J. Math. Kyoto Univ. (JMKYAZ)
22-4 (1983) 555-565

Comparison of measures on the maximal ideal space of $H^\infty(W)$ with applications to the Dirichlet problem

By

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(Received July 7, 1981)

0. Introduction

Let W be any subdomain of $C^n(n\geqq 1)$, the n-dimensional complex Euclidean space. By R^{2n} we denote the underlying real 2n-dimensional Euclidean space of C^n, as usual. Throughout this paper we shall assume that W admits either nonconstant bounded analytic functions, or nonconstant bounded harmonic functions. $H^\infty(W)$ will denote the commutative Banach algebra of all bounded analytic functions on W endowed with the uniform norm. $HB_R(W)$ is the order complete Banach lattice of all real-valued bounded harmonic functions in $W\subset R^{2n}$ with the sup norm topology. $M_{H^\infty(W)}$ will stand for the maximal ideal space of $H^\infty(W)$.

This paper deals with the Dirichlet problem on the Shilov boundary S of $M_{H^\infty(W)}$. Namely, under the appropriate conditions we investigate a positive linear map from $C_R(S)$ into $HB_R(W)$ which acts on Re $H^\infty(W)(\subset C_R(S))$ as the identity map. Since $HB_R(W)$ is an order complete Banach lattice, such a map as above always exists: we can apply techniques of the positive extension in Hahn-Banach's extension theorem to this problem. The solution obtained by this method, however, gives us few information on the maximality in the following sense. Let L be any solution of the problem, and consider the functional $C_R(S)\ni g\rightarrow L(g)(p)$, where p is an arbitrary, but fixed, point of W. Clearly this functional is represented by the probability measure which is uniquely determined, and is supported on S. Denote this measure by $\mathrm{d}v$. Then by the positiveness, and from Harnack's inequality we have the nonnegative kernel $Q(z, \ )$ of $L^\infty(\mathrm{d}v)$ such that $L(g)(z)=\int gQ(z, \ )\mathrm{d}v$ for all $g\in C_R(S)$ and for all $z\in W$. By the maximality for solutions we mean that the induced measure $\mathrm{d}v$ for p as above is a *boundary measure* in the sense of Alfsen [1]. Thus our aim in this paper is to investigate the maximal solutions of the Dirichlet problem on S in terms of Choquet order relation.

In case that W is a subdomain of the complex plane, or more generally, a Rieman surface, the author characterized a representing measure $\mathrm{d}v$ for any point of W such that $\mathrm{d}v$ is a boundary measure and has a positive kernel $Q(z, \ )$ of $L^\infty(\mathrm{d}v)$ with a parameter $z\in W$: $Q(z, \ )\mathrm{d}v$ satisfy the following [3].

1) For every $z\in W$, $Q(z, \ )\mathrm{d}v$ is a representing measure for z with respect to

$H^\infty(W)$.

2) For every $g \in L^\infty(\mathrm{d}v)$, $\int gQ(z,\ )\mathrm{d}v$ is a bounded harmonic function on W with $\left\|\int gQ(z,\ )\mathrm{d}v\right\|_\infty \leqq \|g\|_\infty$.

3) For every $h \in C_R(S)$, $\int hQ(z,\ )\mathrm{d}v$ extends continuously to the Choquet boundary of $M_{H^\infty(\mathrm{d}v)}$ and coincides with h on it. Hence $\left\|\int hQ(z,\ )\mathrm{d}v\right\|_\infty = \|h\|_\infty$ holds.

1′) $Q(z,\ )\mathrm{d}v$ is a Jensen measure for every $z \in W$, if we modify the maximality condition appropriately.

we call such a measure $\mathrm{d}v$ a singular harmonic measure for the point. Now, same results as above are still valid for any closed subalgebra of $H^\infty(W)$ with unit 1. Specifically, in case that W is a bounded plane domain, and that $A(W)$ is a subalgebra of $H^\infty(W)$ defined by $H^\infty(W) \cap C(\overline{W})$, every situation takes the concrete form. Indeed the maximal ideal space of $A(W)$ is just identical with $\overline{W}$, including the topology. Hence all measures involved are actually Baire measures in the complex plane. In such a situation, for some class of domains (e.g. the one constructed by Gamelin [2]) the Dirichlet problem on $S(=\partial W)$ relative to $A(W)$ admits an infinite number of solutions, and a singular harmonic measure is mutually singular with the ordinary harmonic measure on the topological boundary ∂W. This phenomenon motivates the author to extend the preceding results to an arbitrary subdomain of C^n.

For out aim we shall characterize the induced measures from solutions of the Dirichlet problem on S. Proofs will be made in view of the purely real analysis. So the analogous results are always valid for every real or complex linear subspace of $HB(W)$, which denotes the Banach space of all bounded harmonic functions on W endowed with the uniform norm. That is, let $B(W)$ be any, not necessarily closed, real or complex linear subspace of $HB(W)$ with unit 1. Suppose $B(W)$ contains at least one nonconstant function. Then there exists a probability measure with a positive kernel which is also a boundary measure relative to $B(W)$ and satisfies the condition 1), 2) and 3). Further our measure has no mass on (the canonical image of) W.

In section 1, we shall give another proof of Cartier's theorem in the form suitable for our purpose. Section 2~4 will be devoted to proofs of the above statement. In section 5, however, our concerns turn toward a closed subalgebra of $H^\infty(W)$ with unit 1. There we will discuss the condition 1′) about Jensen measures.

The author should like to thank Professor Y. Kusunoki for his valuable suggestions, above all, for the one that results in [3] may be applicable to subdomains of C^n.

1. Comparison of measures

Throughout this paper we follow the usefull terminologies in Alfsen [1], Gamelin [2] and Schaefer [5]. Moreover the term 'measure' shall signify, in all cases, a finite regular Borel measure on some compact Hausdorff space.

In this section we shall assume that X is an arbitrary, but fixed, compact Hausdorff space. $M(X)$ denotes the real linear space of all real measures on X. In particular $M^+(X)$ stands for the set of all positive elements of $M(X)$. We regard $M(X)$ as the dual of $C_R(X)$. In this point of view we shall often use the notation $v(g)$ for $\int g dv$, where $g \in C_R(X)$ and $v \in M(X)$. Let P be a max-stable admissible cone on X. Namely $P \subseteqq C_R(X)$ is the convex cone satisfying the following. P contains the constants and separates the points on X. $f \vee g$ is in P if f and g belong to P. Since P is a convex cone of $C_R(X)$, it defines an order on $M(X)$. This order relation is symbolized by $\prec_P$. Further we will denote by $\partial_P X$ the P-Choquet boundary, and by S_P the Shilov boundary of X with respect to P. S_P coincides with the closure of $\partial_P X$ and is the minimum closed set on which every function of P attains its maximum. Let f be any real valued bounded function with dom $(f) \supseteqq \partial_P X$. The P-lower envelope $\check{f}_P$ of f is defined as a function: $\check{f}_P = \sup \{h \in P: h \leqq f$ on $\operatorname{dom}(f)\}$. The P-upper envelope $\hat{f}_P$ of f is a function $\hat{f}_P = -\widecheck{(-f)}_P$.

Using the P-lower envelope, boundary measures relative to P is defined as follows. A measure dv is called a boundary measure if it satisfies $f = \check{f}_P (= \hat{f}_P)$ a.e. $|dv|$ for all $f \in C_R(X)$. It is known that every boundary measure has no mass on a Baire set disjoint from $\partial_P X$, and hence it is supported on S_P.

Recall that P contains the constants. So the relation $du \prec_P dv$ for positive measures implies $\|du\| = \|dv\|$. Hence we see that the order ($\prec_P$) on $M^+(X)$ is inductive. It is known that every maximal element of $M^+(X)$ is a boundary measure and conversely any positive boundary measure is maximal in $M^+(X)$ under $\prec_P$. These imply, in particular, that for every $du \in M^+(X)$ there exists a positive boundary measure dv with $du \prec_P dv$. Concerning all of the above, the details can be found in Alfsen [1].

Our main object stated early is obtained as a corollary of the general theory on the above order relation. More precisely, it is a special application of the theory on the comparison of measures. Indeed, proofs in [3] heavily rely on the method employed in [1] for the proof of Cartier, Fell and Meyer's theorem. So, we reprove and extend this theorem along the way whose ideas are found implicitly in [3].

Theorem 1.1. *Let v and u_1 be positive measures and u_2 a nonnegative measure on X. Assume that inequalities $u_1(g) + u_2(g) \leqq v(g)$ hold for all nonnegative $g \in P$. Then there exist nonnegative h_1 and h_2 of $L^\infty(dv)$ such that $0 \leqq h_1 + h_2 \leqq 1$ and $du_j \prec_P h_j dv$ $(j = 1, 2)$.*

Proof. The proof will be made in view of separation theorem in $L^1_R(dv)$. Set $Q = \{f \in P: u_1(f) > 1\}$ and $R = \{f \geqq 0: f \in P$ and $v(f) - u_2(f) < 1\}$. Denote by C and U the positive cone and open unit ball of $L^1_R(dv)$, respectively. T denotes the convex hull of $(U - C) \cup R$ where $U - C = \{f - g: f \in U, g \in C\}$. Observe that T is open. Here we require that in $L^1_R(dv)$, T is disjoint from the convex set Q. To see this, assume $T \cap Q \ni f$. Then $f (\in Q)$ takes the form $f = (1-s)h + sg$ where $0 \leqq s \leqq 1$,

$h\in U-C$ and $g\in R$. First of all we have

$$1-s\geqq\int 0\vee(1-s)h\mathrm{d}v=\int 0\vee(f-sg)\mathrm{d}v\geqq\int 0\vee(f-sg)(\mathrm{d}v-\mathrm{d}u_2).$$

On the other hand, the following inequalities hold.

$$\int 0\vee(f-sg)(\mathrm{d}v-\mathrm{d}u_2)=\int\{(sg\vee f)-sg\}(\mathrm{d}v-\mathrm{d}u_2)\geqq\int(sg\vee f)(\mathrm{d}v-\mathrm{d}u_2)-s.$$

From $0\leqq sg\vee f(\in P)$, it follows that

$$\int(sg\vee f)(\mathrm{d}v-\mathrm{d}u_2)\geqq\int(sg\vee f)\mathrm{d}u_1\geqq\int f\mathrm{d}u_1>1.$$

These yield $1-s\geqq\int 0\vee(f-sg)(\mathrm{d}v-\mathrm{d}u_2)>1-s$, a contradiction. Thus we see that $Q\cap T=\emptyset$.

By separation theorem, there exists a continuous linear form Ψ and a constant c such that $\Psi(Q)\geqq c$ and $c\geqq\Psi(f)$ for all $f\in T$. Since $-C\subset T$, Ψ is positive. So we may assume $\Psi(\tau)=1$, where $\tau=1/\|\mathrm{d}u_1\|$. Observe that $\{\tau\}+\varepsilon[Q-\{\tau\}]$ coincides with Q for every positive ε. This yields $\Psi(Q)\geqq\Psi(\tau)=1\geqq c\geqq\Psi(T)$. In particular $\Psi(R)\leqq 1$ and $\Psi(U)\leqq 1$, so that, $\|\Psi\|\leqq 1$. Let $h_1\in L_R^\infty(\mathrm{d}v)$ be the function which corresponds to Ψ. Then we have $0\leqq h_1\leqq 1$ and $\int h_1\mathrm{d}v=\|\mathrm{d}u_1\|=\tau^{-1}$.

Next we verify $\mathrm{d}u_1\prec_P h_1\mathrm{d}v$. For this consider any f of P and any positive ε. A function: $(f+\|f\|+1+\varepsilon)/\int(f+\|f\|+1)\mathrm{d}u_1$ belongs to Q. This implies $\int(f+\|f\|+1+\varepsilon)h_1\mathrm{d}v\geqq\int(f+\|f\|+1)\mathrm{d}u_1$, so that, $\int(f+\varepsilon)h_1\mathrm{d}v\geqq\int f\mathrm{d}u_1$. Letting $\varepsilon\to 0$, we have $\int fh_1\mathrm{d}v\geqq\int f\mathrm{d}u_1$, i.e. $\mathrm{d}u_1\prec_P h_1\mathrm{d}v$. Similarly for any positive ε and for any nonnegative $f\in P$, a function $f/\left(\varepsilon+\int f(\mathrm{d}v-\mathrm{d}u_2)\right)$ is contained in R. This yields $\varepsilon+\int f(\mathrm{d}v-\mathrm{d}u_2)\geqq\int fh_1\mathrm{d}v$, and hence $\int f(1-h_1)\mathrm{d}v\geqq\int f\mathrm{d}u_2$ hold for all nonnegative $f\in P$.

Now, if $\mathrm{d}u_2$ is a zero measure, all is over here. In case that $\mathrm{d}u_2$ is positive, consider $(1-h_1)\mathrm{d}v$, $\mathrm{d}u_2$ and a zero measure as the measures $\mathrm{d}v$, $\mathrm{d}u_1$ and $\mathrm{d}u_2$ of the theorem. Applying the above argument to them, we have a function k of $L^\infty((1-h_1)\mathrm{d}v)$ such that $0\leqq k\leqq 1$ and $\mathrm{d}u_2\prec_P k(1-h_1)\mathrm{d}v$. Here we may view $k(1-h_1)$ as an element of $L^\infty(\mathrm{d}v)$. Setting $h_2=k(1-h_1)$, we establish the assertions.

Corollary 1.2. *Let* $\mathrm{d}u$ *and* $\mathrm{d}v$ *be positive measures on* X. *Suppose that* $u(g)\leqq v(g)$ *holds for every nonnegative* $g\in P$. *Then there exists a nonnegative element* h *of* $L^\infty(\mathrm{d}v)$, $1\geqq h$, *with* $\mathrm{d}u\prec_P h\mathrm{d}v$. *In case that* $\|\mathrm{d}u\|=\|\mathrm{d}v\|$, h *is identical with* 1.

Corollary 1.3. *Let* $\mathrm{d}u$ *and* $\mathrm{d}v$ *be any positive measures with* $\mathrm{d}u\prec_P\mathrm{d}v$. *Then for any finite positive decomposition* $\mathrm{d}u=\sum_{k=1}^n\mathrm{d}u_k$ *of* $\mathrm{d}u$ *there exists a corresponding* *positive decomposition* $\mathrm{d}v=\sum_{k=1}^{n}\mathrm{d}v_k$ *of* $\mathrm{d}v$ *with* $\mathrm{d}u_k\prec_P \mathrm{d}v_k$ $(1\leqq k\leqq n)$.

Proof. The proof is obtained by induction about n. In case $n=2$, Theorem 1.1 guarantees the assertion. To derive the case $n=m+1$ from that of $n=m$, consider any positive decomposition $\mathrm{d}u=\sum_{k=1}^{m+1}\mathrm{d}u_k$ of $\mathrm{d}u$. By Theorem 1.1 there is a positive decomposition $\mathrm{d}v=\mathrm{d}v_1+\mathrm{d}v_2$ of $\mathrm{d}v$ such that $\sum_{k=2}^{m+1}\mathrm{d}u_k\prec_P \mathrm{d}v_2$ and $\mathrm{d}u_1\prec_P \mathrm{d}v_1$. Thus by induction we have the assertion.

Theorem 1.4. *Let* $\mathrm{d}u$ *and* $\mathrm{d}v$ *be positive measures on* X *with* $\mathrm{d}u\prec_P \mathrm{d}v$. *Then there exists one linear map* T *of* $L^p(\mathrm{d}v)$ *into* $L^p(\mathrm{d}u)$ *which satisfies the following conditions* $(1\leqq p\leqq\infty)$.

1) *For any* $f\in L^1(\mathrm{d}v)$ $\int f\mathrm{d}v=\int T(f)\mathrm{d}u$
2) *T is positive, and further* $T(g)\geqq g$ a.e. $\mathrm{d}u$ *for any* $g\in P$.
3) *T is a norm decreasing map,* $\|T\|_p\leqq 1$, $(1\leqq p\leqq\infty)$.

Proof. Let $[\bigcup_{k=1}^{n}E_k]$ be any finite decompsosition of X into Borel sets such that $u(E_k\cap E_j)=0$ for every distinct k and j. Consider the family $\mathfrak{F}$ of all such decompositions. We define an order $(\leqq)$ on $\mathfrak{F}$ as follows. Namely for any $\alpha=[\bigcup_{k=1}^{n}E_k]$ and $\beta=[\bigcup_{j=1}^{m}E'_j]$ the relation: $\alpha\geqq\beta$ holds if any E_k of α is contained in some E'_j of β a.e. $\mathrm{d}u$. (i.e. $u(E_k\backslash E'_j)=0$) (Obviously, the order is well-defined, under trivial modifications.) Every element $\alpha=[\bigcup_{k=1}^{n}E_k]$ of $\mathfrak{F}$ induces a positive decomposition on $\mathrm{d}u$ such that $\mathrm{d}u=\sum_{k=1}^{n}\psi_k\mathrm{d}u$, where ψ_k denotes the characteristic function of E_k $(1\leqq k\leqq n)$. By corollary 1.3, there is a positive decomposition $\mathrm{d}v=\sum_{k=1}^{n}\mathrm{d}v_k$ of $\mathrm{d}v$ with $\psi_k\mathrm{d}u\prec_P \mathrm{d}v_k$ $(1\leqq k\leqq n)$. Fix the one for each $\alpha=[\bigcup_{k=1}^{n}E_k]$ and set $T_\alpha(f)=\sum_{k=1}^{n}\left(\int f\mathrm{d}v_k\right)\psi_k/\|\mathrm{d}v_k\|$ for all $f\in L^\infty(\mathrm{d}v)$. Observe that T_α is a linear map of $L^\infty(\mathrm{d}v)$ into $L^\infty(\mathrm{d}u)$ for each α. We can easily verify the following. 1′) $\int f\mathrm{d}v=\int T_\alpha(f)\mathrm{d}u$ 2′) T_α is positive. 3′) $\|T_\alpha(f)\|_\infty\leqq\|f\|_\infty$ and $\int|T_\alpha(f)|^p\mathrm{d}u\leqq\int|f|^p\mathrm{d}v$ where f is any function of $L^\infty(\mathrm{d}v)$ and $1\leqq p<\infty$.

Set $B_f=\{g\in L^\infty(\mathrm{d}u)\colon \|g\|_\infty\leqq\|f\|_\infty\}$ for every $f\in L^\infty(\mathrm{d}v)$, and consider the direct product space $B=\Pi\{B_f : f\in L^\infty(\mathrm{d}v)\}$. Since each B_f is weak* compact, B is also compact. Here we can identify each T_α with an element $\{T_\alpha(f)\}_{f\in L^\infty(\mathrm{d}v)}$ of B. Observe that $\{T_\alpha\colon \alpha\in\mathfrak{F}\}$ is directed upwards under the induced order from $\mathfrak{F}$. This implies that the family $\{T_\alpha\}$, each of which is viewed as an element of B, forms a filter base in B. Hence it has a cluster point. Pick up the one, say $\{g_f\}_{f\in L^\infty(\mathrm{d}v)}$, and fix it throughout. Define $T(f)=g_f$ for all $f\in L^\infty(\mathrm{d}v)$. Then T is a linear map of $L^\infty(\mathrm{d}v)$ into $L^\infty(\mathrm{d}u)$ because each T_α is linear. Since $T(f)$ is weak* adherent to the subset $\{T_\alpha(f)\colon \alpha\in\mathfrak{F}\}$ of $L^\infty(\mathrm{d}u)$, T also satisfies the above three conditions 1′), 2′) and 3′) for all $f \in L^{\infty}(dv)$. Next we claim that $T(h) \geqq h$ a.e. du for every $h \in P$. Set $h \in P$, and for any positive ε, choose an $\alpha_0 = [\bigcup_{j=1}^{m} K_j]$ such that the oscillation of h on any K_j is less than ε. Take any $\alpha = [\bigcup_{k=1}^{n} E_k]$ so that $\alpha \geqq \alpha_0$. Let $dv = \sum_{k=1}^{n} dv_k$ be the positive decomposition of dv which corresponds to $du = \sum_{k=1}^{n} \psi_k du$. From $\psi_k du \prec_P dv_k$, it follows $\int h dv_k / \|dv_k\| \geqq \int h\psi_k du / \|\psi_k du\|$. Since any E_k is contained in some K_j a.e. du, we have $\int h dv_k / \|dv_k\| \geqq h - \varepsilon$ a.e. du on E_k so that, $T_\alpha(h) \geqq h - \varepsilon$ a.e. du. This yields $T(h) \geqq h - \varepsilon$. Letting $\varepsilon \to 0$, we have $T(h) \geqq h$ a.e. du.

Finally, by the condition 3′) on T, we can extend the map T onto the whole $L^1(dv)$. The extended linear map which is again denoted by T, of course satisfies the conditions of the theorem.

Corollary 1.5. *Under the same assumptions and notations as in the preceding theorem,* $T(\check{h}_P) \geqq \check{h}_P$ *holds for every* $\check{h}_P$ *of bounded real* h *with* $dom(h) \supseteqq \partial_P X$.

Proof. Since du and dv are regular, we can view $\check{h}_P$ as the lattice theoretic supremum of a family $\{g \in P: g \leqq h \text{ on } \mathrm{dom}(h)\}$ with respect to $du + dv$. So there is an increasing sequence $\{g_n\}_{n=1}^{\infty}$ of P such that $g_n \nearrow \check{h}_P$ in $L^1(du + dv)$. *This* implies that $g_n \leqq T(g_n) \nearrow T(\check{h}_P)$, and hence $\check{h}_P \leqq T(\check{h}_P)$ holds.

2. The state space of $B(W)$

Let $HB(W)$ be the Banach space of complex valued bounded harmonic functions on $W(\subset R^{2n})$ endowed with the uniform norm. In the sequel we shall assume that $B(W)$ is a linear subspace of $HB(W)$ with unit 1, and that $B(W)$ contains at least one nonconstant function. We should, however, point out that the quite same, or rather simple, arguments are also applicable when $B(W)$ is contained in $HB_R(W)$.

We will denote by $\mathrm{Re}\, B(W)$ the real linear space of the real parts of functions in $B(W)$. $\mathrm{Re}\, B(W)$ is a normed space with a unit, and has the canonical order ($\leqq$), i.e. $f \leqq g \Leftrightarrow f(z) \leqq g(z)$ for all $z \in W$. Let K be the state space of $\mathrm{Re}\, B(W)$. Under these circumstances, every state of $\mathrm{Re}\, B(W)$ acts on $B(W)$ as a linear form with a unit norm. So we can embed K into the unit ball of the dual $B(W)^*$. Namely $K = \{L \in B(W)^*: \|L\| = L(1) = 1\}$.

We take K as the space X of Section 1 when we view $B(W)$ merely as a linear space. In case that it is further a closed subalgebra of $H^{\infty}(W)$, we consider the maximal ideal space M_B as X. In the latter case, M_B is identified with the closed subset of K, the set of all multiplicative linear forms of K.

Each point of W acts on $B(W)$ as a point evaluation, and hence it can be regarded as a point of K. Under these identifications, every subset of W is also a subset of K which we call the canonical image of the set. Note that such identification is not injective in general. The canonical image of $E(\subseteqq W)$, however, will be denoted by the same symbol E, unless confusions arise. Since the weak* topology on W is coarser than the metric topology, the canonical image of any compact set in W is also compact in K. Specifically $W(\subset K)$ is σ-compact. Observe that the canonical image of W is contained in M_B when $B(W)$ is a closed subalgebra of $H^\infty(W)$.

Let $P(K)$ be the convex cone of $C_R(K)$ which consists of all continuous convex functions on K. $P(K)$ is a max-stable admissible cone. Further each function of $P(K)$ is uniformly approximated from below by a function of the form $\sup\{u_k \in \mathrm{Re}\, B(W)\colon 1 \leqq k \leqq n\}$. So a function $g \mid W (g \in P(K))$ is always a continuous subharmonic function on W with respect to the Euclidean metric.

An ordering over $M(K)$ relative to $P(K)$ is known as the Choquet's ordering on the compact convex set K. The $P(K)$-Choquet boundary then coincides with the usual Choquet boundary of K, the set of all extreme points of K. Following custom, we remove the prefix $P(K)$-, etc. from all notations concerning this order relation. In such a situation, the definition of the lower envelope $\check{f}$ is equivalent to saying that $\check{f} = \sup\{h \in \mathrm{Re}\, B(W)\colon h \leqq f \text{ on } \mathrm{dom}(f)\}$. So by Harnack's inequality, every $\check{f} \mid W$ is a continuous subharmonic function on W.

Now, when $B(W)$ is a closed subalgebra of $H^\infty(W)$, consider a function u on M_B which takes the form $u = \sum_{k=1}^{n} a_k \log|f_k|$, where $a_k \geqq 0$ and $f_k \in B(W)(1 \leqq k \leqq n)$. Set $J' = \{\sup(u_k\colon 1 \leqq k \leqq n)\}$ and $J = J' \cap C_R(K)$. J is a max-stable admissible cone on M_B, because J contains $\mathrm{Re}\, B(W)$. For any function g of J, $g \mid W$ is a continusous subharmonic function on W. Hence every J-lower envelope $\check{f}_J$ has the least harmonic majorant on W.

At the end of this section, one comment should be made concerning the Shilov boundary relative to J. First, observe that $\partial_J M_B$ contains ∂K, the Choquet boundary of K. This implies $S \subseteqq S_J$. On the other hand, every function of J attains its maximum value on S, because every function of $B(W)$ does the maximum modulus on S. Combining these, we conclude $S = S_J$.

3. Harmonic measures on K

The purpose of this section is to construct harmonic measures on K. For this we will consider the real Banach space $HB_R(W)$. It is known that $HB_R(W)$ is an order complete Banach lattice under the canonical order $(\leqq)$. Let H be the state space of $HB_R(W)$. H is of course weak* compact convex set and contains W canonically. The Choquet boundary ∂H of H consists of all lattice homomorphisms of $HB_R(W)$ with unit norm into the scalar. So ∂H is closed. Specifically we have that $HB_R(W) \mid \partial H = C_R(\partial H)$, and that every continuous linear form on $HB_R(W)$ is represented on ∂H by a uniquely determined measure. In other words, the state space H is a Bauer simplex. Let $\mathrm{d}w_z$ be a measure supported on ∂H which represents a point evaluation of $HB_R(W)$ at $z \in W$. Each $\mathrm{d}w_z$ is uniquely determined, and is a probability measure. We need one information about $\{\mathrm{d}w_z\colon z \in W\}$.

Lemma 3.1. (*Harnack's inequality*) *Let $HP(W)$ be a set of all positive harmonic functions on W. For any pair (z, x) of $W \times W$, the Cartesian product of two* *copies of W, define* $\rho(z, x)=\sup\{h(x)/h(z)\vee h(z)/h(x)\colon h\in HP(W)\}$. *Then* ρ *is a finite continuous function on* $W\times W$ *with* $\rho\geqq 1$ *and* $\rho(z, x)=\rho(x, z)$.

Recall that $\mathrm{Re}\,B(W)$ is a subspace of $HB_R(W)$ with a unit. So each point of H acts on $B(W)$ as if it belongs to K. More precisely, there exists a continuous map r of H into K such that $f(q)=f\{r(q)\}$ for all $f\in B(W)$ and any $q\in H$. Note that the map r is surjective. Define a probability measure $\mathrm{d}t_z$ on K by $t_z(E)=w_z\{r^{-1}(E)\}$, where E is any Borel set on K. The measure $\mathrm{d}t_z$ is a regular Borel measure on K for every $z\in W$. Further the family $\{\mathrm{d}t_z\colon z\in W\}$ satisfy the following: $\int \mathrm{d}t_z=\int g\circ r\mathrm{d}w_z$ for all $g\in C_R\{r(\partial H)\}$. In particular, $\int f\mathrm{d}t_z=f(z)$ for all $f\in B(W)$. Moreover, by Harnack's inequality, we have $\rho(z, x)^{-1}\int h\mathrm{d}t_z\leqq\int h\mathrm{d}t_x\leqq\rho(z, x)\int h\mathrm{d}t_z$ for all nonnegative $h\in C_R\{r(\partial H)\}$ and for all (z, x) of $W\times W$. The latter implies that $\rho(z, x)^{-1}\leqq \mathrm{d}t_x/\mathrm{d}t_z\leqq\rho(z, x)$. Choose any point p of W and fix it throughout. Set $H(z,\ )=\mathrm{d}t_z/\mathrm{d}t_p$ and $\mathrm{d}t=\mathrm{d}t_p$. Then $H(z,\ )$ satisfy $H(p,\ )=1$ and $\rho(z, x)^{-1}H(z,\ )\leqq H(x,\ )\leqq\rho(z, x)H(z,\ )$ a.e. $\mathrm{d}t$.

Remark. Harmonic measures $H(z,\ )\mathrm{d}t$ are supported on $r(\partial H)$ and hence weak* adherent to W in K. In case that $B(W)$ is a closed subalgebra of $H^\infty(W)$, this implies that $H(z,\ )\mathrm{d}t$ are supported on M_B.

Proposition 3.2. *For any* $f\in L^\infty(\mathrm{d}t)$, $\int fH(z,\ )\mathrm{d}t$ *is a bounded harmonic function on W whose maximum modulus coincides with* $\|f\|_\infty$. *For any bounded f with* $\mathrm{dom}(f)\supseteqq\partial K$, $\int \check{f}H(z,\ )\mathrm{d}t$ *is the least harmonic majorant of* $\check{f}|W$. *Similarly* $\int \check{f}_J H(z,\ )\mathrm{d}t$ *is the least harmonic majorant of* $\check{f}_J\,|\,W$, *where f is any bounded real function with* $\mathrm{dom}(f)\supseteqq\partial_J M_B$.

Proof. The former assertion is trivial. For the latter observe that $\check{f}\circ r$ is the supremum of the family in $HB_R(W)$: $\{h\in\mathrm{Re}\,B(W)\colon h\leqq f$ on $\mathrm{dom}(f)\}$. Thus we have the conclusion for the assertion on f. For the one about $\check{f}_J$, observe first that the least harmonic majorant of any function f in J is obtained by $\int fH(z,\ )\mathrm{d}t$. This yields the last assertion, because $\check{f}_J$ is the lattice theoretic supremum of a family $\{g\in J\colon g\leqq f$ on $\mathrm{dom}(f)\}$ with respect to $\mathrm{d}t$.

Theorem 3.3. *Every boundary measure on K (resp.* M_B*) with respect to* $\prec$ *(resp.* $\prec_J$*) has no mass on the canonical image of W.*

Proof. Since arguments are same, we will prove only for the order $\prec_J$. For the assertion, it suffices to show that there exists a function $h\in C_R(M_B)$ such that $\hat{h}_J\,|\,F>\check{h}_J\,|\,F$, where F is a given compact set on W.

First of all we shall prove that W is disjoint from ∂K. Assume some point $p\in W$ lies in ∂K. Since $B(W)$ is nontrivial, there are another point $q\in W$ and a function $f\in B(W)$ with $f(p)\neq f(q)$. Let g be a function of $C_R(K)$ such that $0\leqq g\leqq 1$,

$g(p)=1$ and $g(q)=0$. We have then $\check{g}(p)=g(p)=1$. Let $\{f_n\}$ be the sequence of $\mathrm{Re}\,B(W)$ such that $f_n \leqq g$ on K and $f_n(p)\to 1$. Since $\{f_n\}$ forms a normal family, we may assume $f_n \to f$ on every compact set of W, where f is a harmonic function. By $f_n \leqq g \leqq 1 (n \in N)$, we have $f \leqq 1$, $f(p)=1$ and $f(q)\leqq 0$. This is of course a contradiction. Thus we see that W is disjoint from ∂K.

Let F be any compact set of W and consider harmonic measures $H(z,\ )dt$. Then dt can not have full measure on F. In fact, assume the contrary, $t(F)=1$. For any $f \in B(W)$ we have $|f(z)| = \left|\int fH(z,\ )dt\right| \leqq \|f\|_F$, so that, $\|f\| = \|f\|_F$. This yiedls $S \subseteqq F$ and hence $\partial K \subseteqq F$, a contradiction. Thus we have $t(F)<1$. By the regularity of dt, there is a compact set E of M_B such that $t(E)>0$ and $E \cap F = \emptyset$. Let h be the function of $C_R(M_B)$ such that $0 \leqq h \leqq 1$, $h|F=1$ and $h|E=0$. From Harnack's inequalities for $H(z,\ )dt$, it follows $\int_E H(z,\ )dt>0$ so that, $\int hH(z,\ )dt<1$ for all $z \in W$. Hence we have $1 > \int \check{h}_J H(z,\ )dt \geqq \check{h}_J(z)$ for all $z \in W$. Thus the inequalities $\hat{h}_J|F = 1 > \check{h}_J|F$ hold, and these assert the theorem.

4. Singular harmonic measures

Using harmonic measures $H(z,\ )dt$, we define a singular harmonic measures supported on K, K being the state space of $B(W)$. Note that $H(p,\ )=1$ a.e. dt where p is any, but fixed, point of W.

Definition 4.1. We call a positive boundary measure dv a singular harmonic measure for $p \in W$, if it satisfies $dt \prec dv$.

Proposition 4.2. *There always exists a singular harmonic measure for every point of W. In particular it has no mass on W and is supported on S.*

Theorem 4.3. *Let dv be any singular harmonic measure for $p \in W$. Then there exists a positive kernel $Q(z,\ )$ of $L^\infty(dv)$ with a parameter $z \in W$. $Q(z,\ )dv$ satisfy the following.*

1) *For any $g \in L^\infty(dv)$, $\int gQ(z,\ )dv$ is a bounded harmonic function on W whose maximum modulus does not exceed $\|g\|_\infty$ and $Q(p,\ )=1$.*
2) *Every $Q(z,\ )dv$ is a singular harmonic measure for $z \in W$.*
3) *For any $h \in C_R(S)$, $\int hQ(z,\ )dv$ extends continuously to ∂K and coincides with g on it. So we have, in particular, $\left\|\int hQ(z,\ )dv\right\| = \|h\|$.*

Proof. Let T be the linear map of $L^1(dv)$ into $L^1(dt)$ constructed in Theorem 1.3. Since T is a positive map with $\|T\|_1 \leqq 1$, the adjoint map T^* of T is also positive and norm decreasing. Set $T^*(H(z,\ ))=Q(z,\ )$. Then we see $Q(z,\ )\geqq 0$, $Q(p,\ )=1$ and $\rho(z,x)^{-1}Q(z,\ ) \leqq Q(x,\ ) \leqq \rho(z,x)Q(z,\ )$ a.e. dv for all (z,x) of $W \times W$, because $H(z,\ )$ satisfy the same inequalities. We can easily verify that $Q(z,\ )dv$ satisfy the condition (1). For conditions (2) and (3) choose any $g \in C_R(S)$. Recall that $\check{g}|W$

is subharmonic and its least harmonic majorant is given by $\int \check{g}H(z,\ )dt$ (Proposition 3.2.) From Corollary 1.4, it follows $\int gQ(z,\ )dv=\int \check{g}Q(z,\ )dv=\int T(\check{g})H(z,\ )dt \geqq \int \check{g}H(z,\ )dt$ for every $z \in W$. Namely $\int gQ(z,\ )dv \geqq \check{g}(z)$ holds. Specifically if g is in $P(K)$, we have $\int gQ(z,\ )dv \geqq \int gH(z,\ )dt$, so that, $H(z,\ )dt \prec Q(z,\ )dv$. In other words, for every $z \in W$ $Q(z,\ )dv$ is a singular harmonic measure for z. In particular we see $\int fQ(z,\ )dv=\int fH(z,\ )dt=f(z)$ for all $f \in B(W)$, on account of the fact $\pm \mathrm{Re}\, B(W) \subset P(K)$.

From the above inequalities, $\hat{g} \geqq \int gQ(z,\ )dv$ follows, where g denotes any function of $C_R(S)$. Since $\check{g}$(resp. $\hat{g}$) is lower (resp. upper) semi-continuous on K, and coincides with g on ∂K, we conclude that $\int gQ(z,\ )dv$ extends continuously to ∂K and agrees with g on it.

Remark. The terminology of singular harmonic measure is far from suitable in this general situation.

By the similar arguments we can characterize positive measures induced by the Dirichlet problem on S. Let L be any solution of the Dirichlet problem on S. Namely L is a positive linear map from $C_R(S)$ into $HB_R(W)$ with $L(u)=u$ on W for all $u \in \mathrm{Re}\, B(W)$. By a measure for p induced by the Dirichlet problem on S we mean a probability measure on S corresponding to the functional $C_R(S) \ni g \rightarrow L(g)(p)$.

Theorem 4.4. *A probability measure* du *on S is a measure for p induced by the Dirichlet problem on S if and only if it satisfies* $dt \prec du$.

Proof. The sufficiency is contained in the proof of the preceding theorem. For the necessity, observe that $L(g) \geqq g$ on W, where L is the solution corresponding to du and $g \in P(K)$. This implies that $L(g) \geqq \int gH(z,\ )dt$ on W, and hence $dt \prec du$.

We define a J-singular harmonic measure on M_B by means of harmonic measures $H(z,\ )dt$ similarly to the preceding section.

5. J-Singular harmonic measures

In this section we shall assume that $B(W)$ is a closed subalgebra of $H^{\infty}(W)$ with a unit.

Definition 5.1. We call a positive boundary measure dv on M_B relative to $\prec_J$ a J-singular harmonic measure for $p \in W$ if it satisfies $dt \prec_J dv$.

Proposition 5.2. *There always exists a J-singular harmonic measure for every $z \in W$. Specifically it has no mass on W and is supported on S $(=S_J)$.*

Theorem 5.3. *Let* dv *be any singular harmonic measure for $p \in W$. Then there exists a positive kernel of* $L^{\infty}(dv)$ *as follows.*

1) *For any* $g \in L^{\infty}(\mathrm{d}v) \int gQ(z,\ )\mathrm{d}v$ *is a bounded harmonic function on* W *whose maximum modulus does not exceed* $\|g\|_{\infty}$ *and* $Q(p,\ )=1$.
2) *Every* $Q(z,\ )\mathrm{d}v$ *is a* J*-singular harmonic measure for* $z \in W$.
3) *For any* $h \in C_R(S)$, $\int hQ(z,\ )\mathrm{d}v$ *extends continuously to* $\partial_J M_B$ *(hence to* ∂K*), and coincides with* h *on it. In particular,* $\left\|\int hQ(z,\ )\mathrm{d}v\right\| = \|h\|$.
4) $Q(z,\ )\mathrm{d}v$ *is a Jensen measure for every* $z \in W$.

Proof. The same arguments as in the preceding section are available up to the condition 4). The last assertion is, however, easily verified; for every $f \in B(W)$, $(-n) \vee \log|f|\,(n \in N)$ belongs to J, so that, $\int (-n) \vee \log|f|Q(z,\ )\mathrm{d}v \geqq \int (-n) \vee \log\cdot |f|H(z,\ )\mathrm{d}t$. Letting $n \to \infty$, we conclude $\int \log|f|Q(z,\ )\mathrm{d}v \geqq \int \log|f|H(z,\ )\mathrm{d}t$, and hence $\int \log|f|Q(z,\ )\mathrm{d}v \geqq \log|f(z)|$.

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