J. Math. Kyoto Univ. (JMKYAZ)
23-3 (1983) 503-562

The Cauchy problem for effectively hyperbolic equations (a special case)

By

Nobuhisa IWASAKI

(Received June 12, 1982)

§ 0. Introduction.

The results by V. Ya Ivrii and V. M. Petkov [4] have a proposition come to mind. They conjecture that effectively hyperbolic operators must be strongly hyperbolic. The converse of this conjecture is one of their results. In other words a Cauchy problem for a partial differential operator P will be $C^{+\infty}$-well posed independent of the lower order terms if the fundamental (Hamilton) matrix of the principal part P_m has two non-zero real eigenvalues at the singular points of $P_m=0$. (Also refer to L. Hörmander [1].) Some authors have studied related problems after P. A. Oleinik [8]. Her result proves the conjecture in the case that $m=2$ and that some restrictive relations are assumed between an initial surface and the principal part, for example, it is required that the projection of the singular points of characteristics onto the base space should be included in an initial surface. On the other hand V. Ya Ivrii [3] show that the conjecture is true if the characteristics have been union of two simple characteristics. ("Effectively hyperbolic" means that they intersect non-involutively.) In two dimmensional cases the result of O. A. Oleinik implies the complete proof to the conjecture and T. Nishitani [7] has also treated more general types with analytic coefficients. However, it is still open in general.

We shall here prove the conjecture to a standard type of second order equations. It is an equation added a perturbed term of a second order operator with a non-negative principal symbol to one treated by V. Ya Ivrii. Precisely, we consider it to an operator P with a principal symbol P_2 of the form (0.1).

$$P_2=-(\xi_0-\Lambda_1)(\xi_0-\Lambda_0)+b_2, \tag{0.1}$$

where $\Lambda_j=\Lambda_j(x_0, x, \xi)$ $(j=0, 1)$ are real pseudodifferential operators in x of homogeneous order 1 with a parameter x_0 and $b_2=b_2(x_0, x, \xi)$ is a non-negative one of homogeneous order 2. It means that P_2 is hyperbolic with respect to the ξ_0-direction.

Remark. We consider Cauchy problems on domains of $\boldsymbol{R}^{n+1}$ and denote the variables by (x_0, x), where x_0 is one variable of $\boldsymbol{R}$ and $x=(x_1, \cdots, x_n)$ are other n-variables of $\boldsymbol{R}^n$. (ξ_0, ξ) are their dual variables. $X_0=(x_0, \xi_0)$. $X=(x, \xi)$.

Let us put main assumptions (0.2–3) and an additional assumption (0.4).

(0.2) $$\{\xi_0-\Lambda_0, \xi_0-\Lambda_1\}>0$$

on the double characteristic points of P_2, that is, if $\xi_0-\Lambda_0=\xi_0-\Lambda_1=0$ and $b_2=0$ at (X_0, X), where $\{p, q\}$ stands for the Poisson bracket

$$\Sigma_{j=0}^{n}((\partial/\partial\xi_j)p(\partial/\partial x_j)q-(\partial/\partial x_j)p(\partial/\partial\xi_j)q).$$

(0.3) $$\{\xi_0-\Lambda_0, b_2\}=cb_2,$$

where c is a pseudodifferential operator in x of homogeneous order 0.
(0.4) b_2 is uniformly positive out side a bounded set of x.

Remark. All pseudodifferential operators are sufficiently smooth in their variables uniformly in x variables and on bounded sets of x_0 variable in consideration of their order.

Remark. The assumptions (0.1–3) mean that P_2 is effectively hyperbolic.

Let P be a second order operator with the principal symbol P_2, a differential operator in x_0 and a classical pseudodifferential operator in other variables x. Our main result is as follows.

Theorem 1. *If an operator P satisfies the assumptions (0.1–4), then there exists a constant l such that for any function f of $\boldsymbol{H}^S$ vanishing at $x_0<0$, a function u of $\boldsymbol{H}^{S-l}$ vanishing at $x_0<0$ satisfies*

(0.5) $$Pu=f$$

at $x_0<1$ and

(0.6) $$\|u\|_{S-l}\leq C_S\|f\|_S.$$

And also such a function u vanishes at $x_0<1$ if f vanishes at $x_0<1$.

Remark. The constant l depends on the first order term of P. The regularity of data and solutions, that is, the bounds to s of $\boldsymbol{H}^S$ and $\boldsymbol{H}^{S-l}$ depend on the regularity of coefficients of P. Especially if they are infinitely differentiable, the theorem holds for any real s, that is, there exist smooth solutions to smooth data.

We take V. Ya Ivrii's method to prove the theorem. The operator P intertwines to a fractional power of another operator and is reduced to an operator to which the energy method is applicable.

A Simple Example. Let us consider it on $\boldsymbol{R}^3$. The variables and dual variables are denoted by (t,x,y) and (τ, ξ, η), respectively. Let us give a principal symbol such that

$$p_2=\tau^2-(t\eta+\xi)^2-2^{-1}x^2\eta^2.$$

It is clearly hyperbolic with respect to the direction $(\tau, 0, 0)$. It is shown that the canonical type is

$$p_2=2^{-1}\sigma^2-s^2\omega^2-\zeta^2$$

by the canonical transform such that $\sigma=2\tau+x\eta$, $s=t+\xi/\eta$, $\zeta=\tau+x\eta$, $z=t/2-\xi/(2\eta)$, $\omega=\eta$ and $w=y+x\xi/\eta$. The standard type treated in this paper is, however, that

$$p_2=\tau^2-\Lambda_0^2-b_2,$$
$$\Lambda_0=2^{-1/2}(t+\xi/\eta-2^{-1/2}x)|\eta|$$

and

$$b_2=2^{-1}(t\eta+\xi+2^{-1/2}x\eta)^2.$$

Then it holds that

$$\{\tau-\Lambda_0,\ \tau+\Lambda_0\}=2(\partial/\partial t)\Lambda_0=2^{1/2}|\eta|$$

and

$$\{\tau-\Lambda_0,\ b_2\}=0.$$

Here we consider it on a neighborhood of $\{b_2=0\}$, namely, at $\eta\neq 0$. So they should be modified on a conic neighborhood of $\{\eta=0\}$.

Remarks on Notations. 1) Throughout this paper, symbols of pseudodifferential operators are the Weyl symbols because it holds the correspondence between the facts that an operator is symmetric on the natural duality on $\boldsymbol{R}^n$ and that the symbol is real. An operator $q(x, D)$ with the symbol $q(x, \xi)$ is defined by

$$q(x, D)\phi=(2\pi)^{-n}\int e^{i(x-y)\xi}q((x+y)/2, \xi)\phi(y)dxd\xi. \tag{0.7}$$

(For example, refer to C. Iwasaki and N. Iwasaki [5].)

2) When we call q a pseudodifferential operator in x with no note, q is classical one and may depend on x_0-variable as a parameter. In other words q is a classical pseudodifferential operator valued function in x_0-variable.

3) Sometimes we don't distinguish "operator" and "symbol" in terms. For example, an operator A is equal to $q(x, \xi)$. It means that A is equal to a pseudodifferential operator with the symbol $q(x, \xi)$.

§ 1. Transformation by an operator power of operator.

We consider another type of problem, which is equivalent to one stated at the introduction and convenient to proofs. It is global also in x_0-variable and has a parameter λ and a weight function ψ. It includes the equation to $\exp(-\lambda x_0)u$ replacing solutions u after changing the scale as a bounded interval in x_0-variable comes to the whole space $\boldsymbol{R}$.

Let us define a function ψ as (1.1), where ν is a real parameter.

$$\psi=\exp[-(|\nu x_0|^2+1)^{1/2}]. \tag{1.1}$$

The operator P is written as following forms.

$$P=a_1a_0+\psi^2b_2+d_1a_0+a_1d_0+\psi^2b_1+b_0. \tag{1.2}$$

$$a_j=i(\xi_0-\psi\Lambda_j)+\lambda,\ (j=0, 1), \tag{1.3}$$

where $\Lambda_1=\Lambda_1(x_0, x, \xi)$ is a real pseudodifferential operator of order 1 in x, Λ_0 is identically equal to zero and λ is constant.

$$b_2 \geq 0, \tag{1.4}$$

where b_2 is a pseudodifferential operator of order 2 in x and $b_2 \geq \varepsilon \langle D\rangle^2 > 0$ if $|x| \geq M > 0$ for a sufficiently large M. b_1 is a pseudodifferential operator of order 1 in x, and d and b_0 are ones of order 0.

Remark. $\langle D\rangle=(|\xi|^2+1)^{1/2}$.

The assumptions (0.2–3) are replaced by (1.5–6).

$$\{\psi^{-1}\xi_0-\Lambda_0, \psi^{-1}\xi_0-\Lambda_1\} \geq \varepsilon\langle D\rangle > 0, \tag{1.5}$$

if $(\Lambda_0-\Lambda_1)^2+b_2<\delta\langle D\rangle^2$ at (X_0, X) for a positive constant δ.

$$\{a_0, b_2\} = \psi c_0 b_2, \tag{1.6}$$

where c_0 is a pseudodifferential operator of order 0 in x.

Remarks. 1) Any symbol of P is sufficiently smooth. g $(=\Lambda_j, b_j, c_j$ or $d)$ and $\psi^{-1}(\partial/\partial x_0)^\alpha(\partial/\partial X)^\beta g$ are uniformly bounded on $\boldsymbol{R}^{2n+1}$ in consideration of their orders.

2) If (1.5) satisfies for sufficiently large $|\xi|$, it holds for any ξ after some modifications of Λ_j at the bounded set of ξ.

3) There is nothing against assuming for Λ_0 to be identically equal to zero because the Yu. V. Egorov's result [2] assures it.

We shall prove the following theorem instead of Theorem 1.

Theorem 2. *There exist constants l and ν such that for any datum f of $\boldsymbol{H}^S$ a solution u of $\boldsymbol{H}^{S-l}$ uniquely exists and satisfies that $Pu=f$ on the whole space if λ is sufficiently large, which may depends on s. Here P is defined by (1.1–6) and $\boldsymbol{H}^S$ are the Sobolev's spaces on $\boldsymbol{R}^{n+1}$. Moreover it satisfies the estimate*

$$\|u\|_{S-l} \leq C_S(\lambda)\|f\|_S, \tag{1.7}$$

where $C_S(\lambda)$ grows in a polinomial order at most as λ tends to infinity.

(1.8) is one of the usual definitions to the fractional powers of operators if it is well defined.

$$\Lambda^{n-\alpha}=\Gamma(\alpha)^{-1}\int_0^{+\infty} t^{\alpha-1}\exp(-\Lambda t)dt\Lambda^n. \tag{1.8}$$

We use this definition to put a bounded operator into α. The Γ-function, however, has no essential effect to reduce the operator P. Therefore we use an operator excluded the Γ-function from (1.8).

Remark. We know no exact definition of operator powers when an operator put into α of (1.8) and Λ are not commutative to each other. It suffices to define an operator like one because we need only a part of the properties.

Let us consider two one-parameter groups on $\boldsymbol{H}^S$.

$$U(t)=\exp(-a_0 t) \tag{1.9}$$

and

$$W(t)=\exp(ht), \tag{1.10}$$

where a_0 is defined by (1.3) and h is a bounded operator on $\boldsymbol{H}^S$ such that h has norms of $\boldsymbol{H}^S$ with respect to which the operator norms of h on $\boldsymbol{H}^S$ are uniformly bounded for all s. Then it is clear that they are well defined and have the estimates (1.11–12) on $\boldsymbol{H}^S$ because $a_0|_{\lambda=0}$ is formally skew-symmetric and h is uniformly bounded.

$$\|U(t)\|_{S,\lambda}\leq\exp[(\lambda_S-\lambda)t] \quad \text{for } t\geq 0. \tag{1.11}$$

$$\|W(t)\|_{S,\lambda}\leq C_S\exp(N_0|t|). \tag{1.12}$$

Remark. 1) If we choose the suitable norms of $\boldsymbol{H}^S$, we can put $C_S=1$ in (1.12). 2) $\|v\|_{S,\lambda}$ is a norm of $\boldsymbol{H}_S$ defined as

$$\|v\|_{S,\lambda}^2=\|E^{S/2}v\|^2 \tag{1.13}$$

by means of $E=a_0^*a_0+\langle D\rangle^2$.

We define an operator F by (1.14).

$$F=a_0^k\int_0^{+\infty}t^{k-1}U(t)\,W(\log t)dt. \tag{1.14}$$

It is well defined as an operator from $\boldsymbol{H}^S$ to $\boldsymbol{H}^{S-k}$ if

$$k>N_0 \text{ and } \lambda>\lambda_S. \tag{1.15}$$

We want to find an operator $P^\sim$ such that $FPF^\sim=P^\sim$ with another $F^\sim$ and the energy method is applicable to the equation with respect to $P^\sim$. (We shall call such $P^\sim$ a basic type.) So we try the commutation of F and P. Then we have a lemma.

Lemma 1.1. *Let us denote the operator (1.2) by $P=a_1a_0+b$. Then we have,*

$$\begin{aligned}&\int_0^{+\infty}t^{k-1}a_0^kU(t)\,W(\log t)dt[a_1a_0+b]\\ &=[a_1a_0+b-\mathrm{ad}\,a_0(a_1)h]\int_0^{+\infty}t^{k-1}a_0^kU(t)\,W(\log t)dt\\ &\quad+[\mathrm{ad}h(a_1a_0+b)+k\,\mathrm{ad}\,a_0(\mathrm{ad}h(a_1a_0+b))a_0^{-1}]\\ &\qquad\times\int_0^{+\infty}(\log t)t^{k-1}a_0^kU(t)\,W(\log t)dt\\ &\quad+\int_0^{+\infty}Z(t)t^{k-1}a_0^kU(t)\,W(\log t)dt,\end{aligned} \tag{1.16}$$

where $Z(t)$ in the last term is described as (1.17).

$$\begin{aligned}&Z(t)=\textstyle\sum_{j=1}^8 Z_j,\\ &Z_1=-\mathrm{ad}\,a_0(b)t,\\ &Z_2=-\mathrm{ad}\,a_0(a_1)(\mathrm{ad}a_0^kU(t))(h)\,a_0^{-k}U(-t),\\ &Z_3=(\mathrm{ad}a_0^k)(b)a_0^{-k},\end{aligned} \tag{1.17}$$

$$Z_4=(\log t)[(\mathrm{ad}a_0^k U(t))(\mathrm{ad}h(a_1a_0+b))a_0^{-k}U(-t)$$
$$-k\,\mathrm{ad}\,a_0(\mathrm{ad}h(a_1a_0+b))a_0^{-1}],$$
$$Z_5=-t(\mathrm{ad}a_0^k)(\mathrm{ad}a_0(a_1a_0+b))a_0^{-k},$$
$$Z_6=\sum_{j=2}^k C_{kj}(\mathrm{ad}\,a_0)^j(a_1)a_0^{-j+1},$$
$$Z_7=a_0^k U(t)\int_0^{\log t}W(\sigma)(\mathrm{ad}h)^2(a_1a_0+b)W(-\sigma)$$
$$\times(\log t-\sigma)\,d\sigma\,a_0^{-k}U(-t)$$

and

$$Z_8=a_0^k\int_0^t U(\sigma)(\mathrm{ad}\,a_0)^2(a_1a_0+b)\,U(-\sigma)(t-\sigma)d\sigma\,a_0^{-k}.$$

Remark on Notations. We denote the commutator of operators A and B by $\mathrm{ad}\,A(B)=AB-BA$.

Remark. This lemma will be proved mainly by the formula (1.18) of a one-parameter group $V(t)=\exp(-At)$. (Taylor expansion in t.)

(1.18) $$V(-t)BV(t)=\sum_{j=0}^k(j!)^{-1}t^j(\mathrm{ad}A)^j(B)$$
$$+(k!)^{-1}\int_0^t V(-s)(\mathrm{ad}\,A)^{k+1}(B)\,V(s)(t-s)^k ds.$$

From the right hand side of (1.16) we operate G (1.19) and the inverse of H (1.20), which is guaranteed by Lemma 1.3 if k is sufficiently large.

(1.19) $$G=\int_0^{+\infty}W(-\mathrm{logt})\,U(t)\,a_0^k t^{k-1}\,dt.$$

(1.20) $$H=2^{-2k+1}(2k-1)!\int_{-1}^{+1}W(\log[(1+\sigma)/(1-\sigma)])(1-\sigma^2)^{k-1}\,d\sigma.$$

Then the main parts are discribed as (1.21).

Lemma 1.2. *Let F, G and H be defined by* (*1.14*), (*1.19*) *and* (*1.20*), *respectively. Then we have*

(1.21) $$F[a_1a_0+b]GH^{-1}$$
$$=a_1a_0+b-\mathrm{ad}\,a_0(a_1)h$$
$$+\mathrm{ad}h(a_1a_0+b)\,a_0\int_0^{+\infty}(\log t)\,U(t)dt$$
$$+k\,\mathrm{ad}\,a_0(\mathrm{ad}h(a_1a_0+b))\int_0^{+\infty}(\log t)\,U(t)dt$$
$$+a_1a_0+d_1a_0$$
$$+R\quad(\textit{Remainder terms}).$$

The operators F, G and H have to be invertible on suitable Sobolev's spaces in order that the right hand side of (1.21) will be a reduction of the operator P to a basic type. In fact we have a following lemma.

Lemma 1.3. *H (1.20), which is a bounded operator on $\mathbf{H}^S$, is invertible on $\mathbf{H}^S$ for any s if k is sufficiently large. This fact implies the invertibility of F (1.14) and G (1.19) because they have relations (1.22–23) between them.*

$$FG=H+H_1 \tag{1.22}$$

and

$$GF=H+H_2, \tag{1.23}$$

where the norms of $H^{-1}H_j$ and $H_jH^{-1}(j=1,2)$ on $\mathbf{H}^S$ become small as λ tends to infinity.

We next choose the bounded operator h to adjust the first order term of the operator P, which put it hard to apply the energy method. In the same time it also needs to regulate the logarithmic terms which appear at the second and third terms of (1.21) on account of the commutation by the operator powers of operators. We have Lemma 1.4 to the first order term and Lemma 1.5 to the logarithmic terms.

Lemma 1.4. *There exist a real pseudodifferential operator h_0' of order 0 in x and a real constant θ such that, if h_0 is defined by (1.24), h_0 satisfies (1.25), where d_1 is a pseudodifferential operator of order 1 in x satisfying (1.26), c_0, c_1 and d_0 are of order 0 and c_2 is of order -1, respectively.*

$$h_0=h_0'+i\theta. \tag{1.24}$$

$$\psi^2b_1-\mathrm{ad}\,a_0(a_1)h_0 \tag{1.25}$$
$$=c_0a_0+a_1c_1+\psi^2c_2b_2+\psi^2d_1+d_0.$$

$$d_1=d_1^*\geq\varepsilon\langle D\rangle>0 \text{ on } \mathbf{H}^0, \tag{1.26}$$

that is, d_1 is symmetric and exact positive on $\mathbf{H}^0$.

Remark. 1) We can have ε be sufficiently large. 2) h_0' may be chosen such that h_0' is bounded by $2N_0$ on the whole space if $\psi^2b_1/\mathrm{ad}a_0(a_1)$ is bounded by a constant N_0 on the characteristic set Σ of P. We may also put that $\theta=\varepsilon N_1^{-1}$ if $\mathrm{ad}\,a_0(a_1)\leq -N_1\psi^2\langle D\rangle$ on Σ with a positive constant N_1.

Lemma 1.5. *There exists a pseudodifferential operator h_1 of order 0 in x, which is a linear combination of $\Lambda_0-\Lambda_1$, b_2 and ∂b_2, such that*

$$\{p_2, h_1\}=\psi h_2(\xi_0-\psi\Lambda_1)+\psi h_3(\xi_0-\psi\Lambda_0)+\psi^2h_4, \tag{1.27}$$

where $h_j(j=2, 3$ and $4)$ are pseudodifferential operators of order $[j/4]$ in x satisfying with positive constants ε_0, ε_1 and ε_2 that

$$h_j\geq\varepsilon_0(1+\nu^2\Omega)\geq\varepsilon_1\nu|h_1| \quad (j=2, 3) \tag{1.28}$$

and

$$|h_4|^2\leq 2(1+5\varepsilon_2)^{-1}h_2h_3b_2 \tag{1.29}$$

when

$$\Omega=[(\Lambda_0-\Lambda_1)^2+b_2]\langle\xi\rangle^{-2}.$$

Remark. Only arguments with respect to Lemma 1.5 require the restriction to ν of ψ.

We define an operator h by (1.30–31).

$$h_0^{\sim}=h_0'+\nu h_1. \tag{1.30}$$

$$h=h_0^{\sim}E^{-1}\langle D\rangle^2+i\theta, \tag{1.31}$$

where h_0' and θ are h_0' and θ at Lemma 1.4, h_1 is h_1 at Lemma 1.5 and E is an operator defined by $E=a_0^*a_0+\langle D\rangle^2$.

Corollary. 1) *$h_0^{\sim}+i\theta$ also satisfies that*

$$\begin{aligned}\psi^2 b_1-\mathrm{ad}\,a_0(a_1)[h_0^{\sim}+i\theta]\\ =c_0a_0+a_1c_1+\psi^2c_2+\psi^2d_1+d_0,\end{aligned} \tag{1.32}$$

where d_1 is a pseudodifferential operator of order 1 in x satisfying (1.26), c_0, c_1 and d_0 are of order 0 and c_2 is one of order 1 being a linear combination of b_2 and ∂b_2.

2) *There exist pseudodifferential operators $h_j^{\sim}$ ($j=2$, 3 and 4) of order $[j/4]$ such that*

$$\begin{aligned}\{p_2, h_0^{\sim}\}=\psi h_2^{\sim}(\xi_0-\psi\Lambda_1)+\psi h_3^{\sim}(\xi_0-\psi\Lambda_0)+\psi^2h_4^{\sim},\\ h_j^{\sim}\geq\varepsilon_0\nu(1+\nu^2\Omega)\geq\varepsilon_1\nu|h_0^{\sim}| \quad (j=2 \text{ and } 3)\end{aligned} \tag{1.33}$$

and

$$|h_4^{\sim}|^2\leq 2(1+4\varepsilon_2)^{-1}h_2^{\sim}h_3^{\sim}b_2,$$

with some positive constants ε_0, ε_1 and ε_2 if ν is sufficiently large.

We have the following properties on the real part of h. This yields the assumption to h at (1.10).

Lemma 1.6. 1) *E^{-1}, which exists for sufficiently large λ's independent of s, is a bounded operator from H^S to H^{S+2} which satisfies the estimates*

$$\|E^{-1}u\|_{s+2}+\lambda^2\|E^{-1}u\|_s\leq C_0\|u\|_s+C_s\|u\|_{s-1}, \tag{1.34}$$

where C_0 is independent of s, and

$$\|E^{-1/2}a_0u\|_s+\|E^{-1/2}\langle D\rangle u\|_s+\lambda\|E^{-1/2}u\|_s\leq C_0\|u\|_s+C_s\|u\|_{s-1}. \tag{1.35}$$

2) *h (1.31) is a bounded operator on H^S and satisfies that*

$$|\mathrm{Re}(hu, u)_{s,\lambda}|\leq C_0\|u\|_{s,\lambda}^2+C_s\|u\|_{s-1,\lambda}^2 \tag{1.36}$$

with respect to an inner product

$$(u, v)_{s,\lambda}=(E^Su, v),$$

where C_0 is independent of s and θ. $h_R=h-i\theta=h_0^{\sim}E^{-1}\langle D\rangle^2$ also satisfies that

$$\|h_Ru\|_{s,\lambda}\leq C_0\|u\|_{s,\lambda}+C_s\|u\|_{s-1,\lambda}. \tag{1.37}$$

Remark. It is important for C_0 to be independent of s.

According to the above lemmas we get the conclusion of this section.

Theorem 3. *If the bounded operator h is defined by (1.31) and if the parameter ν is sufficiently large, then the right hand side of (1.21) comes to a basic type which is stated at the top of next section after some arrangement of each term.*

Remark. Each proposition stated without proof in this section will be verified at Section 3, 4 and 5.

§ 2. A basic type and the energy method.

Let us define the basic type stated in Theorem 3 at the previous section and apply the energy method to show the existence and uniqueness of solutions for it.

1) We call an operator P_b a basic type if P_b is written as (2.1), where each term is defined by 2)–12).

(2.1)
$$\begin{aligned}P_b=&(a_1+\psi c_1\log a_0+d_1)(a_0+\psi c_0\log a_0+d_0)\\&+\psi^2 b_2+a_0^{-1}\psi^2 b_1+a_0^{-1}(\log a_0)\psi^2 b_3 e_0+d_2\\&+d_3\psi b_0+a_0^{-1}d_4\psi^2 b_2+(\log a_0)\psi^2 c_3 e_0 e_1 b_4\\=&A_1A_0+B.\end{aligned}$$

2) A weight function ψ is also defined by

(2.2)
$$\psi=\exp[-(|\nu x_0|^2+1)^{1/2}].$$

3) e_0 is an operator of order 0 defined by

(2.3)
$$e_0=E^{-1}\langle D\rangle^2=[a_0^*a_0+\langle D\rangle^2]^{-1}\langle D\rangle^2.$$

e_1 is a system of product sums of operators defined as E^{-1} times q_2, a_0q_1 or $a_0^2q_0$, where q_j are suitable pseudodifferential operators of order j in x and $E=a_0^*a_0+\langle D\rangle^2$. Moreover the norms of elements of e_1 on H^0 are uniformly bounded in the parameter ν.

4) a_j $(j=0, 1)$ are operators such that

(2.4)
$$a_j=i(\xi_0-\psi\Lambda_j)+\lambda,$$

where λ is a positive constant, Λ_1 is a real pseudodifferential operator of order 1 in x and Λ_0 is identically equal to zero.

5) b_2 is a pseudodifferential operator of order 2 in x such that

(2.5)
$$b_2\geq\varepsilon\langle D\rangle \quad \text{on } H^0$$

for a positive constant ε, that is,

$$(b_2u, u)\geq\varepsilon(\langle D\rangle u, u)$$

for any u belonging to $\mathcal{S}$.

6) c_0 and c_1 are operators such that, for $j=0$ and 1,

$$c_j=c_{j0}e_0+c_{j1}\mathrm{ad}\,a_j(e_0), \quad (\mathrm{ad}\,a_0(e_0)=0)$$

where c_{jk} are pseudodifferential operators of order 0 in x, and that

$$\mathrm{Re}(c_j u, u)\geq \varepsilon\nu(e_0 u, u) \tag{2.6}$$

for a positive ε, for sufficiently large ν and for any u belonging to $\mathcal{S}$.

7) b_3 is a pseudodifferential operator of order 1 in x such that $b_3+b_3^*$ is a pseudodifferential operator of order 0 in x, that is, the principal part is pure imaginary. Therefore it satisfies

$$|\mathrm{Re}(b_3 u, u)|\leq C(u, u). \tag{2.7}$$

8) c_3 and b_4 are a pseudodifferential operator of order 0 in x and a system of order 1, respectively, such that b_4 is a linear combination of ∂b_2 and b_2 with coefficients of pseudodifferential operators of order 0 and order -1, and that, for a positive ε_2,

$$|(\psi^3 e_0 e_1 b_4 u, e_1 b_4 u)| \tag{2.8}$$
$$\leq 2(1+2\varepsilon_2)^{-1}[\mathrm{Re}(\psi^3 c_0 b_2 u, u)+C(\nu)(u, u)]$$

and

$$|(\psi e_0 c_3 u, c_3 u)|\leq(1+\varepsilon_2)\mathrm{Re}(\psi c_1 u, u).$$

9) b_1 is an operator of order 1.

Remark on Notations. We call L an operator of order m if

$$\psi^\alpha\langle D\rangle^\beta E^\gamma L E^{-\gamma}\langle D\rangle^{-\beta-m}\psi^{-\alpha},$$

for sufficiently many α, β and γ, are uniformly bounded on $\boldsymbol{H}^0$ as λ tends to infinity.

10) $b_0=(b_{0j})$ is a system of operators of order 1 such that

$$|(\psi b_{0j}u, \psi b_{0j}u)|\leq C(\psi^2 b_2 u, u). \tag{2.9}$$

11) The commutator of a_0 and b_2 satisfies

$$\mathrm{ad}\,a_0(b_2)=\psi c_2 b_2+\psi b_5, \tag{2.10}$$

where b_5 is a pseudodifferential operator of order 1 in x and c_2 is a pseudodifferential operator of order 0 in x.

Remark on Notations. Let us say that d an operator of order 0 satisfies the relation (2.11) if there exist d′ and d″ other operators of order 0, and b' and b'' operators of order 1 such that

$$\psi^2 b_2 d=d'\psi^2 b_2+\psi^2 b' \text{ and } d\psi^2 b_2=\psi^2 b_2 d''+\psi^2 b''. \tag{2.11}$$

12) d_j ($j=0, \cdots, 4$) are operators of order 0 or their systems satisfying the relation (2.11).

Remark. 1) The product of such operators is also an operator of order 0 and satisfies the relation (2.11) in which d is replaced with them.

2) λa_0^{-1} and $U(t)$ are operators of order 0 and satisfy the relation (2.11).

Remark on Notations. $\log \Lambda$ is defined by

$$\log \Lambda = \gamma_0 - \int_0^{+\infty} (\log t) \exp(-\Lambda t) dt \Lambda, \tag{2.12}$$

$$\gamma_0 = \int_0^{+\infty} (\log t) \exp(-t) dt.$$

The operator P_b is clearly one of order 2, so that it is an operator from H^S to H^{S-2}. We consider an equation (2.13) for P_b on H^S.

$$P_b u = f. \tag{2.13}$$

At first we deal with it on H^0. We take the inner product of $P_b u$ and $A_0 u$ to estimate it from the below according to the energy method. Then we shall get Theorem 4.

Theorem 4. *If the parameter* ν *is sufficiently large, then* (2.14–15), *therefore* (2.16–17), *hold for any element* u *of* $\mathcal{S}$.

$$\mathrm{Re}(A_0 u, u) \geq (\lambda - C) \|u\|^2 + \varepsilon \|u\|_{eLog}^2. \tag{2.14}$$

$$\mathrm{Re}((A_1 A_0 + B) u, A_0 u) \geq (\lambda - C) \|A_0 u\|^2 + \varepsilon \|A_0 u\|_{cLog}^2 + \varepsilon \|u\|_{bcLog}^2 + (\lambda - C - C|\log \lambda|) \|u\|_b^2. \tag{2.15}$$

$$\|A_0 u\|^2 \geq (\lambda - C)^2 \|u\|^2 + (\lambda - C) \varepsilon \|u\|_{eLog}^2. \tag{2.16}$$

$$\|(A_1 A_0 + B) u\|^2 \geq (\lambda - C)[(\lambda - C) \|A_0 u\|^2 + \varepsilon \|A_0 u\|_{eLog}^2 + (\lambda - C - C|\log \lambda|) \|u\|_b^2 + \varepsilon \|u\|_{bcLog}^2]. \tag{2.17}$$

Remark on Notations. $\|u\|_{eLog}$, $\|u\|_b$, $\|u\|_{cLog}$ and $\|u\|_{bcLog}$ mean (2.18–21), respectively. These notations are guaranteed by Lemma 2.1.

$$\|u\|_{eLog}^2 = \mathrm{Re}(\psi e_0 (\mathrm{Log}\, a_0) u, u) + \delta(u, u), \tag{2.18}$$

$$\|u\|_b^2 = (\psi^2 b_2 u, u) = (b_2 \psi u, \psi u), \tag{2.19}$$

$$\|u\|_{cLog}^2 = \mathrm{Re}(\psi c_1 (\mathrm{Log}\, a_0) u, u) + \delta(u, u) \tag{2.20}$$

and

$$\|u\|_{bcLog}^2 = \mathrm{Re}(\psi^3 [b_2 c_0 \mathrm{Log}\, a_0 + \delta(e_0 \mathrm{Log}\, a_0 + b_2)] u, u), \tag{2.21}$$

where $\mathrm{Log}\, a_0 = \mathrm{Re}(\log a_0)$ and δ is a sufficiently large fixed constant.

Lemma 2.1. $\|u\|_{eLog}$, $\|u\|_b$, $\|u\|_{cLog}$ *and* $\|u\|_{bcLog}$ *by* (2.18–21) *define norms on* $\mathcal{S}$, *respectively, which are stronger than the norm* $\|u\|$ *of* H^0.

Lemma 2.2. *For sufficiently many integer* l, $E^l P_b E^{-l}$ *is also a basic type if* P_b *is a basic type.*

Since Lemma 2.2 is easily verified, the estimate for P_b on H^S is induced from Theorem 4 with the help of Lemma 1.6 for E.

Corollary. *There exist positive constants* ν *independent of* s *and* C_S *such that for any element* u *of* $\mathcal{S}$,

$$\|(A_1A_0+B)u\|_{S,\lambda}^2 \geq (\lambda - C_S)^4 \|u\|_{S,\lambda}^2, \tag{2.22}$$

where $\|v\|_{S,\lambda}$ *is a norm of* $\boldsymbol{H}^S$ *defined by means of* E *as*

$$\|v\|_{S,\lambda}^2 = \|E^{S/2}v\|^2.$$

Remark. If we rewrite (2.22) to one for a fixed norm $\|u\|_S$ of $\boldsymbol{H}^S$, we have

$$\|(A_1A_0+B)u\|_S^2 \geq M_S(\lambda^2+1)^{-|S|}(\lambda - C_S)^4 \|u\|_S^2, \tag{2.23}$$

with another positive constant M_S.

The operator P_b was one reduced from P (1.2) by F, G and H (1.14, 19 and 20) such that

$$FPGH^{-1} = P_b. \tag{2.24}$$

By Lemma 1.3, F, G and H were invertible and had the relation

$$FG = H + H_1. \tag{2.25}$$

Moreover we can prove the estimates (2.26–28) for sufficiently large λ which may depend on s.

$$\|Fu\|_{S,\lambda} \leq M_S \|u\|_{S+k,\lambda}. \tag{2.26}$$

$$\|Gu\|_{S,\lambda} \leq M_S \|u\|_{S+k,\lambda}. \tag{2.27}$$

$$\|H^{-1}u\|_{S,\lambda} \leq M_S \|u\|_{S,\lambda}. \tag{2.28}$$

Commbining (2.22) and these, we get (2.29) and also (2.30) since the formal dual operator P^* of P is the same type as one of P, namely, P^* satisfies (1.1–6) if the variable X_0 is changed to $-X_0$.

Theorem 5. *Let* P *be an operator defined by* (1.1–6). *Then there exist positive constants* ν, k, C_S *and* M_S *such that for any* u *of* $\mathcal{S}$ *and for* $\lambda \geq C_S$, *the estimates* (2.29–30) *hold.*

$$\|Pu\|_{S+2k,\lambda} \geq M_S(\lambda - C_S)^2 \|u\|_{S,\lambda}. \tag{2.29}$$

$$\|P^*u\|_{S+2k,\lambda} \geq M_S(\lambda - C_S)^2 \|u\|_{S,\lambda}. \tag{2.30}$$

Theorem 5 implies Theorem 2 because the following well known lemma is applicable to it.

Lemma 2.3. *Let* T *be an operator from* $\boldsymbol{H}^S$ *to* $\boldsymbol{H}^{S-m}$ *such that for any* u *of* $\mathcal{S}$, *for a fixed* l *and for sufficiently many* s,

$$\|u\|_S \leq C_S \|Tu\|_{S+l} \tag{2.31}$$

and

$$\|u\|_S \leq C_S \|T^*u\|_{S+l}. \tag{2.32}$$

Then there exists a unique solution u *of* (2.33) *belonging to* $\boldsymbol{H}^{\sigma-l}$ *to any* f *of* $\boldsymbol{H}^{\sigma}$ *for sufficiently many* σ.

$$Tu=f. \tag{2.33}$$

Proof of Theorem 4. At first we prove (2.14). Let us consider the inner product on $\boldsymbol{H}^0$ of $A_j u$ and u ($j=0, 1$), and take the real part.

$$\operatorname{Re}(A_j u, u)=\operatorname{Re}(a_j u, u)+\operatorname{Re}(\psi c_j \log a_0 u, u)+\operatorname{Re}(d_j u, u). \tag{2.34}$$

Since $a_j+a_j^*=2\lambda$ and $\|d_j\|\leq C$ by the assumptions, (2.34) is bounded below as

$$\operatorname{Re}(A_j u, u)\geq(\lambda-C)\|u\|^2+\operatorname{Re}(\psi c_j \log a_0 u, u). \tag{2.35}$$

On the other hand we have Lemma 2.4 for $\psi c_j \log a_0$. This implies (2.36) with another constant C and a positive constant ε.

$$\operatorname{Re}(A_j u, u)\geq(\lambda-C)\|u\|^2+\varepsilon(\psi e_0 \log a_0 u, u). \tag{2.36}$$

If we put $j=0$, then we get (2.14).

Lemma 2.4. 1) *Let us denote the real part of* $\log a_0$ *by* $\operatorname{Log} a_0=\left(\frac{1}{2}\right)[\log a_0+(\log a_0)^*]$. *Then we have*

$$\operatorname{Log} a_0 \geq \log\lambda \quad \text{on} \quad \boldsymbol{H}^0 \quad \text{if } \lambda>0. \tag{2.37}$$

2) $\operatorname{Im}\log a_0=\arg a_0$ *the imaginary part of* $\log a_0$ *is uniformly bounded on* $\boldsymbol{H}^0$ *as* λ *tends to infinity. More exact it is an operator of order* 0.

3) *If* λ *is sufficiently large, there exist positive* ε *and* δ *such that*

$$\operatorname{Re}(\psi c_j \log a_0 u, u)\geq\varepsilon \operatorname{Re}(\psi e_0 \operatorname{Log} a_0 u, u)-\delta(u, u), \tag{2.38}$$

where c_j *is the term* c_j *in* A_j.

At (2.35) we put $j=1$ and replace u with $A_0 u$. Then we obtain

$$\operatorname{Re}(A_1 A_0 u, A_0 u)\geq(\lambda-C)\|A_0 u\|^2+\|A_0 u\|_{cLog}^2. \tag{2.39}$$

We next estimate the inner product of Bu and $A_0 u$.

Noting that $a_0^*=-a_0+2\lambda$ and $(\psi^2 b_2)^*=\psi^2 b_2$, we calculate (2.40). According to the assumption (2.10) we have

$$\begin{aligned} a_0^*\psi^2 b_2+\psi^2 b_2 a_0 &= -\operatorname{ad} a_0(\psi^2 b_2)+2\lambda\psi^2 b_2 \\ &= \psi^2 c b_2+\psi^2 c'+2\lambda\psi^2 b_2, \end{aligned} \tag{2.40}$$

where c' is a pseudodifferential operator of order 1 in x. Since $2\operatorname{Re}(\psi^2 b_2 u, a_0 u)=\operatorname{Re}([a_0^*\psi^2 b_2+\psi^2 b_2 a_0]u, u)$, (2.40) implies (2.41) according to Lemma 2.5.

$$\begin{aligned} &\operatorname{Re}(\psi^2 b_2 u, a_0 u) \\ &\geq\lambda(\psi^2 b_2 u, u)-C(\psi^2 b_2 u, u)-C(\psi^2\langle D\rangle u, u) \\ &\geq(\lambda-C')(\psi^2 b_2 u, u)=(\lambda-C')\|u\|_b^2, \end{aligned} \tag{2.41}$$

because of the assumption (2.5) for b_2.

Lemma 2.5. *If q and $q^{\sim}$ are operators of order* 0 *such that*

(2.42) $$qb_2-b_2q^{\sim} \text{ or } q\psi^2b_2-\psi^2b_2q^{\sim} \text{ is an operator of order } 1,$$

then it holds that

(2.43) $$|\mathrm{Re}(\psi^2qb_2u,u)|\leq C(\psi^2b_2u,u).$$

Especially (2.42) *holds if q is a pseudodifferential operator of order* 0 *in x or if q is equal to e_0.*

We obtain (2.44) for the inner product of $a_0^{-1}\psi^2b_1u$ and a_0u since $a_0^*a_0^{-1}=-1+2\lambda a_0^{-1}$.

(2.44) $$\begin{aligned}\mathrm{Re}(a_0^{-1}\psi^2b_1u,a_0u)&=\mathrm{Re}(a_0^*a_0^{-1}\psi^2b_1u,u)\\&=-\mathrm{Re}(\psi^2b_1u,u)+2\mathrm{Re}(\lambda a_0^{-1}\psi^2b_1u,u).\end{aligned}$$

Since b_1 and λa_0^{-1} are operators of order 1 and of order 0, respectively, ψ^2b_1 and $\lambda a_0^{-1}\psi^2b_1$ are operators of order 1 so that it holds that

(2.45) $$\begin{aligned}&|\mathrm{Re}(a_0^{-1}\psi^2b_1u,a_0u)|\\&\leq C(\psi^2\langle D\rangle u,u)\leq C'(\psi^2b_2u,u)=C'\|u\|_b^2,\end{aligned}$$

where we used the assumption (2.5).

In the same way we are able to estimate $\mathrm{Re}(a_0^{-1}d_4\psi^2b_2u,A_0u)$. In fact we have

(2.46) $$\begin{aligned}d_5&\equiv A_0^*a_0^{-1}d_4\\&=-d_4+2\lambda a_0^{-1}d_4+(\log a_0)^*c_0^*\psi a_0^{-1}+d_0^*a_0^{-1}.\end{aligned}$$

d_4, d_0, a_0^{-1}, $a_0^*c_0^*\psi a_0^{-1}$ and $(\log a_0)^*a_0^{*-1}$ are operators of order 0 and satisfy the condition (2.42) according to the relation (2.11) and Remarks after (2.11), so that d_5 is an operator of order 0 satisfying the condition (2.42). Therefore Lemma 2.5 assures the inequality

(2.47) $$|\mathrm{Re}(a_0^{-1}d_4\psi^2b_2u,A_0u)|\leq C(\psi^2b_2u,u)=C\|u\|_b^2.$$

The terms related to d_0, d_2 and d_3 are easily bounded because they are operators of order 0 and satisfy the relation (2.11).

(2.48) $$|\mathrm{Re}(\psi^2b_2u,d_0u)|\leqq C\|u\|_b^2,$$

according to Lemma 2.5.

(2.49) $$|(a_0^{-1}\psi^2b_1u,d_0u)|\leqq C(\psi^2\langle D\rangle u,u)\leqq C'\|u\|_b^2.$$

(2.50) $$|(d_2u,A_0u)|\leq C\|u\|\,\|A_0u\|.$$

(2.51) $$\begin{aligned}&|(d_3\psi b_0u,A_0u)|\\&\leq C\|\psi b_0u\|\,\|A_0u\|\leq C'\|u\|_b\|A_0u\|,\end{aligned}$$

because $(\psi b_0u,\psi b_0u)\leq C(\psi^2b_2u,u)$.

The remainded terms include $\log a_0$. Lemma 2.5 implies (2.52) since $c_0 \mathrm{Im} \log a_0$ satisfies (2.42) by the definition of c_0 and by 2) Lemma 2.4.

$$\begin{aligned} (2.52) \qquad & \mathrm{Re}(\psi^2 b_2 u, \psi c_0(\log a_0)u) \\ & \geq \mathrm{Re}(\psi^3 b_2 c_0(\mathrm{Log}\, a_0)u, u) - C\|u\|_b^2 \\ & \geq \mathrm{Re}(\psi^3[b_2 c_0 \,\mathrm{Log}\, a_0 + \delta(e_0 \,\mathrm{Log}\, a_0 + b_2)]\, u, u) - C\|u\|_b^2 - C\|a_0 u\|\, \|u\|, \end{aligned}$$

because $\|e_0(\mathrm{Log}\, a_0)u\| \leq C\|a_0 u\|$. Lemma 2.1 means that the first term of the right hand side is a positive definite form. Therefore we have

$$(2.53) \qquad \mathrm{Re}(\psi^2 b_2 u, \psi c_0(\log a_0)u) \geq \|u\|_{bcLog}^2 - C\|u\|_b^2 - C\|a_0 u\|\, \|u\|.$$

We also part $\mathrm{Re}(a_0^{-1}(\log a_0)\psi^2 b_3 e_0 u, a_0 u)$ to $\mathrm{Re}(a_0^{-1}(\mathrm{Log}\, a_0)\psi^2 b_3 e_0 u, a_0 u)$ and $\mathrm{Re}(a_0^{-1} i(\arg a_0)\psi^2 b_3 e_0 u, a_0 u)$. Since $a_0^* a_0^{-1} i(\arg a_0)$ and e_0 are operators of order 0 and b_3 is a pseudodifferential operator of order 1 in x, we obtain (2.54) for the latter.

$$(2.54) \qquad |\mathrm{Re}(a_0^{-1} i(\arg a_0)\psi^2 b_3 e_0 u, u)| \leq C(\psi^2\langle D\rangle u, u) \leq C\|u\|_b^2.$$

On the other hand, $b_3 + b_3^*$ is a pseudodifferential operator of order 0 in x and $a_0^{-1}\,\mathrm{Log}\, a_0$ is an operator of order 0 with the bound $C_S(\lambda - C_S)^{-1}(|\log\lambda| + 1)$ on $\boldsymbol{H}^S$ by the assumptions. These facts assure the estimate (2.55) to the former because $\psi^{-2}(\mathrm{ad}\,\psi^2 b_3 e_0)(\mathrm{Log}\, a_0)$ and $\psi^{-2}(\mathrm{ad}\,\psi^2 b_3)(e_0)$ are operators of order 1 and of order 0, respectively.

$$\begin{aligned} (2.55) \qquad & |\mathrm{Re}(a_0^{-1}(\mathrm{Log}\, a_0)\psi^2 b_3 e_0 u, a_0 u)| \\ & \leq |\mathrm{Re}(\mathrm{Log}\, a_0)\psi^2 b_3 e_0 u, u)| + |\mathrm{Re}(2\lambda a_0^{-1}(\mathrm{Log}\, a_0)\psi^2 b_3 e_0 u, u)| \\ & \leq C\|(\mathrm{Log}\, a_0)u\|\, \|u\| + C(|\log\lambda| + 1)|(\psi^2\langle D\rangle u, u)| \\ & \leq C\|a_0 u\|\, \|u\| + C(|\log\lambda| + 1)\|u\|_b^2. \end{aligned}$$

By the inequalities (2.54–55), $\mathrm{Re}(a_0^{-1}(\log a_0)\psi^2 b_3 e_0 u, a_0 u)$ is bounded as

$$(2.56) \qquad |\mathrm{Re}(a_0^{-1}(\log a_0)\psi^2 b_3 e_0 u, a_0 u)| \leq C[\|a_0 u\|\, \|u\| + (|\log\lambda| + 1)\|u\|_b^2].$$

Noting again that $a_0^{-1}\log a_0$ and $(\log a_0)^* a_0^{-1}\log a_0$ are operators of order 0, we obtain that

$$\begin{aligned} (2.57) \qquad & |(a_0^{-1}\psi^2 b_1 u, \psi c_0(\log a_0)u)| \\ & \quad + |(a_0^{-1}(\log a_0)\psi^2 b_3 e_0 u, \psi c_0(\log a_0)u)| \\ & \leq C(\psi^2\langle D\rangle u, u) \leq C\|u\|_b^2, \end{aligned}$$

because b_1 and $b_3 e_0$ are operators of order 1 by the assumption and because $a_0^{*-1}\psi c_0 \log a_0$ and $(\log a_0)^* a_0^{*-1}\psi c_0(\log a_0)$ are shown to be also operators of order 0.

We use Lemma 2.6 to estimate $\mathrm{Re}(B_4 u, A_0 u)$, where

$$B_4 = (\log a_0)\psi^2 c_3 e_0 e_1 b_4.$$

Lemma 2.6 will be proved as well as Lemma 2.1 and 2.5 will be done.

Lemma 2.6. *There exists a positive* ε *such that holds for sufficiently large* ν *that*

(2.58)
$$|\operatorname{Re}(B_4u, v)| \le (1-\varepsilon)[\|u\|^2_{bcLog}+\|v\|^2_{cLog}]+C(\nu)[\|u\|^2_b+\|a_0u\|^2+\|v\|^2].$$

Therefore it holds that

(2.59)
$$\operatorname{Re}(A_1A_0u, A_0u)+\operatorname{Re}(\psi^2b_2u, \psi c_0(\log a_0)u)+\operatorname{Re}(B_4u, A_0u) \ge (\lambda-C)\|A_0u\|^2+\varepsilon[\|A_0u\|^2_{cLog}+\|u\|^2_{bcLog}] -C\|u\|^2_b-C\|a_0u\|^2.$$

Summing up (2.41, 45, 47–51, 56, 57 and 59), we conclude the following lemma.

Lemma 2.7. *There exist positive constants* ν, C *and* ε *such that for any* u *belonging to* $\mathcal{S}$ *and for sufficiently large* λ *it holds that*

(2.60)
$$\operatorname{Re}([A_1A_0+B]u, A_0u) \ge (\lambda-C)\|A_0u\|^2+\varepsilon\|A_0u\|^2_{eLog}+\varepsilon\|u\|^2_{bcLog} +(\lambda-C(|\log\lambda|+1))\|u\|^2_b-C\|a_0u\|^2.$$

Moreover $\|A_0u\|$ and $\|a_0u\|$ are equivalent to each other, that is, if λ is sufficiently large,

(2.61)
$$\|a_0u\|\le C\|A_0u\|\le C'\|a_0u\|.$$

In fact we obtain (2.62) by the natural way and (2.63) by the positivity of $\operatorname{Re}(\psi c_0(\log a_0)u, u)+\delta(u, u)$, namely, (2.38) Lemma 2.4.

(2.62)
$$\|A_0u\|\le\|a_0u\|+C\|(\log a_0)u\|+C\|u\|\le C\|a_0u\|.$$

(2.63)
$$\begin{aligned}\|A_0u\|^2 &\ge\|a_Iu\|^2+\|(\lambda+\psi c_0\log a_0)u\|^2-C\|u\|^2\\ &\ge\|a_Iu\|^2+\lambda^2\|u\|^2+\|\psi c_0(\log a_0)u\|^2+2\varepsilon\lambda\|u\|^2_{eLog}-C\|u\|^2\\ &\ge\|a_0u\|^2-C\|u\|^2\ge C'\|a_0u\|^2,\end{aligned}$$

where $a_I=(a_0-a_0^*)/2$ so that $\|a_0u\|^2=\|a_Iu\|^2+\lambda^2\|u\|^2$.

We apply the equivalence (2.61) to Lemma 2.7 to obtain the complete proof of (2.15) Theorem 4. (2.16–17) are easily deduced from (2.14–15). q.e.d.

§ 3. Various properties related to the operator a_0.

It is a well known result that the closure on H^s of the operator $-a_0$ defined on $\mathcal{S}$ is a generator of a one-parameter group $U(t)$ on H^s for any real number λ. Especially if $s=0$ and $\lambda=0$, then the operator $a_I=a_0|_{\lambda=0}$ is a skew-selfadjoint operator on H^0 with respect to the natural inner product because a_I is skew-symmetric on $\mathcal{S}$ by the definition (1.3). Moreover it permits also the pertubation of any bounded operator on $\boldsymbol{H}^S$. We denote the one parameter group with the generator $-(a_0+c)$ by $U(t, c)$, where c is a bounded operator on $\boldsymbol{H}^S$ for sufficiently many s. If C_S is a bound of c on $\boldsymbol{H}^S$, then $U(t, c)$ is estimated as

$$\|U(t, c)\|_S \leq \exp([C_S-\lambda)t] \tag{3.1}$$

for $t\geq 0$. The relation with another $U(t, c')$ is given by

$$\begin{aligned} U(t, c)-U(t, c') &= \int_0^t U(t-s, c)(c'-c)\, U(s, c')ds \\ &= \int_0^t U(t-s, c')(c'-c)\, U(s, c)ds. \end{aligned} \tag{3.2}$$

Let us consider $\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}a_0E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}$, which is written as for all integers α, β and γ,

$$\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}a_0E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}=a_0+d_{\alpha}, \tag{3.3}$$

where d_{α} is an operator of order 0. (Refer to Remark after 9) Section 2 for notations.) In fact we know the following lemma, which is able to apply inductively to (3.3).

Lemma 3.1. 1) $\mathrm{ad}E(q)=c_0a_0+c_1$ *for any pseudodifferential operator q of order m in x, where c_j are pseudodifferential operators of order $m+j$ ($j=0$, 1).*

2) *$E^{-1}a_0^2$, $E^{-1}a_0\langle D\rangle$ and $E^{-1}\langle D\rangle^2$ are operators of order 0. There exist f_{α} and f_{α}' operators of order -1 such that*

$$\begin{aligned} &\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}(E^{-1})E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha} \\ &\quad =E^{-1}(1+f_{\alpha})=(1+f_{\alpha}')E^{-1}. \end{aligned}$$

3) *Pseudodifferential operators of order m in x are operators of order m. If it is q, then there exists $r_{\alpha\beta\gamma}$ an operator of order $m-1$ such that*

$$\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}qE^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}=q+r_{\alpha\beta\gamma}.$$

4) *If r is an operator of order l, then $\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}rE^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}$ is also an operator of order l.*

Therefore the one-parameter group with the generator $-\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}a_0E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}$ is given by $U(t, d_{\alpha})$. Meanwhile it is also equal to $\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}U(t)E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}$. According to (3.2) we obtain that

$$\begin{aligned} &\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}U(t)E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha} \\ &=U(t, d_{\alpha}) \\ &=U(t)+U(t)\int_0^t U(-s)d_{\alpha}U(s, d_{\alpha})ds \\ &=U(t)+\int_0^t U(s, d_{\alpha})d_{\alpha}U(-s)ds\, U(t). \end{aligned} \tag{3.4}$$

Since $U(t, d_{\alpha})$ is uniformly bounded in $t\geq 0$ for sufficiently large λ on $\boldsymbol{H}^S$, $U(t)$ is an operator of order 0. We have a more pricise lemma.

Lemma 3.2.. 1) *If c is an operator of order* 0, *then there exist constant $C_{\alpha\beta\gamma}$ such that*

$$\|\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}U(t, c)E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}\|\leq\exp[C_{\alpha\beta\gamma}|t|-\lambda t]. \tag{3.5}$$

Especially $U(t, c)$ is an operator of order 0 *for $t\geq 0$ and for sufficiently large* λ.

2) *Let $G(t)$ be a continuous (in the strong sense) function in $t>0$, which is valued on operators of order* 0, *and satisfy the bound such that for sufficient many integers α, β and γ,*

$$\|\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}G(t)E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}\|\leq g_{\alpha\beta\gamma}(t), \tag{3.6}$$

where for constants $C_{\alpha\beta\gamma}$ of (3.5) *and for sufficiently large $\lambda\geq C_{\alpha\beta\gamma}$,*

$$M_{\alpha\beta\gamma}=\int_{0}^{+\infty}g_{\alpha\beta\gamma}(t)\exp[(C_{\alpha\beta\gamma}-\lambda)t]dt \tag{3.7}$$

is uniformly bounded in λ. Then we have that

$$F_{1}=\int_{0}^{+\infty}G(t)\,U(t, c)dt \text{ and } F_{2}=\int_{0}^{+\infty}U(t, c)\,G(t)dt$$

are operators of order 0 *for sufficiently large λ and they satisfy that*

$$\|\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}F_{j}E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}\|\leq M_{\alpha\beta\gamma}\quad (j=1, 2). \tag{3.8}$$

3) *Let H and c_{j} $(j=0, 1)$ be operators of order* 0. *If $G_{l}(t)$ satisfies* (3.10) *for a non-negative integer l, then*

$$G_{l+1}(t)=\int_{0}^{t}U(-s, c_{0})\,HU(s, c_{1})\,G_{l}(s)ds \tag{3.9}$$

is also an operator of order 0 *satisfying* (3.10) *in which l is replaced with $l+1$.*

$$\begin{aligned}&\|\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}G_{l}(t)E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}\|\\ &\leq N_{\alpha\beta\gamma}|t|^{l}\exp(C_{\alpha\beta\gamma}|t|).\end{aligned} \tag{3.10}$$

The same statement holds for

$$G_{l+1}=\int_{0}^{t}U(-s, c_{0})\,G_{l}(s)\,HU(s, c_{1})ds$$

and for

$$G_{l+1}=\int_{0}^{t}G_{l}(s)\,U(-s, c_{0})\,HU(s, c_{1})ds.$$

Remark. If H is an operator of order m at 3), then $G_{l+1}(t)$ is an operator of order m and satisfies a similar estimate.

Proof. The equality (3.3), to the both side of which $c_{\alpha\beta\gamma}=\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}cE^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}$ is added, yields the equality (3.4) in which $U(t)$ and d_{α} are replaced with $U(t, c)$ and $c_{\alpha\beta\gamma}+d_{\alpha}$, respectively. This proves the first statement. The second statement is easily shown because the assumptions imply the boundedness of $\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}F_{j}E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}(j=1, 2)$. If we put $U_{0}(t, c)=U(t, c)|_{\lambda=0}$, then $G_{l+1}(t)$ of (3.9) is equal to

$$\int_{0}^{t}U_{0}(-s, c_{0})\,HU_{0}(s, c_{1})\,G_{l}(s)ds.$$

Hence

$$\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}G_{l+1}(t)E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}$$
$$=\int_0^t U_0(-s, c_0')H_{\alpha\beta\gamma}U_0(s, c_1')G_{l\alpha\beta\gamma}(s)ds$$

with other $c_j'(j=0, 1)$ operators of order 0 and with $H_{\alpha\beta\gamma}$ and $G_{l\alpha\beta\gamma}(s)$ operators of order 0 such that

$$H_{\alpha\beta\gamma}=\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}HE^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}$$

and

$$G_{l\alpha\beta\gamma}(s)=\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}G_l(s)E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}.$$

By (3.5) we get the estimate (3.10). q.e.d.

Lemma 3.2 implies for λa_0^{-1} and $a_0^{-1}\log a_0$ to be operators of order 0. More pricisely we have the following lemma.

Lemma 3.3. 1) *λa_0^{-1} and $a_0^{-1}\log a_0$ are operators of order 0 such that for sufficiently large λ, which may depend on α, β and γ, they satisfy that*

$$\|\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}[\lambda a_0^{-1}]E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}\|\leq C_{\alpha} \tag{3.11}$$

and

$$\|\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}[a_0^{-1}\log a_0]E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}\| \leq C_{\alpha}\lambda^{-1}(|\log\lambda|+1). \tag{3.12}$$

2) *There exist operators d_{α} of order 0 for sufficiently many integers α, β and γ such that*

$$\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}[\log a_0]E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}=\log a_0+d_{\alpha}. \tag{3.13}$$

3) $\arg a_0=\operatorname{Im}\log a_0$ *and* $(\log a_0)^*\, a_0^{-1}\log a_0$ *are operators of order* 0.

Proof. It is trivial because a_0 has the symbol $(i\xi_0+\lambda)$. q.e.d.

We next discuss about properties related to the relations (2.10–11) and the condition (2.42).

We call an operator H on $\boldsymbol{H}^s$ a quasicommutor of an operator K of order l or quasicommutative with K if there exist Q_j $(j=1, 2)$ operators of order 0 and $R_j(j=1, 2)$ operators of order $l-1$ such that

$$\begin{aligned}\mathrm{ad}H(K)&=Q_1K+R_1\\&=KQ_2+R_2.\end{aligned} \tag{3.14}$$

Lemma 3.4. 1) *Let H an operator of order 0 be a quasicommutor of K an operator of order l. If K satisfies that*

$$\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}KE^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}=K+k_{\alpha\beta\gamma}, \tag{3.15}$$

where $k_{\alpha\beta\gamma}$ is an operator of order $l-1$, then $\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}HE^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}$ is also a quasi-

commutor of K.

2) *Let* H *and* H' *operators of order* 0 *be quasicommutors of* K. *Then* $H-H'$ *and* HH' *are also quasicommutors of* K.

3) *Let* H *be a quasicommutor of* K *such that there exist the inverses* H^{-1}, $(H-Q_1)^{-1}$ *and* $(H+Q_2)^{-1}$ *which are operators of order* 0. *Then the inverse* H^{-1} *is also a quasicommutor of* K.

Lemma 3.5. *Let* q *be a pseudodifferential operator of order* m *in* x.

1) *If* r *is a pseudodifferential operator of order* l, *then* $\mathrm{ad}\,r(q)$ *and* $\mathrm{ad}\,a_0(q)$ *are pseudodifferential operators of order* $l+m-1$ *in* x *and of order* m, *respectively.*

2) *Let* F *be one of* $E^{-1}a_0^2$, $E^{-1}a_0\langle D\rangle$ *and* $E^{-1}\langle D\rangle^2$. *Then* $\mathrm{ad}F(q)$ *is an operator of order* $m-1$.

3) *Let us put*

$$d_\alpha=\psi^\alpha\langle D\rangle^\beta E^\gamma a_0E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}-a_0, \tag{3.16}$$

which is an operator of order 0 *by* (3.3). *Then* $\mathrm{ad}d_\alpha(q)$ *is zero.*

4) *If we put*

$$r_{\alpha\beta\gamma}=\psi^\alpha\langle D\rangle^\beta E^\gamma qE^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}-q, \tag{3.17}$$

which is an operator of order $m-1$, *then* $\mathrm{ad}a_0(r_{\alpha\beta\gamma})$ *is an operator of order* $m-1$.

For another pseudodifferential operator q' *of order* l *in* x, $\mathrm{ad}r_{\alpha\beta\gamma}(q')$ *is an operator of order* $l+m-2$.

Proof. 1) It is trivial.

2) Let G be one of a_0^2, $a_0\langle D\rangle$ and $\langle D\rangle^2$. By 2) Lemma 3.1, there exist c_j and $c_j'(j=0, 1)$ pseudodifferential operators of order $m+j$ in x such that

$$\mathrm{ad}E^{-1}(q)=E^{-1}\mathrm{ad}E(q)E^{-1}=E^{-1}(a_0c_0+c_1)E^{-1} \tag{3.18}$$

and

$$\mathrm{ad}G(q)=a_0c_0'+c_1. \tag{3.19}$$

Therefore we have that

$$\begin{aligned}(\mathrm{ad}[E^{-1}G])(q)&=\mathrm{ad}E^{-1}(q)G+E^{-1}\mathrm{ad}G(q)\\&=([E^{-1}a_0\langle D\rangle][\langle D\rangle^{-1}c_0]+[E^{-1}\langle D\rangle^2][\langle D\rangle^{-2}c_1])[E^{-1}G]\\&\quad+[E^{-1}a_0\langle D\rangle][\langle D\rangle^{-1}c_0']+[E^{-1}\langle D\rangle^2][\langle D\rangle^{-2}c_1'].\end{aligned} \tag{3.20}$$

Since $E^{-1}a_0\langle D\rangle$, $E^{-1}\langle D\rangle^{-2}$ and $E^{-1}G$ are operators of order 0 by 3) Lemma 3.1 and since $[\langle D\rangle^{-1}c_0]$, $[\langle D\rangle^{-2}c_1]$, $[\langle D\rangle^{-1}c_0']$ and $[\langle D\rangle^{-2}c_1']$ are pseudodifferential operators of order $m-1$ in x, (3.20) is an operator of order $m-1$.

3) It is trivial because d_α is a function in x_0.

4) Let us consider the commutator of $\psi^\alpha\langle D\rangle^\beta E^\gamma a_0E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}$ and $r_{\alpha\beta\gamma}$. It is equal to

$$(3.21) \qquad [\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}\mathrm{ad}a_0(q)E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}-\mathrm{ad}a_0(q)].$$

Since $\mathrm{ad}a_0(q)$ is a pseudodifferential operator of order m, (3.21) is an operator of order m-1 according to (3.17), q of which is replaced with $\mathrm{ad}a_0(q)$. On the other hand the commutator is also equal to

$$(3.22) \qquad \mathrm{ad}a_0(r_{\alpha\beta\gamma})+[d_\alpha r_{\alpha\beta\gamma}-r_{\alpha\beta\gamma}d_\alpha].$$

The second term of (3.22) is an operator of order $m-1$ because it consists of the products of operators of order $m-1$ and of order 0. Since (3.21) is equal to (3.22), $\mathrm{ad}a_0(r_{\alpha\beta\gamma})$ is an operator of order $m-1$.

Let us prove the last part of the statement 4). If $\gamma=0$, then it is trivial because $r_{\alpha\beta\gamma}$ is a pseudodifferential operator of order $m-1$. 1) of Lemma 3.1 means that $EqE^{-1}=q+F_1$ and $E^{-1}qE=q+F_2$ where F_j ($j=1$, 2) are compositions of pseudodifferential operators of order $m-1$ and one of the operators at 2) of the present lemma. Therefore $\mathrm{ad}q'(F_j)$ ($j=1,2$) are operators of order $l+m-2$ according to 2) so that $\mathrm{ad}q'(\psi^{\alpha}\langle D\rangle^{\beta}E^{\gamma}F_jE^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha})$ are operators of order $l+m-2$ because q' satisfies (3.17), q of which is replaced with q'. These facts imply that $\mathrm{ad}q'(r_{\alpha\beta(\gamma\pm1)})$ is equal to $\mathrm{ad}q'(r_{\alpha\beta\gamma})$ modulo operators of order $l+m-2$. If it is assumed that $\mathrm{ad}q'(r_{\alpha\beta\gamma})$ are operators of order $l+m-2$ if $|\gamma|\leq\gamma_0$, we are able to conclude the same proposition for $|\gamma|\leq\gamma_0+1$. Thus we finish the proof by induction for γ. q.e.d.

Corollary. *Let q_m be a pseudodifferential operator of order m in x and $\{h_j\}_{0\leq j\leq l}$ be finite number of H-type of operators, where h_j is for convenience' sake called a H-type of operator if h_j is a sum of $E^{-1}a_0^2$, $E^{-1}a_0^*a_0$, $E^{-1}a_0\langle D\rangle$ and $E^{-1}\langle D\rangle^2$ multiplied to the right hand side by a pseudodifferential operator of order 0 in x.*

1) $(\Pi_{j=1}^{l}\mathrm{ad}h_j)(h_0q_m)$

is then discribed as the finite sum of product operators, which consist of finite number of H-type of operators and a pseudodifferential operator of order $m-l$ in x at the right end (or at the left end), that is,

$$\sum_{finite}(\Pi_{finite}\tilde{h_j})q_{m-l}$$

where for $l\geq 2$,

$$(\Pi_{j=1}^{l}\mathrm{ad}h_j)(K)=\mathrm{ad}h_l((\Pi_{j=1}^{l-1}\mathrm{ad}h_j)(K)).$$

This statement includes the case that $h_0q_m=q_m$, namely, $h_0=I$ because $I=E^{-1}(a_0^*a_0+\langle D\rangle^2)$.

2) *For any natural number α,*

$$(\mathrm{ad}a_0^{\alpha})(h_0)=\sum_{k=0}^{\alpha-1}h_k'a_0^k=\sum_{k=0}^{\alpha-1}a_0^kh_k''$$

with h_k' and h_k'' H-type of operators. If h_0 contains no term consisting of $E^{-1}a_0^2$ and $E^{-1}a_0^*a_0$, then h_k' and h_k'' also contain no such term.

3) *If h_j for $j\geq 1$ contains no term consisting of $E^{-1}a_0^2$ and $E^{-1}a_0^*a_0$, then*

$$a_0^\beta(\Pi_{j=1}^l \mathrm{ad}\, h_j)(h_0 q_m)\, a_0^\gamma$$

with non-negative integers β *and* γ *is a linear combination of* $q_m^\sim a_0^j$ $(0\leq j\leq \max(\beta+\gamma-l, 0))$ *with coefficients of product-sums of H-type of operators at the left hand side* (*or at the right hand side*), *where* $q_m^\sim$ *is another pseudodifierential operator of order* m.

Proof. 1) In case that $h_0=I$ and that $0\leq m\leq 1$, the second and third terms at (3.20) are already forms required. At the first terms it works effectively in order to rewrite it that $E^{-1}G\langle D\rangle^{1-m}=\langle D\rangle^{1-m}E^{-1}G$. In other cases, the equality (3.20) applies several times to prove the statement of Corollary, namely, by induction with respect to l and m.

2) It is easily proved.

3) In case that $l=1$ and that $\gamma\geq 1$, we have that

$$\begin{aligned} &a_0^\beta\, \mathrm{ad} h_1(h_0 q_m)\, a_0^\gamma \\ &=a_0^\beta(\mathrm{ad} h_1 a_0)(h_0 q_m)a_0^{\gamma-1}-a_0^\beta h_1 \mathrm{ad} a_0(h_0 q_m)a_0^{\gamma-1}. \end{aligned}$$

Since $h_1 a_0=h_1^\sim\langle D\rangle+h^\sim$ with other H-type of operators according to the assumption for h_1, all terms are easily rewritten as desired by 1) and 2) of this corollary. The situations in other cases are same. q.e.d.

Let b be an operator of order 2 such that a_0 is a quasicommutor of b, that is,

$$\mathrm{ad} a_0(b)=q_1 b+r_1=bq_2+r_2. \tag{3.23}$$

with q_j and r_j $(j=1, 2)$ operators of order 0 and of order 1, respectively. For example b_2 defined by (2.5) and (2.10) is such an operator b. If c an operator of order 0 is also a quasicommutor of b, then we have that

$$\begin{aligned} &U(-t, c-q_1-c_1)\, b\, U(t, c)-b \\ &=\int_0^t U(-s, c-q_1-c_1)(r_1+d_1)\, U(s, c)ds \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} &U(-t, c)\, b\, U(t, c+q_2+c_2)-b \\ &=\int_0^t U(-s, c)(r_2+d_2)\, U(s, c+q_2+c_2)ds, \end{aligned} \tag{3.25}$$

where

$$\mathrm{ad} c(b)=c_1 b+d_1=bc_2+d_2 \tag{3.26}$$

with c_j and d_j $(j=1, 2)$ operators of order 0 and of order 1, respectively.

The right hand sides of (3.24–25) are operators of order 1 according to 3) of Lemma 3.2 and satisfy the inequality (3.10) with $l=1$. The definition (2.12) of $\log a_0$ implies the following lemma.

Lemma 3.6. 1) *Let* $G(t)$ *be the same one at* 2) *of Lemma* 3.2 *and be a quasicommutor of* b *such that*

$$-\mathrm{ad} b(G(t))=Q_1(t)b+R_1(t)=bQ_2(t)+R_2(t). \tag{3.27}$$

If $Q_j(t)$ *and* $R_j(t)\langle D\rangle^{-1}$ *operators of order* 0 $(j=1, 2)$ *satisfy the assumptions at* 2) *of Lemma* 3.2 *in which* $G(t)$ *is replaced with them and if* c *an operator of order* 0 *and* a_0 *are quasicommutors of* b, *then*

$$F_1=\int_0^{+\infty} G(t)U(t, c)dt \text{ and } F_2=\int_0^{+\infty} U(t, c)G(t)dt$$

operators of order 0 *are quasicommutors of* b.

2) *Let* $G(t)$ *be the same one as at the above* 1). *If* b *and* c_j $(j=1, 0)$ *are pseudodifferential operators of order* 2 *and of order* 0 *in* x, *respectively, then*

$$G_0(t)=\int_0^t U(s, c_0)G(s)U(-s, c_1)ds$$

is also a quasicommutor of b *and satisfies the same conditions as for* $G(t)$ *at* 1).

3) *If* a_0 *is a quasicommutor of* b *an operator of order* 2, *then* $\log a_0$ *is also a quasicommutor of* b.

4) $$b(a_0+c)^{-1}-(a_0+c-q_1-c_1)^{-1}b$$
$$=(a_0+c-q_1-c_1)^{-1}(r_1+d_1)(a_0+c)^{-1}$$

and

$$b(a_0+c+q_2+c_2)^{-1}-(a_0+c)^{-1}b$$
$$=(a_0+c)^{-1}(r_2+d_2)(a_0+c+q_2+c_2)^{-1}.$$

Proof. 1) By (3.27) and (3.24), we have that

$$\begin{aligned} bF_1&=\int_0^{+\infty} bG(t)U(t, c)dt \\ &=\int_0^{+\infty}[G(t)-Q_1(t)]bU(t, c)dt+\int_0^{+\infty} R_1(t)U(t, c)dt \\ &=\int_0^{+\infty}[G(t)-Q_1(t)]U(t, c-q_1-c_1)dtb \\ &\quad+\int_0^{+\infty}[G(t)-Q_1(t)]S(t)U(t, c)dt \\ &\quad+\int_0^{+\infty} R_1(t)U(t, c)dt \end{aligned} \tag{3.28}$$

where

$$S(t)=\int_0^t U(s, c-q_1-c_1)(r_1+d_1)U(-s, c)ds.$$

By the assumption for $G(t)$ and $Q_1(t)$, 2) of Lemma 3.2 implies that the first term of the right hand side is the product of an operator of order 0 and b, and that the second term is an operator of order 1 because $S(t)$ satisfies (3.10) with $l=1$ so that $\langle D\rangle^{-1}[G(t)-Q_1(t)]S(t)$ satisfies (3.6–7) in which $G(t)$ is replaced with it. The third term is an operator of order 1.

Since the same facts with respect to F_1b, bF_2 and F_2b are proved, it is concluded that F_1 and F_2 are quasicommutors of b.

2) Since $\mathrm{ad}c_j(b)$ is a pseudodifferential operator of order 1 in x, it satisfies (3.26)

with $c_1=c_2=0$. The statement is proved as well as at the above proof of 1) and at the proof of 3) Lemma 3.2.

3) We prove that $F=\int_0^{+\infty}(\log t)U(t)dta_0$ is a quasicommutor of b.

$$\mathrm{ad}F(b)=\int_0^{+\infty}(\log t)U(t)dt\,\mathrm{ad}a_0(b) \tag{3.29}$$
$$-\int_0^{+\infty}(\log t)\mathrm{ad}b(U(t))dta_0.$$

By the substitutions that $Q_1(t)=R_1(t)=c=0$ and $G(t)=\log t$ at (3.28), the second term is equal to

$$\int_0^{+\infty}(\log t)[U(t,-q_1)-U(t)]dtba_0$$
$$+\int_0^{+\infty}(\log t)S(t)U(t)dta_0.$$

which is rewritten by (3.2) and (3.23) as

$$\int_0^{+\infty}(\log t)S_1(t)U(t)dta_0b$$
$$-\int_0^{+\infty}(\log t)S_1(t)YU(t)dt[q_1b+r_1]$$
$$+\int_0^{+\infty}(\log t)S(t)U(t)dta_0,$$

where

$$S_1(t)=\int_0^t U(s,-q_1)q_1U(-s)ds.$$

Since $(d/dt)U(t)=-a_0U(t)$, the sum of the first and third terms of the above is equal to

$$\int_0^{+\infty}(d/dt)[(\log t)S_1(t)]U(t)dtb$$
$$+\int_0^{+\infty}(d/dt)[(\log t)S(t)]U(t)dt,$$

which is written as the form that $qb+r$, where q and r are operators of order 0 and of order 1, respectively, because the integrands are

$$t^{-1}S_1(t)U(t)+(\log t)U(t,-q_1)q_1$$

and

$$t^{-1}S(t)U(t)+(\log t)U(t,-q_1)r_1.$$

Therefore (3.29) is the same form. Since the other form is proved by the same way, it is concluded that F is a quasicommutor of b. Thus $\log a_0$ is a quasicommutor of b.

4) It is trivial from the relations that

$$(a_0+c-q_1-c_1)b-b(a_0+c)=r_1+d_1$$

and

$$(a_0+c)b-b(a_0+c+q_2+c_2)=r_2+d_2. \qquad \text{q.e.d.}$$

§ 4. Proofs of Lemmas.

Proof of Lemma 1.1. Let us consider the commutator of $a_0^k U(t) W(\log t)$ and $(a_1 a_0 + b)$.

(4.1)
$$\begin{aligned} & a_0^k U(t) W(\log t)(a_1 a_0 + b) - (a_1 a_0 + b) a_0^k U(t) W(\log t) \\ & = a_0^k U(t)(\mathrm{ad} W(\log t))(a_1 a_0 + b) \\ & + a_0^k (\mathrm{ad}\, U(t))(a_1 a_0 + b) W(\log t) \\ & + (\mathrm{ad} a_0^k)(a_1 a_0 + b) U(t) W(\log t). \end{aligned}$$

The substitutions of $W(-\log t)$ and $U(-t)$ for $V(t)$ at (1.18) with $k=1$ yields us the equations (3.2–4).

(4.2)
$$\begin{aligned} & W(\log t)(a_1 a_0 + b) - (a_1 a_0 + b) W(\log t) \\ & = (\log t) \mathrm{ad} h (a_1 a_0 + b) W(\log t) + Y_1(t) W(\log t), \end{aligned}$$

where

$$Y_1(t) = \int_0^{\log t} W(\sigma)(\mathrm{ad} h)^2 (a_1 a_0 + b) W(-\sigma)(\log t - \sigma) d\sigma.$$

(4.3)
$$\begin{aligned} & U(t)(a_1 a_0 + b) - (a_1 a_0 + b) U(t) \\ & = -t \mathrm{ad} a_0 (a_1 a_0 + b) U(t) + Y_2(t) U(t), \end{aligned}$$

where

$$Y_2(t) = \int_0^t U(\sigma)(\mathrm{ad} a_0)^2 (a_1 a_0 + b) U(-\sigma)(t - \sigma) d\sigma.$$

Let us multiply (4.1) by t^{k-1} and integrate it in t from zero to infinity after substitutions of (4.2–3) into (4.1). We have there that

$$\begin{aligned} & \int_0^{+\infty} (\log t) t^{k-1} a_0^k U(t) \mathrm{ad} h (a_1 a_0 + b) W(\log t) dt \\ & + \int_0^{+\infty} t^{k-1} a_0^k U(t) Y_1(t) W(\log t) dt \\ & = \int_0^{+\infty} (Z_4 + Z_7) t^{k-1} a_0^k U(t) W(\log t) dt \\ & + [\mathrm{ad} h (a_1 a_0 + b) + k \mathrm{ad} a_0 (\mathrm{ad} h (a_1 a_0 + b)) a_0^{-1}] \\ & \times \int_0^{+\infty} (\log t) t^{k-1} a_0^k U(t) W(\log t) dt. \end{aligned}$$

The first term at the right hand side is one of the last term at (1.16) and the second one is the second term at the right hand side of (1.16).

Since $a_0^k Y_2(t) a_0^{-k} = Z_8$, we have that

$$\int_0^{+\infty} t^{k-1} a_0^k Y_2(t) U(t) W(\log t) dt$$

is one of the last term at (1.16), namely,

$$\int_0^{+\infty} Z_8 t^{k-1} a_0^k U(t) W(\log t) dt.$$

The term including $-t\mathrm{ad}a_0(a_1a_0+b)$ is the sum of ones including Z_5 of the last term at (1.16) and

$$-\int_0^{+\infty} t\mathrm{ad}a_0(a_1a_0+b)\, t^{k-1} a_0^k U(t) W(\log t) dt, \tag{4.4}$$

because

$$-a_0^k t\mathrm{ad}a_0(a_1a_0+b) = -t\mathrm{ad}a_0(a_1a_0+b)\, a_0^k + Z_5 a_0^k.$$

The term consisting of $(\mathrm{ad}a_0^k)(a_1a_0+b)$ is equal to the sum of ones including Z_3 and Z_6 of the last one at (1.16) and

$$k\int_0^{+\infty} \mathrm{ad}a_0(a_1) t^{k-1} a_0^k U(t) W(\log t) dt, \tag{4.5}$$

because

$$(\mathrm{ad}a_0^k)(a_1a_0+b)\, a_0^{-k} = k\mathrm{ad}a_0(a_1) + Z_6 + Z_3.$$

Since the right half at (4.4) is equal to the term including Z_1 of the last one at (1.16), it leaves only the calculation of the left half of (4.4). Since $\mathrm{ad}a_0(a_1a_0) = \mathrm{ad}a_0(a_1)a_0$ and $-a_0 U(t) = (d/dt) U(t)$, we have that

$$\begin{aligned} -\int_0^{+\infty} t\mathrm{ad}a_0(a_1a_0) t^{k-1} a_0^k U(t) W(\log t) dt \\ = \mathrm{ad}a_0(a_1) a_0^k \int_0^{+\infty} t^k [(d/dt)\, U(t)] W(\log t) dt. \end{aligned} \tag{4.6}$$

If k is sufficiently large ($k > N_0 + 1$ with respect to N_0 of (1.12)), then $t^k U(t) W(\log t)$ and $U(t)(d/dt)[t^k W(\log t)] = U(t)(k+h) t^{k-1} W(\log t)$ are integrable so that the integral by part is able to apply to (4.6). It is equal to

$$\begin{aligned} -\mathrm{ad}a_0(a_1) a_0^k \int_0^{+\infty} t^{k-1} U(t)(k+h) W(\log t) dt \\ = -\mathrm{ad}a_0(a_1)(k+h) \int_0^{+\infty} t^{k-1} a_0^k U(t) W(\log t) dt \\ + \int_0^{+\infty} Z_2 t^{k-1} a_0^k U(t) W(\log t) dt. \end{aligned}$$

The sum of the first term at the right hand side and (4.5) makes the right half of the first term at the right hand side of (1.16) and the second term is one of the last term of (1.16). Therefore we obtain the equality (1.16). q.e.d.

Three following lemmas are preparatory ones to shorten the proofs of Lemma 1.2 and 1.3.

Lemma 4.1. *Let us put*

$$\Phi(t, \Psi) = U(t(1+\sigma)/2) \Psi U(t(1-\sigma)/2). \tag{4.7}$$

Then the equalities (4.8–10) *hold.*

(4.8)
$$\begin{aligned}&(d/dt)\Phi(t, \Psi)\\ &=-\Phi(t, a_0\Psi+\Psi a_0)/2-\Phi(t, \mathrm{ad}a_0(\Psi))\sigma/2\\ &=-\Phi(t, a_0\Psi)+\Phi(t, \mathrm{ad}a_0(\Psi))(1-\sigma)/2\\ &=-\Phi(t, \Psi a_0)-\Phi(t, \mathrm{ad}a_0(\Psi))(1+\sigma)/2.\end{aligned}$$

(4.9)
$$\begin{aligned}&(d/dt)^2\Phi(t, \Psi)+\sigma(d/dt)\Phi(t, \mathrm{ad}a_0(\Psi))\\ &=\Phi(t, a_0\Psi a_0)+\Phi(t, (\mathrm{ad}a_0)^2(\Psi))/2.\end{aligned}$$

(4.10)
$$\begin{aligned}&\Phi(t, a_0^k\Psi a_0^k)-(d/dt)^{2k}\Phi(t, \Psi)\\ &=\textstyle\sum_{j=1}^k\sum_{i=0}^j C_{ij}(\sigma d/dt)^i(d/dt)^{2(k-j)}\Phi(t, (\mathrm{ad}a_0)^{2j-i}(\Psi)).\end{aligned}$$

Proof. (4.8–9) are easily shown by the differentiations of (4.7) in t, and (4.10) by induction in $k\geq 1$. q.e.d.

Lemma 4.2. *Let Φ be one used at Lemma 4.1. Ψ stands for $W(\log[(1+\sigma)/(1-\sigma)])$ and $f_i(t)$ $(i=0, 1)$ are sufficiently smooth functions in $t\geq 0$ valued in bounded operators on $\boldsymbol{H}^S$ such that for $0\leq j\leq 2k$*

(4.11)
$$\|(d/dt)^j f_i(t)\|_S\leq C_S\exp\{Mst\}.$$

1)

(4.12)
$$\begin{aligned}&\int_0^{+\infty} t^{k-1}a_0^k U(t)W(\log t)dt\\ &\times\int_0^{+\infty} W(-\log s)\,U(s)a_0^k s^{k-1}ds\\ &=(2k-1)!2^{-2k+1}\int_{-1}^{+1}(1-\sigma^2)^{k-1}\Psi d\sigma\\ &+\textstyle\sum_{j=1}^{2k}\int_0^{+\infty}\int_{-1}^{+1} g_j(1-\sigma^2)^{k-1}\Phi((\mathrm{ad}a_0)^j(\Psi))d\sigma d\tau,\end{aligned}$$

where $\tau^{1-j}g_j$ are polynomials of order at most j in σ.

2)

(4.13)
$$\begin{aligned}&\int_0^{+\infty}(\log t)^l t^{k-1}a_0^k U(t)W(\log t)dt\\ &\times\int_0^{+\infty} W(-\log s)\,U(s)a_0^k s^{k-1}(\log s)^m ds\\ &=2^{-2k+1}\int_0^{+\infty}d\tau\int_{-1}^{+1}g(1-\sigma^2)^{k-1}a_0\Phi(\Psi)d\sigma\\ &+\textstyle\sum_{j=1}^{2k}\int_0^{+\infty}d\tau\int_{-1}^{+1}g_j'(1-\sigma^2)^{k-1}\Phi((\mathrm{ad}a_0)^j(\Psi))d\sigma,\end{aligned}$$

where g and g_j' are functions in (τ, σ) such that

(4.14)
$$\begin{aligned}&g\\ &=(\partial/\partial\tau)^{2k-1}\{\tau^{2k-1}[\log(\tau(1+\sigma)/2)]^l[\log(\tau(1-\sigma)/2)]^m\}\\ &=\textstyle\sum_{0\leq i+j\leq 2k-1}C_{ij}[\log(\tau(1+\sigma)/2)]^{l-i}[\log(\tau(1-\sigma)/2)]^{m-j}\end{aligned}$$

and

(4.15) $$|g_j'| \leq C(|\log(\tau(1+\sigma))|^l+1)(|\log(\tau(1-\sigma))|^m+1)(|\tau|^{2k-1}+1).$$

3) *If f_0 or f_1 vanishes at $t=0$, then there exist finite sets g_{ij} ($i=0, 1$ and $j=0, \cdots, 2k$) of functions valued in bounded operators on $\mathbf{H}^S$ such that for $g_{ij\alpha}$ belonging to g_{ij},*

(4.16) $$\|g_{0j\alpha}(\tau, \sigma)\|_S \leq C_S(|\log(\tau(1+\sigma))|^l+1)\exp(M_S\tau),$$
$$\|g_{1j\alpha}(\tau, \sigma)\|_S \leq C_S(|\log(\tau(1-\sigma))|^m+1)\exp(M_S\tau)$$

and

(4.17) $$\int_0^{+\infty}(\log t)^l f_0(t)t^{k-1}a_0^k U(t)W(\log t)dt \times\int_0^{+\infty}W(-\log s)U(s)a_0^k s^{k-1}f_1(s)(\log s)^m ds = \sum_{j=0}^{2k}\int_0^{+\infty}d\tau\int_{-1}^{+1}(1-\sigma^2)^{k-1}g_{0j}\Phi((\mathrm{ad}a_0)^j(\Psi))g_{1j}d\sigma$$

with abbreviations that $g_{ij}=(g_{ij\alpha})_{0\leq\alpha\leq\beta}$ and

$$g_{0j}\Phi g_{1j}=\sum_{\alpha=0}^{\beta}g_{0j\alpha}\Phi g_{1j\alpha}.$$

Proof. The change of variables (t, s) to (τ, σ) on two functions $t=\tau(1+\sigma)/2$ and $s=\tau(1-\sigma)/2$ yields that the left hand side of (4.16) (also (4.12–13)) is equal to

$$2^{-2k+1}\int_0^{+\infty}\int_{-1}^{+1}(1-\sigma^2)^{k-1}N(\tau, \sigma, \Phi(a_0^k\Psi a_0^k))d\sigma d\tau,$$

where
$$N(\tau, \sigma, R) = \tau^{2k-1}(\log(\tau(1+\sigma)/2))^l f_0(\tau(1+\sigma)/2) \times Rf_1(\tau(1-\sigma)/2)(\log(\tau(1-\sigma)/2))^m.$$

The substitution by the equality (4.10) makes it possible to rewrite this to the form including the derivatives in τ of $\Phi((\mathrm{ad}a_0)^j(\Psi))$. Since in the case 3) it has been assumed that for $0\leq j\leq 2k-1$

$$(\partial/\partial\tau)^j N(\tau, \sigma, R)|_{\tau=0}=0,$$

the integral by part transposes all derivatives in τ of $\Phi((\mathrm{ad}a_0)^j(\Psi))$ to the derivatives of functions except for $\Phi((\mathrm{ad}a_0)^j(\Psi))$, and the arrangement with respect to $\Phi((\mathrm{ad}a_0)^j(\Psi))$ yields (4.16) and (4.17).

In the case 1) the integral in σ at $t=0$, which constitutes the first term at the right hand side of (4.12), appears only if the $2k$-th derivative in τ of $\Phi(\Psi)$ is transposed to the other functions, and the other terms of (4.12) are obtained as well as in the case 3).

In the case 2) it is impossible that all derivatives in τ of $\Phi(\Psi)$ are transposed to the other functions. This part is only discribed as

(4.18) $$-2^{-2k+1}\int_0^{+\infty}\int_{-1}^{+1}(1-\sigma^2)^{k-1}g(\partial/\partial\tau)\Phi(\Psi)d\sigma d\tau$$

with g defined by (4.14). The equality (4.8), however, assures for this to be equal to

$$2^{-2k+1}\int_0^{+\infty}\int_{-1}^{+1}(1-\sigma^2)^{k-1}g\Phi(a_0\Psi)d\sigma d\tau$$
$$-2^{-2k}\int_0^{+\infty}\int_{-1}^{+1}(1-\sigma^2)^{k-1}(1-\sigma)g\Phi(\mathrm{ad}a_0(\Psi))d\sigma d\tau.$$

We obtain (4.13) by the transference of the above second term to the remainder terms of (4.13). q.e.d.

Corollary 1. *All statements of Lemma* 4.2 *are also valid with* $\Psi=W(\log[(1-\sigma)/(1+\sigma)])$ *when* $W(-t)$ *is used instead of* $W(t)$, *namely, when* $W(\log t)$ *is replaced by* $W(-\log t)$ *and* $W(-\log t)$ *by* $W(\log t)$, *respectively.*

Proof. Substitute $-h$ to the generator h of $W(t)$. q.e.d.

Corollary 2. 1) $FG=H+H_1$,

where

$$H_1=\sum_{j=1}^{2k}\int_0^{+\infty}\int_{-1}^{+1}g_j(1-\sigma^2)^{k-1}\Phi((\mathrm{ad}a_0)^j(\Psi))d\sigma d\tau \tag{4.19}$$

with same g_j *as* (4.12).

And also

$$a_0H_1=\int_{-1}^{+1}g_1(0)(1-\sigma^2)^{k-1}\Phi(\mathrm{ad}a_0(\Psi))d\sigma+H_1^{\sim}, \tag{4.20}$$

where

$$H_1^{\sim}=\sum_{j=2}^{2k+1}\int_0^{+\infty}\int_{-1}^{+1}g_j^{\sim}(1-\sigma^2)^{k-1}\Phi((\mathrm{ad}a_0)^j(\Psi))d\sigma d\tau$$

with $g_j^{\sim}$ *such that* $\tau^{2-j}g_j^{\sim}$ *are polynomials in* σ *of order* j.

2)

$$\int_0^{+\infty}(\log t)t^{k-1}a_0^kU(t)W(\log t)dtG \tag{4.21}$$
$$=\int_0^{+\infty}(\log\tau)a_0U(\tau)d\tau H+H_1',$$

where

$$H_1'=\sum_{j=1}^{2k}\int_0^{+\infty}\int_{-1}^{+1}g_j'(1-\sigma^2)^{k-1}\Phi((\mathrm{ad}a_0)^j(\Psi))d\sigma d\tau \tag{4.22}$$
$$+(2k-1)!2^{-2k}\int_0^{+\infty}\int_{-1}^{+1}(\log\tau)(1-\sigma^2)^{k-1}(1-\sigma)U(\tau)N_1d\sigma d\tau$$
$$+\int_{-1}^{+1}g_0(\sigma)(1-\sigma^2)^{k-1}\Psi d\sigma,$$

$$N_1=\tau^{-1}\int_0^{\tau}U(-s)\Phi(s,\mathrm{ad}a_0(\Psi))ds, \tag{4.23}$$
$$|g_0|\le C(|\log(1+\sigma)|+1)$$

and g_j' *satisfy* (4.15) *with* $l=1$ *and* $m=0$.

(*Refer to* (1.14, 19, 20 and 22).)

Proof. The first statement of 1) is only the determination of H_1 at (1.22) by the result 1) of Lemma 4.2. For the second half, it suffices to apply (4.8) to a_0H_1 again and to take same steps as to the first one.

Let us put $l=1$ and $m=0$ at 2) of Lemma 4.2. Then the main part (4.18) appearing in the middle of the proof for 2) of Lemma 4.2 is equal to

$$(4.24)\qquad -(2k-1)!2^{-2k+1}\int_0^{+\infty}\int_{-1}^{+1}(1-\sigma^2)^{k-1}(\log\tau)(\partial/\partial\tau)\Phi(\Psi)d\sigma d\tau$$
$$-\int_0^{+\infty}\int_{-1}^{+1}(1-\sigma^2)^{k-1}[C_k\log(1+\sigma)+C'_k](\partial/\partial\tau)\Phi(\Psi)d\sigma d\tau,$$

and the first term of H'_1 consists of the other terms. The integral by part is able to apply to the second term of (4.24) so that the third term of (4.22) is obtained with $g_0=C_k\log(1+\sigma)+C'_k$. Since the equality (4.8) shows that $\Phi(t, \Psi)$ is equal to

$$U(t)\Psi+U(t)\int_0^{\tau}U(-s)\Phi(s, \mathrm{ad}a_0(\Psi))ds(1-\sigma)/2,$$

the first term of (4.24) splits into two parts as follows.

$$(4.25)\qquad -(2k-1)!2^{-2k+1}\int_0^{+\infty}(\log\tau)(\partial/\partial\tau)\,U(\tau)d\tau\int_{-1}^{+1}(1-\sigma^2)^{k-1}\Psi d\sigma$$
$$-(2k-1)!2^{-2k}\int_0^{+\infty}\int_{-1}^{+1}(\log\tau)(1-\sigma^2)^{k-1}(1-\sigma)$$
$$\times(\partial/\partial\tau)[U(\tau)\int_0^{\tau}U(-s)\Phi(s, \mathrm{ad}a_0(\Psi))ds]d\sigma d\tau.$$

The first term of (4.25) is equal to the first one of (4.21), because $(\partial/\partial\tau)U(\tau)=-a_0U(\tau)$, and the second term of (4.25) to the second one of (4.22) by means of the integral by part. q.e.d.

We define $\theta_\pm(l, t)$ and $I_\pm(\alpha)$ as follows.

$$(4.26)\qquad \theta_+(l, t)=(\mathrm{ad}h)^l(U(t))\,U(-t)$$

and

$$\theta_-(l, t)=U(-t)(\mathrm{ad}h)^l(U(t)).$$

For any natural number m

$$(4.27)\qquad I_+(m, R)=-\int_0^t R(s)\,U(s)(\mathrm{ad}h)^m(a_0)\,U(-s)ds$$

and

$$I_-(m, R)=-\int_0^t U(-s)(\mathrm{ad}h)^m(a_0)\,U(s)R(s)ds,$$

where $R(s)$ is an operator-valued function. $I_\pm(\alpha, R)$ for any multi-index α of natural numbers is inductively defined such that for any multi-indices α and β it holds that

$$(4.28)\qquad I_\pm((\alpha, \beta), R)=I_\pm(\alpha, I_\pm(\beta, R)).$$

For any multi-index α of natural numbers we put

(4.29) $$I_{\pm}(\alpha)=I_{\pm}(\alpha, I),$$

where I is the identity operator.

Lemma 4.3. 1) *For any natural number l it holds that*

(4.30) $$\theta_{\pm}(l, t)=\sum_{|\alpha|=l} C_{l\alpha} I_{\pm}(\alpha),$$

where $\alpha=(\alpha_1, \cdots, \alpha_m)\,(1\leq m\leq l)$ are multi-indices consisting of natural numbers and $C_{l\alpha}$ are positive constants.

2)

(4.31) $$\begin{aligned} &W(t)a_0^k W(-t)\\ &=\sum_{j=0}^{l}(j\,!)^{-1}(t)^j(\mathrm{ad}h)^j(a_0^k)\\ &+(l\,!)^{-1}\int_0^t W(s)\,(\mathrm{ad}h)^{l+1}(a_0^k)W(-s)(t-s)^l ds. \end{aligned}$$

(4.32) $$\begin{aligned} &\int_0^{\log t} W(s)(\mathrm{ad}h)^2(a_1a_0+b)W(-s)(\log t-s)ds\\ &=\sum_{j=2}^{l}(j!)^{-1}(\log t)^j(\mathrm{ad}h)^j(a_1a_0+b)\\ &+(l\,!)^{-1}\int_0^{\log t} W(s)(\mathrm{ad}h)^{l+1}(a_1a_0+b)W(-s)(\log t-s)^l ds. \end{aligned}$$

3)

(4.33) $$\begin{aligned} &W(-\log t)\,U(t)W(\log t)\,U(-t)\\ &=\sum_{j=0}^{l}(j!)^{-1}(-\log t)^j\theta_+(j, t)+Na_+(l)\,U(-t) \end{aligned}$$

and

(4.34) $$\begin{aligned} &U(-t)W(-\log t)\,U(t)W(\log t)\\ &=\sum_{j=0}^{l}(j!)^{-1}(-\log t)^j\theta_-(j, t)+U(-t)Na_-(l), \end{aligned}$$

where

$$\theta_{\pm}(0, t)=I,$$

(4.35) $$\begin{aligned} &Na_+(l)\\ &=(l!)^{-1}(-1)^{l+1}\int_0^{\log t} W(-s)\theta_+(l+1, t)\,U(t)W(s)\\ &\times(\log t-s)^l ds \end{aligned}$$

and

(4.36) $$\begin{aligned} &Na_-(l)\\ &=(l!)^{-1}(-1)^{l+1}\int_0^{\log t} W(-s)\,U(t)\theta_-(l+1, t)W(s)\\ &\times(\log t-s)^l ds. \end{aligned}$$

4) *Let a multi-index α be $(\alpha_1, \cdots, \alpha_m)$. If $\beta+\gamma\leq|\alpha|-m$ for integers β and γ, then $I_{\pm\beta\gamma}=a_0^{\beta}I_{\pm}(\alpha)\,a_0^{\gamma}$ is sufficiently differentiable and of exponential order at infinity in $t\geq 0$*

as functions valued in bounded operators on $\mathbf{H}^S$. *Moreover if* $0\leq j<m$, *then*

$$(d/dt)^j I_{\pm\beta\gamma}|_{t=0}=0,$$

$$\|U(t)(d/dt)^j I_{-\beta\gamma}\|_{S,\lambda}\leq C(1+|t|)^m\exp[(\lambda_S-\lambda)t]$$

and

$$\|(d/dt)^j I_{+\beta\gamma}U(t)\|_{S,\lambda}\leq C(1+|t|)^m\exp[(\lambda_S-\lambda)t]$$

for $t\geq 0$.

Proof. 1) Let us differentiate $\theta_+(l, t)$ in t.

$$(d/dt)\theta_+(l,t) \tag{4.37}$$

$$=[(\mathrm{ad}h)^l(U(t))a_0-(\mathrm{ad}h)^l(U(t)a_0)]\,U(-t).$$

Leibniz formula applies to $(\mathrm{ad}h)^l(U(t)a_0)$ to yield that

$$(\mathrm{ad}h)^l(U(t)a_0)-(\mathrm{ad}h)^l(U(t))a_0$$

$$=\textstyle\sum_{j=0}^{l-1}C_j(\mathrm{ad}h)^j(U(t))(\mathrm{ad}h)^{l-j}(a_0).$$

By means of the substitution to (4.37) we get that

$$(d/dt)\theta_+(l,t)$$

$$=-\textstyle\sum_{j=0}^{l-1}C_j\theta_+(j,t)\,U(t)(\mathrm{ad}h)^{l-j}(a_0)\,U(-t)$$

so that

$$\theta_+(l,t) \tag{4.38}$$

$$=\textstyle\sum_{j=0}^{l-1}C_jI_+(l-j,\theta_+(j,\cdot)),$$

because $\theta_+(l,0)=0$ $(l\geq 1)$. It is shown inductively in $l\geq 1$ that (4.30) for $\theta_+(l,t)$ are solutions of (4.38). (4.30) for $\theta_-(l,t)$ is also obtained in the same way.

2) The direct applications of the formula (1.18) yield the equalities (4.31) and (4.32).

3) We also use (1.18) as $V(s)=W(s)$ and $B=U(t)$ to expand $W(-s)U(t)W(s)$ in s. If $\log t$ is substituted to s, then the first parts of (4.33) and (4.34) consist of the expansion terms multiplied by $U(-t)$, and $Na_\pm(l)$ are the remainer integral terms.

4) It is proved by induction with respect to m. At first we note that $a_0^\beta(\mathrm{ad}a_0)^\delta[(\mathrm{ad}h)^\alpha(a_0)]a_0^\gamma$ is an operator of order 0 if $\beta+\gamma\leq\alpha-1$ and $\delta\geq 0$. In fact, $(\mathrm{ad}h)(a_0)$ is one of H-type of operators used at Corollary of Lemma 3.5 so that Lemma 3.1 and Corollary of Lemma 3.5 imply it if both β and γ are non-negative or non-positive such that $\beta+\gamma\leq\alpha-1$. If $\beta>0>\gamma$, then 3) Corollary of Lemma 3.5 also yields for it to be equal to a sum of the products of operators of order 0 and $a_0^{\gamma-j}$ $(0\leq j\leq\max(\beta-\alpha+1,0))$, which are operators of order 0 because $\gamma-j\leq 0$ by the assumption that $\beta+\gamma\leq\alpha-1$. In case that $\beta>0>\gamma$, it is also proved in the same way. Since the definition (4.27) of $I_\pm(\alpha,I)$ is therefore able to apply 3) of Lemma 3.2 to $I_{\pm\beta\gamma}$, we get the differentiability and the estimate, namely, the statement 4) when $m=1$. Here we use the fact that

$$\|U(t-s)\|_{\sigma,\lambda}\|U(s)\|_{\sigma,\lambda}\leq\exp[(\lambda_\sigma-\lambda)t]$$

for $0\leq s\leq t$. The statements for larger m is also proved by induction according to 3) of Lemma 3.2. q.e.d.

Corollary 1. 1)

$$W(-\log t)U(t)a_0^k = (a_0^k+Q_+)U(t)W(-\log t)+L_+ \tag{4.39}$$

and

$$a_0^k U(t)W(\log t) = W(\log t)\,U(t)\,(a_0^k+Q_-)+L_-, \tag{4.40}$$

where

$$Q_+=R+(a_0^k+R)S_+, \tag{4.41}$$

$$Q_-=R+S_-(a_0^k+R),$$

$$L_+=(a_0^k+R)N_++M_+,$$

$$L_-=N_-(a_0^k+R)+M_-,$$

$$R=\textstyle\sum_{j=1}^{2k-1}(j!)^{-1}(-\log t)^j(\mathrm{ad}h)^j(a_0^k), \tag{4.42}$$

$$S_\pm=\textstyle\sum_{j=1}^{2k-1}(j!)^{-1}(-\log t)^j\theta_\pm(j,t),$$

$$M_+=J_+(t,(\mathrm{ad}h)^{2k}(a_0^k))\,U(t), \tag{4.43}$$

$$M_-=U(t)J_-(t,(\mathrm{ad}h)^{2k}(a_0^k)),$$

$$N_+=J_+(t,\theta_+(2k,t)\,U(t)), \tag{4.44}$$

$$N_-=J_-(t,\,U(t)\theta_-(2k,t)),$$

$$J_+(t,J_0)=(2k-1)!^{-1}\int_0^{\log t}W(s-\log t)J_0W(-s)s^{2k-1}ds \tag{4.45}$$

and

$$J_-(t,J_0)=(2k-1)!^{-1}\int_0^{\log t}W(s)J_0W(\log t-s)s^{2k-1}ds.$$

2) *Let F, G, and H be ones defined by* (1.14), (1.19) *and* (1.20), *and some other operators be defined as*

$$F'=\int_0^{+\infty}W(\log t)\,U(t)a_0^k t^{k-1}dt, \tag{4.46}$$

$$G'=\int_0^{+\infty}t^{k-1}a_0^k U(t)W(-\log t)dt$$

$$Qa_+=\int_0^{+\infty}t^{k-1}\,Q_+U(t)W(-\log t)dt, \tag{4.47}$$

$$Qa_-=\int_0^{+\infty}W(\log t)\,U(t)Q_-t^{k-1}dt$$

and

(4.48) $$La_{\pm}=\int_{0}^{+\infty}t^{k-1}L_{\pm}dt.$$

Then we have that

(4.49) $$GF=G'F'+G'Qa_{-}+Qa_{+}F'+Qa_{+}Qa_{-}$$
$$+GLa_{-}+La_{+}F+La_{+}La_{-}$$

and that

(4.50) $$GF=H+H_{2}$$

if H_2 *stands for* $H_{21}+H_{22}$ *such that*

(4.51) $$H_{21}=\sum_{j=1}^{2k}\int_{0}^{+\infty}d\tau\int_{-1}^{+1}g_{j}(1-\sigma^{2})^{k-1}\Phi((\mathrm{ad}a_{0})^{j}(\Psi'))d\sigma$$
$$+\sum_{j=0}^{2k}\int_{0}^{+\infty}d\tau\int_{-1}^{+1}(1-\sigma^{2})^{k-1}g_{0j}\Phi((\mathrm{ad}a_{0})^{j}(\Psi'))g_{1j}d\sigma$$

and

(4.52) $$H_{22}=GLa_{-}+La_{+}F+La_{+}La_{-},$$

where $\Psi'=W(\log[(1-\sigma)/(1+\sigma)])$, *and* g_j *and* g_{ij} *satisfy the same type of conditions as at* (4.12) *and at* (4.16–17) *with* l *and* $m\leq 4k-2$, *respectively.*

Proof. 1) Let us prove (4.39) because (4.40) is shown in the same way. At first we commute $W(-\log t)$ and a_0^k to get according to (4.31) that

(4.53) $$W(-\log t)U(t)a_{0}^{k}$$
$$=(a_{0}^{k}+R)W(-\log t)U(t)+M_{-}.$$

The next commutation of $W(-\log t)$ and $U(t)$ according to (4.33) implies that

(4.54) $$W(-\log t)U(t)$$
$$=(I+S_{+})U(t)W(-\log t)+N_{+}.$$

The substitution of (4.54) to (4.53) completes the proof of (4.39), where variables of integration should be changed to get the expression with the function J.

2) We get (4.55) by means of integration of (4.39–40) in t from zero to infinity after multiplication by t^{k-1}.

(4.55) $$G-La_{+}=G'+Qa_{+}$$

and

$$F-La_{-}=F'+Qa_{-}.$$

Then (4.49) is the expansion of the product $(G-La_{+})(F-La_{-})=(G'+Qa_{+})(F'+Qa_{-})$ of two above equalities. We use Corollary 1 of Lemma 4.2 in order to prove (4.50). The equality corresponding to (4.12) of Lemma 4.2 implies for $G'F'$ to be equal to

(4.56) $$(2k-1)!2^{-2k+1}\int_{-1}^{+1}(1-\sigma^{2})^{k-1}\Psi'd\sigma$$

plus the first term of H_{21} at (4.51). The change of variable σ to $-\sigma$ proves that (4.56) is equal to H. The other terms except for ones constituting H_{22} are discribed as the second term of H_{21} in virtue of applications of the equalities corresponding to (4.13) and (4.17) of Lemma 4.2. In fact $Q_\pm$ (therefore $Qa_\pm$) consist of two parts R and $(a_0^k+R)S_+$ (or $S_-(a_0^k+R)$). They are regarded as $a_0^k a_0^{-k}R$ (or $Ra_0^{-k}a_0^k$), $a_0^k(I+a_0^{-k}R)S_+$ and $S_-(I+Ra_0^{-k})a_0^k$. (4.17) is applicable to the terms including the parts connected to $(a_0^k+R)S_+$ or $S_-(a_0^k+R)$ because 4) of Lemma 4.3 assures the conditions for f_0 and f_1 at 3) of Lemma 4.2. (4.13) applies to the terms related only to R. A commutation of the first one of gotten terms with a_0 yields expected forms because the coefficients of $a_0^{-k}R$ a polynomial in $(\log t)$ take the shape of $a_0^{-1}R^\sim$ with $R^\sim$ operators of order 0.

q.e.d.

Corollary 2. *Let* $\theta w_\pm(l, t)$, $Iw_\pm(m, R)$ *and* $Iw_\pm(\alpha)$ *stand for ones defined by the replacement of* h *and* $U(t)$ *by* r *and* $W(t)$ *at* (4.26), (4.27) *and* (4.29), *respectively, where* r *is* a_0, *a pseudodifferential operator* q *of order* 1 *in* x *or a H-type operator times* q. (*Refer to Corollary of Lemma* 3.5.) *Then it holds that*

$$\theta w_\pm(l, t)=\sum_{|\alpha|=l}C_{l\alpha}Iw_\pm(\alpha),$$

with α *and* $C_{l\alpha}$ *at* (4.30),

$$(\mathrm{ad}r)^l[W(t)]=\theta w_+(l, t)W(t)=W(t)\theta w_-(l, t)$$

and

$$\|(\mathrm{ad}r)^l[W(t)]\|\le C_{sl}(1+|t|)^l\exp[No|t|].$$

Proof. The first equality is obvious. The inequalities hold for $Iw_+(\alpha)W(t)$ and $W(t)Iw_-(\alpha)$ with $|\alpha|=l$ if $(\mathrm{ad}r)^jh$ for $j\ge 1$ are bounded on $\boldsymbol{H}^s$ with respect to the norms $\|\ \ \|_{s,\lambda}$. In fact it is proved by induction in the number n of the indices α.

q.e.d.

Proof of Lemma 1.2. Let us consider $F(a_1a_0+b)G$, namely, operate G at the right hand side of the equality (1.16). This first term is equal to

$$[a_1a_0+b-\mathrm{ad}a_0(a_1)h]FG.$$

By the notation at Corollary 2 of Lemma 4.2, it is equal to

$$[a_1a_0+b-\mathrm{ad}a_0(a_1)h](H+H_1).$$

$$[a_1a_0+b-\mathrm{ad}a_0(a_1)h]H$$

is the first line at the right hand side of (1.21) if the operator H in it is excluded by the multiplication of H^{-1}.

$$[a_1a_0+b-\mathrm{ad}a_0(a_1)h]H_1H^{-1}$$

is regarded as one of the forth line and the remainder terms because the second half of 1) Corollary 2 of Lemma 4.2 asserts that a_0H_1 is an operator of order 0.

The equality (4.21) at Corollary 2 of Lemma 4.2 rewrites the second term of $F(a_1a_0+b)G$ as

$$[\mathrm{ad}h(a_1a_0+b)+k\mathrm{ad}a_0(\mathrm{ad}h(a_1a_0+b))\,a_0^{-1}]$$
$$\times\int_0^{+\infty}(\log t)t^{k-1}a_0^kU(t)W(\log t)dtH$$
$$+[\mathrm{ad}h(a_1)a_0+a_1\mathrm{ad}h(a_0)+ka_1\mathrm{ad}a_0(\mathrm{ad}h(a_0))a_0^{-1}]H_1'$$
$$+[\mathrm{ad}h(b)+k\mathrm{ad}a_0(a_1)\mathrm{ad}h(a_0)\,a_0^{-1}$$
$$+k\mathrm{ad}a_0(\mathrm{ad}h(a_1))+k\mathrm{ad}a_0(\mathrm{ad}h(b))\,a_0^{-1}]H_1'.$$

The first term of the above is the second and third lines of (1.21) multiplied by H. Since the commutator of a_0 and H_1' is obtained by the replacement of $(\mathrm{ad}a_0)^j$ by $(\mathrm{ad}a_0)^{j+1}$ in H_1' and also $\mathrm{ad}a_0(H^{-1})$ is an operator of order 0, the above second term is included in the forth line and remainder terms. It is clarified at later arguments that the above last term may be regarded as one of the remainder terms. The last term including $Z(t)$ at (1.16) may also be regarded as one of the remainder terms by the application of 3) Lemma 4.2 except for the term consisting of Z_7, for which the statement 3) at Corollary of Lemma 3.5 is used.

We put exactly d_j at the forth line of (1.21) and the remainder terms in the next lemmas for later arguments. q.e.d.

Lemma 4.4. *d_0, d_1 and R at* (1.21) *are discribed as*

(4.57) $$d_0=\mathrm{ad}h(a_1)H_1',$$

(4.58) $$d_1=a_0H_1H^{-1}$$
$$+[\mathrm{ad}h(a_0)+k\mathrm{ad}a_0(\mathrm{ad}h(a_0))\,a_0^{-1}]H_1'$$

and

(4.59) $$R=(b-\mathrm{ad}a_0(a_1)h)H_1H^{-1}$$
$$+\mathrm{ad}h(a_1)\mathrm{ad}a_0(H_1')$$
$$+[\mathrm{ad}h(b)+k\mathrm{ad}a_0(a_1)\,\mathrm{ad}h(a_0)\,a_0^{-1}$$
$$+k\mathrm{ad}a_0(\mathrm{ad}h(a_1))+k\mathrm{ad}a_0(\mathrm{ad}h(b))\,a_0^{-1}]H_1'$$
$$+\int_0^{+\infty}Z(t)t^{k-1}a_0^kU(t)W(\log t)dtGH^{-1},$$

where H_1, H_1' and $Z(t)$ are at (4.19), (4.22) *and* (1.17), *respectively.*

Since we think that h should be exactly defined before proving Lemma 1.3, we deal first of all with Lemma 1.4–6.

Proof of Lemma 1.4. The assumption (1.5) means that

$$\{a_1, a_0\}=\psi^2d+\psi_1(a_1-a_0)$$

with ψ_1 an infinitely differentiable function in x_0 and with d a pseudodifferential operator of order 1 bounded below by $N_1\langle D\rangle>0$ at (X_0, X) such that $(a_1-a_0)^2+\psi^2b_2\leq\psi^2\delta\langle D\rangle^2$. (We denote the set of such (X_0, X) by $\Sigma(\delta)$.) We define h_0' a pseudodifferential operator of order 0 by

$$h_0' = \rho_\delta b_1 d^{-1},$$

where the right hand side means the product as symbols and

$$\rho_\delta = \rho_0([\psi^{-2}(a_1-a_0)^2+b_2]\delta^{-1}\langle D\rangle^{-2})$$

with $\rho_0(t)$ an infinitely differentiable and monotone decreasing function in t such that $\rho_0=1$ when $0\leq t\leq\frac{1}{2}$ and $\rho_0=0$ when $1\leq t$. Since we are able to take sufficiently small $\delta>0$ such that

$$|b_1 d^{-1}|\leq 2N_0 \quad \text{on } \Sigma(\delta)$$

with N_0 a bound of $|b_1 d^{-1}|$ on the characteristic set Σ of P, we may assume that on the whole space

$$|h_0'|\leq 2N_0.$$

Therefore, if h_0 is defined by (1.24), namely,

$$h_0 = h_0' + i\theta,$$

then it holds that

$$\begin{aligned}
&\psi^2 b_1 - i\{a_1, a_0\}h_0\\
&\quad = -\psi_1(a_1-a_0)ih_0\\
&\quad -\psi^2(1-\rho_\delta)(ib_1-N_1\theta)\\
&\quad +\psi^2[\rho_\delta d+N_1(1-\rho_\delta)]\theta.
\end{aligned}$$

Since this is equal to

$$\psi^2 b_1 - \mathrm{ad}a_0(a_1)h_0$$

modulo pseudodifferential operators of order 0 in x, we complete the proof. In fact we put

$$d_1 = [d\rho_\delta + N_1(1-\rho_\delta)\langle D\rangle]\theta + C_1$$

with $\theta = 2N_1^{-1}\varepsilon$ and some large constant C_1. Then it holds that

$$d_1 \geq \varepsilon\langle D\rangle.$$

Since $1-\rho_\delta$ vanishes at $\Sigma(\delta/2)$, it holds that

$$(1-\rho_\delta)(ib_1-N_1\theta) = c_0 b_2 + d_0'$$

with c_0 and d_0' pseudodifferential operators of order -1 and of order 0, respectively.

q.e.d.

Proof of Lemma 1.5. Let us assume the following Lemma 4.5, which will be proved at the end of this section.

Lemma 4.5. *There exists a pseudodifferential operator Λ_3 of order* 0, *which is a linear combination of $\Lambda_0-\Lambda_1$ and ∂b_2, such that*

$$\{p_2, \Lambda_3\} = \psi k_2(\xi_0 - \psi \Lambda_1) + \psi k_3(\xi_0 - \psi \Lambda_0) + \psi^2 k_4,$$

where k_2 and k_3 are uniformly positive ones of order 0 *on a conic neighborhood of the singular points of f and satisfy there that*

$$k_4^2 \leq 2(1+6\varepsilon_2)^{-1} k_2 k_3 b_2,$$

with a positive contant ε_2.

Let Ψ stand for a function in x_0 such that $(\partial/\partial x_0)\Psi(\nu, x_0) = \psi(\nu, x_0)$ and $\Psi(\nu, 0) = 0$. Then $|\Psi| \leq C\nu^{-1}$ because $\Psi(\nu, x_0) = \nu^{-1}\Psi(1, x_0/\nu)$. We also put

$$\Omega = [(\Lambda_0 - \Lambda_1)^2 + b_2]\langle \xi \rangle^{-2}.$$

Since Lemma 4.5 assures that $k_{2+j} (j=0$ and $1)$ are greater than a positive constant ε_0 on $\Sigma(\delta_0)$ for some positive δ_0 (refer $\Sigma(\delta)$ to the proof of Lemma 1.4) and since Ω is greater than another positive ε_1 out side $\Sigma(\delta_0)$, it holds for $j=0$ and 1 that

$$h_{2+j} = k_{2+j} + a[\nu^2\Omega + \nu^2\Psi\{\psi^{-1}\xi_0 - \Lambda_j, \Omega\}] \tag{4.60}$$
$$\geq 3^{-1}(\varepsilon_0 + a\nu^2\Omega),$$

If a is a sufficiently small positive constant and if ν is sufficiently large. In fact we have that with a constant C independent of ν

$$|\nu^2\Psi\{\psi^{-1}\xi_0 - \Lambda_j, \Omega\}| \leq 3^{-1}\nu^2\Omega + C$$

because $|\nu\Psi| \leq C$ and $|\partial\Omega|^2 \leq C\Omega$ in virtue of the positivity of Ω if $\partial\Omega$ stands for

$$(\psi^{-1}(\partial/\partial x_0)\Omega, (\partial/\partial x)\Omega, \langle\xi\rangle(\partial/\partial\xi)\Omega).$$

$$k_{2+j} - aC \geq 3^{-1}\varepsilon_0 \quad \text{on } \Sigma(\delta_0)$$

if a is small and

$$|k_{2+j} - aC| \leq 3^{-1} a\nu^2\Omega$$

out side $\Sigma(\delta_0)$ if ν is large. So we conclude the first statement of the lemma since it is easily shown that there exists a positive ε such that

$$(1+\nu^2\Omega) \geq \varepsilon\nu|h_1|,$$

and that the left hand side of (4.60) is equal to h_{j+2} there if h_1 is defined by

$$h_1 = \Lambda_3 + a\Psi\nu^2\Omega.$$

We next calculate h_4, where

$$h_4 = k_4 + a\Psi\nu^2\{b_2, \Omega\}.$$

Since

$$|a\Psi\nu^2\{b_2, \Omega\}|^2 \leq C\nu^2 b_2\Omega^2,$$

the inequality (4.60) and the result of Lemma 4.5 yield that

$$|h_4|^2 \leq 2(1+5\varepsilon_2)^{-1} h_2 h_3 b_2,$$

if ν is sufficiently large. q.e.d.

Remark. The choise of the parameter ν is able to be fixed independent of perturbations of Λ_j and Ω as far as they keep ε_0, ε_1, δ_0 and the other bounds as noting in the above proof. For example it is possible that they sway in the parameter ν because they are pullbacks $f(\Psi, x, \xi)$ of other functions $f(x_0, x, \xi)$ by Ψ.

Proof of Lemma 1.6. 1) It is trivial.

2) At first we consider it in the case that $s=0$. If h^* stands for the adjoint operator of h on L^2, then

$$2\mathrm{Re}h=h+h^*=h_0^{\sim}E^{-1}\langle D\rangle^2+\langle D\rangle^2E^{-1}h_0^{\sim}.$$

It is well known that the pseudodifferential operator $h_0^{\sim}$ of order 0 in x is a bounded operator on L^2 and that its bound is fixed by ones of the symbol and its derivatives of finite order. Therefore there exists a constant C_0 such that

$$|\mathrm{Re}(hu, u)|=|(\mathrm{Re}hu, u)|\leq C_0\|u\|^2 \tag{4.61}$$

and

$$\|h_Ru\|\leq C_0\|u\|.$$

By Lemma 3.1 or Corollary of Lemma 3.5, we get for a general integer s that

$$E^Sh_RE^{-S}=h_R+k_SE^{-1},$$

where k_S is an operator of order 1. This implies that

$$(hu, u)_{2S,\lambda}=(hE^Su, E^Su)+(k_SE^{S-1}u, E^Su)$$

and

$$\|h_Ru\|_{2S,\lambda}\leq\|h_RE^Su\|+\|k_SE^{S-1}u\|,$$

so that

$$|\mathrm{Re}(hu, u)_{2S,\lambda}|$$
$$\leq|\mathrm{Re}(hE^Su, E^Su)|+\|k_SE^{S-1}u\|\,\|E^Su\|.$$

Since the operator k_S of order 1 is estimated as $\|k_Sv\|\leq C_S\|\langle D\rangle v\|$, we have that

$$\|k_SE^{S-1}u\|\leq C_S\|E^{S-1/2}u\|.$$

This combines with (4.61) to imply that

$$|\mathrm{Re}(hu, u)_{2S,\lambda}|$$
$$\leq C_0\|u\|_{2S,\lambda}^2+C_S\|u\|_{2S-1,\lambda}\|u\|_{2S,\lambda}$$

and

$$\|h_Ru\|_{2S,\lambda}\leq C_0\|u\|_{2S,\lambda}+C_S\|u\|_{2S-1,\lambda}.$$

The interpolation theory will apply to cases for other indices s. q.e.d.

Proof of Lemma 1.3. 1) If λ is taken sufficiently large depending on s at (1.36–37), then it holds with $No=2C_0$ that

$$|\mathrm{Re}(hu, u)_{s,\lambda}| \leq No\|u\|_{s,\lambda}^2$$

and

(4.62) $$\|h_R u\|_{s,\lambda} \leq No\|u\|_{s,\lambda},$$

because $\|v\|_{s,\lambda} \geq \lambda\|v\|_{s-1,\lambda}$. These inequalities imply that $W(t)$, defined by (1.10), satisfies the estimates

$$\|W(t)\|_{s,\lambda} \leq \exp(No|t|),$$

so that

(4.63) $$\|W(\log[(1+\sigma)/(1-\sigma)]\|_{s,\lambda} \leq 4^{No}(1-\sigma^2)^{-No}.$$

Therefore, (1.20) the definition of H is valid if $k>No$, and so H is a bounded operator on $\boldsymbol{H}^s$. The invertibility of H is essentially due to the following facts.

Lemma 4.6. *Let us put*

$$I(\alpha)=\int_0^1 (1-\sigma^2)^\alpha d\sigma$$

and

$$J(\alpha)=\int_{-1}^{+1} (1-\sigma^2)^\alpha \sigma g(\sigma) d\sigma$$

with a differentiable function g such that $(1-\sigma^2)^\alpha g'(\alpha)$ *vanishes at* $\sigma=\pm 1$. *Then it holds that*

$$I(\alpha)=\Pi_{k=1}^{\alpha}[2k/(2k+1)]$$

and

$$J(\alpha)=\int_{-1}^{+1} (1-\sigma^2)^{\alpha+1} g'(\sigma) d\sigma [2(\alpha+1)]^{-1}.$$

Therefore there exists a positive constant γ *such that* $I(\alpha) \sim \gamma\alpha^{-1/2}$ *as* α *tends to infinity and it holds that*

$$\|J(\alpha)\| \leq I(\alpha+1-\beta)(\alpha+1)^{-1}[\sup_{|\sigma|\leq 1}\|(1-\sigma^2)^\beta g'(\sigma)\|].$$

The operator H splits into the sum of two part such that

$$H=C_k[2I(k-1)+J(k-1)],$$

where

$$C_k=2^{-2k+1}(2k-1)!,$$

$$g(\sigma)=\int_0^1 W_l'(\sigma\tau)d\tau$$

and

$$W_l(\sigma)=W(\log[(1+\sigma)/(1-\sigma)]).$$

Since $g'(\sigma)$ satisfies that

$$\|g'(\sigma)\|_{s,\lambda}$$

$$\leq 2^{-1}\sup_{0\leq\tau\leq 1}\|W_l''(\sigma\tau)\|_{s,\lambda}$$
$$\leq C_{No}(1-\sigma^2)^{-No-2}(\|h\|_{s,\lambda}^2+\|h\|_{s,\lambda}),$$

we get that

$$\|g'(\sigma)\|_{s,\lambda}\leq C_{No}(No+|\theta|+1)(No+|\theta|)(1-\sigma^2)^{-No-2},$$

by (4.62–63). Therefore it holds that

$$\|J(a)\|_{s,\lambda}\leq C(No,\theta)I(a+No-1)(a+1)^{-1},$$

where

$$C(No,\theta)=C_{No}(No+|\theta|+1)(No+|\theta|).$$

If k is taken such that

$$2^{-1}C(No,\theta)I(k+No-2)I(k-1)^{-1}k^{-1}<1,$$

then the existence of the inverse of H on $\boldsymbol{H}^S$ is shown by means of Neumann series. Here we should note that k is fixed only by No and θ independent of s and λ, and that the operator norms of H and H^{-1} on $\boldsymbol{H}^S$ are also independent of s and λ.

2) H_1 has been written as (4.19) at Corollary 2 of Lemma 4.2. On the other hand, Corollary 2 of Lemma 4.3 shows that

$$(\mathrm{ad}a_0)^j(\Psi)$$
$$=\theta w_+(l,\log[(1+\sigma)/(1-\sigma)]W(\log[(1+\sigma)/(1-\sigma)]))$$

is estimated as

$$\|(\mathrm{ad}a_0)^j(\Psi)\|_{s,\lambda}\leq C_{sj}(1-\sigma^2)^{-No-1},$$

because $(\mathrm{ad}a_0)$ h are operators of order 0. Therefore (4.19) is estimated as

$$\|H_1\|_{s,\lambda}$$
$$\leq C_s\int_0^{+\infty}(1+\tau)^{2k-1}(1-\sigma^2)^{k-No-2}\exp[(\lambda_s-\lambda)\tau]d\tau,$$

according to the definition of Φ and (1.12). This implies that

$$\|H_1\|_{s,\lambda}\leq C_s\lambda^{-1}$$

as λ tends to infinity if $k-No-1>0$. $H_2=H_{21}+H_{22}$ has also been written as (4.51–52) at Corollary 1 of Lemma 4.3. The first part H_{21} (4.51) has the same integral form as H_1 so that it has the same estimate. It is also proved by the following lemma that the second part H_{22} has the same estimate as the other. q.e.d.

Lemma 4.7. *H_{22}, defined at* (4.52), *has the estimate such that*

$$\|H_{22}\|_{s,\lambda}\leq C\lambda^{-1}$$

as λ tends to infinity.

Proof. Since $H_{22}=GLa_-+La_+F+La_+La_-$ by definition, it suffices to obtain the estimates for Ga_0^{-k}, $a_0^{-k}F$, $a_0^kLa_-$ and $La_+\,a_0^k$. We try to estimate $La_+\,a_0^k$. $a_0^kLa_-$ is also estimated in the same way. The definitions (4.48) and (4.41) show it necessary to bound $a_0^k N_+ a_0^k$, $I+Ra_0^{-k}$ and $M_+ a_0^k$. By the definition (4.44) of N_+,

$$a_0^k N_+ a_0^k$$
$$=(2k-1)!^{-1}\int_0^{\log t} a_0^k W(s-\log t)a_0^{-k} J_1 a_0^{-k} W(-s)a_0^k s^{2k-1}\,ds,$$

where

$$J_1 = a_0^k \theta_+(2k, t)\,U(t)a_0^k.$$

Since

$$a_0^k W(s)a_0^{-k} = \sum_{j=0}^k C_{kj}(\mathrm{ad}a_0)^j(W(s))a_0^{-j}$$

and

$$a_0^{-k} W(s)a_0^k = \sum_{j=0}^k C'_{kj}\, a_0^{-j}(\mathrm{ad}a_0)^j(W(s)),$$

the estimate at Corollary 2 of Lemma 4.3 implies that

$$\|a_0^k W(s-\log t)\, a_0^{-k}\|_{\sigma,\lambda}\|a_0^{-k} W(-s)\, a_0^k\|_{\sigma,\lambda}$$
$$\leq C(1+|s-\log t|)^k(1+|s|)^k \exp[No(|s-\log t|+|s|)]$$
$$\leq C(1+|\log t|)^{2k}\exp[No|\log t|],$$

because $0\leq \pm s\leq \pm\log t$ and $\|a_0^{-1}\|_{\sigma,\lambda}\leq(\lambda_\sigma-\lambda)^{-1}$. There are 1) and 4) of Lemma 4.3 for J_1. It holds that

$$\|J_1\|_{\sigma,\lambda}\leq C(1+|t|)^{2k}\exp[(\lambda_\sigma-\lambda)t].$$

Therefore we obtain that

$$\|a_0^k N_+ a_0^k\|_{\sigma,\lambda}\leq C(1+|t|)^{2k+No}(1+|\log t|)^{4k} t^{-No}\exp[(\lambda_\sigma-\lambda)t]. \tag{4.64}$$

It is easy to see at (4.42) that $(\mathrm{ad}h_0)^j(a_0^k)\,a_0^{-k}$ is an operator of order 0 according to 2) Corollary of Lemma 3.5. Therefore $(I+Ra_0^{-k})$ is estimated as

$$\|I+Ra_0^{-k}\|_{\sigma,\lambda}\leq C(1+|\log t|)^{2k-1}. \tag{4.65}$$

By the definition (4.43) of M_+, $M_+ a_0^k$ is written as

$$M_+ a_0^k = (2k-1)!^{-1}\int_0^{\log t} W(s-\log t)\, J_2 a_0^{-k} W(-s)\, a_0^k s^{2k-1}\,ds\, U(t)$$

and

$$J_2 = (\mathrm{ad}h)^{2k}(a_0^k)\, a_0^k.$$

The combination of 2) and 3) Corollary of Lemma 3.5 shows that J_2 is an operator of order 0. So we get that

$$\|M_+ a_0^k\|_{\sigma,\lambda}\leq C(1+|t|)^{No}(1+|\log t|)^{3k} t^{-No}\exp[(\lambda_\sigma-\lambda)t]. \tag{4.66}$$

By (4.64–66), $L_+ a_0^k$ is bounded as

$$\|L_+ a_0^k\|_{\sigma,\lambda}\leq C(1+|t|)^{2k+No}(1+|\log t|)^{6k-1} t^{-No}\exp[(\lambda_\sigma-\lambda)t].$$

Since the integration of $t^{k-1}L_{+}a_0^k$ in t is equal to $La_{+}a_0^k$ by definition, we get that, if $k>No$ and $\lambda\geq\lambda_\sigma+1$, then

$$\|La_{+}a_0^k\|_{\sigma,\lambda}\leq C\lambda^{-1}.$$

It is clear by the definitions (1.14) and (1.19) for the norms of $a_0^{-k}F$ and Ga_0^{-k} to be bounded by

$$C\int_0^{+\infty}t^{k-1-No}\exp[(\lambda_\sigma-\lambda)t]dt\leq C'\lambda^{-1},$$

if $k>No$ and $\lambda\geq\lambda_\sigma+1$. So we can conclude that

$$\|H_{22}\|_{\sigma,\lambda}\leq C\lambda^{-1}. \qquad \text{q.e.d.}$$

Remark. It is easily checked by the combination of Corollary of Lemma 3.5 and Corollary 2 of Lemma 4.3 that

$$H_{\alpha\beta\gamma}=\psi^\alpha\langle D\rangle^\beta E^\gamma HE^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}-H$$

is an operator of order 0 such that the norm of $H_{\alpha\beta\gamma}$ on $\boldsymbol{H}^0$ is bounded by $C_{\alpha\beta\gamma}\lambda^{-1}$. Therefore we can conclude that H^{-1} is an operator of order 0 because

$$\psi^\alpha\langle D\rangle^\beta E^\gamma H^{-1}E^{-\gamma}\langle D\rangle^{-\beta}\psi^{-\alpha}=(H+H_{\alpha\beta\gamma})^{-1}.$$

Proof of Lemma 4.5. According to the formation of the problem we may omit the function ψ to assume that, with non negative function b,

$$p=-\xi_0^2+f \text{ and } f=\Lambda^2+b, \tag{4.67}$$

because the results are independent of the canonical transform as $\xi_0-(\Lambda_0+\Lambda_1)/2$ is transformed to ξ_0. The conditions are that

$$\{\xi_0, \Lambda\}\neq 0 \text{ at } \Lambda=b=0 \tag{4.68}$$

and

$$\{\xi_0-\Lambda, b\}=cb, \tag{4.69}$$

where c, Λ, f and b are homogeneous order of 0, 1, 2 and 2 in ξ, respectively.

Let $H_p=\nabla^2p$ be the Hesse matrix of p and JH_p be the Hamilton matrix of p, that is, $\sigma(u, JH_pv)=\langle u, H_pv\rangle$ with respect to the canonical form σ. The conditions (4.68–69) imply that JH_p has a real non zero eigenvalue at the singular points Σ of p, namely, p is effectively hyperbolic. In fact the vector $v=J\nabla(\xi_0-\Lambda)$ atains the eigenvector corresponding to the eigenvalue

$$-a=-\{\xi_0-\Lambda, \xi_0+\Lambda\}$$

of JH_p at Σ, that is,

$$(a+JH_p)v=0.$$

The hyperbolicity of H_p implies the existence of another real eigenvalue a, of which eigenvector u satisfies that $\sigma(u, v)\neq 0$ with respect to the canonical form σ. If H_p is defined by the Hesse matrix of p with respect to the fixed coordinate, then a, u and v

may extend to a neighborhood of the singular point Σ as holding their relations. Especially a is a real function.

Let us denote $u=(u_0, u_1, u_2)$ and $v=(v_0, v_1, v_2)$ by the coordinate as

$$w=w_0(\partial/\partial\xi_0)+w_1(\partial/\partial x_0)+w_2(\partial/\partial X),$$

where $X=(x, \xi)$. We normalize them as $u_1=v_1=1$. Then $-u_0=v_0=a/2$.

$$U=(u+v)/2 \text{ and } V=(u-v)/2$$

satisfy that

$$aV=JH_pU \text{ and } aU=JH_pV.$$

Therefore

$$a\sigma(V, U)=\sigma(V, JH_pV)=\langle H_pV, V\rangle.$$

More precisely the component wise expression shows us that

$$\begin{aligned} a\sigma(V_2, U_2)&=\sigma(V_2, JH_pV_2)=\langle H_pV_2, V_2\rangle \\ &=\langle H_fV_2, V_2\rangle \\ &=2[\sigma(V_2, U_2)]^2+\langle H_bV_2, V_2\rangle, \end{aligned}$$

because

$$H_f=2\nabla\Lambda\otimes\nabla\Lambda+\nabla^2 b$$

and

$$\langle V_2, \nabla\Lambda\rangle=\sigma(V_2, v_2)=\sigma(V_2, U_2),$$

where $U_j=(u_j+v_j)/2$ and $V_j=(u_j-v_j)/2$ ($j=0$, 1 and 2) so that $U_0=0$, $U_1=1$, $V_0=-a/2$ and $V_1=0$. The existence of a real solution $\sigma(V_2, U_2)$ for the quadratic equation implies that

$$a^2-8\langle H_bV_2, V_2\rangle\geq 0.$$

The positivity of $\langle H_bV_2, V_2\rangle$ at the singular points $\sum$ implies that there

$$0\leq\sigma(V_2, U_2)\leq a/2.$$

However $\sigma(V_2, U_2)$ is not equal to $a/2$ because $\sigma(V, U)=-a/2+\sigma(V_2, U_2)$ should not be zero. Therefore there exists a positive constant ε_0 such that

$$0\leq(1+\varepsilon_0)\sigma(V_2, U_2)\leq a/2.$$

Let us consider two functions that

$$\Lambda_3=\langle\nabla p, U\rangle=\langle\nabla f, U\rangle=U(f)$$

and

$$\begin{aligned} &a\xi_0+\mu=\langle p, V\rangle=V(p), \\ &(\mu=\langle\nabla f, V\rangle=\langle\nabla f, V_2\rangle=V_2(f)). \end{aligned}$$

Then it holds that

$$\begin{aligned}&\{p, \Lambda_3\}\\&=\langle \nabla p, JH_p U\rangle+\langle \nabla f, \{p, U\}\rangle\\&=a\langle \nabla p, U\rangle+\langle \nabla f, \{p, U\}\rangle\\&=a^2\xi_0+a\mu+L[\nabla f:\nabla p].\end{aligned}$$

Remark on Notations. $L[X_1:X_2:\cdots]$ means that it is a multi-linear combination of each component X_j with coefficients of infinitely differentiable functions.

On the other hand

$$\begin{aligned}\mu=V_2(f)&=2V_2(\Lambda)\Lambda+V_2(b)\\&=2\sigma(V_2, U_2)\Lambda+V_2(b)\end{aligned}$$

and

$$L[\nabla f:\nabla p]=L[\nabla f]\xi_0+L[\nabla b]\Lambda+L[\nabla b:\nabla b].$$

Therefore it holds that

$$\begin{aligned}&\{p, \Lambda_3\}\\&=(a^2/2-a\sigma(V_2, U_2)+L[\nabla f])(\xi_0-\Lambda)\\&+(a^2/2+a\sigma(V_2, U_2)+L[\nabla f])(\xi_0+\Lambda)\\&+aV_2(b)+L[\nabla b:\nabla b].\end{aligned}$$

We denote it as

$$\{p, \Lambda_3\}=a(\alpha_0-\beta_0)(\xi_0-\Lambda)+a(\alpha_0+\beta_0)(\xi_0+\Lambda)+a\gamma_0.$$

The bound for $\sigma(V_2, U_2)$ yields that $\alpha_0-\beta_0$ and $\alpha_0+\beta_0$ are positive at a neighborhood of the singular points Σ. The combination of them with the following lemma yields that

$$\gamma_0^2\leq 2(1+\varepsilon_0/3)^{-1}(\alpha_0^2-\beta_0^2)b$$

at a neighborhood of Σ. In fact, for α, $\beta=\sigma(V_2, U_2)$ and $\gamma=\langle H_b V_2, V_2\rangle$, it holds on Σ that

$$\begin{aligned}2\gamma&=2(\alpha-2\beta)\beta\\&<2(1+\varepsilon_0/2)^{-1}(\alpha/2-\beta)(\alpha/2+\beta)\end{aligned}$$

because $\alpha/2\geq(1+\varepsilon_0)\beta$. And $V_2(V_2(b))=\langle H_b V_2, V_2\rangle+L[\nabla b]$. Since

$$2\varepsilon_0(3+\varepsilon_0)^{-1}(2+\varepsilon_0)^{-1}(\alpha_0^2-\beta_0^2)$$

absorbs the perturbations by $L[\nabla f]$, $L[\nabla b:\nabla b]$ and ε if a neighborhood and ε are small, we get the conclusions. The inequality at Lemma 4.5 is obtained if ε_2 is put as $6\varepsilon_2=\varepsilon_0/3$. q.e.d.

Lemma 4.8. *Let $f(t, x)$ be an infinitely differentiable function on $\boldsymbol{R}^{n+1}$, where t is a variable and x is the others. We assume that f is non negative and denote the zero point set of f by Σ. Then for any positive ε there exists a neighborhood D of Σ such that*

$$|(\partial/\partial t)f(t, x)|^2 \leq 2((\partial/\partial t)^2 f(t, x)+\varepsilon)f(t, x) \quad \text{on } D.$$

§ 5. Verification of the reduction to a basic type.

Proof of Theorem 3. We shall check the definitions from 1) to 12) of a basic type at the top of Section 2, referring to Lemma 1.2. The first line at the left hand side of (1.21) is the main part. $a_1 a_0$, the principal part $\psi^2 b_2$ of b, a part of the first order term $d(a_0+a_1)$ and the term of order 0 b_0 are left as they are. The remainded part of the first order term $\psi^2 b_1$ is changed by means of h, defined at (1.31), as in 1) Corollary of Lemma 1.5. So we have that

$$\begin{aligned} &\psi^2 b_1 - \mathrm{ad}a_0(a_1)h \\ &= \psi^2 b_1 - \mathrm{ad}a_0(a_1)(h_0^{\sim}+i\theta)+E^{-1}\, a_0^* a_0 h_0^{\sim} \\ &= c_0 a_0 + a_1 c_1 + \psi^2 c_2 + \psi^2 d_1 + d_0 \\ &+ E^{-1} a_0^* h_0^{\sim} a_0 + E^{-1} a_0^* \mathrm{ad}a_0(h_0^{\sim}). \end{aligned}$$

Therefore it holds that

$$\begin{aligned} &a_1 a_0 + b - \mathrm{ad}a_0(a_1)h \\ &= a_1 a_0 + \psi^2(b_2+d_1+C) + d_{10}a_0 + a_1 d_{00} + \psi^2 c_2 + d_0^{\sim}, \end{aligned}$$

where

$$\begin{aligned} &d_{00} = d + c_1, \\ &d_{10} = d + c_0 + E^{-1} a_0^* h_0^{\sim}, \\ &d_0^{\sim} = b_0 - \mathrm{ad}a_1(d) + d_0 + E^{-1} a_0^* \mathrm{ad}a_0(h_0^{\sim}) - \psi^2 C \end{aligned}$$

and C is a positive constant. Here if a positive constant C is sufficiently large, then the above b_2+d_1+C is b_2 at (2.5) in virtue of the Gårding type inequality proved by A. Melin, because d_1 defines a norm on $\boldsymbol{H}^{1/2}(\boldsymbol{R}^n)$ according to (1.26).

Lemma (*A. Melin* [6]). *Let p_2 be a pseudodifferential operator of homogeneous order* 2 *and its symbol be real non negative. Then for any constant ε such that ε plus half of positive trace of the fundamental matrix of p_2 is positive at the characteristics of p_2, there exists another constant C such that for any u belonging to $\mathcal{S}$*

$$\mathrm{Re}(p_2 u, u) + \varepsilon \|\langle D\rangle^{1/2} u\|^2 + C\|u\|^2 \geq 0. \tag{5.1}$$

Especially it is valid if ε is positive.

On the other hand d_{00}, d_{10} and $d_0^{\sim}$ are operators of order 0 by Lemma 3.1 because they consist of pseudodifferential operators and $E^{-1}a_0^*$ time pseudodifferential operators. Moreover it is shown by Lemma 3.5 that they satisfy the relation (2.11). Therefore they are able to be constituents of d_0, d_1 and d_2 at (2.1). It is clear that the pseudodifferential operator c_2 of order 1 is a member of b_0 at (2.1), that is, it satisfies (2.9).

We now turn our attention to the second line of (1.21). By the definition (2.12) of $\log a_0$

$$\operatorname{ad}h(a_1a_0+b)a_0\int_0^{+\infty}(\log t)\,U(t)dt$$

is equal to

$$-\operatorname{ad}h(a_1a_0+b)\log a_0+\gamma_0\operatorname{ad}h(a_1a_0+b). \tag{5.2}$$

Since the second term is equal to

$$\gamma_0(\operatorname{ad}h(a_1)a_0+a_1\operatorname{ad}h(a_0)+\operatorname{ad}h(b)),$$

it consists of terms which should be passed to the terms d_1a_0, a_1d_0, d_2 and $d_3\psi b_0$ at (2.1). In fact, $\operatorname{ad}h(a_1)$ and $\operatorname{ad}h(a_0)$ are product sums of H-type operators at Corollary of Lemma 3.5 so that they are operators of order 0 and satisfy the relation (2.11). The terms of $\operatorname{ad}h(b)$, with respect to the terms of b except for the principal part ψ^2b_2, are also combinations of a_j and product sums of H-type operators, which are constituents of d_0, d_1 and d_2 at (2.1). $\operatorname{ad}h(\psi^2b_2)$ is equal to $\psi^2\operatorname{ad}h(b_2)+\operatorname{ad}h(\psi^2)b_2$, which is equal to $g_0\partial b_2\psi^2+g_1b_2\langle D\rangle^{-1}\psi^2$, with g_j product sums of H-type operators, modulo product sums of H-type operators, which are passed to d_2 at (2.1). Here the following Lemma 5.1 shows that

$$g_0\partial b_2\psi^2+g_1g_2\langle D\rangle^{-1}\psi^2$$

has the same properties as members of the term $d_3\psi b_0$ at (2.1) satisfy. Namely it is a linear combination of pseudodifferential operators satisfying (2.9) with coefficients of operators of order 0.

Lemma 5.1. 1) *There exists a positive constant C such that $|\partial b_2|^2\leq Cb_2$ by the assumption that $b_2\geq 0$ if ∂b_2 stands for*

$$(\psi^{-1}\langle D\rangle^{-1}(\partial/\partial x_0)b_2,\ \langle D\rangle^{-1}(\partial/\partial x)b_2,\ (\partial/\partial\xi)b_2).$$

Therefore it holds that with other positive constants ε, which may be small, and C',

$$C(b_2u,u)-(\partial b_2u,\partial b_2u)+\varepsilon\|\langle D\rangle^{1/2}u\|^2+C'\|u\|^2\geq 0.$$

2) *Let p and q be two real pseudodifferential operators of order 0 in x such that for a positive constant ε,*

$$p\geq\varepsilon>0$$

and

$$p\geq q\geq -p.$$

Then there exists a positive constant C such that

$$|(qu,u)|\leq((p+C\langle D\rangle^{-1})u,u).$$

Remark. Both the first and second statements are applications of Gårding type inequality by A. Melin.

We use the results at 2) Corollary of Lemma 1.5 for the first term of (5.2).

(5.3) $$\begin{aligned}&\mathrm{ad}h(a_1a_0+b)\\&=\mathrm{ad}h_0^{\sim}(a_1a_0+b_2)e_0+h_0^{\sim}\mathrm{ad}e_0(a_1a_0+b_2)+(\mathrm{ad}h_0^{\sim}e_0)(b-b_2)\end{aligned}$$

with $h_0^{\sim}$, defined at (1.30), because θ is constant

$$\begin{aligned}&\mathrm{ad}h_0^{\sim}(a_1a_0+b_2)\\&=a_1\psi(h_2^{\sim}+d_2^{\sim})+\psi(h_3^{\sim}+d_3^{\sim})a_0+\psi^2h_4^{\sim}d_4\end{aligned}$$

with $h_j^{\sim}$ ($j=2$, 3 and 4), defined at Corollary of Lemma 1.5, and with pseudodifferential operators $d_j^{\sim}$($j=2$, 3 and 4) of order -1 in x. We put for $j=0$ and 1,

(5.4) $$c_{i0}=h_{j+2}^{\sim}+C\langle D\rangle^{-1}$$

and

$$c_{j1}=h_0^{\sim}.$$

The inequality (1.33) implies that with a positive constant ε,

(5.5) $$\mathrm{Re}(c_{j0}u,u)\geq\varepsilon\nu[(u,u)+|(c_{j1}u,u)|]\quad(j=0\text{ and }1),$$

if the constant C, which may depend on ν, is sufficiently large, because Lemma 5.1 is able to apply it. This inequality implies (2.6) if the constant C at (5.4) is replaced by a large one because $\mathrm{ad}a_0(e_0)=0$ and $\|\mathrm{ad}a_1(e_0)u\|^2$ is uniformly bounded by $C_0(e_0u, u)$ with a constant C_0 independent of the parameter ν. The lower order terms

$$[(d_{j+2}^{\sim}+C\langle D\rangle^{-1})e_0a_ja_0^{-1}\log a_0]a_0\quad(j=0\text{ and }1)$$

are passed to the term d_1a_0 at (2.1), since e_0a_k ($k=0$ and 1) time a pseudodifferential operator of order -1 in x is one of H-type operators so that it is an operator of order 0 satisfying (2.11) and since Lemma 3.3 and 3.7 prove that $a_0^{-1}\log a_0$ is also. The terms of $\mathrm{ad}h(b)$ except for $\mathrm{ad}h(\psi^2b_2)$, and $d_4^{\sim}$ are also passed to the terms d_0, d_1 and d_2 at (2.1) as well as at the second term of (5.2), because they are combinations of a_j and product sums of H-type operators multiplied by $a_0^{-1}\log a_0$, which is an operator of order 0 satisfying (2.11).

$$h_0^{\sim}\mathrm{ad}e_0(\psi^2b_2)=g_0h_0^{\sim}\psi\psi'b_2+g_1h_0^{\sim}\psi^2\partial b_2$$

modulo lower order term, which are operators of order 0 satisfying (2.11), where g_j is product sums of H-type operators. According to the inequality (1.33), it holds that

$$|h_4^{\sim}|^2\leq 2(1+4\varepsilon_2)^{-1}h_2^{\sim}h_3^{\sim}b_2,$$

$$|h_0^{\sim}\psi^{-1}\psi'b_2|^2\leq C|\nu h_0^{\sim}|b_2^2\leq C\nu^{-2}h_2^{\sim}h_3^{\sim}b_2$$

and

$$|h_0^{\sim}\partial b_2|^2\leq C\nu^{-2}h_2^{\sim}h_3^{\sim}b_2.$$

Therefore $h_0^{\sim}\mathrm{ad}e_0(\psi^2b_2)$ is equal to ψ^2gq modulo operator of order 0, where gq is a linear combination, with coefficients $g=(g_l)$ of product sums of H-type operators, of pseudodifferential operators $q=(q_l)$ of order 1 in x such that

$$|q_l|^2 \leq C\nu^{-2} h_2^{\sim} h_3^{\sim} b_2.$$

Moreover g includes the factor e_0. If we put $c_3 = h_3^{\sim 1/2}$, $b_{41} = h_3^{\sim -1/2} q$ and $e_0 e_1^{\sim} = g$, then we obtain that $h_0^{\sim}$ ad$e_0(\psi^2 b_2)$ is equal to

$$\psi^2 c_3 e_0 e_1^{\sim} b_{41}$$

modulo operators of order 0 satisfying (2.11). The operator norm of $e_1^{\sim}$ on H^0 is uniformly bounded in the parameter ν. b_{41} and so $\{a_0, b_{41}\}$ are linear combinations of ∂b_2 and b_2 with coefficients of pseudodifferential operators.

$$|b_{41}|^2 \leq C\nu^{-2} h_2^{\sim} b_2$$

and

$$|c_3|^2 = h_3^{\sim} \tag{5.6}$$

so that

$$\begin{aligned}&(b_{41}u, b_{41}u)\\ &\leq C\nu^{-2}[\mathrm{Re}(h_2^{\sim} b_2 u, u) + C(\langle D\rangle u, u)] + C(\nu)(u, u)\end{aligned}$$

and

$$(c_3 u, c_3 u) \leq \mathrm{Re}(c_{10} u, u)$$

with c_{10} at (5.4) according to Lemma 5.1. Therefore if ν is sufficiently large, then

$$\begin{aligned}&|(\psi^3 e_1^{\sim} b_{41} u, e_1^{\sim} b_{41} u)|\\ &\leq 2(1+4\varepsilon_2)^{-3}\varepsilon_2[\mathrm{Re}(\psi^3 h_2^{\sim} b_2 u, u) + C(\langle D\rangle u, u) + C(\nu)(u, u)]\end{aligned}$$

and

$$|(\psi c_3 v, c_3 v)| \leq \mathrm{Re}(\psi c_{10} v, v).$$

The estimate for $h_4^{\sim} e_0 = c_3 e_0 c_3^{-1} h_4^{\sim}$ modulo lower order terms, is that

$$\begin{aligned}&|(\psi^3 c_3^{-1} h_4^{\sim} u, c_3^{-1} h_4^{\sim} u)|\\ &\leq 4(1+4\varepsilon_2)^{-2}[\mathrm{Re}(\psi^3 h_2^{\sim} b_2 u, u) + C(\langle D\rangle u, u) + C(\nu)(u, u)].\end{aligned}$$

Therefore we get the estimate (5.7) for

$$\psi^2 c_3 e_0 e_1 b_4 = \psi^2 c_3 e_0 (c_3^{-1} h_4^{\sim} + e_1^{\sim} b_{41})$$

that

$$\begin{aligned}&|(\psi^3 e_1 b_4 u, e_1 b_4 u)|\\ &\leq 4(1+3\varepsilon_2)^{-2}[\mathrm{Re}(\psi^3 h_2^{\sim} b_2 u, u) + C(\langle D\rangle u, u) + C(\nu)(u,u)]\end{aligned} \tag{5.7}$$

and

$$|(\psi c_3 v, c_3 v)| \leq \mathrm{Re}(\psi c_{10} v, v).$$

The discussion about the first terms of (5.2) finishes if it is found that the definitions of c_j at 6) Section 2 and (5.4), and the inequalities (2.5–6) and (5.5), deduce the inequalities

(2.8) from the above (5.7). In fact the error terms are absorbed by means of replacement of the constants $2(1+3\varepsilon_2)^{-1}$ to $2(1+2\varepsilon_2)^{-1}$ and 1 to $(1+\varepsilon_2)$ if the parameter ν is as large as needed. The replacement of b_2 in this section by b_2 at (2.5) Section 2 is possible in the same time according to (5.5).

At the third line of (1.21) it is important that the principal part of $k\mathrm{ad}a_0(\mathrm{ad}h(a_1a_0+b))$ is essentially pure imaginary. In fact we can check it as follows. At first we note that

$$\int_0^{+\infty}(\log t)\,U(t)dt=a_0^{-1}(\gamma_0-\log a_0),$$

which is an operator of order 0 and satisfies (2.11) by Lemma 3.3 and 3.7.

$$\begin{aligned}(5.8)\qquad &\mathrm{ad}a_0(\mathrm{ad}h(a_1a_0+b))\\ &\quad=\mathrm{ad}a_0(a_1)\mathrm{ad}h(a_0)-a_1(\mathrm{ad}a_0)^2(h)\\ &\quad+\mathrm{ad}a_0(\mathrm{ad}h(a_1)a_0+\mathrm{ad}a_0(\mathrm{ad}h(b)).\\ &\mathrm{ad}h(a_0)=-\mathrm{ad}a_0(h_0^{\sim}e_0)\\ &\quad=-\mathrm{ad}a_0(h_0^{\sim})e_0.\end{aligned}$$

So the principal part of $\mathrm{ad}h(a_0)$ is discribed as ψg_0e_0 with a real pseudodifferential operator g_0 of order 0 in x. $\mathrm{ad}a_0(a_1)$ is a pseudodifferential operator of order 1 in x with the pure imaginary principal symbol, from which the weight function ψ is taken out. Therefore $\mathrm{ad}a_0(a_1)\mathrm{ad}h(a_0)$ is equal to $\psi b_{30}e_0$ with a pure imaginary pseudodifferential operator b_{30} of order 1 in x modulo product sums of H-type operators. This concludes that

$$\mathrm{ad}a_0(a_1)\mathrm{ad}h(a_0)\int_0^{+\infty}(\log t)\,U(t)dt$$

is equal to

$$\psi b_{30}e_0(\gamma_0a_0^{-1}-a_0^{-1}\log a_0)$$

modulo terms included into the terms d_1a_0 and d_2 at (2.1). It is clear that the second and third terms at the right hand side of (5.8) are passed into the terms a_1d_0, d_1a_0 and d_2 at (2.1) even if they are multiplied by $a_0^{-1}\log a_0$ or a_0^{-1}. The forth term of (5.8) is also equal to

$$\mathrm{ad}a_0(\mathrm{ad}h_0^{\sim}(\psi^2b_2))e_0\quad\text{modulo}\quad a_1g_0+g_1a_0+g_2a_0^*+g_3$$

with product sums of H-type operators g_j $(j=0,\cdots,3)$. Since Lemma 3.3 and 3.6 imply that $a_0^*a_0^{-1}$ is an operator of order 0 satisfying (2.11),

$$(a_1g_0+g_1a_0+g_2a_0^*+g_3)(\gamma_0a_0^{-1}-a_0^{-1}\log a_0)$$

is divided among the terms a_1d_0, d_1a_0 and d_2 at (2.1). The principal symbol of the pseudodifferential operator $\mathrm{ad}a_0(\mathrm{ad}h_0^{\sim}(\psi^2b_2))$ is equal to ψ^2b_{31} with a pure imaginary symbol b_{31} of order 1 in x modulo lower order terms. Therefore the forth term of (5.8) time $\int_0^{+\infty}(\log t)\,U(t)dt$ is equal to

$$\psi^2b_{31}e_0(\gamma_0a_0^{-1}-a_0^{-1}\log a_0)$$

modulo terms passed into the terms a_1d_0, d_1a_0 and d_2 at (2.1). The above results combine to assert that the third line of (1.21)

$$\mathrm{ad}a_0(\mathrm{ad}h(a_1a_0+b))\int_0^{+\infty}(\log t)\,U(t)dt$$

is equal to

$$\psi^2b_3(\gamma_0a_0^{-1}-a_0^{-1}\log a_0)$$

modulo terms passed to the terms a_1d_0, d_1a_0 and d_2 at (2.1), where $b_3=b_{30}+b_{31}$ is a pseudodifferential operator of order 1 in x with a pure imaginary symbol.

Now we consider the forth line and the remainder terms at (1.21), the exact forms of which are there at Lemma 4.4. We assume that H^{-1}, H_1, a_0H_1, H_1' and $\mathrm{ad}a_0(H_1')$ are operators of order 0 satisfying (2.11) and that the commutators of them with q, a pseudodifferential operator of order 1, are written as

$$\mathrm{ad}q(K)=a_0^{-1}K_1\langle D\rangle+K_0$$

where K is one of them and $K_j(j=0,1)$ are operators of order 0 satisfying (2.11). We will prove it after the present proof. Then d_0 and d_1 at (4.57–58), namely, at the forth line are operators of order 0 satisfying (2.11). The first term at (4.59) is included in the terms a_1d_0, d_2 and $a_0^{-1}\psi^2b_1$ at (2.1) except for the part with respect to ψ^2b_2, because it is equal to

$$a_0^{-1}(b-\psi^2b_2-\mathrm{ad}a_0(a_1)h)a_0H_1H^{-1}$$
$$+a_0^{-1}\mathrm{ad}a_0(b-\psi^2b_2-\mathrm{ad}a_0(a_1)h)H_1H^{-1}$$

so that it is divided to a_1 time an operator of order 0 and a_0^{-1} time an operator of order 1 modulo operators of order 0.

$$\psi^2b_2H_1H^{-1}=a_0^{-1}\psi^2b_2a_0H_1H^{-1}+a_0^{-1}(c_0\psi^2b_2+\psi^2c_1)H_1H^{-1}$$

with $c_j(j=0,1)$ pseudodifferential operators of order j in x by the assumption (1.6). The assumption to H^{-1}, a_0H_1 and H_1 yields that

$$\psi^2b_2H_1H^{-1}=a_0^{-1}d_0^{\sim}\psi^2b_2+a_0^{-1}\psi^2d_1^{\sim}$$

with $d_j^{\sim}(j=0,1)$ operators of order j. The second term at (4.59) is an operator of order 0 satisfying (2.11).

There are two types in the third term at (4.59), namely,

$$[\mathrm{ad}h(b)+k\mathrm{ad}a_0(\mathrm{ad}h(a_0))]H_1'$$

and

$$[k\mathrm{ad}a_0(a_1)\mathrm{ad}h(a_0)+k\mathrm{ad}a_0(\mathrm{ad}h(b))]\,a_0^{-1}H_1'.$$

At the first one, $\mathrm{ad}h(b)+k\mathrm{ad}a_0(\mathrm{ad}h(a_1))$ is equal to $\mathrm{ad}h(\psi^2b_2)$ modulo product sums of H-type operators. The consideration after (5.2) or (5.6), and the assumption to H_1' yield that the first one is divided among the term $d_3\psi b_0$ and d_2 at (2.1). The second one is regarded as $\psi^2ga_0^{-1}H_1'$, where g is a linear combination of pseudodifferential operators of order 1 in x with coefficients of product sums of H-type operators. It is equal to

$$a_0^{-1}\psi^2 gH_1' - a_0^{-1}\mathrm{ad}a_0(\psi^2 g)a_0^{-1}H_1'$$

so that it is passed to $a_0^{-1}\psi^2 b_1$ at (2.1).

The term including Z is left. The exact form of Z exists at (1.17). By the assumption (1.6) we have that

$$\mathrm{ad}a_0(b)=\psi^2 b_2 f_2+\psi^2 f_1+\psi f_0 a_0+f_0', \tag{5.9}$$

where f_j and f_j' are pseudodifferential operators of order j in x. The term including Z_1, Z_3, Z_5 and Z_6, that is,

$$\int_0^{+\infty}(Z_1+Z_3+Z_5+Z_6)\,t^{k-1}a_0^k U(t)W(\log t)dtGH^{-1}$$

is equal to

$$[\Sigma_{j=1}^k(\psi^2 b_2 f_{2j}+\psi^2 f_{1j})a_0^{-j}+\Sigma_{j=0}^k f_{0j}a_0^{-j}]FGH^{-1},$$

where f_{ij} are pseudodifferential operators of order j in x. Therefore it is passed to the terms $a_0^{-1}\psi^2 b_1$, $a_0^{-1}d_4\psi^2 b_2$ and d_2 at (2.1) because $FGH^{-1}=I+H_1H^{-1}$ and because 4) Lemma 3.6 are applicable to the change of places between a_0^{-1} and other pseudodifferential operators. Since $(d/dt)U(t)+a_0U(t)=0$, the integral by part implies that

$$\int_0^{+\infty}Z_8 t^{k-1}a_0^k U(t)W(\log t)dt$$
$$=a_0^{-1}\int_0^{+\infty}Z_8'(t)t^{k-1}a_0^k U(t)W(\log t)dt,$$

where

$$Z_8'(t)$$
$$=a_0^{k+1}\int_0^t U(\sigma)(\mathrm{ad}a_0)^2(a_1a_0+b)\,U(-\sigma)(k-(k-1)\sigma/t)d\sigma a_0^{-k-1}.$$

By the assumption (1.6), $Z_8'(t)$ is discribed as

$$\Sigma_{j=0}^{k+1}\int_0^t U(\sigma)g_j U(-\sigma)(k-(k-1)\sigma/t)\,d\sigma a_0^{-j}$$

with g_j which are same types of operators as the right hand side of (5.9). Therefore the relations (3.24–25), 3) of Lemma 3.2 and 2) of Lemma 3.6 yield that

$$\int_0^{+\infty}Z_8(t)t^{k-1}a_0^k U(t)W(\log t)dt$$
$$=a_0^{-1}\psi^2 b_2\int_0^{+\infty}G_2(t)t^{k-1}a_0^k U(t)W(\log t)dt$$
$$+a_0^{-1}\psi^2\langle D\rangle\int_0^{+\infty}G_1(t)t^{k-1}a_0^k U(t)W(\log t)dt$$
$$+\int_0^{+\infty}G_0 t^{k-1}a_0^k U(t)W(\log t)dt.$$

Here G_j ($j=0, 1, 2$) are operators of order 0 and satisfy the conditions for $G(t)$ at Lemma 3.6 and also the conditions for f_0 at 3) of Lemma 4.2 with $l=m=0$ and with $f_1=1$ so that the terms corresponding to g_{ij} at (4.16–17) are operators of order 0 and quasi-commutors of $\psi^2 b_2$. Therefore we conclude that

$$\int_0^{+\infty} Z_8(t)t^{k-1}a_0^k U(t)W(\log t)dtGH^{-1}$$

is passed to the terms $a_0^{-1}\psi^2 b_1$, $a_0^{-1}d_4\psi^2 b_2$ and d_2 at (2.1), if it is shown that (4.17) is an operator of order 0 and a quasicommutor of $\psi^2 b_2$ from the condition that g_{ij} are so.

The terms including Z_2 and Z_4 are passed to $a_0^{-1}\psi^2 b_1$ at (2.1) because Lemma 4.2 is applicable to them.

The term corresponding to Z_7 should be improved such that

$$Z_7=\sum_{j=2}^{2k+1} j!^{-1} a_0^k U(t)(\log t)^j (\mathrm{ad}h)^j (a_1 a_0+b) a_0^{-k} U(-t) \\ +Z_7'$$

and

$$Z_7'=(2k+1)!^{-1} a_0^k U(t)\int_0^{\log t} W(\sigma)(\mathrm{ad}h)^{2k+2}(a_1 a_0+b) \\ \times W(-\sigma)(\log r-\sigma)^{2k+1} d\sigma a_0^{-k} U(-t).$$

At the first term, $(\mathrm{ad}h)^j(a_1a_0+b)$ are H-type operators if $j\geq 2$ so that it is concluded according to Lemma 4.2 that the term corresponding to the first one is one of d_2 at (2.1). The application of 3) Corollary of Lemma 3.5 to $(\mathrm{ad}h)^2(a_1a_0+b)$ yields that

$$a_0^k(\mathrm{ad}h)^{2k+2}(a_1 a_0+b) a_0^k=\phi$$

is a product sum of H-type operators so that it is an operator of order 0 and a quasicommutor of $\psi^2 b_2$. Therefore

$$\int_0^{+\infty} Z_7' t^{k-1} a_0^k U(t) W(\log t) dtGH^{-1} \\ =\int_0^{+\infty} U(t) W_1(t) dt \int_0^{+\infty} t^{k-1} W_2(t) U(t) dtH^{-1}, \tag{5.10}$$

where

$$W_1(t)=(2k+1)!^{-1}\int_0^{\log t} a_0^k W(\sigma) a_0^{-k}\phi a_0^{-k} W(\log t-\sigma) a_0^k(\log t-\sigma)^{2k+1} d\sigma$$

and

$$W_2(t)=a_0^{-k} W(-\log t) a_0^k.$$

Since it is checked by (4.31) that $W_1(t)$ and $W_2(t)$ are operators of order 0 and satisfy the conditions of $G(t)$ at Lemma 3.6, it is concluded that (5.10) is an operator of order 0 and satisfy (2.11), that is, it is passed to d_2 at (2.1). q.e.d.

We give a note on the facts assumed in the previous proof.

Let R and q be an operator of order 0 and a pseudodifferential operator of order 1 in x, respectively. We consider the case that a_0, $\psi^2 b_2$, q and R satisfy the relation that

$$\mathrm{ad}q(R)=a_0^{-1}R_1\langle D\rangle+R_0 \\ \mathrm{ad}a_0(R)=R_2 \\ \psi^2 b_2 R=R_3\psi^2 b_2+R_4\langle D\rangle \tag{5.11}$$

and

$$R\psi^2 b_2 = \psi^2 b_2 R_5 + R_6\langle D\rangle,$$

where $R_j(j=0, \cdots, 6)$ are operators of order 0. We assume that the sets of four a_0, $\psi^2 b_2$, q and one of $R_j(j=0, \cdots, 6)$ also satisfy successively the relation (5.11) sufficiently many times. Then we shall say only that R satisfies the relation (5.11).

Let us consider another relation for a function $R(\sigma, \tau)$ in (σ, τ) valued in operators of order 0. It is that, for a pseudodifferential operator q of order 1 in x,

$$\text{(5.12)} \qquad \begin{aligned} &\mathrm{ad}q(R) = \tau R_1\langle D\rangle + R_0 \\ &\mathrm{ad}a_0(R) = R_2 \\ &\psi^2 b_2 R = R_3 \psi^2 b_2 + R_4\langle D\rangle \end{aligned}$$

and

$$R\psi^2 b_2 = \psi^2 b_2 R_5 + R_6\langle D\rangle$$

with $R_j(j=0, \cdots, 6)$ functions in (σ, τ) valued in operators of order 0. We assume that $R_j(j=0, \cdots, 6)$ also satisfy such a relation. We consider three such functions R, S and T, and ones generated finitely from them by means of the relation (5.12). We denote the sets of such functions by $R^{\sim}$, $S^{\sim}$ and $T^{\sim}$. We assume for any triplet (R', S', T') of $R^{\sim} \times S^{\sim} \times T^{\sim}$ that

$$\text{(5.13)} \qquad \begin{aligned} &\int_0^{+\infty}\int_{-1}^{+1} \tau^{\alpha+\beta+\gamma} \|(\partial/\partial\tau)^\alpha R'\| \, \|(\partial/\partial\tau)^\beta S'\| \, \|(\partial/\partial\tau)^\gamma T'\| \\ &\quad \times \exp[(C-\lambda)\tau] d\sigma d\tau < +\infty \end{aligned}$$

for any non-negative integer α, β and γ.

Lemma 5.2. *Let a triplet* (R, S, T) *of functions in* (σ, τ) *valued in operators of order* 0 *satisfy the above* (5.13). *Then*

$$\text{(5.14)} \qquad G = \int_0^{+\infty}\int_{-1}^{+1} R\Phi_c(S) T d\sigma d\tau$$

satisfies the relation (5.11).

Remark on Notations. $\Phi_c(X)$ means the existence of pseudodifferential operators c_0 and c_1 of order 0 in x such that

$$\Phi_c(t, X) = U(c_0, t(1-\sigma)/2) X U(c_1, t(1+\sigma)/2).$$

Proof. It is trivial for G and $\mathrm{ad}a_0(G)$ to be operators of order 0 and especially for $\mathrm{ad}a_0(G)$ to have the same integral type as G.

$$\text{(5.15)} \qquad \begin{aligned} &\mathrm{ad}q(G) \\ &= \int_0^{+\infty}\int_{-1}^{+1} [\mathrm{ad}q(R)\Phi_c(S)T + R\Phi_c(S)\mathrm{ad}q(T)] d\sigma d\tau \\ &+ \int_0^{+\infty}\int_{-1}^{+1} R(\mathrm{ad}q)(\Phi_c(S)) T d\sigma d\tau. \end{aligned}$$

It holds for the second term that

(5.16)
$$\begin{aligned}&\mathrm{ad}q(\Phi_c(S))\\&=I(\tau(1-\sigma)/2,\ d_0)\Phi_c(S)\\&+\Phi_c(S)I(-\tau(1+\sigma)/2,\ d_1)\\&+\Phi_c(\mathrm{ad}q(S)),\end{aligned}$$

where d_0 and d_1 are pseudodifferential operators of order 1 in x. According to Lemma 4.3 it is easily checked for $\tau^{-1}I(\tau f(\sigma), d)\langle D\rangle^{-1}$ with a continuous function $f(\sigma)$ in σ that it and functions generated by means of the relation (5.12) are bounded by $C\exp(C\tau)$. Therefore $\mathrm{ad}q(G)$ is finite sums of two integral types

$$\int_0^{+\infty}\int_{-1}^{+1}\tau R\Phi_c(S)Td\sigma d\tau\langle D\rangle$$

and

$$\int_0^{+\infty}\int_{-1}^{+1}R\Phi_c(S)Td\sigma d\tau,$$

where R, S and T satisfy (5.13). The second one has the same integral type as G. On the other hand we know the equality that

$$\begin{aligned}&(\partial/\partial\tau)\Phi_c(X)\\&=-a_0\Phi_c(X)+(1+\sigma)\Phi_c(\mathrm{ad}a_0(X))/2\\&-\Phi_c(c_0X(1-\sigma)/2+Xc_1(1+\sigma)/2)\\&+\Phi_c((\partial/\partial\tau)(X)).\end{aligned}$$

This implies that

$$a_0\int_0^{+\infty}\int_{-1}^{+1}\tau R\Phi_c(S)Td\sigma d\tau$$

is equal to

$$-\int_0^{+\infty}\int_{-1}^{+1}\tau R(\partial/\partial\tau)(\Phi_c(S))Td\sigma d\tau$$

modulo the same integral type as G. The integral by part in τ assures that it is also the same integral type as G. We conclude that

$$\mathrm{ad}q(G)=a_0^{-1}G_1\langle D\rangle+G_0,$$

where G_0 and G_1 are finite sums of the same integral type as G. For the commutation of G with ψ^2b_2, (5.15–16) hold if q is replaced by ψ^2b_2 and if the notation $\mathrm{ad}q(X)$ is read as $qX-X'q$ or $qX'-Xq$ with another operator X' of order 0, while $I(t, d)$ at (5.16) should be replaced by

$$I_c(t, d)=\int_0^t U(c_0-s)dU(c_1, s)ds.$$

Therefore $R_j(j=3, \cdots, 6)$ at (5.11) for G are also finite sums of the same integral type of G. This fact applies inductively to assert that G satisfies the relation (5.11).

q.e.d.

Lemma 5.3. 1) *Let f be a function in σ such that*

$$B_f=\int_{-1}^{+1}|f|(1-\sigma^2)^{-No}d\sigma<+\infty$$

with No at (4.63), *and L_j* ($j=0, 1, 2, \cdots$) *be*

$$L_j=\int_{-1}^{+1}f(\mathrm{ad}a_0)^j(\Psi)d\sigma,$$

where $\Psi=W(\log[(1+\sigma)/(1-\sigma)])$. *Then*

$$\Pi_{k=1}^{l}(\mathrm{ad}q_k)(L_j)$$

are operators of order $\sum_{k=1}^{l}m_k-l$ if q_k are pseudodifferential operators of order m_k or a_0 with $m_k=1$. Their bounds are majorated by B_f. Therefore L_j satisfies the relation (5.11).

2) *Let f be a function in* (σ, τ) *such that*

$$\int_0^{+\infty}\int_{-1}^{+1}\tau^\alpha|(\partial/\partial\tau)^\alpha f|(1-\sigma^2)^{-No}\exp[(C-\lambda)\tau]d\tau<+\infty.$$

Then

$$M_j=\int_0^{+\infty}\int_{-1}^{+1}f\Phi((\mathrm{ad}a_0)^j(\Psi))d\sigma d\tau$$

satisfies the relation (5.11).

3) H, H^{-1}, H_1, a_0H_1, H_1' *and* $\mathrm{ad}a_0(H_1')$ *are operators of order* 0 *satisfying* (5.11).

Proof. 1) The results are guaranteed by Corollary 2 of Lemma 4.3.

2) At (5.14), we put $R=f(1-\sigma^2)^{2k}$, $S=\Psi$ and $T=1$. Then they satisfy (5.13) so that M_j has the relation (5.11).

3) They except for H^{-1} are sums of two types of operators at 1) and 2) by their definitions. Therefore they satisfy (5.11). For H^{-1}, it holds that

$$\mathrm{ad}q(H^{-1})=-H^{-1}\mathrm{ad}q(H)H^{-1}$$

with respect to $q=a_0$, ψ^2b_2 or a pseudodifferential operator of order 1 in x. The results for H imply ones for H^{-1}. q.e.d.

§ 6. Positive definite forms.

Here we prove the lemmas at Section 2.

Let A be a generator of one parameter semigroup

$$V(A, t)=V(t)=\exp(-At),$$

and B be a symmetric operator on $\boldsymbol{H}^0$. We assume that A and B have a common core S and that with a positive ε

$$\mathrm{Re}(Au, u)\geq\varepsilon(u, u), \tag{6.1}$$

for any u belonging to $\mathcal{S}$. Then it holds on $\mathcal{S}$ that

(6.2) $$(\mathrm{ad}^* V(t))B=-\int_0^t V^*(t-\tau)(\mathrm{ad}^* A)(B)V(\tau)d\tau$$

and

(6.3) $$(\mathrm{ad}^* V(t))(\mathrm{ad}^* V(s))(B)$$
$$=\int_0^t\int_0^s V^*(t+s-\tau-\sigma)(\mathrm{ad}^* A)^2(B)V(\tau+\sigma)d\sigma d\tau.$$

Remark on Notations.

(6.4) $$(\mathrm{ad}^* X)Y=X^*Y-YX.$$

Let us define $A^{1/2}$ by

(6.5) $$A^{1/2}=\Gamma(1/2)^{-1}\int_0^{+\infty}\sigma^{-1/2}V(\sigma)d\sigma A.$$

Then we have that

(6.6) $$\mathrm{Re}(BAu, u)=(BA^{1/2}u, A^{1/2}u)+\mathrm{Re}(R_1A^{1/2}u, u)$$
$$=(BA^{1/2}u, A^{1/2}u)+(1/2)(R_2u, u)$$

for any u belonging to $\mathcal{S}$, where

(6.7) $$R_1=(\mathrm{ad}^* A^{1/2})(B)$$
$$=\Gamma(1/2)^{-1}2^{-1}\int_0^t\tau^{-3/2}X_1(\tau)d\tau,$$
$$X_1(t)=\int_0^t V^*(t-\tau)(\mathrm{ad}^* A)(B)V(\tau)d\tau$$

and

(6.8) $$R_2=BA+A^*B-2A^{*1/2}BA^{1/2}$$
$$=\Gamma(1/2)^{-2}2^{-2}\int_0^{+\infty}\int_0^{+\infty}t^{-3/2}s^{-3/2}X_2(t, s)dsdt,$$
$$X_2(t, s)=\int_0^t\int_0^s V^*(t+s-\tau-\sigma)(\mathrm{ad}^* A)^2(B)V(\tau+\sigma)d\sigma d\tau.$$

Let us consider two other operators A_0 and A_1 with a common core $\mathcal{S}$ such that

(6.9) $$A_1B-BA_0=Z_1 \text{ on } \mathcal{S}$$

and

$$A=(A_0+A_1^*)/2 \quad \text{on } \mathcal{S}.$$

Then we have that

(6.10) $$\mathrm{Re}(BA_0u, u)=\mathrm{Re}(BAu, u)-2^{-1}\mathrm{Re}(Z_1u, u)$$

for any u belonging to $\mathcal{S}$ and

$$(\mathrm{ad}^* A)(B)=(Z_1-Z_1^*)/2 \quad \text{on } \mathcal{S}.$$

Let B a symmetric operator be non negative and satisfy the relation with X and Y operators of order 0 that

$$BX-YB=Z, \tag{6.11}$$

where Z is an operator of order l. Then $\mathrm{Re}(BXu, u)$ is estimated as with positive constants μ and C,

$$\mathrm{Re}(BXu, u)\leq\mu(Bu, u)+C(\langle D\rangle^{l}u, u) \tag{6.12}$$

for any u belonging to $\mathcal{S}$.

In fact if we put $A_0=\mu-X$ and $A_1=\mu-Y$ and if we take μ sufficiently large, then $A=\mu-(X+Y^*)/2$ is a bounded operator and $\mathrm{Re}(Au, u)\geq\varepsilon(u, u)$. Since $V(A, t)$ and $(\mathrm{ad}^*A)(B)$ are operators of order 0 and of order l, respectively. $R_1A^{1/2}$ is an operator of order l. This implies that

$$|\mathrm{Re}(R_1A^{1/2}u, u)|\leq C(\langle D\rangle^{l}u, u).$$

The fact that $|(BA^{1/2}u, A^{1/2}u)|\geq 0$ yields that

$$\mathrm{Re}(B(\mu-X)u, u)\geq -C(\langle D\rangle^{l}u, u).$$

therefore, the conclusion.

Proof of Lemma 2.5. ψ^2b_2 is positive and symmetric by definition. If we put $X=\psi q\psi^{-1}$, then the above proves Lemma 2.5 because $(\langle D\rangle u, u)\leq C(b_2u, u)$. q.e.d.

Proof of Lemma 2.1. It follows from the assumption (2.10) for ψ^2b_2 and a_0 that

$$\mathrm{ad}a_0(\psi^3b_2c_0)=\gamma_0\psi^3b_2c_0+\delta_0\langle D\rangle,$$

where γ_0 and δ_0 are product sums of H-type operators. Therefore

$$\psi^3b_2c_0(\log a_0)-(\log(a_0-\gamma_0))\psi^3b_2c_0$$

and

$$(\log a_0)\psi^3b_2c_0-\psi^3b_2c_0(\log(a_0+\gamma_0))$$

are operators of order 1 and the same results for $(\log a_0)^*$ hold. Since $\log a_0-\log(a_0\pm\gamma_0)$ and $\mathrm{Im}\log a_0$ are operators of order 0 and satisfy the relation (2.11) from the above, that is, (6.11) with $l=1$, we have that

$$\begin{aligned}&2\mathrm{Re}(\psi^3b_2c_0\,\mathrm{Log}\,a_0u, u)\\ &\geq\mathrm{Re}([\psi^3b_2c_0+(\psi^3b_2c_0)^*]\mathrm{Log}\,a_0u, u)-C(\psi^3b_2u, u).\end{aligned}$$

In fact

$$\mathrm{adLog}\,a_0[(\psi^3b_2c_0)^*]=\gamma_1(\psi^3b_2c_0)^*+\psi^3\delta_1,$$

where γ_1 is an operator of order 0 and satisfies (6.11) of l with $\psi^3b_2c_0$ and δ_1 is an operator of order 1. Therefore (6.12) with $l=1$ applies to $\mathrm{Re}(u, \psi^3b_2c_0\gamma_1u)$ so that $\mathrm{Re}(\psi^3\delta_1u, u)$ and it are bounded by $C(\psi^3b_2u, u)$. Now we put

$$B=\psi^3b_2c_0+(\psi^3b_2c_0)^*+\delta\psi^3$$

with a sufficiently large constant δ. Then B is symmetric and non negative and satisfies the relation with $A_0=\mathrm{Log}\, a_0$ that there exists an operator γ_2 of order 0 such that for $A_1=A_0+\gamma_2$, $Z_1=A_1B-BA_0$ at (6.9) and also $\mathrm{ad}A_j(Z_1)$ ($j=0$, 1) are operators of order 1 multiplied by ψ^3. This implies according to (6.6) and (6.10) that

$$\mathrm{Re}(BA_0u, u)\geq \mathrm{Re}(BA^{1/2}u, A^{1/2}u)-C\|\psi^3\langle D\rangle u\|.$$

Therefore we conclude that $\|u\|^2_{bcLog}$ at (2.21) is positive definite on $\boldsymbol{H}^0$. The results for $\|u\|_{eLog}$ and $\|u\|_{cLog}$ are also shown more simply by means of (6.6). q.e.d.

Proof of Lemma 2.4. The statements 1) and 2) are already proved at Section 3. The statement 3) is proved by applications of the above method to $A=\log a_0$ and $B=\mathrm{Re}(\psi c_j-\varepsilon\psi e_0)$. q.e.d.

Proof of Lemma 2.6. At first we part B_4 into $c_3\psi e_0(\log a_0)\psi e_1 b_4$ and the others. The inner products corresponding to the others are estimated by $\|u\|_b^2+\|v\|^2$ according to the assumption for b_4 and Lemma 2.5. Since $\|W\|_{eLog}$ defines a norm on $\boldsymbol{H}^0$,

$$\begin{aligned}&|\mathrm{Re}(c_3\psi e_0(\log a_0)\psi e_1 b_4 u, v)|\\ &\leq \|\psi e_1 b_4 u\|_{eLog}\|c_3 v\|_{eLog}.\end{aligned}$$

If we substitute $\mathrm{Log}\, a_0$ for A_0 and $\mathrm{Re}[(1+\varepsilon_2)\psi c_1-c_3^* e_0\psi c_3]$ for B at (6.9), then the assumption (2.8) applies the same way as the proof of Lemma 2.1 to prove that

$$\|c_3 v\|_{eLog}\leq(1+\varepsilon_2)\|v\|_{cLog}.$$

On the other hand the form related to e_1b_4 is estimated as follows. Let B at (6.9) be an operator consisting of

$$((\partial/\partial\xi)b_2, (\partial/\partial x)b_2\langle D\rangle^{-1}, (\partial/\partial x_0)b_2\langle D\rangle^{-1}, b_2\langle D\rangle^{-1})$$

on $(\boldsymbol{H}^0)^{2n+2}$. And A_0 is equal to $\mathrm{Log} a_0$ times the identity operator on $(\boldsymbol{H}^0)^{2n+2}$. Then the assumption (2.10) assures that Z is a system of pseudodifferential operators of order 0 in x and that A is a generator of one parameter semigroup and satisfies (6.1) if λ is sufficiently large. Therefore R_1 at (6.7) is an operator of order 0. And $A^{1/2}-A_0^{1/2}$ and $A^{1/2*}-A_0^{1/2}$ are also operators of order 0. If these apply the vectors consisting of all same elements u, it holds that

$$\|\psi e_1 b_4 u\|^2_{eLog}\leq \mathrm{Re}(\psi^3 e_0 e_1 b_4 A^{1/2}u, e_1 b_4 A^{1/2}u)+C(\psi^2 b_2 u, u),$$

where $e_1b_4A^{1/2}$ means $e_2^{\sim}BA^{1/2}$ with some other system $e_1^{\sim}$ of operators. Since

$$e_1 b_4 A_0^{1/2}u=e_1 b_4(\mathrm{Log}\, a_0)^{1/2}u$$

in the above notation, the non negativity of $\mathrm{Re}(e_0 v, v)$ implies that

$$\begin{aligned}&\|\psi e_1 b_4 u\|_{eLog}\\ &\leq \mathrm{Re}(\psi^3 e_0 e_1 b_4(\mathrm{Log}\, a_0)^{1/2}u, e_1 b_4(\mathrm{Log}\, a_0)^{1/2}u)^{1/2}\\ &\quad+\mathrm{Re}(\psi^3 e_0 e_1 b_4(A^{1/2}-A_0^{1/2})u, e_1 b_4(A^{1/2}-A_0^{1/2})u)^{1/2}\\ &\quad+C(\psi^2 b_2 u, u)^{1/2}.\end{aligned}$$

Therefore it holds according to the assumption (2.8) that

$$\begin{aligned}&\|\psi e_1 b_4 u\|_{e\,Log}\\&\leq 2(1+2\varepsilon_2)^{-1}\mathrm{Re}(\psi^3 b_2 c_0(\mathrm{Log}\,a_0)^{1/2}u, (\mathrm{Log}\,a_0)^{1/2}u)^{1/2}\\&\quad+8(n+1)\mathrm{Re}(\psi^3 b_2 c_0(A^{1/2}-A_0^{1/2})u, (A^{1/2}-A_0^{1/2})u)^{1/2}\\&\quad+C(\|a_0 u\|+\|u\|_b).\end{aligned}$$

Since $(A^{1/2}-A_0^{1/2})$ satisfies (6.11) with $B=\mathrm{Re}\psi^3 b_2 c_0$ and $l=1$, it holds that

$$\mathrm{Re}(\psi^3 b_2 c_0(A^{1/2}-A_0^{1/2})u,\ (A^{1/2}-A_0^{1/2})u)\leq C(\|a_0 u\|^2+\|u\|_b^2).$$

We again use the proof of Lemma 2.1 conversely to get that

$$\begin{aligned}&\|\psi e_1 b_4 u\|_{e\,Log}\\&\leq 2(1+2\varepsilon_2)^{-1}\|u\|_{bc\,Log}+C(\|a_0 u\|+\|u\|_b).\end{aligned}$$

Therefore we get the conclusion because

$$\varepsilon=1-(1+2\varepsilon_2)^{-1}(1+\varepsilon_2)>0. \qquad \text{q.e.d.}$$

Research Institute for Mathematical Sciences
Kyoto University, Kyoto 606, Japan

References

[1] L. Hörmander, The Cauchy problem for differential equations with double characteristics, Journal d'analyse math., **32** (1977), 118–196.

[2] Yu. V. Egorov, On canonical transformations of pseudodifferential operators, Uspehi Mat. Nauk., **24** (1969), 235–236.

[3] V. Ya Ivrii, Sufficient conditions for regular and completely regular hyperbolicity, Trudy Moskov. Mat. Obsc., **33** (1976), 3–65, English transl. in Trans. Moscow Math. Soc. Issue, **1** (1978), 1–65.

[4] V. Ya Ivrii and V. M. Petkov, Necessary conditions for the Cauchy problem for non-strictly hyperbolic equations to be well-posed, Uspekhi Mat. Nauk, **29**: 5 (1974), 3–70, English transl. in Russian Math. Surveys, **29**: 5 (1974), 1–70.

[5] C. Iwasaki and N. Iwasaki, Parametrix for a degenerate parabolic equation and its application to the asymptotic behavior of spectral functions for stationary problems, Publ. RIMS, Kyoto Univ., **17** (1981), 577–655.

[6] A. Melin ,Lower bounds for pseudo-differential operators, Ark. Mat., **9** (1971), 117–140.

[7] T. Nishitani, The Cauchy problem for weakly hyperbolic equations of second order, Comm. in P. D. E., **5** (1980), 1273–1296.

[8] O. A. Olejnik, On the Cauchy problem for weakly hyperbolic equations, Comm. Pure Appl. Math., **23** (1970), 569–586.