X, \xi, \mu)\in J^{r}S^{m,l}$. *Denote by* $a_{F}(X, \xi, \mu)$ *the Friedrichs symmetrization of* $a(X, \xi, \mu)$. *Then we have*

$$a_F(X,\xi,\mu)=a(X,\xi,\mu)+b(X,\xi,\mu) \text{ with } b(X,\xi,\mu)\in J^rS^{m-1,l+1}_{1/2}.$$

Proof. (See [9]) In the proof, we omit writing the variable x_0. Take $q(\sigma)\in C_0^\infty(|\sigma|<1)$ and set

$$F(\xi,\zeta)=q((\zeta-\xi)\langle\xi\rangle^{-1/2})\langle\xi\rangle^{-d/4},\ A_F(\xi,\tilde{x},\tilde{\xi},\mu)=\int F(\xi,\zeta)a(\tilde{x},\ \xi,\ \mu)F(\tilde{\xi},\ \zeta)d\zeta.$$

We assume that $\int q(\sigma)^2d\sigma=1$, $q(-\sigma)=q(\sigma)$. First we remark that

$$\partial_\xi^\beta F(\xi,\zeta)=\langle\xi\rangle^{-d/4}\sum_{|\gamma|\leq|\beta|,\gamma_1\leq\gamma}\psi_{\beta,\gamma,\gamma_1}(\xi)((\zeta-\xi)\langle\xi\rangle^{-1/2})^{\gamma_1}\times (\partial_\sigma^\gamma q)((\zeta-\xi)\langle\xi\rangle^{-1/2}), \tag{3.4}$$

with $\psi_{\beta,\gamma,\gamma_1}\in S_{1,0}^{-(|\beta|-|\gamma-\gamma_1|/2)}\subset S_{1,0}^{-|\beta|/2}$. From [9], we have

$$a_F(x,\ \xi,\ \mu)=\sum_{|\alpha|\leq1}\frac{1}{\alpha!}A_{F(\alpha)}^{(\alpha,0)}(\xi,\ x,\ \xi,\ \mu)$$
$$+2\sum_{|\alpha|=2}\int_0^1\frac{(1-\theta)}{\gamma!}\{\iint e^{-iy\eta}A_{F(\gamma)}^{(\gamma,0)}(\xi+\theta\eta,\ x+y,\ \xi,\ \mu)dyd\eta\}d\theta$$

where

$$A_{F(\beta)}^{(\alpha,\gamma)}(\xi,\ \tilde{x},\ \tilde{\xi},\ \mu)=\partial_\xi^\alpha\partial_{\tilde{x}}^\beta\partial_{\tilde{\xi}}^\gamma A_F(\xi,\ \tilde{x},\ \tilde{\xi},\ \mu).$$

Since $q(\sigma)$ is even, it follows that

$$A_F(\xi,\ x,\ \xi,\ \mu)=a(x,\ \xi,\ \mu)$$
$$+2\sum_{|\gamma|=2}\iint_0^1\frac{(1-\theta)}{\gamma!}\langle\xi\rangle a^{(\gamma)}(x,\ \xi+\sigma\langle\xi\rangle^{1/2}\theta,\ \mu)\sigma^\gamma q(\sigma)^2d\sigma d\theta.$$

From this expression, it is easily seen that $A_F(\xi,\ x,\ \xi,\ \mu)-a(x,\ \xi,\ \mu)\in J^rS^{m-1,l+1}$. As for $A_{(\alpha)}^{(\alpha,0)}(\xi,\ x,\ \xi,\ \mu)$ $(|\alpha|=1)$, from (3.4) one can see that these terms belong to $J^rS^{m-1,l+1}$. Finally we observe the oscillatory integral

$$\iint e^{-iy\eta}A_{F(\gamma)}^{(\gamma,0)}(\xi+\theta\eta,\ x+y,\ \xi,\ \mu)dyd\eta.$$

If we remark that

$$c_1J(\tilde{x},\ \xi,\ \mu)\leqq J(\tilde{x},\ \xi+\sigma\langle\xi\rangle^{1/2},\ \mu)\leqq c_2J(\tilde{x},\ \xi,\ \mu),$$

with positive c_i independent of $|\sigma|\leqq1$, μ, using Cauchy-Schwarz inequality, it follows that

$$|A_{F(\beta)}^{(\alpha,\gamma)}(\xi,\ \tilde{x},\ \tilde{\xi},\ \mu)|\leqq C_{\alpha,\beta,\gamma}J(\tilde{x},\ \xi,\ \mu)^r\langle\mu\xi\rangle^{l+|\beta|/2}\langle\xi\rangle^{m-|\alpha|/2}\langle\tilde{\xi}\rangle^{-|\gamma|/2}.$$

Then applying this estimate, the same reasoning as in the proof of proposition 3.5 proves that this oscillatory integral belongs to $J^rS^{m-1,l+1}_{1/2}$. And this completes the proof.

The following two propositions are analogies of the sharp Gårding inequality.

Proposition 3. 8. *Let* $a(X, \xi, \mu)\in J^0S^{m,l}$ *and* $\sup|a(X, \xi, \mu)\times\langle\mu\xi\rangle^{-l}\langle\xi\rangle^{-m}|=\hat{c}$. *Then for any* $L\in\boldsymbol{N}$, *there is* $C(a, L, \mu)$ *such that*

$$||au||^2\leqq(\hat{c}^2+C(a)\mu)||\langle D\rangle^m\langle\mu D\rangle^l u||^2+C(a, L, \mu)||u||^2_{-L}.$$

Proposition 3. 9. *Let* $a(X, \xi, \mu)\in J^0S^{m,l}$, $a(X, \xi, \mu)\geqq\hat{c}\langle\mu\xi\rangle^l\langle\xi\rangle^m$, *with* $\hat{c}>0$ *independent of* μ. *Then for any* $L\in\boldsymbol{N}$, *there is* $C(a, L, \mu)$ *such that*

$$\mathrm{Re}(au, u)\geqq(\hat{c}-C(a)\mu)||\langle\mu D\rangle^{l/2}\langle D\rangle^{m/2}u||^2-C(a, L, \mu)||u||^2_{-L}.$$

Proof. Set $B(X, D, \mu)=a(X, D, \mu)-\hat{c}\langle\mu D\rangle^l\langle D\rangle^m$. Then from proposition 3. 7, it follows that

$$B_F(X, \xi, \mu)=B(X, \xi, \mu)+C(X, \xi, \mu)\ \text{ with }\ C(X, \xi, \mu)\in J^0S^{m-1,l+1}_{1/2}.$$

If we write $C=\langle\mu D\rangle^{l/2}\langle D\rangle^{m/2}\tilde{C}\langle\mu D\rangle^{l/2}\langle D\rangle^{m/2}$with $\tilde{C}\in J^0S^{-1,1}_{1/2}$, according to (3. 2), we see that $\tilde{C}$ belongs to $\mu S^{0,0}_{1/2}$ modulo $S^{-\infty}(\mu)$. Therefore from Calderón-Vaillancourt theorem, it follows that

$$|(\tilde{C}\langle D\rangle^{m/2}\langle\mu D\rangle^{l/2}u, \langle D\rangle^{m/2}\langle\mu D\rangle^{l/2}u)\ |\leqq C(a)\mu||\langle D\rangle^{m/2}\langle\mu D\rangle^{l/2}||^2$$
$$+C(a, L, \mu)||u||^2_{-L}.$$

This proves the proposition.

We shall make some observations on "elliptic" symbol in $J^rS^{m,l}$, that is a symbol $a(X, \xi, \mu)\in J^rS^{m,l}$ satisfying

(3. 5) $\quad a(X, \xi, \mu)\geqq cJ(X, \xi, \mu)^r\langle\mu\xi\rangle^l\langle\xi\rangle^m$, with some positive c independent of μ.

Let $b(X, \xi, \mu)=a(X, \xi, \mu)^{-1}$, then from proposition 3. 5, it follows that

$$a(X, D, \mu)b(X, D, \mu)=1+r(X, D, \mu),\ r(X, \xi, \mu)\in J^0S^{-1,1}.$$

In virtue of (3. 2), we know that $r\in\mu S^{0,0}_{1/2}$ modulo $S^{-\infty}(\mu)$. From [9],

$$q=1+\sum_{j=1}^{\infty}(r(X, D, \mu))^j$$

defines a pseudodifferential operator in $J^0S^{0,0}_{1/2}$ and hence
$a(X, D, \mu)b(X, D, \mu)q(X, D, \mu)\equiv 1$. Here we note that $b(X, D, \mu)\times q(X, D, \mu)$ belongs to $J^{-r}S^{-m,-l}_{1/2}$. By the same way, we can construct a left parametrix.

Proposition 3. 10. *Let* $a(X, \xi, \mu)\in J^rS^{m,l}$ *and satisfy* (3. 5). *Then there exists a parametrix (left and right)* $b(X, \xi, \mu)$ *of* $a(X, \xi, \mu)$ *belonging to* $J^{-r}S^{-m,-l}_{1/2}$.

Proposition 3. 11. *Assume that* $A_i(X, \xi, \mu)\in J^{r_i}S^{m_i,l_i}(i=1, 2)$ *is elliptic in the sense of* (3. 5) *and* $B(X, \xi, \mu)\in J^rS^{m,l}$. *Then one can write*

$$B \equiv A_1(C+R)A_2 \textit{ with } R \in \mu J^{r-(r_1+r_2)} S_{1/2}^{m-(m_1+m_2), l-(l_1+l_2)},$$

where $C(X, \xi, \mu) = B(X, \xi, \mu) A_1(X, \xi, \mu)^{-1} A_2(X, \xi, \mu)^{-1}$.

Proof. With the above choice of $C(X, \xi, \mu)$, it follows that

$$A_1 C A_2 = B + r, \text{ with } r \in J^r S^{m-1, l+1} \subset \mu J^r S^{m,l}.$$

Whereas, using the parametrices $\tilde{A}_i$ of A_i constructed in the above, one can write $r \equiv A_1(\tilde{A}_1 r \tilde{A}_2) A_2$. From proposition 3.10, $\tilde{A}_i$ belongs to $J^{-r_i} S_{1/2}^{-m_i, -l_i}$ and then $R = \tilde{A}_1 r \tilde{A}_2$ belongs to the desired class.

Denote $E(c) = \{(X, \xi);\ |Y(X, \xi, \mu)| \langle \mu\xi \rangle^{1/2} \leqq c\}$. Note that if $a(X, \xi, \mu) \in J^r S^{m,l}$ and $\mathrm{supp}[a] \subset E(c)$, then it is clear that

$$a(X, \xi, \mu) \in J^{r+2q} S^{m, l+q}, \text{ for any } q \in \boldsymbol{R}. \tag{3.6}$$

Let $a_i(X, \xi, \mu) \in J^{r_i} S^{m_i, l_i}$ $(i=1, 2)$ and assume that $a_1(X, \xi, \mu)$ satisfies the following estimates

$$|a_{1(\alpha)}^{(\gamma)}(X, \xi, \mu)| \leqq C_{\alpha,\gamma} J(X, \xi, \mu)^{r_1 - |\alpha+\gamma|} \langle \mu\xi \rangle^{l_1} \langle \xi \rangle^{m_1 - |\gamma|}, \tag{3.7}$$

for any multi-index $\alpha \in \boldsymbol{N}^{d+1}$, $\gamma \in \boldsymbol{N}^d$ with $C_{\alpha,\gamma}$ independent of $\mu (0 < \mu \leqq \mu_1)$. If $\mathrm{supp}[a_1] \cap \mathrm{supp}[a_2]$ is contained in $E(c)$, then the above arguments show that $a_1^{(\alpha)}(X, \xi, \mu) a_{2(\alpha)}(X, \xi, \mu)$ belongs to $J^{r_1+r_2-N} S^{m_1+m_2, l_1+l_2-N/2}$ for any $N \in \boldsymbol{N}$. On the other hand, from proposition 3.5, it follows that

$$a_1(X, D, \mu) a_2(X, D, \mu) - \mathrm{op}\Big\{\sum_{|\alpha|<N} \frac{1}{\alpha!} a_1^{(\alpha)} a_{2(\alpha)}\Big\} \in J^{r_1+r_2-N} S^{m_1+m_2, l_1+l_2-N/2}.$$

Consequently, we have $a_1(X, D, \mu) a_2(X, D, \mu) \in J^{r_1+r_2-N} S^{m_1+m_2, l_1+l_2-N/2}$.

Proposition 3.12. *Let* $a_i(X, \xi, \mu) \in J^{r_i} S^{m_i, l_i}$ $(i=1, 2)$ *and assume that* $a_1(X, \xi, \mu)$ *satisfies the estimates* (3.7). *If* $\mathrm{supp}[a_1] \cap \mathrm{supp}[a_2] \subset E(c)$, *then*

$$a_1(X, D, \mu) a_2(X, D, \mu),\ a_2(X, D, \mu) a_1(X, D, \mu) \in J^{r_1+r_2-N} S^{m_1+m_2, l_1+l_2-N/2},$$

for any $N \in \boldsymbol{N}$.

4. Energy integral.

We start with the following identity.

$$\begin{aligned} -2\,\mathrm{Im}\int &(I_n(n-1/2)(D_0 - i\theta)^2 w,\ I_n(n-1/2)(D_0 - i\theta) w)\, dx_0 \\ &= 2\theta \int \|I_n(n-1/2)(D_0 - i\theta) w\|^2\, dx_0 \\ &\quad - 2\mathrm{Re}\int (\partial_0 I_n(n-1/2)(D_0 - i\theta) w,\ I_n(n-1/2)(D_0 - i\theta) w)\, dx_0 \end{aligned} \tag{4.1}$$

where $I_n(m) = \mathrm{op}(\langle \mu\xi \rangle^n J_-(X, \xi, \mu)^m)$ and $\partial_0 I_n(n-1/2)$ denotes the operator with symbol $\langle\mu\xi\rangle^n\partial_0 J_-(X,\ \xi,\ \mu)^{n-1/2}$. Since

$$\operatorname{supp}[\partial_0 J_-(X,\xi,\mu)^{n-1/2}+(n-1/2)J_-(X,\xi,\mu)^{n-3/2}]\cap\operatorname{supp}[\alpha_n^-]=\varnothing,$$

for $n\geqq 16$ and α_n^- satisfies the estimates (3.3), proposition 3.12 shows that

$$\{\partial_0 I_n(n-1/2)\\ +(n-1/2)I_n(n-3/2)\}\alpha_n^-\equiv I_n(n-1)^*a_1I_n(n-1)\equiv I_n(n-1)^*a_2J_+(-n-1),$$

with $a_i\in\mu S^{0,0}_{1/2}$. On the other hand, proposition 3.11 gives that

$$I_n(n-1/2)^*I_n(n-3/2)\equiv I_n(n-1)^*(1+a_3)I_n(n-1),\ \text{with}\ a_3\in\mu S^{0,0}_{1/2}.$$

Then, taking into account that $\alpha_n^++\alpha_n^-=1$, we have

$$\begin{aligned}(4.2)\quad & I_n(n-1/2)^*\partial_0 I_n(n-1/2)\alpha_n^-\equiv-(n-1/2)I_n(n-1)^*I_n(n-1)\alpha_n^-\\ &+I_n(n-1)^*aI_n(n-1)\alpha_n^-+I_n(n-1)^*bJ_+(-n-1)\alpha_n^+,\ \text{with}\ a,b\in\mu S^{0,0}_{1/2}.\end{aligned}$$

The same reasoning gives that

$$\begin{aligned}(4.3)\quad & I_n(n-1/2)^*\partial_0 I_n(n-1/2)D_0\alpha_n^-+(n-1/2)I_n(n-1)^*I_n(n-1)D_0\alpha_n^-\\ &\equiv I_n(n-1)^*\tilde{a}I_n(n-2)\alpha_n^-+I_n(n-1)^*\tilde{b}J_+(-n-2)\alpha_n^+,\ \text{with}\ \tilde{a},\tilde{b}\in\mu S^{0,0}_{1/2}.\end{aligned}$$

From (4.2) and (4.3), the right hand side of (4.1) with $w=\alpha_n^-u$ is estimated from below by

$$\begin{aligned}(4.4)\quad & 2\theta\int\|I_n(n-1/2)(D_0-i\theta)\alpha_n^-u\|^2dx_0\\ &+(2n-1-C(n)\mu)\int\|I_n(n-1)(D_0-i\theta)\alpha_n^-u\|^2dx_0\\ &-C(n)\mu\int|||(D_0-i\theta)u|||^2_{n,1}dx_0-C(n)\mu\int|||u|||^2_{n,2}dx_0,\end{aligned}$$

modulo $C(n,\mu,L)\int|u|^2_{-L}dx_0$ with $|u|^2_{-L}=\{\|u\|^2_{-L}+\|(D_0-i\theta)u\|^2_{-L}\}$.

Next consider

$$\begin{aligned}-2\,\mathrm{Im}\int(I_n(n-3/2)(D_0-i\theta)w^-,\ I_n(n-3/2)w^-)dx_0&=2\theta\int\|I_n(n-3/2)w^-\|^2dx_0\\ &-2\mathrm{Re}\int(\partial_0 I_n(n-3/2)w^-,\ I_n(n-3/2)w^-)dx_0,\end{aligned}$$

where $w^-=\alpha_n^-u$. From the same reasoning obtaining (4.4), this is estimated from below by

$$\begin{aligned}(4.5)\quad & (2n-3-C(n)\mu)\int\|I_n(n-2)\alpha_n^-u\|^2dx_0\\ &+2\theta\int\|I_n(n-3/2)\alpha_n^-u\|^2dx_0-C(n)\mu\int|||u|||^2_{n,2}dx_0,\end{aligned}$$

modulo $C(n,\mu,L)\int\|u\|^2_{-L}dx_0$. On the other hand, remarking the following expression,

$$I_n(n-3/2)^*I_n(n-3/2)\equiv I_n(n-2)^*(1+a)I_n(n-1),\ \text{with}\ a\in\mu S^{0,0}_{1/2},$$

it follows that

$$\begin{aligned}&2|(I_n(n-3/2)(D_0-i\theta)\alpha_n^-u,\ I_n(n-3/2)\alpha_n^-u)|\\&\quad\leqq n^{-1}(1+C(n)\mu)||I_n(n-1)(D_0-i\theta)\alpha_n^-u||^2\\&\quad+n(1+C(n)\mu)||I_n(n-2)\alpha_n^-u||^2+\theta^{-1/2}C(n,\mu,L)||(D_0-i\theta)u||^2_{-L}\\&\quad+\theta^{1/2}C(n,\mu,L)||u||^2_{-L}.\end{aligned}$$

Then this inequality and (4. 5) imply that

$$\begin{aligned}(4.6)\quad&(1+C(n)\mu)\int||I_n(n-1)(D_0-i\theta)\alpha_n^-u||^2dx_0\\&\geqq 2\theta n\int||I_n(n-3/2)\alpha_n^-u||^2dx_0\\&+n(n-3-C(n)\mu)\int||I_n(n-2)\alpha_n^-u||^2dx_0-C(n)\mu\int|||u|||^2_{n,2}dx_0,\end{aligned}$$

modulo $C(n,\mu,L)\int|u|^2_{-L,\theta}dx_0$, with $|u|^2_{-L,\theta}=\theta^{-1/2}||(D_0-i\theta)u||^2_{-L}+\theta^{1/2}||u||^2_{-L}$.

Similarly, it follows that

$$\begin{aligned}(4.7)\quad&(1+C(n)\mu)\theta\int||I_n(n-1/2)(D_0-i\theta)\alpha_n^-u||^2dx_0\\&\geqq 2\theta^2n\int||I_n(n-1)\alpha_n^-u||^2dx_0+n(n-2-C(n)\mu)\theta\int||I_n(n-3/2)\alpha_n^-u||^2dx_0\\&-C(n)\mu\theta\int|||u|||^2_{n,3/2}dx_0,\end{aligned}$$

modulo $C(n,\mu,L)\theta\int|u|^2_{-L,\theta}dx_0$.

From (4. 4), (4. 6) and (4. 7), we have

Proposition 4. 1.

$$\begin{aligned}&-2\,\mathrm{Im}\int(I_n(n-1/2)(D_0-i\theta)^2\alpha_n^-u,\ I_n(n-1/2)(D_0-i\theta)\alpha_n^-u)\,dx_0\\&\geqq(2n-1)(1-\delta-C(n)\mu)\int||I_n(n-1)(D_0-i\theta)\alpha_n^-u||^2dx_0\\&+c\theta\int||I_n(n-1/2)(D_0-i\theta)\alpha_n^-u||^2dx_0+\delta cn^3\int||I_n(n-2)\alpha_n^-u||^2dx_0\\&+c\theta n^2\int||I_n(n-3/2)\alpha_n^-u||^2dx_0+c\theta^2n\int||I_n(n-1)\alpha_n^-u||^2dx_0-C(n)\mu\int|||u|||^2_{n,2}dx_0\\&-C(n)\mu\int|||(D_0-i\theta)u|||^2_{n,1}dx_0-C(n)\mu\theta\int|||u|||^2_{n,3/2}dx_0,\end{aligned}$$

modulo $C(n,\mu,L)\int\theta|u|^2_{-L,\theta}dx_0$ *where* $0<\mu\leqq\mu_0(n)$.

Replacing $I_n(n-k)$, α_n^- by $J_+(-n-k)=\mathrm{op}(J_+(x,\xi,\mu)^{-n-k})$ $(k=1/2,\ 1,$

3/2, 2), α_n^+ one can proceed exactly in the same way as in the preceding arguments and this procedure gives the inequality of proposition 4.1 where $I_n(n-k)$, α_n^- must be replaced by $J_+(-n-k)$, α_n^+. To sum up, we introduce

$$\Phi_n^-(w,v) = -2\ \mathrm{Im}(I_n(n-1/2)w, I_n(n-1/2)v),$$
$$\Phi_n^+(w,v) = -2\ \mathrm{Im}(J_+(-n-1/2)w, J_+(-n-1/2)v),$$
$$\|u\|_{D,n,k}^2 = \|I_n(n-k)(D_0-i\theta)\alpha_n^-u\|^2 + \|J_+(-n-k)(D_0-i\theta)\alpha_n^+u\|^2$$
$$(k=1, 1/2).$$

Then we have by combining the inequalities for α_n^-u and α_n^+u,

Proposition 4.2.

$$\sum\int\Phi_n^\pm((D_0-i\theta)^2\alpha_n^\pm u, (D_0-i\theta)\alpha_n^\pm u)\,dx_0 \geqq (2n-1)(1-\delta-C(n)\mu)\int\|u\|_{D,n,1}^2dx_0$$
$$+c\theta\int\|u\|_{D,n,1/2}^2dx_0 + \delta cn^3(1-C(n)\mu)\int|||u|||_{n,2}^2dx_0$$
$$+c\theta n^2(1-C(n)\mu)\int|||u|||_{n,3/2}^2dx_0 + c\theta^2 n\int|||u|||_{n,1}^2dx_0$$
$$-C(n)\mu\int|||(D_0-i\theta)u|||_{n,1}^2dx_0,$$

modulo $C(n,\mu,L)\theta\int|u|_{-L,\theta}^2dx_0$, *where* $0<\mu\leqq\mu_0(n)$.

5. Some basic estimates.

Let $B(\mu)$ be one of $\mu\phi q_p$, $\mu^2 Y_i^? q_i$, $0\leqq i\leqq p$, $\Gamma_i^? r_i$, $1\leqq i\leqq p$. From the arguments in section 2, we know that

$$(5.1)\qquad \partial_x^\alpha\partial_{\xi''}^\gamma B(\mu)\in(Es)_{1/2,1/2}^{2-|\gamma|,1}\ \text{for}\ |\alpha|+|\gamma|\leqq 2.$$

Following section 3, we may suppose that

$$(5.2)\qquad \partial_x^\alpha\partial_{\xi''}^\gamma B(\mu)\in J^0S^{1-|\gamma|,1}\ \text{if}\ |\alpha|+|\gamma|\leqq 2.$$

We begin with the following egality.

$$(5.3)\quad 2\ \mathrm{Im}\int(I_n(n-1/2)Bw, I_n(n-1/2)(D_0-i\theta)w)\,dx_0$$
$$=2\theta\mathrm{Re}\int(BI_n(n-1/2)w, I_n(n-1/2)w)\,dx_0$$
$$-\int\{(B\partial_0 I_n(n-1/2)w, I_n(n-1/2)w)$$
$$+(BI_n(n-1/2)w,\ \partial_0 I_n(n-1/2)w)\}\,dx_0$$
$$-\int((\partial_0 B)I_n(n-1/2)w, I_n(n-1/2)w)\,dx_0$$
$$+\mathrm{Im}\int((B-B^*)I_n(n-1/2)w, I_n(n-1/2)(D_0-i\theta)w)\,dx_0$$
$$+2\ \mathrm{Im}\int([I_n(n-1/2), B]w, I_n(n-1/2)(D_0-i\theta)w)\,dx_0$$

$$-\theta\int(I_n(n-1/2)w,\ (B-B^*)I_n(n-1/2)w)dx_0.$$

In the following, for the simplicity of notations, sometimes we write w_n^-, w_n^+ instead of $J_-(-1/2)I_n(n-1/2)\alpha_n^- u$, $J_+(-1/2)J_+(-n-1/2)\alpha_n^+ u$. We also denote

$$||\partial^{\pm}(n)Bu||^2=\sum_{j=p+1}^{d}||B^{(j)}w_n^{\pm}||^2+\sum_{j=1}^{d}||\langle D\rangle^{-1}B_{(j)}w_n^{\pm}||^2,$$

where $B^{(j)}$, $B_{(j)}$ denotes the operator with symbol $\partial_{\xi_j}B(\mu)$, $\frac{1}{i}\partial_{x_j}B(\mu)$.

We study the second term of the right hand side of (5. 3).

Proposition 5. 1. *Suppose that* $B=B(\mu)$ *satisfies* (5. 2) *and* $B-B^*\in J^0S^{0,1}$. *Then for any* $\delta>0$, *we have*

$$\begin{aligned}&-\mathrm{Re}\{(B\partial_0 I_n(n-1/2)w^-,\ I_n(n-1/2)w^-)+(BI_n(n-1/2)w^-,\ \partial_0 I_n(n-1/2)w^-)\}\geqq\\&\geqq(2n-1)\mathrm{Re}(Bw_n^-,w_n^-)-(\delta^{-1}+C(n)\mu)||\partial^-(n)Bu||^2-|(R_1Bw_n^-,w_n^-)|-\\&-|(R_2Bw_n^-,w_n^+)|-C(n)(\delta+\mu)|||u|||_{n,2}^2-C(n)\mu^{1/2}|||u|||_{n+1,0}^2,\end{aligned}$$

with $R_i\in\mu S_{1/2}^{0,0}$, *modulo* $C(n,\mu,L)||u||_{-L}^2$.

Proof. Since one can write

$$\begin{aligned}&\partial_0 I_n(n-1/2)\alpha_n^-\equiv-(n-1/2)J_-(-1/2)J_-(-1/2)I_n(n-1/2)\alpha_n^- -\\&\quad -J_-(-1/2)^*R_1J_-(-1/2)I_n(n-1/2)\alpha_n^- -\\&\quad -J_-(-1/2)^*R_2J_+(-1/2)J_+(-n-1/2)\alpha_n^+,\end{aligned}$$

$J_-(-1/2)^*I_n(n-1/2)\equiv(1+R_3)J_-(-1/2)I_n(n-1/2)$, with $R_i\in\mu S_{1/2}^{0,0}$, it follows that

$$\begin{aligned}(5.4)\quad &-(B\partial_0 I_n(n-1/2)w^-,\ I_n(n-1/2)w^-)\equiv(n-1/2)(Bw_n^-,\ (1+R_3)w_n^-)+\\&+(R_1w_n^-+R_2w_n^+,\ Bw_n^-)+(n-1/2)([B,J_-(-1/2)]w_n^-,\ I_n(n-1/2)w^-)\\&+(R_1w_n^-+R_2w_n^+,\ [J_-(-1/2),B]I_n(n-1/2)w^-)+\\&+(R_1w_n^-+R_2w_n^+,\ J_-(-1/2)(B-B^*)I_n(n-1/2)w^-).\end{aligned}$$

If we note that $J_-(X,\xi,\mu)$ does not depend on x', it follows from the assumption (5. 2) that

$$\begin{aligned}(5.5)\quad [J_-(-1/2),B]&\equiv\sum_{j=1}^{d}B_{(j)}\mathrm{op}(J_-^{-1/2(j)})-\sum_{j=p+1}^{d}B^{(j)}\mathrm{op}(J_{-(j)}^{-1/2})+rJ_-(-1/2)\equiv\\&\equiv\sum_{j=1}^{d}\mathrm{op}(J_-^{-1/2(j)})B_{(j)}-\sum_{j=p+1}^{d}\mathrm{op}(J_{-(j)}^{-1/2})B^{(j)}+\tilde{r}J_-(-1/2)\end{aligned}$$

with r, $\tilde{r}\in J^{-1}S_{1/2}^{-1/2,1}$. Writing
$\mathrm{op}(J_-^{-1/2(j)})\equiv L^jJ_-(-1/2)$, $\mathrm{op}(J_{-(j)}^{-1/2})\equiv L_jJ_-(-1/2)$, $L^j\in J^{-1}S_{1/2}^{-1,0}$, $L_j\in J^{-1}S_{1/2}^{0,0}$, and consider $[B_{(j)},L^j](1\leqq j\leqq d)$, $[B^{(j)},L_j](p+1\leqq j\leqq d)$.

Noting again that L^j, L_j does not depend on x', we see that these commutators belong to $J^{-1}S_{1/2}^{-1/2,1}$. Consequently we get

$$(5.6)\quad [J_-(-1/2), B] \equiv \sum_{j=1}^{d} L^j B_{(j)} J_-(-1/2) - \sum_{j=p+1}^{d} L_j B^{(j)} J_-(-1/2) + r J_-(-1/2),$$

with $r \in J^{-1} S_{1/2}^{-1/2,1}$. On the other hand, for any $r \in J^{-1} S_{1/2}^{-1/2,1}$, we have

$$\begin{aligned} &I_n(n-1/2)^* J_-(-1/2)^* r J_-(-1/2) I_n(n-1/2) \\ &\equiv I_n(n)^* \langle \mu D \rangle A \langle \mu D \rangle I_n(n), \ A \in \mu^{1/2} S_{1/2}^{0,0}, \\ &J_+(-n-1/2)^* J_+(-1/2)^* r J_-(-1/2) I_n(n-1/2) \\ &\equiv J_+(-n)^* \langle \mu D \rangle \tilde{A} \langle \mu D \rangle I_n(n), \ \tilde{A} \in \mu^{1/2} S_{1/2}^{0,0}. \end{aligned}$$

Also, the following expressions follow by the same arguments.

$$\begin{aligned} &(L^j \langle D \rangle)^* J_-(-1/2) I_n(n-1/2) \equiv A_1^j I_n(n-2), \\ &L_j^* J_-(-1/2) I_n(n-1/2) \equiv A_2^j I_n(n-2), \\ &\mathrm{op}(J_-^{-1/2(j)})^* I_n(n-1/2) \equiv \tilde{A}_1^j I_n(n-2), \\ &\mathrm{op}(J_{-(j)}^{-1/2})^* I_n(n-1/2) \equiv \tilde{A}_2^j I_n(n-2), \end{aligned}$$

with A_i^j, $\tilde{A}_i^j \in S_{1/2}^{0,0}$.

Then taking (5.5), (5.6) into account, and using the above expressions, the third and fourth term of (5.4) is estimated from below by

$$\begin{aligned} &-C(n) \sum_{j=1}^{d} \|\tilde{A}_1^j I_n(n-2) w^-\| \|\langle D \rangle^{-1} B_{(j)} w_n^-\| - C(n) \sum_{j=p+1}^{d} \|\tilde{A}_2^j I_n(n-2) w^-\| \|B^{(j)} w_n^-\| - \\ &-C(n) \mu \||u\||_{n,2}^2 - C(n) \mu \|\partial^-(n) Bu\|^2 - C(n) \mu^{1/2} \||u\||_{n+1,0}^2. \end{aligned}$$

In virtue of the fact that $R_i \in \mu S_{1/2}^{0,0}, B^* - B \in J^0 S^{0,1}$, it is easy to see that the last term of the right hand side of (5.4) is estimated from below by $-C(n) \mu \||u\||_{n,2}^2 - C(n) \mu \||u\||_{n+1,0}^2$.

Analogous arguments give the same estimates for $-(BI_n(n-1/2) w^-, \partial_0 I_n(n-1/2) w^-)$, and this completes the proof.

Next consider the term $([I_n(n-1/2), B]w, I_n(n-1/2)(D_0 - i\theta) w)$.

Proposition 5.2. *Suppose that $B = B(\mu)$ satisfies (5.2). Then for any $\delta > 0$, the following inequality is valid.*

$$\begin{aligned} &|([I_n(n-1/2), B] w^-, I_n(n-1/2)(D_0 - i\theta) w^-)| \leqq \delta^{-1} \|\partial^-(n) Bu\|^2 + \\ &+ C(n)(\delta + \mu^{1/2}) \|I_n(n-1)(D_0 - i\theta) w^-\|^2 + C(n) \mu^{1/2} \||u\||_{n+1,0}^2 \end{aligned}$$

modulo $C(n, \mu, L) |u|_{-L}^2$.

Proof. The same arguments deriving (5.5) gives that

$$\begin{aligned} [I_n(n-1/2), B] \equiv &\sum_{j=1}^{d} \mathrm{op}(\sigma(I_n(n-1/2)^{(j)}) B_{(j)} - \\ &- \sum_{j=p+1}^{d} \mathrm{op}(\sigma(I_n(n-1/2))_{(j)}) B^{(j)} + r, \end{aligned}$$

with $r \in J^{n-3/2} S^{-1/2, 1+n}$. Here we note that

$$I_n(n-1/2)^*\mathrm{op}(\sigma(I_n(n-1/2))^{(j)})$$
$$\equiv I_n(n-1)^*L^jJ_-(-1/2)I_n(n-1/2),\ L^j\in S^{-1,0}_{1/2},$$
$$I_n(n-1/2)^*\mathrm{op}(\sigma(I_n(n-1/2))_{(j)})$$
$$\equiv I_n(n-1)^*L_jJ_-(-1/2)I_n(n-1/2),\ L_j\in S^{0,0}_{1/2}.$$

Admitting the following proposition 5. 3, it follows that

$I_n(n-1/2)^*[I_n(n-1/2),\ B]\equiv\sum_{j=1}^{d}I_n(n-1)^*L^j\langle D\rangle(\langle D\rangle^{-1}B_{(j)})J_-(-1/2)I_n(n-1/2)-\sum_{j=p+1}^{d}I_n(n-1)^*L_jB^{(j)}J_-(-1/2)I_n(n-1/2)+r$, with $r\in J^{2n-1}S^{-1/2,2n+3/2}$. Whereas, for $r\in J^{2n-1}S^{-1/2,2n+3/2}$, we can write

$$r\equiv I_n(n-1)^*R\langle\mu D\rangle I_n(n),\ R\in\mu^{1/2}S^{0,0}_{1/2},$$

and this proves the proposition.

Proposition 5. 3. *Let* $B=B(\mu)$ *satisfy* (5. 2). *Then*

$$[J_-(-1/2)I_n(n-1/2),\ B_{(j)}]\in J^{n-1}S^{0,n+3/2}_{1/2},\ 1\leqq j\leqq d,$$
$$[J_-(-1/2)I_n(n-1/2),\ B^{(j)}]\in J^{n-1}S^{-1,n+3/2}_{1/2},\ p+1\leqq j\leqq d.$$

Proof. We show the second assertion. Write

$$J_-(-1/2)I_n(n-1/2)\equiv I_n(n-1)+T,\ \text{with}\ T\in J^{n-3}S^{-1,n}\subset J^{n-1}S^{-1,n+1},$$

where T does not depend on x'. Since $[T,\ B^{(j)}]\in J^{n-1}S^{-2,n+5/2}\subset\mu J^{n-1}S^{-1,n+3/2}$, it suffices to consider $[I_n(n-1),\ B^{(j)}]$. But it is also easy to see that

$$[I_n(n-1),\ B^{(j)}]\in J^{n-2}S^{-1,n+1}\subset J^{n-1}S^{-1,n+3/2},$$

and this completes the proof.

Proposition 5. 4. *Suppose that* $B=B(\mu)$ *is positively homogeneous of degree* 2 *in* ξ *for large* ξ, *non-negative and satisfies* (5. 1). *Then we have*

$$\|\partial^{\pm}(n)Bu\|^2\leqq C\mu\,\mathrm{Re}(Bw_n^{\pm},\ w_n^{\pm})+C(n)\mu^{1/2}|||u|||^2_{n+1,0}+$$
$$+C(n)\mu^{1/2}|||u|||^2_{n,2}+C(n,\ \mu)|||u|||^2_{n,1},$$

modulo $C(n,\mu,L)\|u\|^2_{-L}$.

Proof. From the assumption, it follows that

$$|B_{(j)}(X,\xi,\mu)\langle\xi\rangle^{-1}|^2\leqq C\mu B(X,\xi,\mu),\ 1\leqq j\leqq d,\ |B^{(j)}(X,\ \xi,\ \mu)|^2\leqq C\mu B(X,\ \xi,\ \mu),$$

$p+1\leqq j\leqq d$, with C independent of μ. The arguments in section 2 show that $B_{(j)}\langle D\rangle^{-2}B_{(j)}-\mathrm{op}(|\langle\xi\rangle^{-1}B_{(j)}|^2)\in(Es)^{1,2}_{1/2,1/2}$, modulo $S^{-\infty}(\mu)$, hence it belongs to $\mu J^0S^{0,1}$. On the other hand, we know that $B^*_{(j)}-B_{(j)}$ belongs to $(Es)^{1,1/2}_{1/2,1/2}$, modulo $S^{-\infty}(\mu)$. Therefore it follows that

$$B^*_{(j)}\langle D\rangle^{-2}B_{(j)}\equiv\mathrm{op}(|\langle\xi\rangle^{-1}B_{(j)}|^2)+r_1,\ r_1\in\mu^{1/2}J^0S^{0,1},\ 1\leqq j\leqq d.$$

Similarly, we get

$$B^{(j)*}B^{(j)}\equiv \mathrm{op}(|B^{(j)}|^2)+r_2,\ r_2\in\mu^{1/2}S^{0,1},\ p+1\leqq j\leqq d.$$

Then, by Melin's inequality [10], one has

$$(5.7)\quad \begin{aligned} &C\mu\mathrm{Re}(Bv,v)+\nu\|\langle\mu D\rangle^{1/2}v\|^2+C(\nu,\mu)\|v\|^2\geqq\|\langle D\rangle^{-1}B_{(j)}v\|^2-|(r_1v,v)|,\\ &C\mu\mathrm{Re}(Bv,v)+\nu\|\langle\mu D\rangle^{1/2}v\|^2+C(\nu,\mu)\|v\|^2\geqq\|B^{(j)}v\|^2-|(r_2v,v)|,\end{aligned}$$

for any positive ν. Here note the following expressions.

$$\begin{aligned} &I_n(n-1/2)^*J_-(-1/2)^*\langle\mu D\rangle J_-(-1/2)I_n(n-1/2)\\ &\quad\equiv I_n(n-2)^*A_1\langle\mu D\rangle I_n(n),\ A_1\in S^{0,0}_{1/2},\\ &\quad J_-(-1/2)I_n(n-1/2)\equiv A_2I_n(n-1),\ A_2\in S^{0,0}_{1/2},\\ &\quad I_n(n-1/2)^*J_-(-1/2)^*rJ_-(-1/2)I_n(n-1/2)\\ &\quad\equiv I_n(n-2)^*\tilde{A}\langle\mu D\rangle I_n(n),\ \tilde{A}\in\mu^{1/2}S^{0,0}_{1/2},\end{aligned}$$

for $r\in\mu^{1/2}J^0S^{0,1}$. In (5.7), taking $\nu=\mu^{1/2}$, $v=w_n^-$, and using the above expressions, (5.7) yields the desired inequalities.

Proposition 5.5. *Let $B=B(\mu)$ be as in proposition 5.4. We assume that $B-B^*\in J^0S^{0,1}$ and $A=A(\mu)\in\mu S^{0,0}_{1/2}$. Then for any $\varepsilon>0$, we have*

$$\begin{aligned}\textstyle\sum^{\pm}|(ABw_n^{\pm},w_n^{\pm})|\leqq&(c\varepsilon+C(n)\mu^{3/2})\mathrm{Re}\textstyle\sum(Bw_n^{\pm},w_n^{\pm})+(c\varepsilon+C(n)\mu^{1/2})|||u|||^2_{n+1,0}+\\ &+(c\varepsilon+C(n)\mu)|||u|||^2_{n,2}+C(n,\mu,\varepsilon)|||u|||^2_{n,1},\end{aligned}$$

for $0<\mu\leqq\mu_0(n,\varepsilon)$, modulo $C(n,\mu,L)\|u\|^2_{-L}$, where $\sum^{\pm}$ denotes that the sum is taken over all combinations of $(\pm,\pm)$.

Proof. Since $B-B^*\in J^0S^{0,1}$, $A^*\in\mu S^{0,0}_{1/2}$, it suffices to estimate $(BAw_n^{\pm}, w_n^{\pm})$.
For any given $\varepsilon>0$, we set

$$X(\mu)=\varepsilon\pm(A+A^*)/2.$$

Then for $0<\mu\leqq\mu_0(\varepsilon)$, there exists $X(\mu)^{1/2}\in S^{0,0}_{1/2}$ which is defined by

$$X(\mu)^{1/2}(X,\xi,\mu)=\frac{1}{2\pi i}\int_{\Gamma}\zeta^{1/2}(\zeta-X(\mu))^{-1}d\zeta,$$

where Γ is a contour containing the spectrum of $X(\mu)$ and lies in $\mathrm{Im}\,\zeta>0$. From [9], it follows that

$$(\zeta-X(\mu))^{-1}\in J^0\mathrm{S}^{0,0}_{1/2},\ \partial_x^\alpha\partial_\xi^\gamma(\zeta-X(\mu))^{-1}\in\mu J^0S^{(|\alpha|-|\gamma|)/2,0}_{1/2}\ \text{if}\ |\alpha+\gamma|\geqq1.$$

This implies that

$$(5.8)\quad \partial_x^\alpha\partial_\xi^\gamma X(\mu)^{1/2}(X,\xi,\mu)\in\mu J^0S^{(|\alpha|-|\gamma|)/2,0}_{1/2}\ \text{for}\ |\alpha+\gamma|\geqq1.$$

Since $(X(\mu)^{1/2})^*=X(\mu)^{1/2}$, we get

$$(5.9)\quad \begin{aligned}\mathrm{Re}(BX^{1/2}u,X^{1/2}u)=&\varepsilon\,\mathrm{Re}(Bu,u)\pm\mathrm{Re}(BAu,u)\pm2^{-1}\mathrm{Re}([A,B]u,u)-\\ &-\mathrm{Re}(Xu,(B-B^*)u)-\end{aligned}$$

$$-\mathrm{Re}(X^{1/2}u, [X^{1/2}, B]u) \pm 2^{-1}\mathrm{Re}(u, A(B^* - B)u).$$

Similarly, if we set $Y(\mu) = i\varepsilon \pm (A - A^*)/2$, then it follows that $Y(\mu)^{1/2} \in J^0 S^{0,0}_{1/2}$ and $\partial_x^\alpha \partial_\xi^\gamma Y(\mu)^{1/2}(X, \xi, \mu) \in \mu J^0 S^{(|\alpha|-|\gamma|)/2,0}_{1/2}$ for $|\alpha+\gamma| \geqq 1$. Taking into account that $(Y^{1/2})^* = -iY^{1/2}$, the same arguments give that

$$\text{(5.10)} \quad \begin{aligned} \mathrm{Re}(BY^{1/2}u, Y^{1/2}u) &= \varepsilon\, \mathrm{Re}(Bu, u) \pm \mathrm{Im}(BAu, u) \pm \\ &\pm 2^{-1}\,\mathrm{Im}([A, B]u, u) - \mathrm{Im}(Yu, (B^* - B)u) - \\ &- \mathrm{Re}(Y^{1/2}u, [Y^{1/2}, B]u) \pm 2^{-1}\mathrm{Im}(A(B^* - B)u, u). \end{aligned}$$

On the other hand, Melin's inequality shows that

$$\text{(5.11)} \quad \begin{aligned} \mathrm{Re}(BX^{1/2}u, X^{1/2}u) + \mathrm{Re}(BY^{1/2}u, Y^{1/2}u) &\geqq -\nu\{\|\langle\mu D\rangle^{1/2}X^{1/2}u\|^2 + \\ &+ \|\langle\mu D\rangle^{1/2}Y^{1/2}u\|^2\} - C(\nu, \mu)\{\|X^{1/2}u\|^2 + \|Y^{1/2}u\|^2\}. \end{aligned}$$

Using the expressions

$$\langle\mu D\rangle^{1/2}X^{1/2} \equiv A\langle\mu D\rangle^{1/2},\ \langle\mu D\rangle^{1/2}Y^{1/2} \equiv \tilde{A}\langle\mu D\rangle^{1/2},\ A, \tilde{A} \in S^{0,0}_{1/2},$$

it follows from (5.9), (5.10) and (5.11) that

$$\text{(5.12)} \quad \begin{aligned} 2\varepsilon\mathrm{Re}(Bu, u) &\geqq |(BAu, u)| - |([A, B]u, u)| - |(X^{1/2}u, [X^{1/2}, B]u)| - \\ &- |(Xu, (B^* - B)u)| - |(Yu, (B^* - B)u)| - |(A(B^* - B)u, u)| - \\ &- |(Y^{1/2}u, [Y^{1/2}, B]u)| - \nu C(\varepsilon)\|\langle\mu D\rangle^{1/2}u\|^2 - C(\nu, \mu)\|u\|^2. \end{aligned}$$

From the hypotheses on B and (5.8), the following estimate is easily verified,

$$\text{(5.13)} \quad \begin{aligned} &\textstyle\sum^{\pm} |(Xw_n^{\pm}, (B^* - B)w_n^{\pm})| + \sum^{\pm} |(Yw_n^{\pm}, (B^* - B)w_n^{\pm})| + \\ &+ \textstyle\sum^{\pm} |(A(B^* - B)w_n^{\pm}, w_n^{\pm})| \leqq (c\varepsilon + C(n)\mu)\{|||u|||^2_{n+1,0} + \\ &+ |||u|||^2_{n,2}\},\ \text{modulo } C(n, \mu, L)\|u\|^2_{-L}. \end{aligned}$$

Next we shall estimate $(X^{1/2}u, [X^{1/2}, B]u)$. Here we remark that $X^{1/2}$ does not depend on x'. From the expression

$$[X^{1/2}, B] \equiv \sum_{j=1}^{d} A_1^j\langle D\rangle^{-1}B_{(j)} - \sum_{j=p+1}^{d} A_2^j B^{(j)} + r,\ A_i^j \in \mu^{1/2}S^{0,1/2}_{1/2},\ r \in \mu S^{0,1}_{1/2},$$

we see that

$$\begin{aligned} &I_n(n-1/2)^* J_-(-1/2)^*[X^{1/2}, B]J_-(-1/2)I_n(n-1/2) \equiv \\ &\equiv I_n(n)^*\langle\mu D\rangle \sum_{j=1}^{d} \tilde{A}_1^j\langle D\rangle^{-1}B_{(j)}J_-(-1/2)I_n(n-1/2) - \\ &I_n(n)^*\langle\mu D\rangle \sum_{j=p+1}^{d} \tilde{A}_2^j B^{(j)}J_-(-1/2)I_n(n-1/2) + I_n(n)^*\langle\mu D\rangle A I_n(n-2), \end{aligned}$$

with $\tilde{A}_i^j \in \mu^{1/2}S^{0,0}_{1/2}$, $A \in \mu S^{0,0}_{1/2}$. Hence one can easily estimate $\sum^{\pm}(X^{1/2}w_n^{\pm}, [X^{1/2}, B]w_n^{\pm})$ by

$$C(n)\mu^{1/2}\sum\|\partial^{\pm}(n)Bu\|^2 + C(n)\mu^{1/2}|||u|||^2_{n+1,0} + C(n)\mu|||u|||^2_{n,2}.$$

Applying proposition 5.4 to $\|\partial^{\pm}(n)Bu\|^2$, this is also estimated by

$$C(n)\mu^{3/2}\mathrm{Re}\ \sum(Bw_n^{\pm}, w_n^{\pm})+C(n)\mu^{1/2}|||u|||_{n+1,0}^2 +C(n)\mu|||u|||_{n,2}^2+C(n,\mu)|||u|||_{n,1}^2. \tag{5.14}$$

It is clear that the same estimate holds for $\sum^{\pm}|(Y^{1/2}w_n^{\pm}, [Y^{1/2}, B]w_n^{\pm})|$ and $\sum^{\pm}|([A, B]w_n^{\pm}, w_n^{\pm})|$.

Then, in virtue of the identity,
$4(BAu, v)=(BA(u+v), (u+v))-(BA(u-v), (u-v))+i(BA(u+iv), (u+iv))-i(BA(u-iv), (u-iv))$, the estimate (5.12) with $\nu=\mu$ yields the required inequality in view of (5.13) and (5.14).

6. Estimates of commutators with $\alpha_n^{\pm}$.

We start with the identity,

$$[(D_0-i\theta)^2-B]\alpha_n^{\pm}=\alpha_n^{\pm}[(D_0-i\theta)^2-B]+2D_0\alpha_n^{\pm}(D_0-i\theta)+D_0^2\alpha_n^{\pm}+[\alpha_n^{\pm}, B].$$

We need to estimate the last three terms of the right hand side of the above equality. From proposition 3.12, we know that

$$I_n(n-1/2)^*I_n(n-1/2)D_0^2\alpha_n^-\in J^{2n-1-N}S^{0,2n+1-N/2},\ \text{for any } N\in\boldsymbol{N},$$

and hence one can write

$$I_n(n-1/2)^*I_n(n-1/2)D_0^2\alpha_n^-\equiv nI_n(n-1)^*(A+b)I_n(n-2)\equiv \equiv nI_n(n-1)^*(\tilde{A}+\tilde{b})J_+(-n-2), \tag{6.1}$$

with b, $\tilde{b}\in\mu S_{1/2}^{0,0}$, where

$$A(X,\xi,\mu)=J_-(X,\xi,\mu)^2\langle\mu\xi\rangle\chi^{(2)}(-n^{1/2}Y(X,\xi,\mu)\langle\mu\xi\rangle^{1/2}),$$
$$\tilde{A}(X,\xi,\mu)=(\langle\mu\xi\rangle J_-(X,\xi,\mu)J_+(X,\xi,\mu))^nJ_+(X,\xi,\mu)^2\langle\mu\xi\rangle\times \times\chi^{(2)}(-n^{1/2}Y(X,\xi,\mu)\langle\mu\xi\rangle^{1/2}). \tag{6.2}$$

To handle A, $\tilde{A}$, we prepare the following proposition.

Proposition 6.1. *Let $n\geqq 16$. Then there exist positive constants c_i independent of n such that the inequalities*

$$c_1\leqq\{\langle\mu\xi\rangle J_+(X,\xi,\mu)J_-(X,\xi,\mu)\}^{\pm n}\leqq c_2,$$

hold on $\{(X,\xi);\ |Y(X,\xi,\mu)|\langle\mu\xi\rangle^{1/2}\leqq n^{-1/2}\}$.

Proof. When $|Y(X,\xi,\mu)|\langle\mu\xi\rangle^{1/2}\leqq 4^{-1}$, we know from the definitions that $J_+(X,\xi,\mu)J_-(X,\xi,\mu)=\langle\mu\xi\rangle^{-1}-Y(X,\xi,\mu)^2$. Then the inequalities

$$c_1\leqq(1-1/n)^{\pm n}\leqq c_2,\ \text{with positive } c_i \text{ independent of } n,$$

prove this proposition.

From the above proposition, it follows that

$$|A(X,\xi,\mu)|,\ |\tilde{A}(X,\xi,\mu)|\leqq C,\ \text{with } C \text{ independent of } n \text{ and } \mu.$$

Therefore, applying proposition 3. 8 to A, $\tilde{A}$, it follows from (6. 1) that

$$|(I_n(n-1/2)D_0^2\alpha_n^-u, I_n(n-1/2)(D_0-i\theta)\alpha_n^-u)| \leqq \\ \leqq ||I_n(n-1)(D_0-i\theta)\alpha_n^-u||^2+n^2(c+C(n)\mu)|||u|||_{n,2}^2.$$

By exactly the same way, we have

$$|\Phi_n^+(D_0^2\alpha_n^+u, (D_0-i\theta)\alpha_n^+u)| \leqq ||J_+(-n-1)(D_0-i\theta)\alpha_n^+u||^2+ \\ +n^2(c+C(n)\mu)|||u|||_{n,2}^2.$$

Summing up, we get

Proposition 6. 2.

$$\sum |\Phi_n^\pm(D_0^2\alpha_n^\pm u, (D_0-i\theta)\alpha_n^\pm u)| \leqq ||u||_{D,n,1}^2+n^2(c+C(n)\mu)|||u|||_{n,2}^2, \\ \sum |\Phi_n^\pm(D_0\alpha_n^\pm(D_0-i\theta)u, (D_0-i\theta)\alpha_n^\pm u)| \\ \leqq n^{1/2}||u||_{D,n,1}^2+n^{1/2}(c+C(n)\mu)|||(D_0-i\theta)u|||_{n,1}^2,$$

modulo $C(n, \mu, L)|u|_{-L}^2$.

Proposition 6. 3. *Assume that* $B=B(\mu)$ *satisfies the conditions in proposition* 5. 4. *Then we have*

$$\sum |\Phi_n^\pm([\alpha_n^\pm, B]u, (D_0-i\theta)\alpha_n^\pm u)| \leqq C\mu^{1/4}\mathrm{Re}\sum(Bw_n^\pm, w_n^\pm)+C(n)\mu^{1/4}|||u|||_{n+1,0}^2+ \\ +C(n)\mu^{1/4}|||u|||_{n,2}^2+C(n)\mu^{1/4}||u||_{D,n,1}^2+C(n,\mu)|||u|||_{n,1}^2,$$

modulo $C(n, \mu, L)|u|_{-L}^2$.

Proof. Since the arguments of deriving the estimates for $\Phi_n^-(\cdots)$, $\Phi_n^+(\cdots)$ are parallel each other, we shall estimate only $\Phi_n^-(\cdots)$. Consider $I_n(n-1/2)^*I_n(n-1/2)[\alpha_n^-, B]$. Remarking that α_n^- does not depend on x', we see that

$$[\alpha_n^-, B] \equiv \sum_{j=1}^{d} \alpha_n^{-(j)}B_{(j)} - \sum_{j=p+1}^{d} \alpha_{n(j)}^-B^{(j)}+r, \quad r\in J^0S^{-1,2}. \tag{6.3}$$

In addition, the fact that $r\in J^{-N}S^{-1,2-N/2}$ for any $N\in \boldsymbol{N}$ follows from proposition 3. 12. Taking (3. 6) into account, we get

$$I_n(n-1/2)^*I_n(n-1/2)\alpha_{n(j)}^- \equiv I_n(n-1)^*L_1^jJ_-(-1/2)I_n(n-1/2) \equiv \\ \equiv I_n(n-1)^*L_2^jJ_+(-1/2)J_+(-n-1/2),$$

with $L_i^j\in S_{1/2}^{0,0}$. Here using proposition 5. 3, one can write

$$I_n(n-1/2)^*I_n(n-1/2)\alpha_{n(j)}^-B^{(j)} \equiv I_n(n-1)^*L_1^jB^{(j)}J_-(-1/2)I_n(n-1/2)+ \\ +I_n(n-1)^*B_1^j\langle\mu D\rangle I_n(n) \\ \equiv I_n(n-1)^*L_2^jB^{(j)}J_+(-1/2)J_+(-n-1/2) \\ +I_n(n-1)^*B_2^j\langle\mu D\rangle J_+(-n),$$

with $B_i^j\in \mu S_{1/2}^{0,0}$. Thus, from $\alpha_n^-+\alpha_n^+=1$, we have

$$|\Phi_n^-(\alpha_{n(j)}^- B^{(j)}u, (D_0-i\theta)\alpha_n^- u)| \leqq \delta^{-1}\sum\|\partial^{\pm}(n)Bu\|^2 + \\ +C(n)(\delta+\mu)\|u\|_{D,n,1}^2 + C(n)\mu\|\|u\|\|_{n+1,0}^2,$$

modulo $C(n,\mu,L)|u|_{-L}^2$, for any $\delta>0$. If we take $\delta=\mu^{1/4}$ in the above inequality, by virtue of proposition 5. 4, the right hand side of the above inequality is estimated by

$$C\mu^{1/4}\sum \mathrm{Re}(Bw_n^{\pm}, w_n^{\pm}) + C(n)\mu^{1/4}\|\|u\|\|_{n+1,0}^2 + C(n)\mu^{1/4}\|\|u\|\|_{n,2}^2 + \\ +C(n)\mu^{1/4}\|u\|_{D,n,1}^2 + C(n,\mu)\|\|u\|\|_{n,1}^2. \tag{6. 4}$$

The same arguments show that $|\Phi_n^-(\alpha_n^{-(j)}B_{(j)}u, (D_0-i\theta)\alpha_n^- u)|$ is estimated also by (6. 4) modulo $C(n,\mu,L)|u|_{-L}^2$.

As for $r\in J^0S^{-1,2}$ in (6. 3), recalling that $r\in J^{-N}S^{-1,2-N/2}$, we have

$$I_n(n-1/2)^*I_n(n-1/2)r \equiv I_n(n-1)^*b\langle\mu D\rangle I_n(n) \equiv \\ \equiv I_n(n-1)^*\tilde{b}\langle\mu D\rangle J_+(-n),\ b,\ \tilde{b}\in\mu S_{1/2}^{0,0}.$$

Now, the rest of the proof is clear.

Next, we consider $T_i(X,\xi,\mu)$. Since

$$I_n(n-1/2)^*I_n(n-1/2)[\alpha_n^-, T_1] \in J^{2n-1-N}S^{-1,2+2n-N/2}, \text{ for any } N\in \boldsymbol{N},$$

one can express it as follows

$$I_n(n-1)^*a\langle\mu D\rangle I_n(n)\alpha_n^- + I_n(n-1)^*b\langle\mu D\rangle J_+(-n)\alpha_n^+, \text{ with } a, b\in\mu S_{1/2}^{0,0}.$$

Thus this implies that

$$|\Phi_n^-([\alpha_n^-, T_1]u, (D_0-i\theta)\alpha_n^- u)| \leqq C(n)\mu\|u\|_{D,n,1}^2 + C(n)\mu\|\|u\|\|_{n+1,0}^2. \tag{6. 5}$$

On the other hand, in view of the fact $[T_1, I_n(n-1/2)]\in J^{n-3/2}S^{-1,n+3/2}$, it follows that

$$|([T_1, I_n(n-1/2)]\alpha_n^- u, I_n(n-1/2)(D_0-i\theta)\alpha_n^- u)| \leqq \\ \leqq C(n)\mu\|\|u\|\|_{n+1,0}^2 + C(n)\mu\|u\|_{D,n,1}^2. \tag{6. 6}$$

Combining (6. 5) and (6. 6) we get

$$\Phi_n^-(\alpha_n^- T_1 u, (D_0-i\theta)\alpha_n^- u) \\ = -2\ \mathrm{Im}(T_1 I_n(n-1/2)\alpha_n^- u, I_n(n-1/2)(D_0-i\theta)\alpha_n^- u) + K_n^-(u),$$

where $|K_n^-(u)| \leqq C(n)\mu\|u\|_{D,n,1}^2 + C(n)\mu\|\|u\|\|_{n+1,0}^2$. After having done the same arguments for $\Phi_n^+(\cdots)$, we have

Proposition 6. 4.

$$\sum\Phi_n^{\pm}(\alpha_n^{\pm}T_1 u, (D_0-i\theta)\alpha_n^{\pm}u) \\ = -2\ \mathrm{Im}\{(T_1 I_n(n-1/2)\alpha_n^- u, I_n(n-1/2)(D_0-i\theta)\alpha_n^- u) + \\ +(T_1 J_+(-n-1/2)\alpha_n^+ u, J_+(-n-1/2)(D_0-i\theta)\alpha_n^+ u)\} + K_n(u),$$

where $|K_n(u)|\leqq C(n)\mu\|u\|^2_{D,n,1}+C(n)\mu|||u|||^2_{n+1,0}$, *modulo* $C(n,\mu,L)\,|u|^2_{-L}$.

Assume that $T_0\in J^0S^{0,0}$. Then the similar arguments to that of obtaining (6.5) show that

$$\sum|\Phi_n^\pm([\alpha_n^\pm,T_0](D_0-i\theta)u,(D_0-i\theta)\alpha_n^\pm u)|\leqq$$
$$\leqq C(n)\mu\|u\|^2_{D,n,1}+C(n)\mu|||(D_0-i\theta)u|||^2_{n,1/2}.$$

Therefore it follows that

$$(6.7)\qquad \sum|\Phi_n^\pm(\alpha_n^\pm T_0(D_0-i\theta)u,(D_0-i\theta)\alpha_n^\pm u)|\leqq$$
$$\leqq(c+C(n)\mu)\|u\|^2_{D,n,1/2}+(c+C(n)\mu)|||(D_0-i\theta)u|||^2_{n,1/2}.$$

Here, using proposition 6.2, we rewrite the estimate in proposition 4.2.

Proposition 6. 5.

$$\sum\int\Phi_n^\pm(\alpha_n^\pm(D_0-i\theta)^2u,(D_0-i\theta)\alpha_n^\pm u)\,dx_0\geqq(2n-1)(1-\delta-n^{-1/2}-$$
$$-C(n)\mu)\int\|u\|^2_{D,n,1}dx_0+c\theta\int\|u\|^2_{D,n,1/2}dx_0+\delta cn^3(1-cn^{-1}-$$
$$-C(n)\mu)\int|||u|||^2_{n,2}dx_0+c\theta n^2(1-C(n)\mu)\int|||u|||^2_{n,3/2}dx_0+$$
$$+c\theta^2n\int|||u|||^2_{n,1}dx_0-(cn^{1/2}+C(n)\mu)\int|||(D_0-i\theta)u|||^2_{n,1}dx_0,$$

for $0<\mu\leqq\mu_0(n)$, *modulo* $C(n,\mu,L)\theta\int|u|^2_{-L,\theta}dx_0$.

7. Energy integral (continued).

In this section, we specialize $B=B(\mu)$ in section 5. First we sum up the results in sections 5 and 6. Assume that B satisfies the conditions in proposition 5.4 and $B^*-B\in J^0S^{0,1}$. We take $\delta=\mu^{1/4}$ in propositions 5.1 and 5.2, then from propositions 5.1 through 5.5 and 6.3, the estimate (5.3) yields

$$(7.1)\qquad -\sum\int\Phi_n^\pm(\alpha_n^\pm Bu,(D_0-i\theta)\alpha_n^\pm u)\,dx_0\geqq2\theta\sum\mathrm{Re}\int(Bw^\pm_{n-1/2},w^\pm_{n-1/2})\,dx_0+$$
$$+(2n-1-C(n)\varepsilon-C(n)\mu^{1/4})\sum\mathrm{Re}\int(Bw_n^\pm,w_n^\pm)\,dx_0+$$
$$+\mathrm{Im}\int\{((B-B^*)I_n(n-1/2)\alpha_n^-u,I_n(n-1/2)(D_0-i\theta)\alpha_n^-u)+$$
$$+((B-B^*)J_+(-n-1/2)\alpha_n^+u,J_+(-n-1/2)(D_0-i\theta)\alpha_n^+u)\}\,dx_0-$$
$$-\sum\mathrm{Re}\int((\partial_0B)w^\pm_{n-1/2},w^\pm_{n-1/2})\,dx_0-C(n)(\varepsilon+\mu^{1/4})\int|||u|||^2_{n+1,0}dx_0-$$
$$-C(n)(\varepsilon+\mu^{1/4})\int|||u|||^2_{n,2}dx_0-C(n)\mu^{1/4}\int\|u\|^2_{D,n,1}dx_0-$$

$$-C(n,\mu,\varepsilon,L)\int|||u|||_{n,1}^2dx_0-\theta\sum\mathrm{Re}\int(w_{n-1/2}^{\pm},(B-B^*)w_{n-1/2}^{\pm})dx_0,$$

where $w_{n-1/2}^-=I_n(n-1/2)\alpha_n^-u$, $w_{n-1/2}^+=J_+(-n-1/2)\alpha_n^+u$, $0<\mu\leqq\mu_0(n,\varepsilon)$, modulo $C(n,\mu,L)\int|u|_{-L,\theta}^2dx_0$.

Let B be one of $\mu^2Y_i(X,\xi,\mu)^2q_i(X,\xi,\mu)$, $0\leqq i\leqq p$. From (2.9) and (3.1), one may suppose that $B-B^*\in\mu^{1/2}J^0S^{0,1}$, whereas, this implies that

$$I_n(n-1/2)^*(B-B^*)I_n(n-1/2)\equiv I_n(n-1)^*A\langle\mu D\rangle I_n(n)\equiv$$
$$\equiv I_n(n-3/2)^*\tilde{A}\langle\mu D\rangle I_n(n+1/2),$$

with A, $\tilde{A}\in\mu^{1/2}S_{1/2}^{0,0}$. Therefore, the integrand of the third and the last term of the right hand side of (7.1) is estimated by

$$C(n)\mu^{1/2}||u||_{D,n,1}^2+C(n)\mu^{1/2}|||u|||_{n+1,0}^2,\ C(n)\theta\mu^{1/2}|||u|||_{n,3/2}^2+C(n)\theta\mu^{1/2}|||u|||_{n+1,-1/2}^2$$

respectively, modulo $C(n,\mu,L)\theta|u|_{-L,\theta}^2$.

Next consider ∂_0B. Suppose that $1\leqq i\leqq p$, then since Y_i does not depend on x_0 and $|q_i(X,\xi,\mu)|\geqq c|\xi|^2$, $c>0$, it follows, with some $\theta_0>0$, that $\theta_0B-\partial_0B\geqq0$. Then from the inequality of Melin, we have

$$\text{(7.2)}\quad \mathrm{Re}((\theta_0B-\partial_0B)w,w)+\nu||\langle\mu D\rangle^{1/2}w||^2+C(\nu,\mu)||w||^2\geqq0,\ \text{for any } \nu>0.$$

Remarking the following relation,

$$I_n(n-1/2)^*\langle\mu D\rangle I_n(n-1/2)\equiv I_n(n-1)^*A\langle\mu D\rangle I_n(n),\ \text{with } A\in S_{1/2}^{0,0},$$

it follows from (7.2) with $\nu=\mu$, $w=w_{n-1/2}^-$, that

$$\mathrm{Re}((\theta_0B-\partial_0B)w_{n-1/2}^-,w_{n-1/2}^-)\geqq-\mu|||u|||_{n+1,0}^2-C(n,\mu)|||u|||_{n,1}^2.$$

Similarly, $((\theta_0B-\partial_0B)w_{n-1/2}^+,w_{n-1/2}^+)$ has the same estimate as above from below.

We proceed to consider ∂_0B when $B=\mu^2Y_0(X)^2q_0(X,\xi,\mu)$. We know that B belongs to $(Es)_{1/2,0}^{2,2}$ and then we see that

$$|\langle\mu\xi\rangle^{-1}\partial_0B(X,\xi,\mu)|^2\leqq\bar{c}B(X,\xi,\mu),$$

with $\bar{c}$ independent of μ. We note that one can take $\bar{c}=2|q_0(0,\hat{\xi})|+1$, if δ, defining $V(\delta\mu^{1/2},\hat{\xi})$, is sufficiently small and $0<\mu\leqq\mu_0$. Noting the expression

$$(\partial_0B)^*\langle\mu D\rangle^{-2}\partial_0B\equiv\mathrm{op}(|\langle\mu\xi\rangle^{-1}\partial_0B(X,\xi,\mu)|^2)+r,\ r\in\mu^{1/2}J^0S^{0,1},$$

and again applying the Melin's inequality, it follows that

$$\bar{c}\,\mathrm{Re}(Bw,w)+\nu||\langle\mu D\rangle^{1/2}w||^2+C(\nu,\mu)||w||^2\geqq||\langle\mu D\rangle^{-1}\partial_0Bw||^2-|(rw,w)|,$$

with any $\nu>0$. On the other hand, it is easy to see that

$$I_n(n-1/2)^*J_-(-1/2)^*\langle\mu D\rangle J_-(-1/2)I_n(n-1/2)\equiv I_n(n)^*\langle\mu D\rangle AI_n(n-2),$$
$$I_n(n-1/2)^*J_-(-1/2)^*rJ_-(-1/2)I_n(n-1/2)\equiv I_n(n)^*\langle\mu D\rangle\tilde{A}I_n(n-2),$$

with $A\in S^{0,0}_{1/2}$, $\tilde{A}\in\mu^{1/2}S^{0,0}_{1/2}$, where $r\in\mu^{1/2}J^0S^{0,1}$. Then taking $\nu=\mu^{1/2}$, in the above inequality, we get

$$(7.3)\qquad \bar{c}\,\mathrm{Re}(Bw_n^-,w_n^-)\geqq\|\langle\mu D\rangle^{-1}\partial_0Bw_n^-\|^2-C(n)\mu^{1/2}|||u|||^2_{n+1,0}-$$
$$-C(n)\mu^{1/2}|||u|||^2_{n,2}-C(n,\mu)|||u|||^2_{n,1}.$$

We turn to $((\partial_0B)w^-_{n-1/2},w^-_{n-1/2})$. Using the expression

$$I_n(n-1/2)\equiv J_-(-1/2)^*(1+R)I_n(n),\ R\in\mu S^{0,0}_{1/2},$$

this is reduced to

$$([J_-(-1/2),\ \partial_0B]w^-_{n-1/2},\ (1+R)I_n(n)\alpha_n^-u)+$$
$$+(\langle\mu D\rangle^{-1}\partial_0Bw_n^-,\langle\mu D\rangle(1+R)I_n(n)\alpha_n^-u).$$

Since $[J_-(-1/2),\ \partial_0B]\in J^{-3/2}S^{-1,2}$, the first term is estimated by $C(n)\mu|||u|||^2_{n+1,0}+C(n)\mu|||u|||^2_{n,2}$. Whereas the second term is estimated by

$$\|\langle\mu D\rangle^{-1}\partial_0Bw_n^-\|^2+(1+C(n)\mu)|||u|||^2_{n+1,0}.$$

Then from (7.3), it follows that

$$(7.4)\qquad |((\partial_0B)w^-_{n-1/2},\ w^-_{n-1/2})|\leqq\bar{c}\,\mathrm{Re}(Bw_n^-,w_n^-)+(1+C(n)\mu^{1/2})|||u|||^2_{n+1,0}+$$
$$+C(n)\mu^{1/2}|||u|||^2_{n,2}+C(n,\mu)|||u|||^2_{n,1}.$$

Similarly we have

$$|((\partial_0B)w^+_{n-1/2},w^+_{n-1/2})|\leqq\bar{c}\,\mathrm{Re}(Bw_n^+,w_n^+)+(1+C(n)\mu^{1/2})|||u|||^2_{n+1,0}+$$
$$+C(n)\mu^{1/2}|||u|||^2_{n,2}+C(n,\mu)|||u|||^2_{n,1}.$$

Summing up, from (7.1), we have

Proposition 7. 1. *Let B be one of $\mu^2Y_i(X,\xi,\mu)^2q_i(X,\xi,\mu)$, $0\leqq i\leqq p$. Then*

$$-\sum\int\Phi_n^\pm(\alpha_n^\pm Bu,(D_0-i\theta)\alpha_n^\pm u)dx_0\geqq2(\theta-\theta_0)\sum\mathrm{Re}\int(Bw^\pm_{n-1/2},w^\pm_{n-1/2})dx_0+$$
$$+(2n-1-\bar{c}-C(n)\varepsilon-C(n)\mu^{1/4})\sum\mathrm{Re}\int(Bw_n^\pm,w_n^\pm)dx_0-$$
$$-(1+C(n)(\varepsilon+\mu^{1/4}))\int|||u|||^2_{n+1,0}dx_0-C(n)(\varepsilon+\mu^{1/4})\int|||u|||^2_{n,2}dx_0-$$
$$-C(n)\mu^{1/4}\int||u||^2_{D,n,1}dx_0-C(n)\theta\mu^{1/2}\int|||u|||^2_{n,3/2}dx_0-$$
$$-C(n)\theta\mu^{1/2}\int|||u|||^2_{n+1,-1/2}dx_0-C(n,\mu,L)\int|||u|||^2_{n,1}dx_0,$$

modulo $C(n,\mu,L)\int|u|^2_{-L,\theta}dx_0$.

Now we assume that B is one of $\Gamma_i(\xi, \mu)^2 r_i(X, \xi, \mu)$, $1 \leqq i \leqq p$. Obviously, with some $\theta_0 > 0$, we have $\theta_0 B - \Gamma_i^2 \geqq 0$. Then it follows that (see (7.2)) $\mathrm{Re}(\theta_0 B w_{n-1/2}^-, w_{n-1/2}^-) \geqq \|\Gamma_i w_{n-1/2}^-\|^2 - \mu\||u\||_{n+1,0}^2 - C(n, \mu)\||u\||_{n,1}^2$. On the other hand, from (2.10) we can write

$$B - B^* \equiv L_i \Gamma_i + r_i, \text{ with } L_i \in \mu^{1/2} J^0 S^{0,0}, r_i \in J^0 S^{0,0},$$

and therefore $|((B-B^*) w_{n-1/2}^-, I_n(n-1/2)(D_0 - i\theta)\alpha_n^- u)|, \theta|((B-B^*) w_{n-1/2}^-, w_{n-1/2}^-)|$ is estimated by

$$\begin{aligned} (7.6) \qquad & \mathrm{Re}(\theta_0 B w_{n-1/2}^-, w_{n-1/2}^-) + C(n)\mu^{1/2}\|u\|_{D,n,1}^2 + \\ & + C(n)\mu\||u\||_{n+1,0}^2 + C(n, \mu)\||u\||_{n,1}^2, \\ & \mathrm{Re}(\theta_0 B w_{n-1/2}^-, w_{n-1/2}^-) + \theta^2(C(n)\mu^{1/2} + \\ & C(n, \mu)\theta^{-1})\||u\||_{n,1}^2 + C(n)\mu\||u\||_{n+1,0}^2, \end{aligned}$$

respectively. Thus we have

Proposition 7. 2. *Let B be one of $\Gamma_i(\xi, \mu)^2 r_i(X, \xi, \mu)$, $1 \leqq i \leqq p$. Then we have*

$$\begin{aligned} & -\sum \int \Phi_n^\pm (\alpha_n^\pm B u, (D_0 - i\theta)\alpha_n^\pm u) dx_0 \geqq 2(\theta - \theta_0) \sum \mathrm{Re} \int (B w_{n-1/2}^\pm, w_{n-1/2}^\pm) dx_0 + \\ & + (2n - 1 - C(n)\varepsilon - C(n)\mu^{1/4}) \sum \mathrm{Re} \int (B w_n^\pm, w_n^\pm) dx_0 - C(n)(\varepsilon + \mu^{1/4}) \int \||u\||_{n+1,0}^2 dx_0 \\ & - C(n)(\varepsilon + \mu^{1/4}) \int \||u\||_{n,2}^2 dx_0 - C(n)\mu^{1/4} \int \|u\|_{D,n,1}^2 dx_0 - \theta^2(C(n)\mu^{1/2} + \\ & + C(n, \mu)\theta^{-1}) \int \||u\||_{n,1}^2 dx_0, \end{aligned}$$

modulo $C(n, \mu, L)\theta \int |u|_{-L,\theta}^2 dx_0$.

Now we are in a position to consider $B = \mu\phi(x'', \xi, \mu) q_p(X, \xi, \mu)$. Set

$$\tilde{K}(X, \xi, \mu) = i\mu \sum_{j=1}^{d} C_j \partial_{x_j} \partial_{\xi_j} \phi_p(0, \hat{\xi}) |\xi''|^{-1} q_p,$$

then we know from (2.12) that $B - B^* = \tilde{K} + r$, with $r \in \mu^{1/2} J^0 S^{0,1}$. If we note that $\tilde{K}^* + \tilde{K} \in \mu^{1/2} J^0 S^{0,0}$, the following estimate is immediate.

$$\theta |\mathrm{Re}((B - B^*) w_{n-1/2}^-, w_{n-1/2}^-)| \leqq C(n)\mu^{1/2}\||u\||_{n+1,0}^2 + C(n)\theta^2\mu^{1/2}\||u\||_{n,1}^2.$$

Consequently, taking proposition 6.4 into account, we have

$$\begin{aligned} (7.6) \qquad & -\sum \int \Phi_n^\pm (\alpha_n^\pm (B - \mu T_1) u, (D_0 - i\theta)\alpha_n^\pm u) dx_0 \geqq \\ & \geqq 2(\theta - \theta_0) \sum \mathrm{Re} \int (B w_{n-1/2}^\pm, w_{n-1/2}^\pm) dx_0 + (2n - 1 - C(n)\varepsilon - \\ & - C(n)\mu^{1/4}) \sum \mathrm{Re} \int (B w_n^\pm, w_n^\pm) dx_0 + \\ & + 2\,\mathrm{Im} \int \{((2^{-1}\tilde{K} - \mu T_1) w_{n-1/2}^-, I_n(n - 1/2)(D_0 - i\theta)\alpha_n^- u) + \end{aligned}$$

$$+((2^{-1}\tilde{K}-\mu T_1)w^+_{n-1/2}, J_+(-n-1/2)(D_0-i\theta)\alpha^+_n u)\}dx_0-$$
$$-C(n)(\varepsilon+\mu^{1/4})\int|||u|||^2_{n+1,0}dx_0-C(n)(\varepsilon+\mu^{1/4})\int|||u|||^2_{n,2}dx_0-$$
$$-C(n)\mu^{1/4}\int||u||^2_{D,n,1}dx_0-\theta^2(C(n)\mu^{1/2}+C(n,\mu)\theta^{-1})\int|||u|||^2_{n,1}dx_0,$$

modulo $C(n, \mu, L)\theta\int|u|^2_{-L,\theta}dx_0$.

From the definition of $T_1(X, \xi, \mu)$, we see that

$$2^{-1}\tilde{K}-\mu T_1=2^{-1}i\sum_{j=1}^{d}\partial_{x_j}\partial_{\xi_j}\phi_p(0, \hat{\xi}'')q_p|\xi|^{-1}-\mu T_{1,1}(X, \xi, \mu)-\mu T_1(0, \hat{\xi})|\xi|.$$

In addition, taking δ in the definition of q_p and $T_{1,1}$, sufficiently small, it follows that

$$\hat{c}_0=\sup|(2^{-1}\tilde{K}-\mu T_1)\langle\mu\xi\rangle^{-1}|\leqq|P^s(0,0,0,\hat{\xi})|+1. \tag{7.7}$$

Then using this $\hat{c}_0$, the third term of (7.6) is estimated by

$$n^{-1}\delta_1^{-1}(\hat{c}_0^2+C(n)\mu)|||u|||^2_{n+1,0}+n\delta_1||u||^2_{D,n,1}.$$

Let us set

$$\mathscr{B}(X, \xi, \mu)=\mu^2\sum_{i=0}^{p}Y_i(X, \xi, \mu)^2q_i(X, \xi, \mu)+ \tag{7.8}$$
$$+\sum_{i=1}^{p}\Gamma_i(\xi, \mu)^2r_i(X, \xi, \mu)+\mu\phi(x'', \xi, \mu)q_p(X, \xi, \mu).$$

From the definition of $Y(X, \xi, \mu)$, it follows that

$$\sum_{i=0}^{p}Y_i(X, \xi, \mu)=Y(X, \xi, \mu).$$

Hence, with positive constants $\hat{c}_i>0$, independent of μ, we get

$$\mathscr{B}(X, \xi, \mu)\geqq\hat{c}_1Y(X, \xi, \mu)^2\langle\mu\xi\rangle^2+\hat{c}_2\sum_{i=1}^{p}\Gamma_i(\xi, \mu)^2$$

Here $\hat{c}_1$ is determined by $\{q_i(0, \hat{\xi})\}_{i=0}^{p}$. Set $K(X, \xi, \mu)=Y(X, \xi, \mu)\langle\mu\xi\rangle$, then

$$K^*K=\mathrm{op}(Y(X, \xi, \mu)^2\langle\mu\xi\rangle^2)+r,\ r\in\mu J^0S^{0,1}.$$

(in virtue of $Y(X, \xi, \mu)\in J^1S^{0,0}$ and (3.3)). Using the expression

$$I_n(n-1/2)^*J_-(-1/2)^*rJ_-(-1/2)I_n(n-1/2)\equiv$$
$$\equiv I_n(n)^*\langle\mu D\rangle AI_n(n-2),\ A\in\mu S^{0,0}_{1/2},$$

Melin's inequlity shows that

$$\mathrm{Re}(\mathscr{B}w_n^-, w_n^-)+\nu||\langle\mu D\rangle^{1/2}w_n^-||^2+C(\nu, \mu)||w_n^-||^2\geqq\hat{c}_1||Kw_n^-||^2+ \tag{7.9}$$
$$+\hat{c}_2\sum_{i=1}^{p}||\Gamma_iw_n^-||^2-C(n)\mu|||u|||^2_{n+1,0}-C(n)\mu|||u|||^2_{n,2},$$

for any $\nu>0$. Remarking that

$$I_n(n-1/2)^*J_-(-1/2)^*\langle\mu D\rangle J_-(-1/2)I_n(n-1/2)\equiv$$
$$\equiv I_n(n)^*\langle\mu D\rangle AI_n(n-2),\ A\in S^{0,0}_{1/2},$$
$$J_-(-1/2)I_n(n-1/2)\equiv\tilde{A}I_n(n-1),\ \tilde{A}\in S^{0,0}_{1/2},$$

the inequality (7.9) with $\nu=\mu$ gives that

Proposition 7. 3. *Notations being as above,*

$$\sum\mathrm{Re}(\mathscr{B}w_n^{\pm}, w_n^{\pm})\geqq\hat{c}_1\sum||Kw_n^{\pm}||^2+\hat{c}_2\sum\sum_{i=1}^{p}||\Gamma_iw_n^{\pm}||^2-C(n)\mu|||u|||^2_{n+1,0}-$$
$$-C(n)\mu|||u|||^2_{n,2}-C(n,\mu)|||u|||^2_{n,1},$$

modulo $C(n,\mu,L)\,|u|^2_{-L}$.

Similarly, we have

Proposition 7. 4.

$$\sum\mathrm{Re}(\mathscr{B}w^{\pm}_{n-1/2}, w^{\pm}_{n-1/2})\geqq\hat{c}_1\sum||Kw^{\pm}_{n-1/2}||^2+\hat{c}_2\sum\sum_{i=1}^{p}||\Gamma_iw^{\pm}_{n-1/2}||^2-$$
$$-C(n)\mu|||u|||^2_{n+1,-1/2}-C(n)\mu|||u|||^2_{n,3/2}-C(n,\mu)|||u|||^2_{n,1},$$

modulo $C(n,\mu,L)\,|u|^2_{-L}$.

Here we prepare the following lemma.

Lemma 7. 1. *Notations being as above,*

$$\sum||Kw_n^{\pm}||^2+|||u|||^2_{n,2}\geqq(\hat{c}_3-C(n)\mu)|||u|||^2_{n+1,0}-C(n,\mu,L)||u||^2_{-L},$$
$$\sum||Kw^{\pm}_{n-1/2}||^2+|||u|||^2_{n,3/2}\geqq(\hat{c}_3-C(n)\mu)|||u|||^2_{n+1,-1/2}-C(n,\mu,L)||u||^2_{-L}.$$

Proof. We shall prove the first assertion. Noting that $K\in J^1S^{0,1}$, we see that

$$\text{(7.10)}\quad I_n(n-1/2)^*J_-(-1/2)^*K^*KJ_-(-1/2)I_n(n-1/2)+$$
$$+I_n(n-2)^*I_n(n-2)\equiv I_n(n)^*\langle\mu D\rangle(A+b)\langle\mu D\rangle I_n(n),\ b\in\mu S^{0,0}_{1/2},$$

where

$$A(X,\xi,\mu)=K(X,\xi,\mu)^2\langle\mu\xi\rangle^{-2}J_-(X,\xi,\mu)^{-2}+\langle\mu\xi\rangle^{-2}J_-(X,\xi,\mu)^{-4}=$$
$$=(x_0-\phi(x'',\xi,\mu))^2J_-(X,\xi,\mu)^{-2}+\langle\mu\xi\rangle^{-2}J_-(X,\xi,\mu)^{-4},\ (\in J^0S^{0,0}).$$

As for $A(X,\xi,\mu)$, it is obvious that

$$A(X,\xi,\mu)=J_-(X,\xi,\mu)^{-4}\{(x_0-\phi(x'',\xi,\mu))^2J_-(X,\xi,\mu)^2+\langle\mu\xi\rangle^{-2}\}\geqq$$
$$\geqq cJ_-(X,\xi,\mu)^{-4}\{(x_0-\phi(x'',\xi,\mu))^4+\langle\mu\xi\rangle^{-2}\}\geqq\hat{c}_3>0.$$

Thus, from proposition 3. 9, it follows that

$$\text{(7.11)}\qquad \mathrm{Re}(Aw,w)+C(\mu,L)||w||^2_{-L}\geqq(\hat{c}_3-C(A)\mu)||w||^2.$$

Putting $w=\langle\mu D\rangle I_n(n)\alpha_n^- u$ in (7.11), in virtue of (7.10), the estimate (7.11) yields

$$||Kw_n^-||^2+||I_n(n-2)\alpha_n^- u||^2+C(\mu, L)||u||_{-L}^2\geqq(\hat{c}_3-C(n)\mu)||\langle\mu D\rangle I_n(n)\alpha_n^- u||^2.$$

By the same way, we get

$$||Kw_n^+||^2+||J_+(-n-2)\alpha_n^+ u||^2+C(\mu, L)||u||_{-L}^2\geqq(\hat{c}_3-\\ -C(n)\mu)||\langle\mu D\rangle J_+(-n)\alpha_n^+ u||^2,$$

and these prove this lemma.

Finally, from proposition 6.5, lemma 7.1 and the arguments of this section, we have

Proposition 7. 5.

$$\sum\int\Phi_n^\pm(\alpha_n^\pm P_{(\mu),\theta}u, (D_0-i\theta)\alpha_n^\pm u)dx_0\geqq\hat{c}_3(2n-1-\hat{c}_1-\hat{c}_3\delta_1^{-1}n^{-1}\hat{c}_0^2-C(n)\varepsilon-$$
$$-C(n)\mu^{1/4})\int|||u|||_{n+1,0}^2dx_0+c\theta(\hat{c}_3-C(n)\mu^{1/4})\int|||u|||_{n+1,-1/2}^2dx_0+$$
$$+(2n-1)(1-\delta-\delta_1-n^{-1/2}-C(n)\mu^{1/4})\int||u||_{D,n,1}^2dx_0+c\theta\int||u||_{D,n,1/2}^2dx_0+$$
$$+\delta cn^3(1-cn^{-1}-C(n)\varepsilon-C(n)\mu^{1/4})\int|||u|||_{n,2}^2dx_0+c\theta n^2(1-C(n)\mu^{1/4})\int|||u|||_{n,3/2}^2dx_0$$
$$+c\theta^2n(1-C(n)\mu^{1/2}-C(n,\mu)\theta^{-1})\int|||u|||_{n,1}^2dx_0-(cn^{1/2}+C(n)\mu)\int|||(D_0-$$
$$-i\theta)u|||_{n,1}^2dx_0-C(n)\mu\int|||(D_0-i\theta)u|||_{n,1/2}^2dx_0, \text{ modulo } C(n, \mu, L)\theta\int|u|_{-L,\theta}^2dx_0,$$

where $P_{(\mu),\theta}=(D_0-i\theta)^2-\mathscr{B}+\mu T_1+\mu^{1/2}T_0(D_0-i\theta)$.

Next we handle the term $||u||_{D,n,k}$, $(k=1, 1/2)$.

Proposition 7. 6.

$$n||u||_{D,n,1}^2\geqq 2^{-1}n|||(D_0-i\theta)u|||_{n,1}^2-n^2(c+C(n)\mu)|||u|||_{n,2}^2,$$
$$\theta||u||_{D,n,1/2}^2\geqq 2^{-1}\theta|||(D_0-i\theta)u|||_{n,1/2}^2-n\theta(c+C(n)\mu)|||u|||_{n,3/2}^2,$$

modulo $C(n,\mu,L)\theta|u|_{-L}^2$.

Proof. We write

$$I_n(n-1)D_0\alpha_n^-\equiv n^{1/2}(A+a)I_n(n-2)\alpha_n^-+n^{1/2}(\tilde{A}+\tilde{a})J_+(-n-2)\alpha_n^+,\ a,\ \tilde{a}\in\mu S_{1/2}^{0,0},$$

where

$$A(X,\xi,\mu)=J_-(X,\xi,\mu)\langle\mu\xi\rangle^{1/2}\chi^{(1)}(-n^{1/2}Y(X,\xi,\mu)\langle\mu\xi\rangle^{1/2}),\ (\in J^0S^{0,0}),$$
$$\tilde{A}(X,\xi,\mu)=(\langle\mu\xi\rangle J_+(X,\xi,\mu)J_-(X,\xi,\mu))^nJ_+(X,\xi,\mu)^2J_-(X,\xi,\mu)^{-1}\langle\mu\xi\rangle^{1/2}\times$$
$$\chi^{(1)}(-n^{1/2}Y(X,\xi,\mu)\langle\mu\xi\rangle^{1/2}),\ (\in J^0S^{0,0}).$$

From proposition 6.1, it follows that

$$|A(X,\xi,\mu)|,\ |\tilde{A}(X,\xi,\mu)|\leqq C, \text{ with } C \text{ independent of } \mu,\ n.$$

Hence proposition 3.8 implies that $||I_n(n-1)D_0\alpha_n^+u||^2 \leqq n(C+C(n)\mu)|||u|||^2_{n,2}$. Similarly, we get $||J_+(-n-1)D_0\alpha_n^+u||^2 \leqq n(C+C(n)\mu)|||u|||^2_{n,2}$. Combining these, we obtain the first assertion. The proof of the second assertion is similar.

Consider $\Phi_n^\pm(\alpha_n^\pm P_{(\mu),\theta}u,\ (D_0-i\theta)\alpha_n^\pm u)$. Since one can express $I_n(n-1/2)^*I_n(n-1/2) \equiv I_n(n-1)^*(1+a)I_n(n)$, $J_+(-n-1/2)^*J_+(-n-1/2) \equiv J_+(-n-1)^*(1+b)J_+(-n)$, with $a,\ b \in \mu S^{0,0}_{1/2}$, it is clear that $\sum |\Phi_n^\pm(\alpha_n^\pm P_{(\mu),\theta}u,\ (D_0-i\theta)\alpha_n^\pm u)|$ is estimated by

$$(7.12)\qquad |||P_{(\mu),\theta}u|||^2_{n,0} + (1+C(n)\mu)||u||^2_{D,n,1} + C(n,\mu,L)||P_{(\mu),\theta}u||^2_{-L},$$

modulo $C(n,\mu,L)\,|u|^2_{-L}$.

Now using proposition 7.6 and (7.12), we rewrite the estimate in proposition 7.5. First we fix n so that $n \geqq C(q_i)\hat{c}_0$, where $C(q_i)$ is a suitably chosen constant depending only on $\{q_i(0,\hat{\xi})\}$. Next we take $\varepsilon>0$ so that $C(n)\varepsilon$ is sufficiently small. Then we have

Lemma 7.2.

$$\int |||P_{(\mu),\theta}u|||^2_{n,0}dx_0 + C(n,\mu,L)\int ||P_{(\mu),\theta}u||^2_{-2L}dx_0 \geqq c_1 n\int |||u|||^2_{n+1,0}dx_0 +$$
$$+c_2 n\int |||(D_0-i\theta)u|||^2_{n,1}dx_0 + c_3\theta\int |||u|||^2_{n+1,-1/2}dx_0 + c_3\theta\int |||(D_0-i\theta)u|||^2_{n,1/2}dx_0 +$$
$$+c_4 n^3\int |||u|||^2_{n,2}dx_0 + c_4\theta n^2\int |||u|||^2_{n,3/2}dx_0 + c_4\theta^2 n\int |||u|||^2_{n,1}dx_0,$$

for $n \geqq C(q_i)\hat{c}_0$, $0<\mu\leqq\mu_0(n)$, $\theta\geqq\theta_0(n,\mu)$, *modulo* $C(n,\mu,L)\theta\int |u|^2_{-L,\theta}dx_0$.

8. Estimates of error terms.

From the relations

$$1 \equiv A_1J_+(-n-1)\alpha_n^+ + A_2I_n(n-1)\alpha_n^-,$$
$$1 \equiv \tilde{A}_1J_+(-n-1/2)\alpha_n^+ + \tilde{A}_2I_n(n-1/2)\alpha_n^-,\ A_i,\ \tilde{A}_i \in S^{0,0}_{1/2},$$

it follows that

$$(8.1)\qquad ||u||^2 \leqq (c+C(n)\mu)|||u|||^2_{n,1} + C(n,\mu,L)||u||^2_{-L},$$
$$||(D_0-i\theta)u||^2 \leqq (c+C(n)\mu)|||(D_0-i\theta)u|||^2_{n,1/2} + C(n,\mu,L)||(D_0-i\theta)u||^2_{-L}.$$

On the other hand, the following inequalities are immediate.

$$(8.2)\qquad 2\theta^3||u||^2_{-L} \leqq \theta^2||u||^2 + \theta^4||u||^2_{-2L},$$
$$2\theta^{3/2}||(D_0-i\theta)u||^2_{-L} \leqq \theta||(D-i\theta)u||^2 + \theta^2||(D_0-i\theta)u||^2_{-2L}.$$

Therefore, from (8.1) and (8.2), we have

$$(8.3)\qquad \theta^3(2-C(n,\mu,L)\theta^{-1})||u||^2_{-L} \leqq \theta^2(c+C(n)\mu)|||u|||^2_{n,1} + \theta^4||u||^2_{-2L},$$
$$\theta^{3/2}(2-C(n,\mu,L)\theta^{-1/2})||(D_0-i\theta)u||^2_{-L} \leqq$$
$$\theta(c+C(n)\mu)|||(D_0-i\theta)u|||^2_{n,1/2} + \theta^2||(D_0-i\theta)u||^2_{-2L}.$$

A rough energy estimate gives that

$$\int \|P_{(\mu),\theta}u\|^2 dx_0 \geqq c_0\theta^2 \int \|(D_0-i\theta)u\|^2 dx_0 + c_0\theta^4 \int \|u\|^2 dx_0 - C(\mu)\theta^2 \int \|\langle D\rangle u\|^2 dx_0.$$

Then if we note that

$$\langle D\rangle^{-2L} P_{(\mu),\theta} = P_{(\mu),\theta}\langle D\rangle^{-2L} + q\langle D\rangle^{-2L} + r(D_0-i\theta)\langle D\rangle^{-2L},$$

with $q\in S^1_{1,0}$, $r\in S^{-1}_{1,0}$ depending on μ, we see easily that

$$(8.4)\qquad \int \|P_{(\mu),\theta}u\|^2_{-2L} dx_0 \geqq c_0(\theta^2 - C(\mu)) \int \|(D_0-i\theta)u\|^2_{-2L} dx_0 + \\ + c_0\theta^4 \int \|u\|^2_{-2L} dx_0 - C(\mu)\theta^2 \int \|\mu\|^2_{-2L+1} dx_0.$$

Proposition 8. 1. *For* $0<\mu\leqq\mu_0(n)$, $\theta\geqq\theta_0(n,\mu,L)$, $L\geqq 1$, *we have*

$$\int \|P_{(\mu),\theta}u\|^2_{-2L} dx_0 + \theta \int |||(D_0-i\theta)u|||^2_{n,1/2} dx_0 + \theta^2 \int |||u|||^2_{n,1} dx_0 \geqq c_1\theta^2 \int |u|^2_{-L,\theta} dx_0.$$

Proof. Since $-L\geqq -2L+1$, and $\theta^3\|u\|^2_{-L} + \theta^{3/2}\|(D_0-i\theta)u\|^2_{-L} \geqq \theta^2|u|^2_{-L,\theta}$, this proposition follows from (8. 3) and (8. 4) immediately.

In view of this proposition, one can absorb the error terms. Then

Lemma 8. 1. *For any* $L\in N$,

$$\int |||P_{(\mu),\theta}u|||^2_{n,0} dx_0 + C(n,\mu,L)\int \|P_{(\mu),\theta}u\|^2_{-2L} dx_0 \geqq c_1 n \int |||u|||^2_{n+1,0} dx_0 + \\ + c_2 n \int |||(D_0-i\theta)u|||^2_{n,1} dx_0 + c_3\theta \int |||u|||^2_{n+1,-1/2} dx_0 + c_3\theta \int |||(D_0-i\theta)u|||^2_{n,1/2} dx_0 + \\ + c_4 n^3 \int |||u|||^2_{n,2} dx_0 + c_4\theta n^2 \int |||u|||^2_{n,3/2} dx_0 + c_4\theta^2 n \int |||u|||^2_{n,1} dx_0 + \\ + c_5\theta^{3/2} \int \|(D_0-i\theta)u\|^2_{-L} dx_0 + c_5\theta^3 \int \|u\|^2_{-L} dx_0,$$

for $n\geqq C(q_i)\hat{c}_0$, $0<\mu\leqq\mu_0(n)$, $\theta\geqq\theta_0(n,\mu,L)$.

Now, to prove theorem 1. 1, it suffices to note that $(D_0-i\theta)e^{-x_0\theta} = e^{-x_0\theta}D_0$. Next, we shall prove theorem 1. 2. Observe $M_1 = [\langle D\rangle^{s+1}, \mathscr{B}]\langle D\rangle^{-s-1}$, where $\mathscr{B}$ is defined by (7. 8). From (2. 7), (2. 9) and (2. 13), it is clear that M_1 belongs to $J^0 S^{0,1}$. Moreover it is also immediate that

$$(8.5)\qquad M_1 \equiv \sum_{i=1}^{p} [\langle D\rangle^{s+1}, \mathrm{op}(\Gamma_i^2 r_i)]\langle D\rangle^{-s-1} + R_1, \text{ with } R_1\in\mu^{1/2}J^0S^{0,1}, \\ [\langle D\rangle^{s+1}, \mathrm{op}(\Gamma_i^2 r_i)]\langle D\rangle^{-s-1} \equiv L_i\Gamma_i + r_i,\ L_i\in\mu^{1/2}J^0S^{0,0},\ r_i\in\mu^{1/2}J^0S^{0,0}.$$

From proposition 6. 4, it follows that $\sum |\Phi_n^{\pm}(\alpha_n^{\pm}M_1 u, (D_0-i\theta)\alpha_n^{\pm}u)|$ is estimated by

$$2|(M_1 w^-_{n-1/2}, I_n(n-1/2)(D_0-i\theta)\alpha_n^- u)| + \\ + 2|(M_1 w^+_{n-1/2}, J_+(-n-1/2)(D_0-i\theta)\alpha_n^+ u)| + C(n)\mu\|u\|^2_{D,n,1} + C(n)\mu|||u|||^2_{n+1,0},$$

modulo $C(n, \mu, L)|u|^2_{-L}$. In addition, in view of the expression (8.5) (cf. (7.6)), it is also estimated by

$$\sum \mathrm{Re}(\theta_0 \mathscr{B} w^{\pm}_{n-1/2}, w^{\pm}_{n-1/2}) + C(n)\mu^{1/2}\|u\|^2_{D,n,1} + C(n)\mu^{1/2}|||u|||^2_{n+1,0} + C(n,\mu)|||u|||^2_{n,1}.$$

Denote $\tilde{R}_1 = [\langle D\rangle^{s+1}, \mu T_1]\langle D\rangle^{-s-1}$, $\tilde{R}_0 = [\langle D\rangle^{s+1}, \mu^{1/2}T_0]\langle D\rangle^{-s-1}$, then taking proposition 6.4 and (6.7) into account, it is easily seen that $\sum|\Phi^{\pm}_n(\alpha^{\pm}_n \tilde{R}_1 u, (D_0 - i\theta)\alpha^{\pm}_n u)|$, $\sum|\Phi^{\pm}_n(\alpha^{\pm}_n \tilde{R}_0(D_0 - i\theta)u, (D_0 - i\theta)\alpha^{\pm}_n u)|$ is estimated by

$$C(n)\mu\|u\|^2_{D,n,1} + (c + C(n)\mu)\|u\|^2_{D,n,1/2} + C(n)\mu|||u|||^2_{n+1,0} + \\ + (c + C(n)\mu)|||(D_0 - i\theta)u|||^2_{n,1/2}.$$

If we operate $\langle D\rangle^{s+1}$ to $P_{(\mu)}$, it results that

$$\langle D\rangle^{s+1}P_{(\mu)} = \tilde{P}_{(\mu)}\langle D\rangle^{s+1},\ \tilde{P}_{(\mu)} = D_0^2 - \mathscr{B} + \mu T_1 + \mu^{1/2}T_0 D_0 + M_1 + \tilde{R}_1 + \tilde{R}_0 D_0.$$

Thus the above arguments show that lemma 8.1 holds also for $\tilde{P}_{(\mu)}$ without any change. On the other hand, since $I_n(n) \in J^n S^{0,n} \subset S^n_{1/2,1/2}$, $J_+(-n) \in J^{-n}S^{0,0} \subset S^{n/2}_{1/2,1/2}$, it follows that $|||\langle D\rangle^{s+1}P_{(\mu)}u|||^2_{n,0} \leqq C(n)\|P_{(\mu)}u\|^2_{n+s+1}$, and hence

$$|||\tilde{P}_{(\mu)}\langle D\rangle^{s+1}u|||^2_{n,0} \leqq C(n)\|P_{(\mu)}u\|^2_{n+s+1},\ \|\tilde{P}_{(\mu)}\langle D\rangle^{s+1}u\|^2_{-2L} = \|P_{(\mu)}u\|^2_{s+1-2L}.$$

Then from lemma 8.1 with $L=1$, applied to $\tilde{P}_{(\mu)}$ and $\langle D\rangle^{s+1}u$, it follows that

$$C(n,\mu,L)\int e^{-2x_0\theta}\|P_{(\mu)}u\|^2_{n+s+1}dx_0 \geqq c_5\theta^{3/2}\int e^{-2x_0\theta}\|D_0 u\|^2_s dx_0 + c_5\theta^3\int e^{-2x_0\theta}\|u\|^2_s dx_0.$$

After changing a scale of variables, we get theorem 1.2.

9. Some remarks.

In this section, we make some observations on operators of type $(1)_p$ with $\mathrm{grad}\,\phi_p(0, \hat{\xi}^{(p+1)}) \neq 0$. If $d\phi_p$ is not proportional to $\sum_{i=p+1}^{d} \xi_i dx_i$ at $(0, \hat{\xi}^{(p+1)})$, then one can construct a homogeneous canonical transformation $\{X_j(x^{(p+1)}, \xi^{(p+1)}), \Xi_j(x^{(p+1)}, \xi^{(p+1)})\}^d_{j=p+1}$ such that

$$X_{p+1} = \phi_p,\ X_j(0, \hat{\xi}^{(p+1)}) = 0,\ p+2 \leqq j \leqq d,\ \Xi_j(0, \hat{\xi}^{(p+1)}) = 0, \\ p+1 \leqq j \leqq d-1,\ \Xi_d(0, \hat{\xi}^{(p+1)}) \neq 0.$$

When $d\phi_p$ is proportional to $\sum_{i=p+1}^{d} \xi_i dx_i$ at $(0, \hat{\xi}^{(p+1)})$, making a linear change of coordinates $x^{(p+1)}$, if necessary, we may suppose that

$$(9.1)\quad \begin{cases} \partial\phi_p/\partial x_{p+1} \neq 0,\ \partial\phi_p/\partial x_j = 0, p+2 \leqq j \leqq d,\ \partial\phi_p/\partial\xi_j = 0, p+1 \leqq j \leqq d, \\ \partial^2\phi_p/\partial\xi^2_{p+1} = 0 \text{ at } (0, \hat{\xi}),\ \hat{\xi} = (0, \cdots, 0, \hat{\xi}_{p+1}, 0, \cdots 0). \end{cases}$$

Therefore, we are led to the following operators.

$$(9.2)\quad \begin{cases} \xi_0^2-\sum_{i=0}^{p}(x_i-x_{i+1})^2q_i(X,\xi)-\sum_{i=1}^{p}\xi_i^2r_i(X,\xi)-\psi_p(x^{(p+1)},\xi^{(p+1)})q_{p+1}(X,\xi) \\ \text{with } (\partial^2\psi_p/\partial\xi_{p+1}^2)(0,\hat{\xi}^{(p+1)})=0,\ \hat{\xi}=(0,\cdots,\hat{\xi}_d), \end{cases}$$

$$(9.3)\quad \begin{cases} \xi_0^2-\sum_{i=0}^{p-1}(x_i-x_{i+1})^2q_i(X,\xi)-\sum_{i=1}^{p}\xi_i^2r_i(X,\xi)-\{(x_p-\phi_p(x^{(p+1)},\xi^{(p+1)}))^2+ \\ +\psi_p(x^{(p+1)},\xi^{(p+1)})\}q_p(X,\xi), \text{ with } \phi_p,\ \psi_p \text{ satisfying (9.1).} \end{cases}$$

We consider the operators of type (9.3). Set

$$x''=(x_{p+1},\tilde{x}),\ \xi''=(\xi_{p+1},\tilde{\xi}),\ x''(\mu)=(\mu x_{p+1},\mu^{1/2}\tilde{x}),\ \xi''(\mu)=(\xi_{p+1},\mu^{1/2}\tilde{\xi}),$$

and make a change of scale of coordinates; $y_0=\mu x_0$, $y'=\mu x'$, $y_{p+1}=\mu x_{p+1}$, $\tilde{y}=\mu^{1/2}\tilde{x}$.

Then it results

$$\begin{aligned} &\xi_0^2-\mu^2\sum_{i=0}^{p-1}(x_i-x_{i+1})^2\tilde{q}_i(X,\xi,\mu)-\sum_{i=1}^{p}\xi_i^2\tilde{r}_i(X,\xi,\mu)- \\ &-\mu^2(x_p-\tilde{\phi}(x'',\xi'',\mu))^2\tilde{q}_p(X,\xi,\mu)-\tilde{\psi}(x'',\xi'',\mu)\tilde{q}_p(X,\xi,\mu)+ \\ &+\mu\tilde{T}_1(X,\xi,\mu)+\mu\tilde{T}_0(X,\xi,\mu)\xi_0, \end{aligned}$$

where $\tilde{\phi}(x'',\xi'',\mu)=\mu^{-1}\phi_p(x''(\mu),\ \xi''(\mu))$, $\tilde{\psi}(x'',\xi'',\mu)=\psi_p(x''(\mu),\xi''(\mu))$ and $\tilde{f}(X,\xi,\mu)=f(\mu x_0,\mu x',x''(\mu),\xi',\xi''(\mu))$.

When we define extensions of $\tilde{q}_i$ and others following section 2, a little modifications are needed, but for extended symbols, exactly the same reasoning as in section 3 through 8 are applicable to this case and we obtain theorems 1. 1 and 1. 2.

Here we shall indicate the needed modifications briefly. We define $q_{i,2}(X,\xi,\mu)=\tilde{q}_{i,1}(x'',\xi,\mu)+q_{i,2}(X,\xi,\mu)$ by

$$\begin{aligned} q_{i,2}(X,\xi,\mu)=\{q_i(X,\xi,\mu)-q_i(0,x''(\mu),0,\xi''(\mu))\}\eta_1(\delta^{-2}\mu^{-2}|\xi'|^2|\xi|^{-2})\times& \\ \times\eta_2(\delta^{-2}(\xi''|\xi|^{-1}-\hat{\xi}''))&, \end{aligned}$$

$$\begin{aligned} &q_{i,1}(x'',\xi,\mu)=\{\tilde{q}_i(0,x''(\mu),0,\xi''(\mu))- \\ &\quad q_i(0,\hat{\xi})|\xi|^2\}\eta_2(\delta^{-1}(\xi''|\xi|^{-1}-\hat{\xi}''))+q_i(0,\hat{\xi})|\xi|^2. \end{aligned}$$

It is clear that $q_{i,1}\in(Es)^{2,0}_{0,0}$, $q_{i,2}\in(Es)^{2,1}_{1,0}$, $\partial_x^\alpha\partial_{\xi''}^\gamma q_{i,2}\in(Es)^{2,1}_{1,0}$. Similarly, $\tilde{r}_i$ will be extended to $r_i(X,\xi,\mu)=r_{i,1}(x'',\xi,\mu)+r_{i,2}(X,\xi,\mu)$. ξ_i is extended to $\tilde{\Gamma}_i(\xi,\mu)=\mu\eta_3(\mu^{-1}(\xi_i|\xi|^{-1}-\hat{\xi}_i))|\xi|\in(Es)^{1,1}_{1,0}$. Then the following properties are easily verified.

$$\begin{aligned} &\mu^2Y_i(X)^2q_i,\ \tilde{\Gamma}_i^2r_i\in(Es)^{2,2}_{1,0},\ \partial_x^\alpha\partial_{\xi''}^\gamma(\mu^2Y_i(X)^2q_i),\ \partial_x^\alpha\partial_{\xi''}^\gamma(\tilde{\Gamma}_i^2r_i)\in(Es)^{2-|\gamma|,2}_{1,0}, \\ &\quad \mathrm{op}(\mu^2Y_i(X)^2q_i)-\mathrm{op}(\mu^2Y_i(X)^2q_i)^*\in(Es)^{1,2}_{1,0}, \\ &\quad \mathrm{op}(\tilde{\Gamma}_i^2r_i)-\mathrm{op}(\tilde{\Gamma}_i^2r_i)^*-L_i\cdot\tilde{\Gamma}_i\in(Es)^{0,0}_{1,0},\ \text{with } L_i\in(Es)^{0,0}_{1,0}. \end{aligned}$$

Next, we define $\psi(X,\xi,\mu)=\psi_1(x'',\xi,\mu)+\psi_2(x'',\xi,\mu)$ which is an extension of $\tilde{\psi}(x'',\xi'',\mu)$ by

$$\psi_1(x'',\xi,\mu)=\mu\sum_{|\alpha+\beta|=2}\frac{1}{\alpha!\beta!}Z(x,\mu)^\beta\Gamma(\xi,\mu)^\alpha\partial_x^\beta\partial_\xi^\alpha\psi_p(0,\hat{\xi}'')|\xi|^{-|\alpha|},$$

$$\psi_2(x'',\xi,\mu)=\{\psi\,(x''(\mu),\xi''(\mu))-\sum_{|\alpha+\beta|=2}\frac{1}{\alpha!\beta!}(x''(\mu))^{\beta}(\xi''(\mu)\,|\xi|^{-1}-$$
$$-\hat{\xi}'')^{\alpha}\partial_x^{\beta}\partial_\xi^{\alpha}\psi_p(0,\hat{\xi}'')\}\,\eta_1(\mu^{-1}|\tilde{\xi}|^2|\xi|^{-2})\,\eta_1(\mu^{-1}(\xi_{p+1}|\xi|^{-1}-$$
$$-\hat{\xi}_{p+1}))\,\eta_1(\mu^{-1}|\tilde{x}|^2),$$

where $Z_i(x,\mu)$, $\Gamma_i(\xi,\mu)$, $p+2\leqq i\leqq d$ are the same ones in section 2 and $Z_{p+1}(x,\mu)=\mu^{1/2}x_{p+1}$. Since $(\partial^2\psi_p/\partial\xi_{p+1}^2)(0,\hat{\xi}^{(p+1)})=0$, it follows that $\psi_1\in(Es)^{0,2}_{1/2,1/2}$, $\psi\in(Es)^{0,2}_{1,1/2}$ and $\partial_x^{\alpha}\partial_{\xi''}^{\gamma}\psi\in(Es)^{-|\gamma|,1}_{1,1/2}$ for $|\alpha+\gamma|\leqq 2$. Moreover, for instance, we have

$$\mathrm{op}(\psi q_p)-\mathrm{op}(\psi q_p)^*-i\sum_{j=1}^{d}\mathrm{op}(C_j\partial_{x_j}\partial_{\xi_j}\psi_p(0,\hat{\xi}'')\,|\xi|^{-1}q_p)\in(Es)^{1,3/2}_{1,1/2},$$

with $C_j(x'',\xi,\mu)\in(Es)^{0,0}_{1/2,1/2}$ being equal to 1 in $\{|\tilde{\xi}|<2\mu^{1/2}|\xi|,\ |\tilde{x}|<2\mu^{1/2}\}$. In the same way, we extend $\tilde{\phi}(x'',\xi'',\mu)$ to $\phi(x'',\xi,\mu)$ which belongs to $(Es)^{0,0}_{1,1/2}$. If we remark the following expression,

$$\phi(x'',\xi,\mu)=ax_{p+1}+\hat{\phi}(x'',\xi,\mu),\ \text{with}\ a=(\partial\phi_p/\partial x_{p+1})(0,\hat{\xi}^{(p+1)}),$$
$$\hat{\phi}(x'',\xi,\mu)\in(Es)^{0,1}_{1,1/2},$$

it follows that $\phi^{(\gamma)}_{(\alpha)}(x'',\xi,\mu)\in(Es)^{-|\gamma|,0}_{1,1/2}$ if $|\alpha+\gamma|\leqq 1$.

Now it is easy to verify that the extended symbols have the required properties listed in section 2.

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References

[1] R. Beals and C. Fefferman, Spatially inhomogeneous pseudo-differential operators, I, Comm. Pure Appl. Math., **27** (1974), 1-24.
[2] L. Boutet de Monvel, Hypoelliptic operators with double characteristics and related pseudo differential operators, Comm. Pure Appl. Math., **27** (1974), 585-639.
[3] A. P. Calderón and R. Vaillancourt, On the boundedness of pseudo-differential operators, J. Math. Soc. Japan, **23** (1971), 374-378.
[4] L. Hörmander, The Cauchy problem for differential equations with double characteristics, Journal d'analyse math. **32** (1977), 118-196.
[5] V. Ja. Ivrii and V. M. Petkov, Necessary conditions for the Cauchy problem for non-strictly hyperbolic equations to be well-posed, Russian Math. Surveys, **29** (1974), 1-70.
[6] V. Ja. Ivrii, Sufficient conditions for regular and completely regular hyperbolicity, Trans. Moscow Math. Soc., 1 (1978), 1-65.
[7] N. Iwasaki, Cauchy problem for effectively hyperbolic equations (a special case), preprint (1983).
[8] N. Iwasaki, Cauchy problem for effectively hyperbolic equations (a standard type), preprint (1983).
[9] H. Kumanogo, Pseudo Differential Operators, M. I. T. press (1982).
[10] A. Melin, Lower bounds for pseudo-differential operators, Ark. Mat., **9** (1971), 117-140.
[11] T. Nishitani, Sur les opérateurs fortement hyperboliques qui ne dédendent que du temps, Séminaires sur les équations aux dérivées partielles hyperboliques et holomorphes, Univ. de Paris VI, (1981-82).

[12] T. Nishitani, Weakly hyperbolic Cauchy problem for second order operators, to appear in Publ. RIMS, Kyoto Univ., 21 (1985)

[13] T. Nishitani, A note on reduced forms of effectively hyperbolic operators and energy integrals, Osaka Jour. Math., 21 (1984), 843-850.

[14] K. Taniguchi, Multi-products of Fourier integral operators and the fundamental solution for a hyperbolic system with involutive characteristics, to appear.