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The Hopf algebra structure of $MU_*(\Omega Sp(n))$

By

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§ 0. Introduction

Let $Sp(n)$ be the n-th symplectic group and $\Omega Sp(n)$ its loop space. In [12], the Hopf algebra structure of $H_*(\Omega Sp(n))$ and $h_*(\Omega Sp(n))\otimes Z\,[\frac{1}{2}]$ were determined where $h_*(\ )$ is a complex oriented homology theory. Moreover, F. Clarke [9] and the author [13] determined that of $K_*(\Omega Sp(n))$ independently.

The purpose of this paper is to determine $MU_*(\Omega Sp(n))$ as a Hopf algebra over $MU_*(pt)$ where MU is complex cobordism.

Let C be a $MU_*(pt)$-algebra and $f(x)=\sum_{i\geq 0} f_i x^i$, $g(x)=\sum_{i\geq 0} g_i x^i \in C[[x]]$. Define $(f\Box g)(x)\in(C\otimes_{MU_*(pt)}C)[[x]]$ to be $\sum_{i\geq 0}(\sum_{\substack{j+k=i\\ j,k\geq 0}} f_j\otimes g_k)x^i$. Then the main result of this paper is

Theorem 2.14.
There are $r_{2i-1}\in MU_*(\Omega Sp(n))$ $(1\leqq i\leqq n)$ such that $MU_*(\Omega Sp(n))=MU_*(pt)[r_1, r_3,\dots, r_{2n-1}]$ as a Hopf algebra and there exists $P(x)\in MU_*(pt)[[x]]$ such that the diagonal ϕ is given by

$$\phi(r_{2k-1})=\left[\frac{(1\Box r_n)(x)+(r_n\Box 1)(x)+P(x)\cdot(r_n\Box r_n)(x)}{1\otimes 1+(r_n\Box r_n)(x)}\right]_{2k-1}$$

where $r_n(x)=\sum_{i=1}^{n} r_{2i-1}x^{2i-1}$ and $[\sum a_i x^i]_j$ denotes the coefficient of x^j in $\sum a_i x^i$.

The paper is organized as follows:
In §1, we recall some general results in [12] for $MU_*(\Omega Sp(n))$.
In §2, we introduce some algebraic notations and prove the main result. The proof is similar as one in [13], but more systematic.

We use the quite similar notation as in [12] or [13], so the definitions of some usual notations are omitted in this paper.

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§1. The algebra $MU_*(\Omega Sp(n))$

First, recall some notations (See [12] and [13]).

Let $U(n)$, $Sp(n)$ be the n-th unitary and symplectic groups, and U, Sp the infinite groups $U(\infty)$, $Sp(\infty)$, respectively.

Let q: $U(n) \hookrightarrow Sp(n)$ and c: $Sp(n) \hookrightarrow U(2n)$ be the natural inclusions in [12].

Let i_n: $Sp(n) \hookrightarrow Sp$ be the natural inclusion.

The H-structure of ΩSU is given by the loop product λ: $\Omega SU \times \Omega SU \to \Omega SU$ and the diagonal map is denoted by Δ: $\Omega SU \to \Omega SU \times \Omega SU$. Futhermore, let J: $\Omega SU \to \Omega SU$ be the loop inverse of ΩSU.

Define the conjugation I: $U \to U$ by $I(A) = \bar{A}$. Then I induces a map BI: $BU \to BU$.

Let g: $BU \simeq \Omega SU$ be the Bott map. For simplicity, we define $\swarrow$: $\Omega SU \to \Omega SU$ to be $g \circ BI \circ g^{-1}$.

Let $\iota(x)$ be the formal inverse of the formal group F_{MU} (for detail, see [2] and [17]).

Put $R = MU_*(pt)$.

Under the notation, we can quote the results from [12] and [13].

Theorem 1.1.

(i) *There exist* $\beta_i \in MU_{2i}(\Omega SU)$ $(i \geqq 1)$ *such that* $MU_*(\Omega SU) = R[\beta_1, \beta_2, \ldots, \beta_n, \ldots]$ *as an algebra and* $\tilde{\phi}(\beta_i) = \sum_{\substack{j+k=1 \\ j,k>0}} \beta_j \otimes \beta_k$ *where* $\tilde{\phi}$ *is the reduced diagonal defined by* Δ.

(ii) $\Omega c \circ \Omega q = \lambda \circ (id \times (J \circ \swarrow)) \circ \Delta$ *holds and if we put* $\beta(x) = \sum_{i \geqq 0} \beta_i x^i$ $(\beta_0 = 1)$ *and extend* J_*, $\swarrow_*$ *and* $\Omega(c \circ q)_*$ *over* $MU_*(\Omega SU)[[x]]$ *by the natural way, then*

$$J_* \beta(x) = 1/\beta(x), \ \swarrow_* \beta(x) = \beta(\iota(x)) \quad and$$

$$\Omega(c \circ q)_* \beta(x) = \beta(x)/\beta(\iota(x)).$$

(iii) *There are* $z_{2k-1} \in MU_{4k-2}(\Omega Sp)$ *such that* $MU_*(\Omega Sp) = R[z_1, z_3, \ldots, z_{2k-1}, \ldots]$ *as an algebra and* $\Omega c_* z_{2k-1} \equiv \beta_{2k-1}$ *modulo the subalgebra generated by* $\beta_1, \beta_2, \ldots, \beta_{2k-2}$ *over* R. *Thus* Ωc_* *is a split monomorphism.*

(iv) $(\Omega i_n)_*$: $MU_*(\Omega Sp(n)) \to MU_*(\Omega Sp)$ *is a split monomorphism and* $\mathrm{Im}\,(\Omega i_n)_*$ *is generated by* $z_1, z_3, \ldots, z_{2n-1}$ *as a subalgebra of* $MU_*(\Omega Sp)$.

For the proofs, see [12] and [13].

§2. Algebraic notation and the main result

Put $R = MU_*(pt)$, $A = MU_*(\Omega SU)$ and $B = MU_*(\Omega Sp)$, for simplicity.

We need some algebraic notations.

Let C be an R-algebra and $C[[x]]$ the formal power series ring over C. Then clearly $C[[x]]$ has a natural R- or $R[[x]]$-algebra structure.

Let C, D be R-algebras and $f: C \to D$ be an R-algebra homomorphism. Then we define $f: C[[x]] \to D[[x]]$ by $f(\sum_i c_i x^i) = \sum_i f(c_i) x^i$ where $c_i \in C$. Also, if $f(x) = \sum_i f_i x^i \in C[[x]]$ and $g(x) = \sum_j g_j x^j \in D[[x]]$, then we define $(f \square g)(x) \in (C \otimes_R D)[[x]]$ to be $\sum_k (\sum_{\substack{i+j=k \\ i,j \geqq 0}} (f_i \otimes g_j)) x^k$.

If C is a Hopf algebra over R, then $\phi: C \to C \otimes_R C$ is an R-algebra homomorphism. So we can obtain $\phi: C[[x]] \to (C \otimes_R C)[[x]]$.

Let $C[[x]]_{ev}$ be all even functions in $C[[x]]$ and $C[[x]]_{od}$ all odd functions in $C[[x]]$ where C is an R-algebra.

Definition 2.1.

Define $bev(x)$, $bod(x) \in A[[x]]$ to be $1 + \sum_{k \geqq 1} m_k^{ev}(x) \cdot \beta_k$ and $\sum_{k \geqq 1} m_k^{od}(x) \cdot \beta_k$, respectively, where $m_k^{ev}(x) \in R[[x]]_{ev}$ and $m_k^{od}(x) \in R[[x]]_{od}$.

Of course, if we change $m_k^{ev}(x)$ and $m_k^{od}(x)$, then we get various $bev(x) \in A[[x]]_{ev}$ and $bod(x) \in A[[x]]_{od}$.

Let $p(x) = \sum_{i \geqq 1} p_{2i-1} x^{2i-1} \in R[[x]]_{od}$.

Definition 2.2.

We call the pair (bev, bod) to be a nice pair for $p(x)$, if $\phi bev = bev \square bev + bod \square bod$ and $\phi bod = bev \square bod + bod \square bev + p \cdot (bod \square bod)$ hold.

Then we have the following lemma.

Lemma 2.3.

The pair (bev, bod) *is a nice pair for* $p(x)$ *if and only if*

$$(2.4) \qquad \begin{aligned} m_k^{ev} &= m_1^{ev} \cdot m_{k-1}^{ev} + m_1^{od} \cdot m_{k-1}^{od} \\ m_k^{od} &= m_1^{ev} \cdot m_{k-1}^{od} + m_1^{od} \cdot m_{k-1}^{ev} + p \cdot m_1^{od} \cdot m_{k-1}^{od} \end{aligned}$$

hold for all $k \geqq 2$.

Proof. By the Definition 2.1.,

$$\phi bev(x) = \phi(1 + \textstyle\sum_{k>0} m_k^{ev}(x) \cdot \beta_k) = \sum_k m_k^{ev}(x) (\sum_{s+t=k} \beta_s \otimes \beta_t) \quad \text{and}$$

$$\phi bod(x) = \phi(\textstyle\sum_{k>0} m_k^{od}(x) \cdot \beta_k) = \sum_k m_k^{od}(x) (\sum_{s+t=k} \beta_s \otimes \beta_t).$$

On the other hand, if (bev, bod) is nice, then we have

$$\begin{aligned} \phi bev(x) &= (bev \square bev + bod \square bod)(x) \\ &= (1 + \textstyle\sum_{s>0} m_s^{ev}(x) \cdot \beta_s) \square (1 + \sum_{t>0} m_t^{ev}(x) \cdot \beta_t) \\ &\quad + (\textstyle\sum_{s>0} m_s^{od}(x) \beta_s) \square (\sum_{t>0} m_t^{od}(x) \beta_t) \qquad \text{and} \\ \phi bod(x) &= (bev \square bod + bod \square bev + p \cdot (bod \square bod))(x) \\ &= (1 + \textstyle\sum_{s>0} m_s^{ev}(x) \cdot \beta_s) \square (\sum_{t>0} m_t^{od}(x) \cdot \beta_t) \\ &\quad + (\textstyle\sum_{s>0} m_s^{od}(x) \cdot \beta_s) \square (1 + \sum_{t>0} m_t^{ev}(x) \cdot \beta_t) \\ &\quad + p(x) \cdot (\textstyle\sum_{s>0} m_s^{od}(x) \cdot \beta_s) \square (\sum_{t>0} m_t^{od}(x) \cdot \beta_t). \end{aligned}$$

If we check the coefficients at $\beta_1 \otimes \beta_{k-1}$, then the only if part is easily seen.

To prove the converse, we have only to show the following two equations for all s, t such that $s+t=k$ under (2.4):

$$m_s^{ev} \cdot m_t^{ev} + m_s^{od} \cdot m_t^{od} = m_{s-1}^{ev} \cdot m_{t+1}^{ev} + m_{s-1}^{od} \cdot m_{t+1}^{od} \quad \text{and}$$

$$m_s^{ev} \cdot m_t^{od} + m_s^{od} \cdot m_t^{ev} + p \cdot m_s^{od} \cdot m_t^{od}$$
$$= m_{s-1}^{ev} \cdot m_{t+1}^{od} + m_{s-1}^{od} \cdot m_{t+1}^{ev} + p \cdot m_{s-1}^{od} \cdot m_{t+1}^{od}.$$

We can easily show this by the induction for s and omit details. □

Thus, the nice pair for $p(x)$ has one to one correspondence with the pair (m_1^{ev}, m_1^{od}) for the fixed $p(x)$. So, we denote the nice pair for $p(x)$ decided with (m_1^{ev}, m_1^{od}) by $(bev(m_1^{ev}, m_1^{od}), bod(m_1^{ev}, m_1^{od}))$. Also, we define $m(m_1^{ev}, m_1^{od})_k^{ev}$ (resp. $m(m_1^{ev}, m_1^{od})_k^{od}$) to be the coefficient of $bev(m_1^{ev}, m_1^{od})$ (resp. $bod(m_1^{ev}, m_1^{od})$) at β_k.

Example.
If we put $m_1^{ev}(x)=0$ and $m_1^{od}(x)=x$, then (2.4) gives

$$m(0, x)_1^{ev}=0, \quad m(0, x)_2^{ev}=x^2, \quad m(0, x)_3^{ev}=x^3 p(x)$$

$$m(0, x)_1^{od}=x, \quad m(0, x)_2^{od}=x^2 p(x) \quad \text{and} \quad m(0, x)_3^{od}=x^3+x^3(p(x))^2.$$

We put $Bev(x)=bev(0, x)(x)$ and $Bod(x)=bod(0, x)(x)$.

Now we consider $\iota_* bev$, $\iota_* bod$. Since $\iota_*: A \to A$ is a Hopf algebra homomorphism over R, if (bev, bod) is nice for $p(x)$, then $(\iota_* bev, \iota_* bod)$ is so.

We denote $\pi: A[[x]] \to R[[x]]$ corresponding $f \in A[[x]]$ to the coefficient at β_1.

Put $\iota(x)=\sum_{i \geq 1} g_i x^i$ where $\iota(x)$ the formal inverse of the formal group of complex cobordism theory. Then, as is well-known, $g_1=-1$ (see [2]).

Lemma 2.5. $\pi(\iota_* \beta_k)=g_k$.

Proof. $[\pi(\iota_* \beta(x))]_k=[\pi(\beta(\iota(x)))]_k=[\iota(x)]_k=g_k$. Since π and $[\ ]_k$ commutes, the result follows. □

Thus, we have

$$\pi(\iota_* bev)=\pi(\iota_*(1+\textstyle\sum_{k \geq 1} m_k^{ev} \cdot \beta_k))=\sum_{k \geq 1} m_k^{ev} \cdot g_k \quad \text{and}$$

$$\pi(\iota_* bod)=\pi(\iota_*(\textstyle\sum_{k \geq 1} m_k^{od} \cdot \beta_k))=\sum_{k \geq 1} m_k^{od} \cdot g_k.$$

Proposition 2.6.
There is a $P(x)=\sum_{i \geq 1} P_{2i-1} x^{2i-1} \in R[[x]]_{od}$ *such that*

$$\pi(\iota_* Bev(x))=x \cdot P(x). \tag{2.7}$$

Proof. Using (2.5), we have $\pi(\iota_* Bev(x))=\sum_{k \geq 1} m(0, x)_k^{ev} \cdot g_k$. Since $m_1^{ev}(0, x)=0$, we obtain $\pi(\iota_* Bev(x))=\sum_{k \geq 2} m(0, x)_k^{ev} \cdot g_k$.

We need the following lemma.

Lemma 2.8.

(i) $m(0, x)_k^{ev}$ *and* $m(0, x)_k^{od} \in x^k \cdot \boldsymbol{Z}[[x, p(x)]]$,

(ii) $m(0, x)_{2k}^{ev} = x^{2k} + higher$ *and*

$m(0, x)_{2k-1}^{od} = x^{2k-1} + higher$ *for all positive integer* k.

Proof. All follows from (2.4) and by an easy induction. □

Then we have $[\pi(\ell_* Bev(x))]_{2i+1} = 0$ and $P_{2i-1} = [\pi(\ell_* Bev(x))]_{2i} = [\sum_{2i \geqq k \geqq 2} m(0, x)_k^{ev} g_k]_{2i}$. Since $k \geqq 2$, the last can be written by $g_1, g_2, \ldots, g_{2i}$ and by $P_1, P_2, \ldots, P_{2i-3}$. So (2.7) gives an inductive formula for the definition of P_{2i-1}. □

Define $f(x) = \sum_{i \geqq 1} f_{2i-1} x^{2i-1} \in R[[x]]_{od}$ to be $\sum_{k \geqq 1} m(0, x)_k^{od} \cdot g_k$. Since $m(0, x)_1^{od} = x$ and $g_1 = -1$, we obtain $f_1 = -1$.

Lemma 2.9.

If $f(x) \equiv -x \mod x^{2n} \cdot R[[x]]$, *then* $m(f(x), x \cdot P(x))_k^{od} \equiv -m(0, x)_k^{od} \mod x^{2n+2} \cdot R[[x]]$ *for* $k \geqq 2$.

This lemma is a key formula. It is easy but tedious to show (2.9). So we defer this to the appendix.

Proposition 2.10.

$f(x) = -x$.

Proof. We prove this by the induction. Let assume $f(x) \equiv -x \mod x^{2n} \cdot R[[x]]$ for $n \geqq 1$.

Since $\ell_* \circ \ell_* = id$, we have

$$\sum_{k \geqq 1} m(f, xP)_k^{od} \cdot g_k = m(0, x)_1^{od} = x.$$

So, for $n \geqq 1$, we obtain the following equations

$$0 = [\sum_{k \geqq 1} m(f, xP)_k^{od} \cdot g_k]_{2n+1} = [g_1 \cdot f + \sum_{k \geqq 2} m(f, xP)_k^{od} \cdot g_k]_{2n+1}$$

$$= -f_{2n+1} + [\sum_{k \geqq 2} - m(0, x)_k^{od} \cdot g_k]_{2n+1}. \quad \text{Thus we have}$$

$$f_{2n+1} = -[\sum_{k \geqq 2} m(0, x)_k^{od} \cdot g_k]_{2n+1}.$$

But since $f(x) = \sum_k m(0, x)_k^{od} g_k$, we have also $f_{2n+1} = [\sum_{\geqq 2} m(0, x)_k^{od} \cdot g_k]_{2n+1}$. Since R is torsion free, $f_{2n+1} = 0$. Thus the induction argument asserts the result. □

Thus we have

$$(\ell_* Bev, \ell_* Bod) = (1 + \sum_{k \geqq 1} m(xP, -x)_k^{ev} \cdot \beta_k, \sum_{k \geqq 1} m(xP, -x)_k^{od} \cdot \beta_k).$$

But, if we put $Bev' = Bev + P \cdot Bod$ and $Bod' = -Bod$, then we have the following proposition by an easy calculation.

Proposition 2.11.

(Bev', Bod') is a nice pair for P.

Since $\pi(Bev')=xP$, $\pi(Bod')=-x$, one can show easily $\ell_* Bev=Bev'$, $\ell_* Bod=Bod'$.

Since $Bev(x)$ is unit in $A[[x]]$, we can put

$$r(x)=\sum_{i\geq 1} r_{2i-1}x^{2i-1}=Bod(x)/Bev(x).$$

As in [13], we can calculate $\phi r(x)$ and $\Omega(c\circ q)_* r(x)$.

Proposition 2.12.

(i)
$$\phi r=\frac{r\square 1+1\square r+P\cdot r\square r}{1\otimes 1+r\square r},$$

(ii)
$$\Omega(c\circ q)_* r=\frac{2r+P\cdot r^2}{1+r^2}.$$

Proof. Since $A[[x]]\xrightarrow{\phi}(A\otimes_R A)[[x]]$ and $A[[x]]\otimes_R A[[x]]\xrightarrow{\square}(A\otimes_R A)[[x]]$ are R-algebra homomorphisms, and since (Bev, Bod) is nice, (i) of (2.12) is clear.

Since $\lambda\circ(1\times J)\circ\Delta\colon \Omega SU\to\Omega SU$ is null-homotopic, we have $\lambda_*\circ(1\otimes J_*)\circ\phi r=0$. By this equation and (i) of (2.12), we obtain easily the following equation: $J_* r=-r/(1+P\cdot r)$. On the other hand, we have $\ell_* r=\ell_* Bod/\ell_* Bev=Bod'/Bev'=-Bod/(Bev+P\cdot Bod)=-r/(1+P\cdot r)$. So we obtain

$$J_*\circ\ell_* r=J_*\circ J_* r=(J\circ J)_* r=r.$$

Then, by (ii) of (1.1), we have the following equations:

$$\Omega(c\circ q)_* r=\lambda_*\circ(1\otimes J_*\circ\ell_*)\circ\phi r=\frac{r+J_*\circ\ell_* r+P\cdot r\cdot J_*\circ\ell_* r}{1+r\cdot J_*\circ\ell_* r}$$

$$=\frac{2r+P\cdot r^2}{1+r^2}. \quad \square$$

Put $\Gamma=R[r_1, r_3,\dots, r_{2k-1},\dots]\subset A$.
Then, as in [13], we can now prove,

Theorem 2.13. $\mathrm{Im}\,(\Omega c)_*=\Gamma$.

Proof. First, we prove $\Gamma\subset\mathrm{Im}\,(\Omega c)_*$. By the definition of $r(x)$, $Bev(x)$ and $Bod(x)$, $r_1=\beta_1$ is easily seen. On the other hand, (iii) of (1.1) implies $(\Omega c)_* z_1=\beta_1$. So, $r_1\in\mathrm{Im}\,(\Omega c)_*$.

Assume that $r_1, r_3,\dots, r_{2k-1}\in\mathrm{Im}\,(\Omega c)_*$. Note that

$$\left[\frac{2\cdot r(x)+P(x)\cdot(r(x))^2}{1+(r(x))^2}\right]_{2k+1}\equiv 2r_{2k+1}$$

modulo $R[r_1, r_3,\dots, r_{2k-1}]$. Since $R[r_1, r_3,\dots, r_{2k-1}]\subset\mathrm{Im}\,(\Omega c)_*$ by the assumption, we have $2r_{2k+1}\in\mathrm{Im}\,(\Omega c)_*$. But, by (iii) of (1.1), $\mathrm{Im}\,(\Omega c)_*$ is a split submodule of A. Thus, $r_{2k+1}\in\mathrm{Im}\,(\Omega c)_*$ and we have $\Gamma\subset\mathrm{Im}\,(\Omega c)_*$.

By (ii) of (2.8), $r_{2k-1}\equiv\beta_{2k-1} \mod R[\beta_1, \beta_2,\dots, \beta_{2k-2}]$ is easily seen. Then (iii) of (1.1) asserts the following equation:

$$r_{2k-1} \equiv (\Omega c)_* z_{2k-1}$$

modulo $R[(\Omega c)_* z_1, (\Omega c)_* z_3, \ldots, (\Omega c)_* z_{2k-1}]$. So $R[r_1, r_3, \ldots, r_{2k-1}] = R[(\Omega c)_* z_1, (\Omega c)_* z_3, \ldots, (\Omega c)_* z_{2k-1}]$ can be obtained by an easy induction. If we put $k=\infty$, then we have (2.13). □

We have also

Theorem 2.14.

There are $r_{2i-1} \in MU_*(\Omega Sp(n))$ $(1 \leqq i \leqq n)$ such that $MU_*(\Omega Sp(n)) = MU_*(pt)[r_1, r_3, \ldots, r_{2n-1}]$ as a Hopf algebra and there exists $P(x) \in MU_*(pt)[[x]]$ such that the diagonal ϕ is given by

$$\phi(r_{2k-1}) = \left[\frac{(1 \square r_n)(x) + (r_n \square 1)(x) + P(x) \cdot (r_n \square r_n)(x)}{1 \otimes 1 + (r_n \square r_n)(x)} \right]_{2k-1}$$

where $r_n(x) = \sum_{i=1}^{n} r_{2i-1} x^{2i-1}$ and $[\sum a_i x^i]_j$ denotes the coefficient of x^j in $\sum a_i x^i$.

Appendix.

First, we prove that if $f(x) \equiv -x \mod x^{2n} \cdot R[[x]]$, then the following equations hold for all $k \geqq 2$:

$$\text{(A.1)} \quad \begin{aligned} &\text{(a)} \quad m(xP, f)_k^{ev} \equiv m(xP, -x)_k^{dv} \mod x^{2n+1} \cdot R[[x]] \\ &\text{(b)} \quad m(xP, f)_k^{od} \equiv m(xP, -x)_k^{od} \mod x^{2n+2} \cdot R[[x]]. \end{aligned}$$

We prove this by the induction. Using (2.4), we have easily $m(xP, f)_2^{ev} = f^2 + x^2 P$ and $m(xP, f)_2^{od} = 2xPf + Pf^2$. So, (A.1) is directly seen for $k=2$. For $k \geqq 3$, we obtain

$$\begin{aligned} m(xP, f)_k^{od} &= m(xP, f)_1^{od} \cdot m(xP, f)_{k-1}^{ev} + m(xP, f)_1^{ev} \cdot m(xP, f)_{k-1}^{od} \\ &\quad + P \cdot m(xP, f)_1^{od} \cdot m(xP, f)_{k-1}^{od} \\ &= f \cdot m(xP, f)_{k-1}^{ev} + xP \cdot m(xP, f)_{k-1}^{od} + f \cdot P \cdot m(xP, f)_{k-1}^{od}. \end{aligned}$$

By the assumption of the induction, and by the fact that $\deg(m(xP, -x)_{k-1}^{ev}) \geqq 2$ $(k \geqq 3)$ and $\deg(P) \geqq 1$, we obtain

$$\begin{aligned} m(xP, f)_k^{od} &\equiv (-x) \cdot m(xP, -x)_{k-1}^{ev} + xP \cdot m(xP, -x)_{k-1}^{od} + (-x) \cdot P \cdot m(xP, -x)_{k-1}^{od} \\ &\equiv m(xP, -x)_k^{od} \mod x^{2n+2} \cdot R[[x]]. \end{aligned}$$

The case (a) is obtained by the similar method.

Next, we prove that

$$\text{(A.2)} \quad \begin{aligned} &\text{(a)} \quad x \cdot m(xP, -x)_k^{ev} = m(0, x)_{k+1}^{od} \\ &\text{(b)} \quad m(xP, -x)_k^{od} = -m(0, x)_k^{od} \qquad (k \geqq 1). \end{aligned}$$

Again, we prove this by the induction on k.

The results are clear for $k=1$, for

$$x\cdot m(xP,\ -x)_1^{od}=x\cdot xP=P\cdot(m(0,\ x)_1^{od})^2=m(0,\ x)_2^{od}\quad\text{and}$$

$$m(xP,\ -x)_1^{od}=-x=-m(0,\ x)_1^{od}.$$

Assume the results for k. Then we have

$$\begin{aligned}m(xP,\ -x)_{k+1}^{od}&=m(xP,\ -x)_1^{od}\cdot m(xP,\ -x)_k^{ev}+m(xP,\ -x)_1^{ev}\cdot m(xP,\ -x)_k^{od}\\&\quad+P\cdot m(xP,\ -x)_1^{od}\cdot m(xP,\ -x)_k^{od}\\&=(-x)\cdot m(xP,\ -x)_k^{ev}+xP\cdot m(xP,\ -x)_k^{od}-xP\cdot m(xP,\ -x)_k^{od}\\&=(-x)\cdot m(xP,\ -x)_k^{ev}=-m(0,\ x)_{k+1}^{od}.\end{aligned}$$

The case (a) for $k+1$ is proved more easily. (b) of (A.1) and (A.2) assert the key lemma (2.9). □

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