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Invariant β and uniruled threefolds

Dedicated to Yoshikazu Nakai on his sixtieth birthday

By

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§ 0. Introduction.

In this paper, introducing a bimeromorphic invariant β, we shall make a systematic study of compact Kähler threefolds with $\kappa=-\infty$: For a compact complex manifold X, we put $\beta(X)=\mathrm{Max}\{\dim Y\,;\ Y$ is a compact complex manifold with $\kappa(Y)\geqq 0$ and there exists a generically surjective meromorphic map $f: X\to Y\}$. Then, if $\dim X\leqq 4$, one can naturally find a generically surjective meromorphic map $b_X: X\to B(X)$ such that $\beta(X)=\dim B(X)$ and that $f: X\to Y$ above always factors through $B(X)$, (cf. 1.1.3 and 1.1.4 of § 1). We now assume that X is a compact Kähler threefold. Then there exists a Zariski open dense subset U (resp. U') or $B(X)$ (resp. X) such that:

(a) $b_{X|U'}: U'\to b_X(U')=U$ is a proper morphism, and furthermore
(b) for every $u\in U$, the fibre $b_X^{-1}(u)$ is irreducible and $\beta(b_X^{-1}(u))=0$.

Thus, if $\beta(X)=1$ or 2, general fibres of b_X are rational. Therefore, it is naturally expected that problems of β can be translated into those of degenerations of rational curves or surfaces. In fact, via such translations, we shall prove:

(1) If $\pi: \tilde{X}\to X$ is a finite étale cover, then $\beta(\tilde{X})=\beta(X)$, (cf. Theorem 3.1.1).
(2) Let $g: Z\to S$ be a proper smooth morphism of Kähler manifolds such that $g^{-1}(s)=X$ for some $s\in S$. Assume that $\beta(X)=1$ or 2. Then $\beta(g^{-1}(s'))=\beta(X)$ for every $s'\in S$, (cf. Theorem 4.1.1).

Further results we obtained are the following:

(I) Let X be a compact Kähler uniruled threefold. Then
(I-a) $\beta(X)=0$ if and only if $q(X)=h^0(X, S^2(\Omega_X^2))=0$, (cf. Theorem 2.1.5);
(I-b) $\beta(X)=1$ if and only if $q(X)>h^0(X, S^{12}(\Omega_X^2))=0$;
(I-c) $\beta(X)=2$ if and only if $h^0(X, S^{12}(\Omega_X^2))\neq 0$;
(I-d) if $\pi: \tilde{X}\to X$ is a finite étale cover, then π naturally induces an étale cover $b(\pi): B(\tilde{X})\to B(X)$ with $\deg b(\pi)=\deg \pi$, (cf. Proposition 3.1.4);
(I-e) if $g: Z\to S$ is a proper smooth morphism of Kähler manifolds such that

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$g^{-1}(s)=X$ for some $s\in S$, then $\beta(g^{-1}(s'))=\beta(X)$ for every $s'\in S$, (cf. Theorem 4.1.1).

(II) (cf. Theorm 8.4.1). Let X be a compact Kähler threefold with $\kappa(X)\leqq 0$. Put $l=h^{2,0}(X)$. Let r be the rank of the subsheaf of Ω_X^2 generated by the global sections $H^0(X, \Omega_X^2)$, and we denote by $\Phi: X\rightarrow Gr(l, l-r)$ the meromorphic map defined generically by

$$\begin{aligned} \Phi: X &\longrightarrow Gr(l, l-r) \\ x &\longmapsto \{\omega\in H^0(X, \Omega_X^2)\,;\, \omega(x)=0\}\,. \end{aligned}$$

Then we have at least one of the following:

(a) $\beta(X)=2$; (b) $r=l=3$ and $\kappa(X)=0$; (c) $r=l\leqq 2$; (d) X is not uniruled with $r=2<l$ and the meromorphic image $\mathrm{Im}\,\Phi$ of Φ has dimension at least 2.

Notation and Convention.

(0.1) $\boldsymbol{Z}$=the set of all integers, $\boldsymbol{Z}_+$=the set of positive integers.

(0.2) $g.c.d.(\cdots)$ (resp. $l.c.m.(\cdots)$)=the greatest common divisor (resp. the least common multiple) of $\cdots$.

(0.3) A *complex variety* is an irreducible reduced complex space, and a *manifold* is a nonsingular complex variety. Note that manifolds are always connected. For a complex space X, we denote by X_{red} its underlying reduced complex space, and an analytic subset (resp. a subvariety) of X means a reduced (resp. an irreducible reduced) analytic subspace of X.

(0.4) For a compact complex manifold X, a triple $(f: W\rightarrow Y, g: W\rightarrow X, Y^0)$ is called a *covering family of rational curves on* X if the following conditions are satisfied:

i) both f and g are surjective morphisms of compact complex manifolds;
ii) Y^0 is a Zariski open dense subset of Y, and f is smooth over Y^0;
iii) for every $y\in Y^0$, the fibre $W_y(=f^{-1}(y))$ is isomorphic to $\boldsymbol{P}^1$ and $\dim g(W_y)=1$.

A compact complex manifold X is called *uniruled* if there exists a covering family of rational curves on X. It is known that a uniruled compact complex manifold X which is either Moishezon or a threefold of class $\mathcal{C}$ (cf. 1.2.1) always admits a covering family of rational curves $(f: W\rightarrow Y, g: W\rightarrow X, Y^0)$ on X such that g is generically finite.

(0.5) All modifications in this paper are assumed to be proper.

(0.6) For a (possibly open) n-dimensional complex manifold X, we write $\omega_X=\mathcal{O}_X(K_X)$ $(=\Omega_X^n)$, where K_X denotes the canonical bundle of X. If there is no fear of confusion, we use locally free sheaves and vector bundles interchangeably.

(0.7) Let $f: X\rightarrow Y$ be a proper morphism of complex varieties. For every analytic subspace F of Y, we denote by $f^{-1}(F)$ the analytic subspace of X obtained as the ideal-theoretic inverse image of F. In particular for every $y\in Y$, $f^{-1}(y)$ is the ideal-theoretic fibre of f over y. The set-theoretic inverse image of F is, however, denoted also by $f^{-1}(F)$, if no confusion seems likely to result. For every compact analytic cycle γ on Y (resp. X), we denote by $f^*(\gamma)$ (resp. $f_*(\gamma)$) the cycle-theoretic inverse (resp. direct) image of γ under the mapping f.

(0.8) Let $f: X \to S$ be a proper flat morphism of complex manifolds. Fix an arbitrary point s of S, and we write the cycle $f^*(s)$ as $\sum_{i=1}^{r} \mu_i \gamma_i$ with multiplicities $\mu_i \in \boldsymbol{Z}_+$ and irreducible reduced cycles γ_i on X. Then the fibre $X_s(=f^{-1}(s))$ is called a *multiple singular fibre* if $g.c.d.(\mu_1, \mu_2, \cdots, \mu_r)>1$.

(0.9) Let $f: X \to Y$ be a meromorphic map of compact complex varieties, and let F be a closed subvariety of X which is not contained in the set $S(f)$ of points of indeterminacy of f. Then the meromorphic image of F under the meromorphic map f denotes the closure of $f(F-S(f))$ in Y. The meromorphic image of X under the meromorphic map f is sometimes called the meromorphic image of f.

(0.10) "Closed" (resp. "open") means "closed (resp. open) in Euclidean topology" and is distinguished from "Zariski closed" (resp. "Zariski open").

(0.11) Let X and Y be complex spaces. Then $X \approx Y$ means that X and Y are bimeromorphic, and $X \cong Y$ means that X and Y are isomorphic, i.e., biholomorphic.

(0.12) We understand that the Kodaira dimension of a point is 0.

(0.13) Let $\mathcal{E}$ be a locally free sheaf on a complex variety X, and L be a linear subspace of the global sections $H^0(X, \mathcal{E})$ of $\mathcal{E}$. Then the subsheaf of $\mathcal{E}$ generated by L denotes the sheaf of $\mathcal{O}_X$-modules whose stalk at each point x of X is the $\mathcal{O}_{X,x}$-submodule of $\mathcal{E}_x$ generated by L.

(0.14) Let $\mathcal{Z}=\{Z_t\}_{t \in T}$ be an analytic family (of divisors, cycles, fibres, etc.) parametrized by a reduced complex space T. We say that general elements of $\mathcal{Z}$ have a property, if the property is possessed by every Z_t whose index t belongs to some countable intersection of Zariski open dense subsets of T.

In concluding this introduction, I wish to record my indebtedness to Professors A. Fujiki, M. Miyanishi and K. Ueno; their constant suggestions largely improved this paper. In particular, I learned from Fujiki several interesting facts on nonalgebraic compact complex manifolds and also on Douady spaces, (cf. 1.2.1, Step 2 of 8.3.1, and § 9); and following a suggestion of Miyanishi, I rewrote § 1 from a relative point of view, which fairly simplified the proofs of 1.3.4 and 1.3.9; I am heartily grateful to Ueno, and numerous stimulating discussions with him immensely influenced this paper, (cf. 2.3.1, 8.1.1, and § 5~§ 9). Note that the study of holomorphic 2-forms on threefolds was proposed by him [32], (cf. § 5~§ 8).

Thanks go also to Doctors T. Fujita and M. Ishida, with whom I was able to have fruitful conversations at Montréal. In particular, Fujita pointed out a gap in our original version of § 5 and 6, and Ishida gave me several interesting comments on § 2.

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§ 1. Basic fibrations.

The main purpose of this section is to define a couple of fibrations for compact complex manifolds of dimension 3 or 4. Our construction of fibrations essentially depends on the following: (i) a general reduction theory, (see for instance Fujiki [6]), and (ii) the subadditive property of the invariant κ of algebraic varieties, (see Iitaka [12] and Ueno [30] for standard facts on κ). In our actual treatment, the problem will be discussed more generally from a relative point of view. Throughout this section, S is assumed to be a (possibly open) complex variety.

Definition 1.1.1. (a) A complex variety X with a proper surjective morphism $\pi_X: X\to S$ whose general fibres are irreducible is said to be an *S-variety.* An S-variety X is called *nonsingular* if X is, just as a complex variety, nonsingular. For S-varieties $X, Y, Z, \cdots$, we denote the corresponding morphisms onto S by $\pi_X, \pi_Y, \pi_Z, \cdots$, and a morphism (resp. a meromorphic map, a bimeromorphic map) $f: X\to Y$ is called an *S-morphism* (resp. an *S-meromorphic map*, an *S-bimeromorphic map*) if $\pi_Y\circ f=\pi_X$. Generically surjective S-meromorphic maps $f: X\to Y$ and $\tilde{f}: \tilde{X}\to\tilde{Y}$ are called *S-bimeromorphically equivalent* if there exist S-bimeromorphic maps $i: X\to\tilde{X}$ and $j: Y\to\tilde{Y}$ such that $\tilde{f}\circ i=j\circ f$.

(b) Let X be an S-variety. We define: $\dim(X/S)=\dim X-\dim S$, $\kappa(X/S)=\kappa$(general fibre of π_X). If $\dim(X/S)=\kappa(X/S)$, then X is said to be *of general type over S.* We furthermore define:

$$\Xi_{X/S}=\left\{(Y, f)\,\middle|\, \begin{array}{l} f: X\to Y \text{ is a generically surjective} \\ S\text{-meromorphic map to an } S\text{-variety } Y \end{array}\right\},$$

$$\Lambda_{X/S}=\{(Y, f)\in\Xi_{X/S}\,;\,\kappa(Y/S)\geqq 0\},$$

$$\beta(X/S)=\operatorname{Max}\{\dim(Y/S)\,;\,(Y, f)\in\Lambda_{X/S}\},$$

$$\Lambda'_{X/S}=\{(Y, f)\in\Xi_{X/S}\,;\,Y \text{ is of general type over } S\},$$

$$\beta'(X/S)=\operatorname{Max}\{\dim(Y/S)\,;\,(Y, f)\in\Lambda'_{X/S}\}.$$

(c) For any two elements (Y_1, f_1), (Y_2, f_2) of $\Xi_{X/S}$, let $f_1*f_2: X\to Y_1\times_S Y_2$ be the S-meromorphic map defined generically by $(f_1*f_2)(x)=(f_1(x), f_2(x))\in Y_1\times_S Y_2$ with $x\in X$. We furthermore denote by Y_1*Y_2 the meromorphic image of f_1*f_2. Then (Y_1*Y_2, f_1*f_2) is naturally an element of $\Xi_{X/S}$.

Proposition 1.1.2. *Let X be an S-variety* with $\dim(X/S)\leqq 4$.

(a) *If (Y_1, f_1), $(Y_2, f_2)\in\Lambda_{X/S}$, then $(Y_1*Y_2, f_1*f_2)\in\Lambda_{X/S}$.*

(b) *If (Y_1, f_1), $(Y_2, f_2)\in\Lambda'_{X/S}$, then $(Y_1*Y_2, f_1*f_2)\in\Lambda'_{X/S}$.*

Proof. The assertion is straightforward in the following three cases: (1) $\dim(Y_1*Y_2/S)=\dim(Y_1/S)$ or $\dim(Y_2/S)$, (2) either $\dim(Y_1/S)$ or $\dim(Y_2/S)$ is 0, (3) $\dim(Y_1*Y_2/S)=\dim(Y_1/S)+\dim(Y_2/S)$ (and therefore $Y_1*Y_2=Y_1\times_S Y_2$).

In view of $\dim(X/S) \leqq 4$, the only remaining possibility is

$$\dim(Y_1 * Y_2/S) = \dim(Y_1/S)+1 \text{ or } \dim(Y_2/S)+1 .$$

Then by symmetry, we have only to consider the case $\dim(Y_1*Y_2/S)=\dim(Y_1/S)+1$. Replacing X, Y_1, Y_2 by their suitable nonsingular models, we may assume that both f_1 and f_2 are morphisms of complex manifolds. Let $\sigma : \widetilde{Y_1*Y_2} \to Y_1*Y_2$ be a desingularization of Y_1*Y_2, and $\overline{\overline{Y_1*Y_2}} \xrightarrow{\nu} \bar{\bar{Y}}_1 \to Y_1$ be the Stein factorization of $pr_1 \circ \sigma$, where $pr_i : Y_1*Y_2 \to Y_i$ $(i=1, 2)$ denote the natural projections. Note that both (Y_1, f_1) and (Y_2, f_2) are elements of $\Lambda_{X/S}$ (resp. $\Lambda'_{X/S}$). Applying Viehweg's theorem [26] to the morphism obtained by restricting ν to fibres over a general point of S, we see that

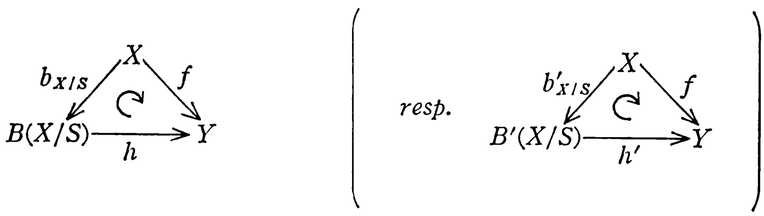
$$\kappa(Y_1*Y_2/S) \geqq \kappa(Y_1/S) + \kappa(\text{general fibre } F \text{ of } \nu),$$

and therefore the proof of (a) (resp. (b)) is reduced to showing

$$\kappa(F) \geqq 0 \text{ (resp. } F \text{ is of general type)}.$$

Since F sits over a single point of S, without loss of generality we may assume that S consists of a point. Choose an analytic slice Σ in $\bar{\bar{Y}}$ such that $\tilde{\Sigma} := \nu^{-1}(\Sigma)$ has the same dimension as Y_2 and that $pr_2 \circ \sigma_{|\tilde{\Sigma}} : \tilde{\Sigma} \to Y_2$ is of maximal rank at least at a point of $\tilde{\Sigma}$. Since $\kappa(Y_2)$ $(=\kappa(Y_2/S)) \geqq 0$ (resp. since Y_2 is of general type), there exists an $m \in \boldsymbol{Z}_+$ such that the complete linear system $|mK_{Y_2}|$ on Y_2 is nonempty (resp. such that the meromorphic image of $\Phi_m : Y_2$ $\boldsymbol{P}^N$ $(N=\dim |mK_{Y_2}|)$ associated with $|mK_{Y_2}|$ has the same dimension as Y_2). For simplicity, we put $\lambda = pr_2 \circ \sigma_{|\tilde{\Sigma}}$. Taking Σ general enough, we may assume that for a general point y of Σ, the fibre $\nu^{-1}(y)$ is smooth with the properties $(\lambda^*|mK_{Y_2}|)_{|\nu^{-1}(y)} \neq \emptyset$ (resp. $\dim \Phi_m(\lambda(\nu^{-1}(y))) = \dim \nu^{-1}(y)$) and $\kappa(\nu^{-1}(y)) = \kappa(F)$. Now by abuse of terminology, we have $\lambda^*|mK_{Y_2}| \subseteqq |mK_{\tilde{\Sigma}}|$, and hence in view of $(K_{\tilde{\Sigma}})_{|\nu^{-1}(y)} \cong K_{\nu^{-1}(y)}$, we conclude that: $\kappa(F) = \kappa(\nu^{-1}(y)) \geqq 0$ (resp. F as well as $\nu^{-1}(y)$ is of general type). Q.E.D.

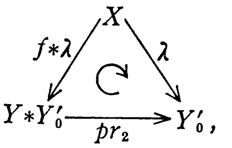
Proposition 1.1.3. *Let X be an S-variety with* $\dim(X/S) \leqq 4$. *Then there exists an element $(B(X/S), b_{X/S})$ (resp. $(B'(X/S), b'_{X/S})$) of $\Lambda_{X/S}$ (resp. $\Lambda'_{X/S}$), unique up to S-bimeromorphic equivalence, such that for every element (Y, f) of $\Lambda_{X/S}$ (resp. $\Lambda'_{X/S}$), we can find a generically surjective S-meromorphic map $h : B(X/S) \to Y$ (resp. $h' : B'(X/S) \to Y$) which makes the following diagram commutative:*



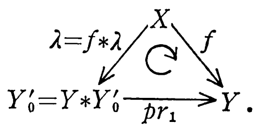
In particular, $\beta(X/S) = \dim(B(X/S)/S)$ (resp. $\beta'(X/S) = \dim(B'(X/S)/S)$).

Proof. Since the proofs are similar, we just consider the case $(Y, f) \in \Lambda_{X/S}$.

Choose $(Y_0, f_0) \in \Lambda_{X/S}$ such that $\dim(Y_0/S) = \beta(X/S)$. Replacing X by its suitable nonsingular model, we may assume that f and f_0 are S-morphisms. Let $X \xrightarrow{\lambda} Y'_0 \to Y_0$ be the Stein factorization of $f_0: X \to Y_0$. Then for every $(Y, f) \in \Lambda_{X/S}$, we have the commutative diagram of S-morphisms:



where $pr_2: Y*Y'_0 \to Y'_0$ denotes the natural projection. By 1.1.2, $(Y*Y'_0, f*\lambda) \in \Lambda_{X/S}$. Hence by our choice of Y_0, $\dim Y*Y'_0 = \dim Y'_0$, i.e., pr_2 is generically finite. Now for a general point y of Y'_0, the fibre $\lambda^{-1}(y)$ $(=(f*\lambda)^{-1}(pr_2^{-1}(y)))$ is irreducible, and therefore (degree of pr_2)$=1$. Thus, identifying $Y*Y'_0$ and Y'_0 via the bimeromorphic map pr_2 and denoting by $pr_1: Y*Y'_0 \to Y$ the natural projection, we obtain the following commutative diagram (modulo S-bimeromorphic equivalence):



Then $B(X/S) := Y'_0$ and $b_{X/S} := \lambda$ have the required properties. Q. E. D.

Remark 1.1.4. (a) If S is a single point then we denote $\beta(X/S)$, $\beta'(X/S)$, $(B(X/S), b_{X/S})$ and $(B'(X/S), b'_{X/S})$ simply by $\beta(X)$, $\beta'(X)$, $(B(X), b_X)$ and $(B'(X), b'_X)$ respectively. Note that $\beta(X) = \dim B(X)$ and $\beta'(X) = \dim B'(X)$ for $\dim X \leqq 4$. Throughout our paper, for compact complex varieties X of dimension $\leqq 4$, we assume the following rules:

i) *If* $\kappa(X) \geqq 0$, *then* $(B(X), b_X) = (X, id_X)$.

ii) *If* X *is of general type, then* $(B'(X), b'_X) = (X, id_X)$.

iii) $B(X)$ *is nonsingular unless* X *is a singular variety with* $\kappa(X) \geqq 0$.

iv) $B'(X)$ *is nonsingular unless* X *is at the same time singular and of general type.*

v) *If* $\dim X > \beta(X) = 2$, *then* $B(X)$ *is an absolutely minimal complex surface.*

vi) *If* $\dim X > \beta'(X) = 2$, *then* $B'(X)$ *is an absolutely minimal complex surface.*

(b) Let $g: V \to W$ be a surjective morphism of Moishezon manifolds with connected fibres. Then there are the following conjectures of Iitaka:

1) $\kappa(V) \geqq \kappa(W) + \kappa$(general fibre of g).

2) If $\kappa(W) \geqq 0$ and κ(general fibre of g)$\geqq 0$, then $\kappa(V) \geqq 0$.

3) If W and a general fibre of g are both of general type, then so is V.

Note that 1) implies 2) and 3). Let X be an S-variety such that general fibres of $\pi_X: X \to S$ (cf. 1.1.1) are Moishezon. In view of the proofs of Propositions 1.1.2 and 1.1.3, one easily sees that $(B(X/S), b_{X/S})$ (resp. $(B'(X/S), b'_{X/S})$) as in 1.1.3 exists even for $\dim(X/S)>4$, if the conjecture 2) (resp. 3)) above is true. Recently we hear that Viehweg [29] refined the result of Kawamata [17] and proved 1) above under the condition that W is of general type, though some of his arguments aren't clear. If we assume his result, then 3) is true, and in particular $(B'(X/S), b'_{X/S})$ as in 1.1.3 always exists (independently of $\dim(X/S)$) as long as general fibres of π_X are Moishezon.

(c) For simplicity, we assume that S is a single point. For every (possibly open) algebraic variety X, we put:

$$\bar{\Xi}_X = \left\{ (Y, f) \,\middle|\, \begin{array}{l} f: X \to Y \text{ is a dominant strictly rational} \\ \text{map of algebric varieties} \end{array} \right\},$$

$$\bar{\Lambda}_X = \{(Y, f) \in \bar{\Xi}_X \,;\, \bar{\kappa}(Y) \geqq 0\}, \qquad \bar{\Lambda}'_X = \{(Y, f) \in \bar{\Xi}_X \,;\, \bar{\kappa}(Y) = \dim Y\},$$

$$\bar{\beta}(X) = \operatorname{Max}\{\dim Y \,;\, (Y, f) \in \bar{\Lambda}_X\}, \quad \bar{\beta}'(X) = \operatorname{Max}\{\dim Y \,;\, (Y, f) \in \bar{\Lambda}'_X\},$$

where $\bar{\kappa}(Y)$ denotes the logarithmic Kodaira dimension of Y, (cf. Iitaka [13]). Since Viehweg's theorem [27] plays an essential role in the proofs of 1.1.2 and 1.1.3, in view of Kawamata's generalization [16] of this Viehweg's result to open varieties, we naturally expect that a theorem similar to 1.1.3 still holds even if we replace $\Lambda_{X/S}$, $\Lambda'_{X/S}$, $\beta(X/S)$, $\beta'(X/S)$ by $\bar{\Lambda}_X$, $\bar{\Lambda}'_X$, $\bar{\beta}(X)$, $\bar{\beta}'(X)$, Such a theorem actually holds, though at present we can show the uniqueness of $(\bar{B}(X), \bar{b}_X)$ and $(\bar{B}'(X), \bar{b}'_X)$ only up to strictly birational equivalence (not proper birational equivalence).

Proposition 1.1.5. *Let $f: X \to Y$ be a generically surjective S-meromorphic map of S-varieties. Then there exists a generically surjective S-meromorphic map $b(f): B(X/S) \to B(Y/S)$ (resp. $b'(f): B'(X/S) \to B'(Y/S)$), unique up to S-bimeromorphic equivalence, such that the following diagram commutes*:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle b_{X/S}}\downarrow & \circlearrowright & \downarrow{\scriptstyle b_{Y/S}} \\ B(X/S) & \xrightarrow{\ b(f)\ } & B(Y/S) \end{array} \qquad \left(\text{resp.} \quad \begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle b'_{X/S}}\downarrow & \circlearrowright & \downarrow{\scriptstyle b'_{Y/S}} \\ B'(X/S) & \xrightarrow{\ b'(f)\ } & B'(Y/S) \end{array} \right)$$

Corollary 1.1.6. *Let X be an S-variety with* $\dim(X/S) \leqq 4$. *Then the generically surjective S-meromorphic map $b_{X/S}: X \to B(X/S)$ naturally induces an S-bimeromorphic map $b'(b_{X/S}): B'(X/S) \xrightarrow{\approx} B'(B(X/S)/S)$.*

Proof of 1.1.5. The existence of $b(f)$ (resp. $b'(f)$) is straightforward from the universality of $B(X/S)$ (resp. $B'(X/S)$), (cf. 1.1.3), and their uniqueness easily follows from the commutativity of the diagrams. Q.E.D.

Proof of 1.1.6. Since $\kappa(B'(X/S)/S) \geqq 0$, the universality of $B(X/S)$ says that there exists a generically surjective S-meromorphic map $g: B(X/S)\to B'(X/S)$ with $b'_{X/S}=g\circ b_{X/S}$. Then the naturally induced meromorphic map $b'(g): B'(B(X/S)/S)\to B'(B'(X/S)/S)$ $(\approx B'(X/S))$ is the inverse of $b'(b_{X/S})$. Q.E.D.

(1.2) We shall next study basic properties of $\beta(X/S)$ and $\beta'(X/S)$. Recall the following definition of Fujiki [3]:

Definition 1.2.1. (a) A compact complex variety X is called *of class* $\mathcal{C}$ (or shortly $X\in\mathcal{C}$) if there exists a surjective morphism of a compact Kähler manifold onto X. This class of varieties is known to have good functorial properties, and furthermore for every compact complex manifold of class $\mathcal{C}$, its each complex cohomology group has a Hodge decomposition.
(b) A proper surjective morphism $f: Y\to Z$ is called *of class* $\mathcal{C}_{loc}$ if for every point of Z, there exist its open neighbourhood U, a Kähler manifold M_U, a proper morphism $g: M_U\to U$, and a surjective morphism $h: M_U\to f^{-1}(U)$ such that $g=f\circ h$. An S-variety X is called *of class* $\mathcal{C}_{loc}$ if the morphism $\pi_X: X\to S$ (cf. 1.1.1) is of class $\mathcal{C}_{loc}$.

Proposition 1.2.2. *Let X be an n-dimensional compact complex variety. Then*
(a) $0\leqq\beta'(X)\leqq\beta(X)\leqq n$. (b) *If* $\kappa(X)=-\infty$, $\beta(X)<n$. (c) *If* $\kappa(X)\geqq 0$, $\beta(X)=n$.
(d) *Let X be Moishezon. If* $\kappa(X)=0$, *then* $\beta'(X)=0$.
(e) *Let* $X\in\mathcal{C}$ *and* $\dim X\leqq 3$. *If* $\kappa(X)=0$, *then* $\beta'(X)=0$.

Proof. (a), (b), and (c) are obvious. Assume that $\kappa(X)=0$, and we consider (d) and (e) at the same time. Let $f: X\to Y$ be a generically surjective meromorphic map to a compact complex manifold Y of general type. Replacing X by its suitable nonsingular model, we may assume that f is a morphism. If X is Moishezon (resp. If $X\in\mathcal{C}$ and $\dim X\leqq 3$), then a theorem of Kawamata [17] (resp. Viehweg [28]) asserts that

$$0=\kappa(X)\geqq\kappa(Y)+\kappa(\text{general fibre of } f)=\dim Y+\kappa(\text{general fibre of } f)\,.$$

Since we have an easy inequality $\kappa(X)\leqq\dim Y+\kappa(\text{general fibre of } f)$, it now follows that $\dim Y=0$. Thus $\beta'(X)=0$. Q.E.D.

Proposition 1.2.3. *Let $g: X\to Z$ be a surjective S-morphism with connected fibres between nonsingular S-varieties X and Z. Suppose that β (general fibre of g) (resp. β'(general fibre of g)) is 0. Then for every element (Y, f) of $\Lambda_{X/S}$ (resp. $\Lambda'_{X/S}$), the S-meromorphic map f factors through Z, i.e., $f=h\circ g$ for some generically surjective S-meromorphic map $h: Y\to Z$. In particular, we have:*
(a) $\beta(X/S)=\beta(Y/S)$ (*resp.* $\beta'(X/S)=\beta'(Y/S)$);
(b) *if* $\dim(X/S)\leqq 4$, *then* $b(g): B(X/S)\to B(Y/S)$ (*resp.* $b'(g): B'(X/S)\to B'(Y/S)$) *is an S-bimeromorphic map.*

Proof. Replacing X by its suitable S-bimeromorphic model, we may assume that f is a morphism. Since both (Z, g) and (Y, f) belong to $\Lambda_{X/S}$ (resp. $\Lambda'_{X/S}$), so does $(Z*Y, g*f)$, (cf. 1.1.2). Let $\sigma: \overline{\overline{Z*Y}} \to Z*Y$ be a desingularization of $Z*Y$, and we denote by $\overline{\overline{Z*Y}} \xrightarrow{\nu} \bar{\bar{Z}} \xrightarrow{\mu} Z$ the Stein factorization of $pr_1 \circ \sigma$, where $pr_1: Z*Y \to Z$ is the natural projection. Note that, for a general point z of Z, $pr_1^{-1}(z) = \{(z, y) \in Z \times Y; y \in f(g^{-1}(z))\}$ $(\cong f(g^{-1}(z)))$ is irreducible. Therefore μ is a modification, and furthermore $\nu^{-1}(\mu^{-1}(z))$ and $f(g^{-1}(z))$ are bimeromorphic for z as above. Thus

$$\kappa(f(\text{general fibre of } g)) = \kappa(\text{general fibre } F \text{ of } \nu),$$

where the right-hand side was already shown to be nonnegative (resp. $\dim F$) in the proof of 1.1.2. Then in view of the fact that β(general fibre of g) (resp. β'(general fibre of g)) is 0, it follows that f(general fibre of g) consists of a single point. We hence conclude that f factors S-meromorphically through Z. (a) and (b) above are now straightforward. Q. E. D.

(1.3) Our last task in this section is to study "general fibres" of $b_X: X \to B(X)$ and $b'_X: X \to B'(X)$ for compact complex manifolds X of class $\mathcal{C}$ with $\dim X \leqq 3$.

Lemma 1.3.1. *Let X be a compact complex manifold with $\beta(X)=0$.*
(a) *If* $\dim X=1$, *then* $X \cong \boldsymbol{P}^1$. (b) *If* $X \in \mathcal{C}$ *and* $\dim X=2$, *then X is rational.*
(c) *If* $X \notin \mathcal{C}$ *and* $\dim X=2$, *then X is a surface of class* VII *with* $\kappa=-\infty$.

proof. (a) is obvious. Note that, by $\beta(X)=0$, we have $\kappa(X)=-\infty$. Then (b) and (c) are straightforward from the classification table of Kodaira [21].

Proposition 1.3.2. *Let $g: V \to W$ be a generically surjective meromorphic map of compact complex varieties, where V is nonsingular and W satisfies $\kappa(W) \geqq 0$. Assume that at least one of the following holds*:
(a) $\dim W \leqq 1$, (b) $\dim V - \dim W \leqq 1$, (c) $\dim V \leqq 3$.
Then there exists a Zariski open dense subset U (resp. U^*) of W (resp. V) such that $g_{|U^*}: U^* \to g(U^*) = U$ is a proper morphism.

Proof. If (c) holds, then so does at least one of (a) and (b). On the other hand, it is a standard fact that if (a) holds, then $g: V \to W$ itself is a morphism. Hence we have only to consider the case (b). Now by Hironaka [10], there exists a finite sequence of monoidal transformations with nonsingular centres

$$V_n \xrightarrow{\mu_n} V_{n-1} \xrightarrow{\mu_{n-1}} V_{n-2} \xrightarrow{\mu_{n-2}} \cdots \xrightarrow{\mu_2} V_1 \xrightarrow{\mu_1} V_0 = V$$

such that the composite $g \circ \mu$ of g with $\mu := \mu_n \circ \mu_{n-1} \circ \cdots \circ \mu_1$ is a morphism of V_n onto W. Let $E \subseteqq V_n$ be the exceptional locus for the modification μ, i.e., E is a purely 1-codimensional closed analytic subset of V_n with $\operatorname{codim}_V \mu(E) \geqq 2$ and $\mu_{|V_n - E}: V_n - E \cong V - \mu(E)$. Since every irreducible component of E is uniruled, and since $\kappa(W) \geqq 0$, we have $(g \circ \mu)(E) \subsetneqq W$. Then $U := W - (g \circ \mu)(E)$ and $U^* := \mu((g \circ \mu)^{-1}(U))$ have the required property. Q. E. D.

Theorem 1.3.3. *Let X be a nonsingular S-variety of class $\mathcal{C}_{loc}$. Assume that* $\dim(X/S)\leqq 2$. *Let $b_{X/S}: X\to B(X/S)$ be as in* 1.1.3, *and for each $s\in S$, we put $X_s=\pi_X^{-1}(s)$ and $B(X/S)_s=(\pi_{B(X/S)})^{-1}(s)$,* (cf. 1.1.1). *Then there exists a Zariski open dense subset U of S having the following properties:*

(a) *π_X is smooth over U;*
(b) *for every point s of U, the fibre X_s is not contained in the set of points of indeterminacy of $b_{X/S}$;*
(c) *there is a natural bimeromorphic identification of $B(X/S)_s$ with $B(X_s)$ such that the restriction $(b_{X/S})_{|X_s}: X_s\to B(X/S)_s$ coincides with $b_{X_s}: X_s\to B(X_s)$.*

In particular $\beta(X_s)=\beta(X/S)$ for every $s\in U$.

Proof. Fix a smooth fibre X_o of π_X, $(o\in S)$. First, let $\beta(X_o)=0$: Then by $X_o\in\mathcal{C}$, X_o is rational. Since π_X is of class $\mathcal{C}_{loc}$, every smooth fibre X_s of π_X is again rational, and therefore we may set $B(X/S)=S$ and $b_{X/S}=\pi_X$. Secondly, we consider the case $\beta(X_o)=\dim X_o$: Then $\kappa(X_o)\geqq 0$, and by Iitaka [11], $\kappa(X_s)\geqq 0$ for every smooth fibre X_s of π_X. Hence in this case, we may set $B(X/S)=X$ and $b_{X/S}=id_X$. Thus, in both cases, the assertion of 1.3.3 is straightforward. We now consider the following remaining case:

X_o is a surface with $\beta(X_o)=1$, (i.e., X_o is an irrational ruled surface).

Let $U=\{s\in S; X_s \text{ is smooth}\}$, Then every X_s $(s\in U)$ is also an irrational ruled surface. Let $\alpha_s: X_s\to \mathrm{Alb}(X_s)$ be the Albanese map. Since $\dim(X/S)\leqq 2$, a theorem of Fujiki [4] says that there exist an S-bimeromorphic model X' of X and a relative Albanese variety $\mathrm{Alb}(X'/S)$ such that

(1) over U, the S-variety X' contains $\pi_X^{-1}(U)$ as a Zariski open subset,
(2) $A:=\mathrm{Alb}(X'/S)$ is an S-variety, and furthermore
(3) there is an S-morphism $\alpha: X'\to A$ satisfying $\pi_A^{-1}(s)=\mathrm{Alb}(X_s)$ and $\alpha_{|X_s}=\alpha_s$ for all $s\in U$.

Then the fibration $\pi_{A|\alpha(X')}:\alpha(X')\to S$ has a nonsingular curve of positive genus as a general fibre. Since general fibres of $\alpha: X'\to\alpha(X')$ are isomorphic to $\boldsymbol{P}^1$, we may now assume that $(B(X/S), b_{X/S})=(\alpha(X'), \alpha)$, where α is regarded as a meromorphic map from X $(\approx X')$ to $\alpha(X')$. One can easily check that (a), (b), and (c) above are all satisfied. Q. E. D.

Theorem 1.3.4. *Let X be a compact complex manifold of class $\mathcal{C}$ with* $\dim X\leqq 3$. *Then there exists a Zariski open dense subset U (resp. U^*) of $B(X)$ (resp. X) such that:*

(a) *$b_{X|U^*}: U^*\to b_X(U^*)=U$ is a proper smooth morphism, and furthermore*
(b) *for every $u\in U$, the fibre $X_u=(b_{X|U^*})^{-1}(u)$ is irreducible and $\beta(X_u)=0$.*

Proof. In view of 1.3.2, it suffices to show, assuming b_X to be a morphism, that X_u is irreducible and satisfies $\beta(X_u)=0$ for all points u of some Zariski open dense subset U of $B(X)$. This is obvious if $\beta(X)=0$, and hence we may assume that $\beta(X)>0$. Note that, by the universality of $B(X)$, every smooth fibre of b_X is irreducible. Putting $S=B(X)$, we have $\dim(X/S)\leqq 2$. Then by Theorem 1.3.3, there exists a Zariski open dense subset U of S such that $\beta(X_u)=\beta(X/S)$ for every $u\in U$. Now by Viehweg [28], $\kappa(B(X/S))\geqq\kappa(S)\geqq 0$, and hence $B(X/S)$ is bimeromorphic to S. Thus $\beta(X/S)=0$, and this completes the proof. Q. E. D.

Definition 1.3.5. (i) Let X be an S-variety. Choose a nonsingular model $\tilde{S}$ (resp. $\tilde{X}$) of S (resp. X) such that $\pi_X: X\to S$ induces a morphism $\tilde{\pi}_{\tilde{X}}: \tilde{X}\to\tilde{S}$. Denoting by $\omega_{\tilde{X}/\tilde{S}}$ the relative canonical sheaf $\omega_{\tilde{X}}\otimes\tilde{\pi}_{\tilde{X}}^*(\omega_{\tilde{S}}^{-1})$ of $\tilde{X}$ over $\tilde{S}$, we naturally have a meromorphic map $\Phi_m: \tilde{X}\to \boldsymbol{P}((\tilde{\pi}_{\tilde{X}})_*(\omega_{\tilde{X}/\tilde{S}}^{\otimes m}))$ for each $m\in\boldsymbol{Z}_+$. Let s be a general point of S. Identifying s with the corresponding point in $\tilde{S}$, we have the restriction of Φ_m to the fibres over s:

$$\Phi_{m|\tilde{X}_s}: \tilde{X}_s \longrightarrow \boldsymbol{P}((\tilde{\pi}_{\tilde{X}})_*(\omega_{\tilde{X}/\tilde{S}}^{\otimes m}))_s(=\boldsymbol{P}(H^0(\tilde{X}_s, \omega_{\tilde{X}_s}^{\otimes m}))).$$

There now exists an m_0 such that, for all general points s of S, $\Phi_{m_0|\tilde{X}_s}$ is bimeromorphically equivalent to the Iitaka fibration of X_s. Let Y be the meromorphic image of Φ_{m_0}. Then the generically surjective S-meromorphic map $\Phi_{m_0}: X(\approx\tilde{X})\to Y$ is called the *relative Iitaka fibration of X/S* (see, for instance, Ashikaga-Ueno [1]).

(ii) A compact complex variety X is said to be *purely non-hyperbolic* if there exists a sequence of surjective morphisms of compact complex manifolds

$$f_i: X_{i-1} \longrightarrow X_i, \qquad i=1, 2, \cdots, r,$$

such that the following conditions are satisfied:

1) X_0 is bimeromorphic to X, and X_r is a point;
2) general fibres of each f_i are irreducible;
3) if i is such that κ(general fibre of f_i)$\neq 0$, then β(general fibre of f_i)$=0$.

Lemma 1.3.6. *Let X be a purely non-hyperbolic compact complex variety. Then $\beta'(X)$ is 0 if either X is Moishezon or X satisfies both $X\in\mathcal{C}$ and* $\dim X\leqq 3$.

Proof. In view of (a), (d), and (e) of 1.2.2, the assertion is an immediate consequence of (a) of Proposition 1.2.3. Q. E. D.

Proposition 1.3.7. *Let X be an S-variety such that one of the following conditions is satisfied:*

(a) $\dim(X/S)\leqq 4$ *and general fibres of π_X are Moishezon;*
(b) $\dim(X/S)\leqq 3$ *and X is, as an S-variety, of class $\mathcal{C}_{loc}$.*

Then there exists a sequence of generically surjective S-meromorphic maps

$$f_i: X_{i-1} \longrightarrow X_i, \qquad i=1, 2, \cdots, r,$$

of S-varieties such that

1) $X_0=X$ *and* $\dim X_{i-1}>\dim X_i$ *for each i,*

2) *X_r is of general type over S,*
3) *if i is such that $\kappa(X_{i-1}/S)=-\infty$, then $(X_i, f_i)=(B(X_{i-1}/S), b_{X_{i-1}/S})$, and*
4) *if i is such that $\kappa(X_{i-1}/S)\geqq 0$, then $f_i: X_{i-1}\to X_i$ is the relative Iitaka fibration of X_{i-1}/S.*

Such a sequence of meromorphic maps is unique up to bimeromorphic equivalence. Furthermore, for each i, $f_r\circ f_{r-1}\circ\cdots\circ f_i: X_{i-1}\to X_r$ is bimeromorphically equivalent to $b'_{X_{i-1}/S}: X_{i-1}\to B'(X_{i-1}/S)$.

Proof. Note that, by 3) and 4) above, a sequence as above is well-defined and unique up to bimeromorphic equivalence, (cf. 1.1.3 and (i) of 1.3.5). Since $\dim X<+\infty$, such a sequence stops exactly when a variety $(=X_r)$ of general type over S first comes up. Applying Corollary 1.1.6 to the case $\kappa(X_{i-1}/S)=-\infty$ and Proposition 1.2.3 (b) to the case $\kappa(X_i/S)\geqq 0$ (see also (d) and (e) of 1.2.2), we see that $B'(X_0/S)\approx B'(X_1/S)\approx\cdots B'(X_r/S)$ $(\approx X_r)$ S-bimeromorphically. Thus $f_r\circ f_{r-1}\circ\cdots\circ f_i$ and $b'_{X_{i-1}/S}$ are bimeromorphically equivalent for each i.

Q. E. D.

Tneorem 1.3.8. *Let X be a compact complex manifold of class C with* $\dim X\leqq 3$. *Then $\beta'(X)=0$ if and only if X is purely non-hyperbolic.*

Proof. This is straightforward from Lemma 1.3.6 and also from Proposition 1.3.7 applied to the case that both S and $B'(X)$ $(\approx X_r)$ consist of a point.

Q. E. D.

Theorem 1.3.9. *Let X be a compact complex manifold of class C with* $\dim X\leqq 3$. *Then there exists a Zariski open dense subset U (resp. U^*) of $B'(X)$ (resp. X) such that:*

(a) *$b'_{X|U^*}: U^*\to b'_X(U^*)=U$ is a proper smooth morphism, and furthermore*
(b) *for every $u\in U$, the fibre $X_u=(b'_{X|U^*})^{-1}(u)$ is irredubible and $\beta'(X_u)=0$.*

Proof. By the same argument as in the former half of the proof of 1.3.4, we may assume that 1) b'_X is a morphism, and that 2) $\beta'(X)>0$. Moreover, it is sufficient to show that $\beta'(X_u)=0$ for all points u of some Zariski open dense subset U of $B'(X)$. We now apply Proposition 1.3.7 to the case: S is a point. Replacing X by its suitable bimeromorphic model, we may assume that every f_i in 1.3.7 is a morphism of compact complex manifolds. Since $\beta'(X)>0$, we furthermore obtain $\dim X_i-\dim X_{i-1}\leqq 2$ for all i. In view of Theorem 1.3.4 and also of the deformation invariance of κ for compact complex surfaces, (cf. Iitaka [11]), there now exists a Zariski open dense subset U of $B'(X)$ such that X_u is purely non-hyperbolic for every point u of U. Then by 1.3.8, the proof of 1.3.9 is complete. Q. E. D.

Theorem 1.3.10. *Let X be a nonsingular S-variety of class C_{loc}. Assume that* $\dim(X/S)\leqq 2$. *Considering the meromorphic map $b'_{X/S}: X\to B'(X/S)$, we put $X_s=\pi_X^{-1}(s)$ and $B'(X/S)_s=(\pi_{B(X/S)})^{-1}(s)$ for each $s\in S$. Then there exists a Zariski* *open dense subset U of S having the following properties:*

(a) *π_X is smooth over U;*

(b) *for every point s of U, the fibre X_s is not contained in the set of points of indeterminacy of $b'_{X/S}$;*

(c) *there is a natural bimeromorphic identification of $B'(X/S)_s$ with $B'(X_s)$ such that the restriction $(b'_{X/S})_{|X_s}: X_s \to B'(X/S)_s$ coincides with $b'_{X_s}: X_s \to B'(X_s)$.*

In particular $\beta'(X_s)=\beta'(X/S)$ for every $s \in U$.

Proof. In view of 1.3.3 and (i) of 1.3.5, this is straightforward from 1.3.7 above. Q.E.D.

In the appendix (cf. §10), we shall generalize Theorems 1.3.3 and 1.3.10 to the case $\dim(X/S) \leqq 3$ with a slight modification of the statement.

§2. Holomorphic differential forms on compact complex threefolds.

For a compact complex threefold X of class $\mathcal{C}$, we consider the fibration $b_X: X \to B(X)$ defined in §1. One then naturally asks how many of the holomorphic differential forms on X come from $B(X)$. The answer is

Theorem 2.1.1. *Let X be a 3-dimensional compact complex manifold of class $\mathcal{C}$ with $\beta(X)>0$. Then for all $m, p \in \boldsymbol{Z}_+$, the meromorphic map $b_X: X \to B(X)$ induces the following isomorphisms:*

(1) $b_X^*: H^0(B(X), \overset{m}{\otimes} \Omega^p_{B(X)}) \cong H^0(X, \overset{m}{\otimes} \Omega^p_X)$,

(2) $b_X^*: H^0(B(X), S^m(\Omega^p_{B(X)})) \cong H^0(X, S^m(\Omega^p_X))$.

Remark 2.1.2. (i) For the case $m \geqq 2=\beta(X)$, the original statement of our theorem was much weaker than the above. We owe the present improvement to Professor Ueno, (cf. 2.3.1).

(ii) The case $m>1=\beta(X)$ remained open until recently. Thanks to a result of Kawamata [18], we managed to finish this case.

(iii) Let $m, p \in \boldsymbol{Z}_+$, and we put $m'=m \cdot p$. Since both $\overset{m}{\otimes} \Omega^p_X$ and $S^m(\Omega^p_X)$ are vector subbundles of $\overset{m'}{\otimes} \Omega^1_X$, the proof of Theorem 2.1.1 is reduced to showing just the case $p=1$ of (1) above.

It is plausible that Theorem 2.1.1 is true even in the case $\beta(X)=0$. Thus we raise the following:

Conjecture 2.1.3. *Let X be a 3-dimensional compact complex manifold of class $\mathcal{C}$. If $\beta(X)=0$, then $h^{2,0}(X)=0$.*

This conjecture is true, for instance, if one can prove:

Conjecture 2.1.4. *Let X be a 3-dimensional compact complex manifold of* *class C with $\kappa(X)=-\infty$. Then X is uniruled.*

Because we generally have:

Theorem 2.1.5. *Let X be a 3-dimensional uniruled compact complex manifold of class C, and let $q(X)$ denote the irregularity of X. Then:*

i) *If $\beta(X)=0$, we have $h^0(X, S^m(\Omega_X^2))=0$ for all $m\in \boldsymbol{Z}_+$.*
ii) *$\beta(X)=0$ if and only if $q(X)=h^0(X, S^2(\Omega_X^2))=0$.*
iii) *$\beta(X)=1$ if and only if $q(X)>h^0(X, S^{12}(\Omega_X^2))=0$.*
iv) *$\beta(X)=2$ if and only if $h^0(X, S^{12}(\Omega_X^2))\neq 0$.*

(2.2) In order to prove 2.1.1 and 2.1.5, we shall study holomorphic differential forms in a more general setting:

Definition 2.2.1 Let $f: X\to Y$ be a surjective morphism of compact complex manifolds with connected fibres. Fix an arbitrary $m\in \boldsymbol{Z}_+$. For every purely 1-codimensional closed analytic subset $\Delta(=\bigcup_{i=1}^r \Delta_i)$ of Y with its irreducible components Δ_i, we define an effective divisor $\Delta_{f;m}\in \mathrm{Div}(Y)$ as follows: Express each $f^*(\Delta_i)$ as $\sum_{j=1}^{n_i}\mu_{ij}E_{ij}$ with multiplicities $\mu_{ij}\in \boldsymbol{Z}_+$ and prime divisors $E_{ij}\in \mathrm{Div}(X)$. We number E_{ij}; $j=1, 2, \cdots, n_i$, so that $\{j; f \text{ maps } E_{ij} \text{ onto } \Delta_i\}=\{j\in \boldsymbol{Z}; 1\leqq j\leqq k_i\}$ for some integer k_i with $1\leqq k_i\leqq n_i$. Let $\nu_{i;m}$ be $\mathrm{Min}\{[m(\mu_{ij}-1)/\mu_{ij}]; 1\leqq j\leqq k_i\}$, where for every real number λ, the symbol $[\lambda]$ denotes the largest integer which does not exceed λ. We now put $\Delta_{f;m}=\sum_{i=1}^r \nu_{i;m}\Delta_i$.

Proposition 2.2.2. *Let $f: X\to Y$ be a surjective morphism of compact complex manifolds with connected fibres, and Δ be a purely 1-codimensional closed analytic subset of Y. Let $m, p\in \boldsymbol{Z}_+$. Suppose $\omega\in H^0(X, S^m(\Omega_X^p))$ (resp. $\omega\in H^0(X, \overset{m}{\otimes}\Omega_X^p)$) is such that $\omega_{|X-f^{-1}(\Delta)}$ is expressible as $f^*(\theta)$ for some $\theta\in H^0(Y-\Delta, S^m(\Omega_Y^p))$ (resp. $\theta\in H^0(Y-\Delta, \overset{m}{\otimes}\Omega_Y^p)$). Then θ extends to $\theta'\in H^0(Y, S^m(\Omega_Y^p)(\Delta_{f;m}))$ (resp. $\theta'\in H^0(Y, (\overset{m}{\otimes}\Omega_Y^p)(\Delta_{f;m}))$) with $\omega=f^*(\theta')$, where $H^0(Y, S^m(\Omega_Y^p)(\Delta_{f;m}))$ (resp. $H^0(Y, (\overset{m}{\otimes}\Omega_Y^p)(\Delta_{f;m}))$) denotes the space of all those meromorphic sections to $S^m(\Omega_Y^p)$ (resp. $\overset{m}{\otimes}\Omega_Y^p$) on Y whose possible poles are only along Δ and of order at most $\nu_{i;m}$ at each Δ_i.*

Corollary 2.2.3. *Let $f: X\to Y$ be a surjective morphism of compact complex manifolds with connected fibres. Let U be a Zariski open nonempty subset of Y, and $\theta\in H^0(U, \Omega_Y^p)$ ($p\in \boldsymbol{Z}_+$) be such that $f^*(\theta)$ ($\in H^0(f^{-1}(U), \Omega_X^p)$) extends to a global holomorphic section $\omega\in H^0(X, \Omega_X^p)$. Then θ extends to a global section $\theta'\in H^0(Y, \Omega_Y^p)$ such that $\omega=f^*(\theta')$.*

Proof of 2.2.2. Since $S^m(\Omega_X^p)$ and $S^m(\Omega_Y^p)$ are regarded as vector subbundles of $\overset{m}{\otimes}\Omega_X^p$ and $\overset{m}{\otimes}\Omega_Y^p$ respectively, we may just consider the case $\omega\in H^0(X, \overset{m}{\otimes}\Omega_X^p)$.

Now, using the notation in 2.2.1, we fix a general point e_{ij} of E_{ij} with $1\leqq i\leqq r$ and $1\leqq j\leqq k_i$. Choose a sufficiently small open neighbourhood V of e_{ij} in X (resp. U of $f(e_{ij})$ in Y) with a system of local coordinates $(x_1, \cdots, x_l)$ (resp. $(y_1, \cdots, y_n)$) such that (1) $x_1=0$ (resp. $y_1=0$) is the local equation of E_{ij} in V (resp. Δ_i in U), (2) $f(V)=U$, (3) $f^*(y_1)=(1/\mu_{ij})\cdot x_1^{\mu_{ij}}$, and (4) $f^*(y_\alpha)=x_\alpha$ for $2\leqq\alpha\leqq n$, (where $n=\dim Y$ and $l=\dim X$). Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the set of all subsets of $\{1, 2, \cdots, l\}$ (resp. $\{1, 2, \cdots, n\}$) of cardinality p, and $\mathcal{A}^m$ (resp. $\mathcal{B}^m$) be the product $\mathcal{A}\times\cdots\times\mathcal{A}$ (resp. $\mathcal{B}\times\cdots\times\mathcal{B}$) of m-copies of $\mathcal{A}$ (resp. $\mathcal{B}$). For each $A=\{\alpha_1, \cdots, \alpha_p\}\in\mathcal{A}$ (resp. $B=\{\beta_1, \cdots, \beta_p\}\in\mathcal{B}$) with $\alpha_1<\cdots<\alpha_p$ (resp. $\beta_1<\cdots<\beta_p$), we put

$$dx_A=dx_{\alpha_1}\wedge dx_{\alpha_2}\wedge\cdots\wedge dx_{\alpha_p}\quad (\text{resp. } dy_B=dy_{\beta_1}\wedge dy_{\beta_2}\wedge\cdots\wedge dy_{\beta_p}).$$

We finally define $dX_A\in H^0(V, \overset{m}{\otimes}\Omega_X^p)$ (resp. $dY_B\in H^0(U, \overset{m}{\otimes}\Omega_Y^p)$) by

$$dX_A=dx_{A_1}\otimes dx_{A_2}\otimes\cdots\otimes dx_{A_m}\quad (\text{resp. } dY_B=dy_{B_1}\otimes dy_{B_2}\otimes\cdots\otimes dy_{B_m})$$

for every $\boldsymbol{A}=(A_1, A_2, \cdots, A_m)\in\mathcal{A}^m$ (resp. $\boldsymbol{B}=(B_1, B_2, \cdots, B_m)\in\mathcal{B}^m$). Then $\{dX_{\boldsymbol{A}}; \boldsymbol{A}\in\mathcal{A}^m\}$ (resp. $\{dY_{\boldsymbol{B}}; \boldsymbol{B}\in\mathcal{B}^m\}$) forms a local base over U for the vector bundle $\overset{m}{\otimes}\Omega_X^p$ (resp. $\overset{m}{\otimes}\Omega_Y^p$), and the Laurent expansion of θ is given by

$$\theta=\sum_{\boldsymbol{B}\in\mathcal{B}^m}\sum_{k=-\infty}^{+\infty}h_{\boldsymbol{B};k}\cdot y_1^k\cdot dY_{\boldsymbol{B}},$$

where each $h_{\boldsymbol{B};k}\in\boldsymbol{C}[[y_2, y_3, \cdots, y_n]]$ is a holomorphic function on U. Note that, by $l\geqq n$, we have the inclusion $\mathcal{B}\subseteq\mathcal{A}$. Since $f^*(dy_1)=x_1^{\mu_{ij}-1}\cdot dx_1$, one can associate, to each $\boldsymbol{B}\in\mathcal{B}^m$, an integer $\gamma(\boldsymbol{B})$ with $0\leqq\gamma(\boldsymbol{B})\leqq m(\mu_{ij}-1)$ such that $f^*(dY_{\boldsymbol{B}})=x_1^{\gamma(\boldsymbol{B})}\cdot dX_{\boldsymbol{B}}$. Thus

$$\omega=f^*(\theta)=\sum_{\boldsymbol{B}\in\mathcal{B}^m}\sum_{k=-\infty}^{+\infty}(1/\mu_{ij})^k\cdot f^*(h_{\boldsymbol{B};k})\cdot x_1^{k\mu_{ij}+\gamma(\boldsymbol{B})}\cdot dX_{\boldsymbol{B}}.$$

In view of $\omega\in H^0(X, \overset{m}{\otimes}\Omega_X^p)$, the holomorphic function $h_{\boldsymbol{B};k}$ on U must vanish for those $(k, \boldsymbol{B})\in\boldsymbol{Z}\times\mathcal{B}^m$ which satisfy $k\mu_{ij}+\gamma(\boldsymbol{B})<0$. In particular, if $k<-m(\mu_{ij}-1)/\mu_{ij}$, then $h_{\boldsymbol{B};k}=0$. Hence

$$(y_1^{[m(\mu_{ij}-1)/\mu_{ij}]})\cdot\theta\in H^0(U, \overset{m}{\otimes}\Omega_Y^p).$$

Since this holds for every $j\in\{1, 2, \cdots, k_i\}$, it follows that $y_1^{\nu_{i;m}}\cdot\theta\in H^0(U, \overset{m}{\otimes}\Omega_Y^p)$. Varying i, we now conclude that $\theta\in H^0(Y, (\overset{m}{\otimes}\Omega_Y^p)(\Delta_{f;m}))$. Q. E. D.

Proof of 2.2.3. Let Δ be the union of all those irreducible components of $Y-U$ which are 1-codimensional in Y. Since $\mathrm{codim}_Y((Y-\Delta)-U)\geqq 2$, every $\omega\in H^0(U, \Omega_Y^p)$ extends to a holomorphic section in $H^0(Y-\Delta, \Omega_Y^p)$. We now apply 2.2.2 to $m=1$. By $\Delta_{f;1}=0$, the assertion of 2.2.3 immediately follows.

Q. E. D.

Before getting into the proof of 2.1.1, we here define a few notions which feature rational curves or surfaces.

Definition 2.2.4. (i) A compact complex manifold X is said to have *Property* Π, if $h^0(X, \overset{m}{\otimes}\Omega_X^1)=0$ for all $m\in \boldsymbol{Z}_+$.
(ii) Fix an arbitrary positive integer p. A compact complex manifold X is said to have *Property* $(\Pi-p)$ (resp. $(\Pi'-p)$), if $h^0(X, \Omega_X^q)=0$ for all $q=1, 2\cdots, p$ (resp. $h^0(X, S^{m_1}(\Omega_X^1)\otimes S^{m_2}(\Omega_X^2)\otimes\cdots\otimes S^{m_p}(\Omega_X^p))=0$ for all nonnegative integers $m_1, m_2, \cdots, m_p$ such that $(m_1, m_2, \cdots, m_p)\neq(0, 0, \cdots 0)$).

Remark 2.2.5. Straightforward consequences of our definition are:
(i) *every unirational compact complex manifold has Property* Π;
(ii) *every nonsingular* $K3$ *surface has Property* $(\Pi'-1)$, (cf. Kobayashi [19]);
(iii) *Property* Π *implies* $(\Pi'-p)$ *(and in particular* $(\Pi-p)$*) for all* $p\in \boldsymbol{Z}_+$.

Now we come to the following theorem which is a main ingredient of 2.1.1.

Theorem 2.2.6. *Let* $f: X\rightarrow Y$ *be a surjective morphism of compact complex manifolds with connected fibres. Fixing positive integers* p *and* m *arbitrarily, we assume that general fibres of* f *have Property* Π *(resp.* $(\Pi-p)$, $(\Pi'-p)$*). Then every element* ω *in* $H^0(X, \overset{m}{\otimes}\Omega_X^p)$ *(resp.* $H^0(X, \Omega_X^p)$, $H^0(X, S^m(\Omega_X^p))$*) is expressible as* $f^*(\theta)$ *for some* θ *in* $H^0(Y, (\overset{m}{\otimes}\Omega_Y^p)(\Delta_{f;m}))$ *(resp.* $H^0(Y, \Omega_Y^p)$, $H^0(Y, S^m(\Omega_Y^p)(\Delta_{f;m}))$*), where* Δ *is the analytic subset of* Y *consisting of all* 1*-codimensional components of* $f(\{x\in X; f$ *is not of maximal rank at* $x\})$ *in* Y, *(and see* 2.2.1 *for the definition of* $\Delta_{f;m}$*). In particular, if* $\dim Y<p$, *then* $h^0(X, \overset{m}{\otimes}\Omega_X^p)$ *(resp.* $h^{p,0}(X)$, $h^0(X, S^m(\Omega_X^p))$*) is* 0.

Corollary 2.2.7. *Let* $f: X\rightarrow Y$ *be a surjective morphism of compact complex manifolds with connected fibres. Put* $p_0=\dim X-\dim Y$. *Suppose general fibres of* f *have Property* $(\Pi-p_0)$. *Then* f *induces the isomorphism* $f^*: H^0(Y, \Omega_Y^p)\cong H^0(X, \Omega_X^p)$ *for each* $p\in \boldsymbol{Z}_+$. *In particular, if* X *is of class* C, *then* $\chi(X, \mathcal{O})=\chi(Y, \mathcal{O})$.

Corollary 2.2.8. *Let* $f: X\rightarrow Y$ *be a surjective morphism of compact complex manifolds with connected fibres. Put* $p_0=\dim X-\dim Y$. *Suppose that general fibres of* f *have Property* Π *(resp.* $(\Pi'-p_0)$*) and that, in terms of the notation in* 2.2.1, *the analytic subset* Δ *of* Y *as in* 2.2.6 *satisfies the following condition*:

($\sharp$) *For every* $i\in\{1, 2, \cdots, r\}$, *there exists* $j\in\{1, 2, \cdots, k_i\}$ *such that* $\mu_{ij}=1$.

Then f *induces an isomorphism* $f^*: H^0(Y, \overset{m}{\otimes}\Omega_Y^p)\cong H^0(X, \overset{m}{\otimes}\Omega_X^p)$ *(resp.* $f^*: H^0(Y, S^m(\Omega_Y^p))\cong H^0(X, S^m(\Omega_X^p))$*) for each* m, $p\in \boldsymbol{Z}_+$.

Proof of 2.2.6. Since proofs are similar, we just consider the case $\omega\in H^0(X, S^m(\Omega_X^p))$. Fix an arbitrary smooth fibre $f^{-1}(o)$, $o\in Y$, and let V be an open neighbourhood of o in $Y-\Delta$ with a system of local coordinates $(y_1, \cdots, y_n)$ $(n=\dim Y)$. Choosing V small enough, we write $f^{-1}(V)$ as a union $\bigcup_{\lambda\in\Lambda} U_\lambda$ of its coordinate open subsets U_λ with local coordinates $(f^*(y_1), f^*(y_2), \cdots, f^*(y_n), x_{\lambda;1}, x_{\lambda;2}, \cdots, x_{\lambda;l})$ ($l=\dim X-\dim Y$). Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the set of all (possibly empty) subsets of $\{1,2,\cdots,l\}$ (resp. $\{1,2,\cdots,n\}$), and for each $A=\{\alpha_1, \alpha_2, \cdots, \alpha_s\}\in\mathcal{A}$ (resp. $B=\{\beta_1, \beta_2, \cdots \beta_t\}\in\mathcal{B}$) with $\alpha_1<\alpha_2<\cdots<\alpha_s$ (resp. $\beta_1<\beta_2<\cdots<\beta_t$), we put

$$|A|=s=\text{cardinality of } A \quad (\text{resp. } |B|=t=\text{cardinality of } B)$$

and

$$dx_{\lambda;A}=dx_{\lambda;\alpha_1}\wedge dx_{\lambda;\alpha_2}\wedge\cdots\wedge dx_{\lambda;\alpha_s} \quad (\text{resp. } dy_B=dy_{\beta_1}\wedge dy_{\beta_2}\wedge\cdots\wedge dy_{\beta_t}),$$

where $dx_{\lambda;\phi}$ (resp. dy_ϕ) denotes the constant function 1 on U_λ (resp. V). Let $\mathcal{F}$ be the set $\{(A, B)\in\mathcal{A}\times\mathcal{B}\,;\,|A|+|B|=p\}$, and $\mathcal{F}_q$ ($q=0, 1, \cdots, p$) be its subset defined by $\mathcal{F}_q=\{(A, B)\in\mathcal{F}\,;\,|A|=q\}$. We denote by Ψ the set of all set-theoretic maps ψ of $\mathcal{F}$ into $\boldsymbol{Z}_+\cup\{0\}$ such that $\sum_{(A,B)\in\mathcal{F}}\psi(A, B)=m$. For each $\psi\in\Psi$ and $q\in\{0, 1, \cdots p\}$, put $|\psi|_q=\sum_{(A,B)\in\mathcal{F}_q}\psi(A, B)$. Then Ψ is endowed with the following partial order:

Let $\psi, \psi'\in\Psi$. Then $\psi>\psi'$ if and only if $\mathrm{Max}\{q\,;\,|\psi|_q\neq|\psi'|_q\}=\mathrm{Max}\{q\,;\,|\psi|_q>|\psi'|_q\}\geqq 0$, where Max(empty set ϕ) denotes -1.

We now write $0\neq\omega\in H^0(X, S^m(\Omega_X^p))$ in the form

$$\omega=\sum_{\psi\in\Psi}\Big(g_{\lambda;\psi}\prod_{(A,B)\in\mathcal{F}}(dx_{\lambda;A}\wedge f^*(dy_B))^{\psi(A,B)}\Big)$$

for some $g_{\lambda;\psi}\in H^0(U_\lambda, \mathcal{O})$. Let Ξ be the subset $\{\psi\in\Psi\,;\,g_{\lambda;\psi}\neq 0\in H^0(U_\lambda,\mathcal{O})$ for some $\lambda\in\Lambda\}$ of Ψ, and we fix a maximal element ξ_o of Ξ. For each $q\in\boldsymbol{Z}_+$, we put

$$m_q=\begin{cases}|\xi_o|_q & \text{if } q\leqq p\,,\\ 0 & \text{if } q>p\,.\end{cases}$$

At each point v of V, we define $\sigma_{v;\lambda}\in H^0(X_v\cap U_\lambda, S^{m_1}(\Omega^1_{X_v})\otimes\cdots\otimes S^{m_n}(\Omega^n_{X_v}))$ (where $X_v=f^{-1}(v)$) by

$$\sigma_{v;\lambda}=(g_{\lambda;\xi_o|X_v})\cdot\prod_{(A,B)\in\mathcal{F}}(dx_{\lambda;A})^{\xi_o(A,B)}\,.$$

Then the local sections $\{\sigma_{v;\lambda}\,;\,\lambda\in\Lambda\}$ are glued together to define a global section $\sigma_v\in H^0(X_v, S^{m_1}(\Omega^1_{X_v})\otimes\cdots\otimes S^{m_n}(\Omega^n_{X_v}))$. For a general $v\in V$, the fibre X_v has Property $(\Pi'-p)$, and hence $m_1=m_2=\cdots=m_n=0$. Thus, σ_v is a constant holomorphic function for all $v\in V$, and furthermore the maximality of ξ_o implies that every element of Ξ is again maximal in Ξ. Then, repeating the same argument as above, we can write $\omega_{|f^{-1}(V)}=f^*(\theta)$ for some $\theta\in H^0(V, S^m(\Omega_V^p))$. Since $Y-\Delta$ is covered by such V's outside an analytic subset of codimension $\geqq 2$, it now follows that $\omega_{|X-f^{-1}(V)}=f^*(\theta)$ for some $\theta\in H^0(Y-\Delta, S^m(\Omega_Y^p))$. The assertion of 2.2.6 is hence straightforward from 2.2.2. Q.E.D.

Proof of 2.2.7. Since p_0 is $\dim X-\dim Y$, general fibres of f have Properties $(\Pi-p)$ for all $p\in\boldsymbol{Z}_+$. Then $f^*: H^0(Y, \Omega_Y^p)\cong H^0(X, \Omega_X^p)$ $(p\in\boldsymbol{Z}_+)$ immediately follows from 2.2.6. If X is of class $\mathcal{C}$, then so is Y, and hence $\chi(X, \mathcal{O})=\sum(-1)^p h^{p,0}(X)=\sum(-1)^p h^{p,0}(Y)=\chi(Y, \mathcal{O})$. Q.E.D.

Proof of 2.2.8. Our assumption on μ_{ij}'s implies that $\Delta_{f;m}=0$. Then 2.2.8 is a straightforward consequence of 2.2.6. Q.E.D.

(2.3) We shall now prove 2.1.1. The first important observation is due to Ueno and says that the condition (♯) of 2.2.8 is satisfied for those f whose general fibres are isomorphic to $\boldsymbol{P}^1$ (cf. the proof of 2.3.1 below). In particular,

Theorem 2.3.1.[*)] *Let $f: X\to Y$ be a surjective morphism of compact complex manifolds such that (general fibre)$\cong\boldsymbol{P}^1$. Then for all $m, p\in\boldsymbol{Z}_+$, we have the isomorphisms $f^*: H^0(Y, \overset{m}{\otimes}\Omega_Y^p)\cong H^0(X, \overset{m}{\otimes}\Omega_X^p)$ and $f^*: H^0(Y, S^m(\Omega_Y^p))\cong H^0(X, S^m(\Omega_X^p))$.*

Proof. In the below, we use the notation in 2.2.6 (and also 2.2.1). Fix a general point y_i of Δ_i, and choose a holomorphic curve $\Gamma=\{\gamma(t); |t|<1\}$ embedded in Y so that Γ intersects Δ transversally at just one point $y_i=\gamma(0)=\Gamma\cap\Delta$. Then $f: f^{-1}(\Gamma)\to\Gamma$ is a proper morphism of complex manifolds with $\dim\Gamma=1$ and (general fibre)$\cong\boldsymbol{P}^1$. Hence the divisor $f^*(\gamma(0))\in\mathrm{Div}(f^{-1}(\Gamma))$ corresponding to the central fibre $f^{-1}(\gamma(0))$ has a component of multiplicity 1. Since the restriction of $f^*(\Delta)\in\mathrm{Div}(X)$ to $f^{-1}(\Gamma)$ is exactly $f^*(\gamma(0))\in\mathrm{Div}(f^{-1}(\Gamma))$, it then follows that $\mu_{ij}=1$ for some $j\in\{1, 2, \cdots, k_i\}$. Noting that $\boldsymbol{P}^1$ has Properties Π and $(\Pi'-1)$, we now conclude from 2.2.8 that f^* induces the required isomorphisms. Q.E.D.

Next, using Kawamata's improvement [18] of Manin's results, we shall show the following:

Theorem 2.3.2. *Let X (resp. Y) be a compact complex 3-dimensional manifold of class $\mathcal{C}$ (resp. a nonsingular projective curve), and $f: X\to Y$ be a surjective morphism whose general fibre is an irreducible nonsingular rational surface. Then for all $m, p\in\boldsymbol{Z}_+$, we have the isomorphisms $f^*: H^0(Y, \overset{m}{\otimes}\Omega_Y^p)\cong H^0(X, \overset{m}{\otimes}\Omega_X^p)$ and $f^*: H^0(Y, S^m(\Omega_Y^p))\cong H^0(X, S^m(\Omega_X^p))$.*

Proof. Since Y is algebraic, and since general fibres of f are Moishezon with irregularity 0, it follows that X is also Moishezon. Then by Kawamata [18], every singular fibre of f contains an irreducible component of multiplicity 1. Thus, the condition (♯) of 2.2.8 is satisfied. In view of (i) and (iii) of 2.2.5, we now conclude from 2.2.8 that f^* induces the required isomorphisms.

Q.E.D.

Combining Theorems 2.3.1 and 2.3.2, we thus obtain:

Proof of 2.1.1. Replacing X by its suitable bimeromorphic model, we may

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*) We here thank Professor Ueno who pointed out that this theorem is true for $m>1$.

assume that $b_X: X\to B(X)$ is a morphism. If $\beta(X)=3$, then $b_X=id_X$, and hence the isomorphisms (1) and (2) are obvious. Therefore we consider the remaining cases: $\beta(X)=1, 2$. Then by 1.3.4, general fibres of b_X are rational. Theorems 2.3.1 and 2.3.2 now finish the proof. Q.E.D.

(2.4) We shall finally prove 2.1.5. For later purposes, a little more general situation will be considered.

Definition 2.4.1. Let m, n, and p be positive integers such that $p\leqq n$. Let X be an n-dimensional compact complex manifold with $h^0(X, S^m(\Omega_X^p))\neq 0$.
i) A nonzero element θ in $H^0(X, S^m(\Omega_X^p))$ is said to be *neatly* (resp. *very neatly*) *foliated* if there exist a p-dimensional compact complex manifold Y, a generically surjective meromorphic map $f: X\to Y$, and a meromorphic (resp. holomorphic) m-ple p-form ψ in $H^0_{mero}(Y, \omega_Y^{\otimes m})$ (resp. $H^0(Y, \omega_Y^{\otimes m})$) such that $\theta=f^*(\psi)$.
ii) Let $T^*(X)$ denote the cotangent bundle of X. (Hence $\Omega_X^p=\mathcal{O}_X(\wedge^p T^*(X))$.) Consider the mapping $\mu_m: \wedge^p T^*(X)\to S^m(\wedge^p T^*(X))$ defined by $\mu_m(v)=v^m$ for each $v\in\wedge^p T_x^*(X)$ and $x\in X$. A nonzero element θ in $H^0(X, S^m(\Omega_X^p))$ is said to be *of purely multiple type* if one of the following equivalent conditions is satisfied:
(ii-a) $\theta(x)\in \text{Image}\,\mu_m$ for every $x\in X$.
(ii-b) There exists a Zariski open dense subset U of X such that, for each $x\in U$, θ is locally written as η^m for some germ $\eta\in\Omega_{X,x}^p$.

Remark 2.4.2. It is easily seen that, in 2.4.1 above, a nonzero element θ in $H^0(X, S^m(\Omega_X^p))$ is of purely multiple type, for instance, if either $m=1$ or θ is neatly foliated.

Theorem 2.4.3. *Let X be an n-dimensional uniruled compact complex manifold such that $h^0(X, S^m(\Omega_X^{n-1}))\neq 0$ for some $m\in\boldsymbol{Z}_+$. Then any nonzero element $\theta\in H^0(X, S^m(\Omega_X^{n-1}))$ which is of purely multiple type is very neatly foliated.*

Proof. Step 1: Let D be the Douady space of X which parametrizes the closed analytic subspaces of X. Let $\rho_1: Z\to D$ be the corresponding universal family with a natural embedding $Z\subseteqq D\times X$, where ρ_1 coincides with (resp. $\rho_2: Z\to X$ denotes) the restriction to Z of the natural projection $D\times X\to D$ (resp. $D\times X\to X$). Now, by the uniruledness of X, there exists a covering family of rational curves $(f: W\to Y, g: W\to X, Y^0)$ on X, (cf. (0.4)). We define a complex variety Γ to be the image of the morphism $g\times f: W\to X\times Y$ which sends each $w\in W$ to $(g(w), f(w))\in X\times Y$. Let $\pi:\Gamma\to Y$ be the restriction to Γ of the natural projection $X\times Y\to Y$. Then, making Y^0 smaller if necessary, we may assume that π is flat over Y^0 and furthermore that the fibre $\pi^{-1}(y)$ is reduced for every $y\in Y^0$. Note that $\pi^{-1}(y)=g(W_y)\times\{y\}$ for $y\in Y^0$. Hence we have a natural morphism $\lambda: Y^0\to D$ which sends each $y\in Y^0$ to $\lambda(y):=g(W_y)\in D$, where $g(W_y)$ is regarded as an irreducible reduced rational curve on X. Moreover, this λ extends to a generically surjective meromorphic map of Y to some compact subvariety S of D. We now put $Z_S=\rho_1^{-1}(S)$, $\rho'_S=\rho_{1|Z_S}$, and $\rho''_S=\rho_{2|Z_S}$. Note that general fibres of $\rho'_S: Z_S\to S$ are (possibly singular) irreducible rational curves. In the next step, we shall show that the surjective morphism $\rho''_S: Z_S\to X$ is a modification.

Step 2. Since θ is of purely multiple type, we can choose a Zariski open dense subset U of X as in (ii-b) of 2.4.1. Making U smaller if necessary, we may assume that $U\subseteqq g(f^{-1}(Y^0))\cap\{x\in X;\theta(x)\neq 0$ and g is smooth over $x\}$. Consider the set $Z_S^0:=\rho_S''^{-1}(U)\cap\rho_1^{-1}(\lambda(Y^0))$ which contains a Zariski open dense subset of Z_S. Note that, if we can show the injectivity of $\rho''_{S|Z_S^0}: Z_S^0\to U$, then $\rho''_S: Z_S\to X$ is a modification. Thus, for contradiction, we assume that:

(*) $\rho''_{S|Z_S^0}: Z_S^0\to U$ is not injective, i.e., there exist points y', y'' of Y^0 such that $g(W_{y'})$ and $g(W_{y''})$ are distinct curves on X which pass through a common point u_0 of U.

Put $C=g(W_{y'})$ and let C^0 be its Zariski open dense subset $\{c\in C\cap U; c$ is a nonsingular point of C and $g_{|W_{y'}}$ is unramified over $c\}$. Take an open neighbourhood V $(\subseteqq U)$ of u_0 with a system of local coordinates $(x_1, x_2, \cdots, x_n)$. For a small enough V, $\theta_{|V}$ is expressible as η^m for some $\eta\in H^0(V,\Omega_X^{n-1})$. We write

$$\eta=\sum_{i=1}^n h_i dx_1\wedge dx_2\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n,\qquad \text{with}\quad h_i\in H^0(V,\mathcal{O}),$$

and then define a nonvanishing vector field $\tau\in H^0(V, T(X))$ on V by

$$\tau=\sum_{i=1}^n h_i(\partial/\partial x_i).$$

Fix an arbitrary point c of $C^0\cap V$, and choose an open neighbourhood N $(\subseteqq V)$ of c in X with a system of local coordinates $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ such that C is defined by $\tilde{x}_2=\tilde{x}_3=\cdots=\tilde{x}_n=0$ on N. Fix furthermore a point $b\in W_{y'}$ with $g(b)=c$, and let N' (resp. N'') be an open neighbourhood of b (resp. y') with a system of local coordinates $(w_0, w_1, \cdots, w_l)$ (resp. $(y_1, y_2, \cdots y_l)$) (where $l=\dim Y$) such that:

(1) $f(N')\subseteqq N''$, $g(N')\subseteqq N$, and
(2) $g^*(\tilde{x}_i)=w_{i-1}$ for $1\leqq i\leqq n$ and $f^*(y_j)=w_j$ for $1\leqq j\leqq l$.

Let $\mathcal{J}$ be the set of all those subsets of $\{1, 2, \cdots, l\}$ whose cardinality is $n-1$, and for each $J=\{j_1, j_2, \cdots, j_{n-1}\}\in\mathcal{J}$ (where $j_1<j_2<\cdots<j_{n-1}$), we put $dW_J=dw_{j_1}\wedge dw_{j_2}\wedge\cdots\wedge dw_{j_{n-1}}$. Since every fibre of f over Y^0 is isomorphic to $\boldsymbol{P}^1$, and since $\boldsymbol{P}^1$ has Property $(\Pi'-(n-1))$, (cf. 2.2.4 and 2.2.5), Theorem 2.2.6 asserts that $g^*(\theta)_{|f^{-1}(Y^0)}$ is written as $f^*(\xi)$ for some $\xi\in H^0(Y^0, S^m(\Omega_Y^{n-1}))$. Hence, on N', we have

$$g^*(\theta)=\sum_{J_1,J_2,\cdots,J_m} q_{J_1J_2\cdots J_m}\cdot dW_{J_1}dW_{J_2}\cdots dW_{J_m},\quad \text{with } q_{J_1J_2\cdots J_m}\in H^0(N',\mathcal{O}),$$

where the summation is taken over all $(J_1, J_2, \cdots, J_m)\in\mathcal{J}\times\mathcal{J}\times\cdots\times\mathcal{J}$. Thus, there are no dw_0's in the expression of $g^*(\theta)$, and therefore in view of the equalities

$$g^*(d\tilde{x}_1)=dw_0 \quad \text{and} \quad g^*(\theta)_{|N'}=(g^*(\eta))^m{}_{|N'},$$

we can write the value $\eta(c)$ of η at c in the form

$$\eta(c)=a\cdot d\tilde{x}_2\wedge d\tilde{x}_3\wedge\cdots\wedge d\tilde{x}_n, \qquad a\in\boldsymbol{C}^*.$$

Since C is locally defined by $\tilde{x}_2=\tilde{x}_3=\cdots=\tilde{x}_n=0$ on N, it follows that $\tau(c)$ is tangent to C at the point c. Note that c is an arbitrary point of $C^0\cap V$. Thus, in a neighbourhood of u_0 in V, the curve C $(=g(W_{y'}))$ sits in a single orbit of the local 1-parameter complex analytic group generated by τ. Similarly, the same thing is also true of the curve $g(W_{y''})$. Since both $g(W_{y'})$ and $g(W_{y''})$ pass through u_0, we then have $g(W_{y'})=g(W_{y''})$ in contradiction to our assumption (*). We now conclude that ρ''_S is a modification.

Step 3. Choosing suitable desingularizations $\iota_1:\tilde{S}\to S$ and $\iota_2:\tilde{Z}_S\to Z_S$ of S and Z_S respectively, we obtain a morphism $\tilde{\rho}'_S:=\iota_1^{-1}\circ\rho'_S\circ\iota_2$ of $\tilde{Z}_S$ onto $\tilde{S}$. Since general fibres of ρ'_S are irreducible rational curves, (cf. Step 1), the same thing is true of $\tilde{\rho}'_S$. Then by Theorem 2.3.1, we have an isomorphism $(\tilde{\rho}'_S)^*: H^0(\tilde{S}, S^m(\Omega_{\tilde{S}}^{n-1}))\cong H^0(\tilde{Z}_S, S^m(\Omega_{\tilde{Z}_S}^{n-1}))$. On the other hand, by Step 2, $\rho''_S\circ\iota_2:\tilde{Z}_S\to X$ is a modification. It is now straightforward that our ω is expressible as $(\tilde{\rho}'_S\circ(\rho''_S\circ\iota_2)^{-1})^*(\phi)$ for some $\phi\in H^0(\tilde{S}, S^m(\Omega_{\tilde{S}}^{n-1}))$. Thus, ω is very neatly foliated.

Q. E. D.

Corollary 2.4.4. *Let X be an n-dimensional uniruled compact complex manifold such that $h^0(X, S^m(\Omega_X^{n-1}))\neq 0$ for some $m\in\boldsymbol{Z}_+$. Suppose one of the following conditions is satisfied*: (a) *X is Moishezon*; (b) *X is of class C together with $n=3$. Then $\beta(X)=n-1$.*

Proof. In view of (0.4), there exists a covering family of rational curves $(f: W\to Y,\ g: W\to X,\ Y^0)$ such that g is generically finite. Fix a nonzero element θ of $H^0(X, S^m(\Omega_X^{n-1}))$. Since $\dim Y=n-1$, and since general fibres of f are isomorphic to $\boldsymbol{P}^1$, Theorem 2.3.1 shows that the element $g^*(\theta)$ of $H^0(W, S^m(\Omega_W^{n-1}))$ is of purely multiple type. Note that g is étale over a Zariski open dense subset of X. Hence θ is also of purely multiple type. Then by Theorem 2.4.3, θ is very neatly foliated. From the definition of $\beta(X)$, we now obtain $\beta(X)=n-1$.

Q. E. D.

Once we have this corollary, the preceding 2.1.5 easily follows from Castelnuovo-Enriques criteria of rationality or ruledness of surfaces:

Proof of 2.1.5. i) is straightforward from Corollary 2.4.4 above. Next note that $\beta(X)=0$, 1 or 2 by the uniruledness of X. ii): Suppose $q(X)=h^0(X, S^2(\Omega_X^2))=0$. Then by $q(X)=0$, we cannot have $\beta(X)=1$. Moreover one obtains $\beta(X)\neq 2$, because otherwise Theorem 2.1.1 would imply that $q(B(X))=h^0(B(X), \omega_{B(X)}^{\otimes 2})=0$ in contradiction to $\kappa(B(X))\geqq 0$. Thus we have $\beta(X)=0$. The other implication of ii) is immediate from i) above and 2.1.1. iii): Suppose $q(X)>0=h^0(X, S^{12}(\Omega_X^2))$. Then by $q(X)>0$, the case $\beta(X)=0$ does not occur. Furthermore $\beta(X)\neq 2$, because otherwise 2.1.1 would again imply that $h^0(B(X), \omega_{B(X)}^{\otimes 12})=0$ in contradiction to $\kappa(B(X))\geqq 0$. Thus $\beta(X)=1$. The other implication of iii) is immediate from 2.1.1. iv): Suppose $h^0(X, S^{12}(\Omega_X^2))\neq 0$. Then $\beta(X)\neq 0$ by i), and moreover $\beta(X)\neq 1$ by iii). Thus $\beta(X)=2$. The other implication of iv) is immediate from 2.1.1 and Enriques criterion. Q. E. D.

§ 3. Étale invariance of β.

Theorem 3.1.1. *Let $f:\tilde{X}\to X$ be an étale cover of compact complex manifolds of dimension* $\leqq 3$. *Assume that X is of class $\mathcal{C}$. Then we have*:

either (1) $\beta(\tilde{X})=\beta(X)=0$ *and* $\dim X=3$

or (2) *the natural meromorphic map $b(f): B(\tilde{X})\to B(X)$ defined in* 1.1.5 *is an étale morphism satisfying the equality* $\deg b(f)=\deg f$.

In particular, the equality $\beta(\tilde{X})=\beta(X)$ always holds.

In this theorem, we very reasonably expect that (2) above holds even in the situation (1). Thus

Conjecture 3.1.2. *Let $f:\tilde{X}\to X$ be an étale cover of compact complex manifolds of dimension* $\leqq 3$. *Assume that X is of class $\mathcal{C}$. Then the meromorphic map $b(f): B(\tilde{X})\to B(X)$ is again an étale morphism and satisfies* $\deg b(f)=\deg f$.

The only open case of this says that any compact complex threefold X of class $\mathcal{C}$ with $\beta(X)=0$ admits conjecturally no nontrivial finite étale coverings. Now concerning this 3.1.2, we can show the following:

Proposition 3.1.3. *If Conjecture* 2.1.3 *is true, then so is* 3.1.2.

Proposition 3.1.4. *Conjecture* 3.1.2 *is true for all those X which satisfy one of the following conditions* (a) $\dim X\leqq 2$, (b) $\kappa(X)\geqq 0$, (c) *X is uniruled.*

(3.2) Before proving 3.1.1, we here give a basic information of the degeneration of certain compact complex manifolds which include, for instance, unirational ones.

Theorem 3.2.1. *Let V (resp. S) be a compact complex manifold of class $\mathcal{C}$ (resp. an irreducible nonsingular projective curve). Put $p=\dim V-1$, and let $g: V\to S$ be a surjective morphism whose general fibre is irreducible and has Property ($\Pi-p$),* (cf. 2.2.4 (ii) *and* 2.2.5 (i)). *Then g has no multiple singular fibres.*

Proof. For contradiction, we assume that a multiple singular fibre $g^{-1}(s_0)$ ($s_0\in S$) exists, i.e., the largest positive integer e dividing $g^*(s_0)$ in $\mathrm{Div}(V)$ satisfies $e\geqq 2$. Fixing a general smooth fibre $F=g^{-1}(s_1)$ ($s_1\in S$), we have an e-fold abelian covering $\pi:\tilde{S}\to S$ which is unramified over $S-\{s_0, s_1\}$ and has just one point $\tilde{s}_i\in\tilde{S}$ with ramification index e over each s_i ($i=0, 1$), where $\tilde{S}$ is nonsingular. Let $\nu:\tilde{V}\to V\times_S\tilde{S}$ be the normalization of $V\times_S\tilde{S}$, and $\tilde{\pi}:\tilde{V}\to V$

(resp. $\tilde{g}: \tilde{V}\to\tilde{S}$) be the composite of ν with the natural projection $V\times_S\tilde{S}\to V$ (resp. $V\times_S\tilde{S}\to\tilde{S}$). Then one easily checks that 1) $\tilde{V}$ is nonsingular and 2) $\tilde{\pi}$ is ramified just over the branch locus F with multiplicity e. Now, for every coherent sheaf $\mathcal{E}$ on V (resp. $\tilde{V}$), we denote by $td(\mathcal{E})=\sum_{i=0}^{\infty} td_i(\mathcal{E})$ (resp. $\widetilde{td}(\mathcal{E})=\sum_{i=0}^{\infty}\widetilde{td}_i(\mathcal{E})$) the total Todd class of $\mathcal{E}$ on V (resp. $\tilde{V}$). Let $\mathcal{I}_{\tilde{F}}$ be the sheaf of ideals defining $\tilde{F}=\tilde{g}^{-1}(\tilde{s}_1)$ in $\tilde{V}$. Then by the exact sequence $0\to g^*(\Omega_V^1)\to\Omega_{\tilde{V}}^1\to\mathcal{I}_{\tilde{F}}/(\mathcal{I}_{\tilde{F}})^e\to 0$, we obtain $\widetilde{td}(\Omega_{\tilde{V}}^1)=g^*(td(\Omega_V^1))\cdot\widetilde{td}(\mathcal{I}_{\tilde{F}}/(\mathcal{I}_{\tilde{F}})^e)$. Since $\mathcal{I}_{\tilde{F}}$ is the pull back $\tilde{g}^*(\mathcal{I}_{\tilde{s}_1})$ of the ideal sheaf $\mathcal{I}_{\tilde{s}_1}$ of $\tilde{s}_1$, $\mathcal{I}_{\tilde{F}}/(\mathcal{I}_{\tilde{F}})^e$ is expressible as a direct sum of $(e-1)$-copies of $\mathcal{O}_{\tilde{F}}$. Hence $\widetilde{td}(\mathcal{I}_{\tilde{F}}/(\mathcal{I}_{\tilde{F}})^e)=1+((e-1)/2)\cdot c_1([\tilde{F}])$, where $c_1([\tilde{F}])\in H^2(\tilde{V}\,;\boldsymbol{Z})$ denotes the Poincaré dual of $\tilde{F}$ in $\tilde{V}$. Put $n=\dim V$. Since, for the tangent bundle $T(\tilde{V})$ (resp. $T(V)$) of $\tilde{V}$ (resp. V), one has $\widetilde{td}_i(\Omega_{\tilde{V}}^1)=(-1)^i\,\widetilde{td}_i(T(\tilde{V}))$ (resp. $td_i(\Omega_V^1)=(-1)^i td_i(T(V))$), it then follows that:

$$\widetilde{td}_n(T(\tilde{V}))=g^*(td_n(T(V)))-((e-1)/2)\cdot c_1([\tilde{F}])\cdot g^*(td_{n-1}(T(V)))\,.$$

Rewriting this by Riemann-Roch's formula, we obtain

$$\begin{aligned}\chi(\tilde{V},\mathcal{O})=(\widetilde{td}_n(T(\tilde{V})))[\tilde{V}]&=(td_n(T(V)))(g_*[\tilde{V}])-((e-1)/2)(td_{n-1}(T(V)))(g_*[\tilde{F}])\\&=e\cdot\chi(V,\mathcal{O})-((e-1)/2)(td_{n-1}(T(V)))[F]\,.\end{aligned}$$

Since F has trivial normal bundle in V, $c_i(F)=c_i(V)_{|F}$ holds for all $i=1,2,\cdots,n-1$. Hence $(td_{n-1}(T(V)))[F]=(td_{n-1}(T(F)))[F]=\chi(F,\mathcal{O})=\sum(-1)^i h^{i,0}(F)=1$. Thus $\chi(\tilde{V},\mathcal{O})=e\cdot\chi(V,\mathcal{O})-((e-1)/2)$. On the other hand, 2.2.7 shows that $\chi(\tilde{V},\mathcal{O})=\chi(\tilde{S},\mathcal{O})$ and $\chi(V,\mathcal{O})=\chi(S,\mathcal{O})$. We now have $\chi(\tilde{S},\mathcal{O})=e\cdot\chi(S,\mathcal{O})-((e-1)/2)$, in contradiction to the formula of Hurwitz $2\chi(\tilde{S},\mathcal{O})=2e\chi(S,\mathcal{O})-2(e-1)$ applied to the ramified cover π. Q, E. D.

If $p=1$, a little more general statement is possible. In fact, in the proof of Theorem 2.3.1, we have already (implicitly) shown the following:

Proposition 3.2.2. *Let $g: V\to S$ be a proper surjective morphism of complex manifolds with (general fibre)$\cong\boldsymbol{P}^1$. Then there exists a Zariski open dense subset S^0 of S such that* $\operatorname{codim}_S(S-S^0)\geqq 2$ *and that g over S^0 is flat and has no multiple singular fibres.*

(3.3) We next prove a lemma which is a key to the proof of Theorem 3.1.1.

Lemma 3.3.1. *Let a finite group G act holomorphically on compact complex manifolds X and Y so that the action on X is free. Let $f: X\to Y$ be a G-equivariant surjective morphism whose general fibre is irreducible and has Property $(\Pi-p)$, where $p=\dim X-\dim Y$. Assume that either $p=1$ with (general fibre of f)$\in\mathcal{C}$ or, if $p\neq 1$, Y is projective algebraic with $X\in\mathcal{C}$. Then G acts freely also on Y.*

Proof. Let $H=\{g\in G\,;\,g$ acts identically on $Y\}$, and choose a sufficiently general smooth fibre F of f. Since H acts freely on F, we have $\chi(F/H,\mathcal{O})\cdot\deg H$

$=\chi(F, \mathcal{O})=\sum(-1)^i h^{i,0}(F)=1$. Hence $H=\{1\}$, i.e., G acts effectively on Y. For contradiction, we now assume that the G-action on Y is not free. Then there exists an element $1\neq\gamma\in G$ which fixes a point y^0 of Y. Let $\Gamma=\{1, \gamma, \gamma^2, \cdots, \gamma^{r-1}\}$ be the cyclic subgroup of G generated by γ (where $r=|\Gamma|=$the order of γ) and let

$$\begin{aligned}\rho: \Gamma &\longrightarrow GL(T_{y^0}(Y))\\ \gamma &\longmapsto \rho(\gamma)\end{aligned}$$

be the isotropy representation of Γ on the tangent space $T_{y^0}(Y)$ at y^0. Put $\zeta=\exp(2\pi\sqrt{-1}/r)$ and $\boldsymbol{Z}_r=\{0, 1, \cdots, r-1\}$. Then for a suitable $\boldsymbol{C}$-basis $\{e_1, e_2, \cdots e_n\}$ of $T_{y^0}(Y)$, the element $\rho(\gamma)$ is expressible as $\Delta(i_1, i_2, \cdots, i_n)$ for some $(i_1, i_2, \cdots i_n)$ $\in\boldsymbol{Z}_r\times\boldsymbol{Z}_r\times\cdots\times\boldsymbol{Z}_r-\{(0, 0, \cdots, 0)\}$ with $n=\dim Y$, where $\Delta(i_1, i_2, \cdots, i_n)\in GL(n, \boldsymbol{C})$ denotes the diagonal matrix with each α-th diagonal element equal to ζ^{i_α}. Here we may assume that $i_1\leqq i_2\leqq\cdots\leqq i_n$, and let Y^Γ denote the fixed point set of the Γ-action on Y. Then an irreducible component W of the nonsingular analytic set Y^Γ passes through y^0 so that $T_{y^0}(W)$ is the eigen space of $\rho(\gamma)$ corresponding to the eigen value 1. Now the following two cases are possible:

Case 1: $i_1=i_2=\cdots=i_{n-1}=0$. Then W $(\subseteq Y^\Gamma)$ is a divisor$\in\mathrm{Div}(Y)$ passing through y^0.

Case 2: $i_{n-1}\neq 0$. In this case, let $\sigma_1: Y_1\to Y$ be the (Γ-equivariant) blowing-up of Y along W, and let $q=\mathrm{Min}\{\alpha; i_\alpha\neq 0\}$. We then denote by y^1 the point on $\sigma_1^{-1}(y^0)$ which corresponds to the line e_q in $T_{y^0}(Y)$. Since y^1 is a fixed point of the Γ-action on Y_1, considering the isotropy representation

$$\begin{aligned}\rho': \Gamma &\longrightarrow GL(T_{y^1}(Y_1))\\ \gamma &\longmapsto \rho'(\gamma)\end{aligned}$$

of Γ at the point y^1, we can express the image $\rho'(\gamma)$ of γ as the diagonal matrix $\Delta(i_1', i_2', \cdots, i_n')$ for a suitable $\boldsymbol{C}$-basis of $T_{y^1}(Y_1)$, where $i_\alpha'=i_\alpha$ (if $1\leqq\alpha\leqq q$) and $i_\alpha'=i_\alpha-i_q$ (if $q<\alpha\leqq n$). Note that $\sum_{\alpha=1}^n i_\alpha'<\sum_{\alpha=1}^n i_\alpha$.

Thus in view of Cases 1 and 2, one always obtains a finite sequence $\sigma=\sigma_1\circ\sigma_2\circ\cdots\circ\sigma_m: Y_m\xrightarrow{\sigma_m}Y_{m-1}\xrightarrow{\sigma_{m-1}}\cdots\xrightarrow{\sigma_2}Y_1\xrightarrow{\sigma_1}Y_0=Y$ of Γ-equivariant blowing-ups such that an irreducible divisor $0\neq D\in\mathrm{Div}(Y_m)$ is contained in the fixed point set Y_m^Γ of the Γ-action on Y_m. By a theorem of Hironaka [10], corresponding to the Γ-equivariant meromorphic map $\sigma^{-1}\circ f: X\to Y_m$, there exists a Γ-equivariant modification $\mu: X'\to X$ such that $h\overset{defn}{=\!=}\sigma^{-1}\circ f\circ\mu: X'\to Y_m$ is a (Γ-equivariant) morphism of compact complex manifolds. This h naturally induces a morphism $\bar{h}: X'/\Gamma\to Y_m/\Gamma$, and together with the canonical quotient morphisms $\pi: Y_m\to Y_m/\Gamma$ and $\pi': X'\to X'/\Gamma$, our h and $\bar{h}$ form the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\ \pi'\ } & X'/\Gamma \\ {\scriptstyle h}\downarrow & \circlearrowright & \downarrow{\scriptstyle \bar{h}} \\ Y_m & \xrightarrow{\ \pi\ } & Y_m/\Gamma. \end{array}$$

Here Γ acts freely on X', and in particular X'/Γ is nonsingular. On the other hand, since D is an irreducible component of the nonsingular analytic set Y_m^{Γ}, the finite morphism π is ramified along D with ramification index r ($\geqq 2$) and the corresponding branch $\pi(D)$ ($\cong D$) is contained in the regular locus $(Y_m/\Gamma)_{reg}$ of Y_m/Γ. We now choose a sufficiently general holomorphic curve $S=\{\sigma(t); |t|<1\}$ embedded in $(Y_m/\Gamma)_{reg}$ such that the following conditions are satisfied:

i) S intersects $\pi(Y_m^{\Gamma})$ transversally at only one point $\sigma(0)=S\cap\pi(D)$;
ii) a general fibre of $\bar{h}_{|\bar{h}^{-1}(S)}: \bar{h}^{-1}(S)\to S$ is a p-dimensional compact complex manifold with Property $(\Pi-p)$;
iii) if $p\neq 1$, then S is a part of a nonsingular projective curve sitting in Y_m.

Note that, by i), $\bar{h}^{-1}(S)$ is a complex submanifold of $(Y_m/\Gamma)_{reg}$. In view of the above commutative diagram, ii) shows that $\bar{h}_{|\bar{h}^{-1}(S)}$ has a multiple fibre over $\sigma(0)$ with multiplicity divisible by r. But then this contradicts 3.2.1 and 3.2.2. We now conclude that G acts freely on Y. Q.E.D.

(3.4) We shall now prove 3.1.1, 3.1.3, and 3.1.4.

Proof of 3.1.1. Choose a finite étale cover $h: X^*\to\tilde{X}$ of $\tilde{X}$ such that $f\circ h: X^*\to X$ is a normal covering. Denoting by G the group of covering transformations of X^* over X, we obtain $X^*/G=X$. Note that there exists a subgroup H of G satisfying $X^*/H=\tilde{X}$. We shall first eliminate the following obvious cases:

Case 1. $\beta(X^*)=0$ with $\dim X^*\leqq 2$: Then either $\dim X^*=0$ or X^* is rational (and hence simply connected). Hence $X^*=\tilde{X}=X$. (2) of 3.1.1 now holds.
Case 2. $\beta(X^*)=0$ with $\dim X^*=3$: In this case, (1) of 3.1.1 is obviously satisfied.
Case 3. $\beta(X^*)=\dim X^*$: Then $\kappa(X)=\kappa(\tilde{X})=\kappa(X^*)\geqq 0$. Hence $B(X)=X$, $B(\tilde{X})=X$, and $b(f)=f$, (cf. 1.1.4). We consequently have (2) of 3.1.1.

Thus we have only to consider the remaining case $1\leqq\beta(X^*)<\dim X^*\leqq 3$. Then $B(X^*)$ is either a nonsingular curve of genus$\geqq 1$ or an absolutely minimal complex surface of $\kappa\geqq 0$, (cf. 1.1.4). In particular, every bimeromorphic transformation of $B(X^*)$ is biholomorphic. Hence each $g\in G(\subseteqq \mathrm{Aut}(X^*))$ induces a biholomorphic automorphism $b(g)$ of $B(X^*)$ (cf. 1.1.5) in such a way that the corresponding G-action on $B(X^*)$ makes the meromorphic map $b_{X^*}: X^*\to B(X^*)$ G-equivariant. We now choose a G-equivariant modification $\mu: X^{**}\to X^*$ such that $b_{X^*}\circ\mu: X^{**}\to B(X^*)$ is a (G-equivariant) morphism of compact complex manifolds. Since G acts freely on X^{**}, and since general fibres of $b_{X^*}\circ\mu$ have Property $(\Pi-p)$ with $p=\dim X^{**}-\dim B(X^*)$ (cf. 1.3.4), Lemma 3.3.1 then asserts that G acts freely on $B(X^*)$. Since the morphism $f\circ h: X^*\to X$ (resp. $h: X^*\to\tilde{X}$) is G-invariant (resp. H-invariant), the corresponding G-invariant (resp. H-invariant) meromorphic map $b(f\circ h): B(X^*)\to B(X)$ (resp. $b(h): B(X^*)\to B(\tilde{X})$) naturally induces a generically surjective meromorphic map $j_G: B(X^*)/G\to B(X)$ (resp. $j_H: B(X^*)/H\to B(\tilde{X})$). On the other hand, since the canonical quotient morphism $q_G: B(X^*)\to B(X^*)/G$ (resp. $q_H: B(X^*)\to B(X_*)/H$) is unramified, the inequality $\kappa(B(X^*))\geqq 0$ implies $\kappa(B(X^*)/G)\geqq 0$ (resp. $\kappa(B(X^*)/H)\geqq 0$). Hence by the universality of $B(X)$ (resp. $B(\tilde{X})$), there naturally exists a generically surjective meromorphic map from $B(X)$ (resp. $B(\tilde{X})$) to $B(X^*)/G$ (resp. $B(X^*)/H$). Thus j_G (resp. j_H) is bimeromorphic. Now the following two cases are possible:

Case (a). $\beta(X^*)=1$: Then j_G and j_H are both isomorphisms.
Case (b). $\beta(X^*)=2$: Note that $B(X)$, $B(\tilde{X})$, $B(X^*)$ are all minimal models. In particular, j_G (resp. j_H) is a modification. Since q_G (resp. q_H) is unramified, the nonsingular surface $B(X^*)/G$ (resp. $B(X^*)/H$) is again a minimal model, and therefore j_G and j_H are isomorphisms.

Thus in both cases, we have $j_G: B(X^*)/G \cong B(X)$ and $j_H: B(X^*)/H \cong B(\tilde{X})$. Via these isomorphisms, $b(f): B(\tilde{X}) \to B(X)$ coincides with the natural quotient morphism $q: B(X^*)/H \to B(X^*)/G$. Hence $b(f)$ is an étale morphism satisfying the equality

$$\deg b(f) = \deg q = \frac{|G|}{|H|} = \deg f. \qquad \text{Q. E. D.}$$

In the above proof, one can easily see that if $\beta(\tilde{X})=2$, we don't need the assumption that X is of class $\mathcal{C}$. Thus we have:

Corollary 3.4.1. *Let $f: \tilde{X} \to X$ be an étale cover of compact complex 3-dimensional manifolds. Assume that $\beta(\tilde{X})=2$. Then $\beta(X)=2$ and furthermore the natural meromorphic map $b(f): B(\tilde{X}) \to B(X)$ is an étale morphism satisfying* $\deg b(f) = \deg f$.

As to 3.1.3 and 3.1.4, we need the following lemma:

Lemma 3.4.2. *Conjecture 3.1.2 is true if $h^{2,0}(\tilde{X})=0$.*

Proof. In view of 3.1.1, we may assume $\beta(\tilde{X})=\beta(X)=0$ and $\dim \tilde{X}$ ($=\dim X$) $=3$. Then $\kappa(\tilde{X})=-\infty$ and $\dim \mathrm{Alb}(\tilde{X})=0$. Hence $h^{3,0}(\tilde{X})=h^{1,0}(\tilde{X})=0$. Since $\tilde{X} \in \mathcal{C}$, it follows that $1=1-h^{1,0}(\tilde{X})+h^{2,0}(\tilde{X})-h^{3,0}(\tilde{X})=\chi(\tilde{X}, \mathcal{O})=(\deg f)\cdot\chi(X, \mathcal{O})$. Thus $\deg f=1$, i.e., $\tilde{X}=X$ and $f=id_X$. We now obtain $\deg b(f)=1=\deg f$.
Q. E. D.

Proof of 3.1.3. In view of 3.1.1, we may assume $\beta(\tilde{X})=0$ and $\dim \tilde{X}=3$. If 2.1.3 is true, then $h^{2,0}(\tilde{X})=0$ and hence Lemma 3.4.2 finishes the proof.
Q. E. D.

Proof of 3.1.4. In view of 3.1.1, the proof is reduced to showing 3.1.2 on the following assumption: X is uniruled with $\beta(\tilde{X})=0$ and $\dim \tilde{X}=3$. Now, since $\boldsymbol{P}^1$ is simply connected, every rational curve in X can be lifted to another in $\tilde{X}$, and in particular the uniruledness of X implies that $\tilde{X}$ is also uniruled. Then 2.1.5 asserts that $h^{2,0}(\tilde{X})=0$. By Lemma 3.4.2, the proof is now complete. Q. E. D.

§ 4. Deformation invariance of β.

The purpose of this section is to prove the following:

Theorem 4.1.1. *Let $f: X\to S$ be a proper smooth surjective morphism of complex manifolds with irreducible fibres. For each $s\in S$, we put $X_s=f^{-1}(s)$. Fixing an arbitrary point $o\in S$, we obtain:*

I) *If* $\dim X_o\leqq 2$, *then* $\beta(X_s)=\beta(X_o)$ *for every* $s\in S$.

II) *Assume that* $\dim X_o=3$ *and furthermore that the morphism f is of class $\mathcal{C}_{loc}$*, (cf. 1.2.1). *Then*:

II-a) *If* $\beta(X_o)=1$ *or* 2, $\beta(X_s)=\beta(X_o)$ *for every* $s\in S$.

II-b) *More generally, if X_o is uniruled, then* $\beta(X_s)=\beta(X_o)$ *for every* $s\in S$.

II-c) *If* $\beta(X_o)=0$ *and if* $s\in S$ *is such that* $\kappa(X_s)=-\infty$, *then* $\beta(X_s)=0$.

Remark 4.1.2. A proper smooth surjective morphism $f: X\to S$ of complex manifolds with irreducible fibres is of class $\mathcal{C}_{loc}$, for instance, if every fibre of f is Moishezon or if X is Kähler, (cf. Fujiki [5]).

(4.2) We first consider I) of Theorem 4.1.1: Since the assertion is clear for $\dim X_o\leqq 1$, we may assume that $\dim X_o=2$. Now, three cases are possible.
Case 1. $\beta(X_o)=2$ (i.e., $\kappa(X_o)\geqq 0$): In this case, by the deformation invariance of κ, (cf. Iitaka [11]), we see that $\kappa(X_s)\geqq 0$ (i.e., $\beta(X_s)=2$) for every $s\in S$.
Case 2. $\beta(X_o)=1$: Then X_o is an irrational ruled surface (i.e., $\kappa(X_o)=-\infty$ and $b_1(X_o)=\text{even}>0$). By the deformation invariance of b_1 and κ, every X_s is again an irrational ruled surface, i.e., $\beta(X_s)=1$.
Case 3. $\beta(X_o)=0$: If $\beta(X_s)\neq 0$ for some $s\in S$, then in view of Cases 1 and 2 above, we should obtain $\beta(X_o)\neq 0$ in contradiction. Thus $\beta(X_s)=0$ for every $s\in S$. These now complete the proof of I) of Theorem 4.1.1.

(4.3) We next consider II) of Theorem 4.1.1.

Proposition 4.3.1. *Let X and S be complex manifolds with* $\dim X=4$ *and* $\dim S=1$. *Let $f: X\to S$ be a proper surjective smooth morphism of class $\mathcal{C}_{loc}$ with irreducible general fibres. Assume further that a fibre X_o ($o\in S$) of f satisfies $\beta(X_o)=2$. Then there exist complex manifolds X', Y, a modification $\mu: X'\to X$, and proper surjective morphisms $g: Y\to S$, $h: X'\to Y$ such that*:
(i) $f\circ\mu=g\circ h$, (ii) *general fibres of h are isomorphic to $\boldsymbol{P}^1$, and* (iii) *every smooth fibre of g is a surface of nonnegative Kodaira dimension.*

Proof. Step 1: Let $\pi: D_{X/S}\to S$ be the relative Douady space of X over S which parametrizes the compact analytic subspaces of X contained in the fibres of f, (cf. Fujiki [3]). Now by Theorem 1.3.4, there exists a Zariski open dense subset V (resp. V') of $B(X_o)$ (resp. X_o) such that $b_{X_o|V'}: V'\to b_{X_o}(V')=V$ is a proper smooth morphism with irreducible fibres isomorphic to $\boldsymbol{P}^1$. Corresponding to this smooth family of rational curves parametrized by V, there exists an irreducible component D_α of $D_{X/S}$ such that we have a natural embedding $i: V \subsetneq D_\alpha$ by sending each $v \in V$ to $i(v) = b_{X_o}^{-1}(v) \in D_\alpha$. Note that, by Fujiki [3; Theorem 4.5], $\pi_\alpha \overset{defn}{=\!=} \pi_{|D_\alpha}: D_\alpha \to S$ is proper. We now consider the universal family $\rho_1: Z_\alpha \to D_\alpha$ with the natural embedding $Z_\alpha \subseteq D_\alpha \times_S X$, where ρ_1 coincides with (resp. $\rho_2: Z_\alpha \to X$ denotes) the restriction to Z_α of the natural projection $D_\alpha \times_S X \to D_\alpha$ (resp. $D_\alpha \times_S X \to X$). For each $v \in V$, regarding $i(v)$ as a rational curve on X, one easily sees that the normal bundles $N_{X_o/X}$ and $N_{i(v)/X_o}$ are trivial. Hence $N_{i(v)/X}$ is again trivial and in particular we obtain $h^0(i(v), N_{i(v)/X}) = 3$ and $h^1(i(v), N_{i(v)/X}) = 0$. Then $\dim D_\alpha = 3$, and every $i(v)$ $(v \in V)$ is a nonsingular point of D_α. Thus ρ_2 is a generically finite surjective morphism, and general fibres of ρ_1 are isomorphic to $\boldsymbol{P}^1$. Choose a desingularization $j: D'_\alpha \to D_\alpha$ of D_α such that j restricts to an isomorphism over the regular locus of D_α. Let $D'_\alpha \xrightarrow{\lambda} S' \xrightarrow{\nu} S$ be the Stein factorization of the morphism $\pi_\alpha \circ j: D'_\alpha \to S$. We claim that the proof of 4.3.1 is reduced to showing the following:

$$\text{(a)} \quad \kappa(\text{general fibre of } \lambda) \geqq 0\,; \qquad \text{(b)} \quad \deg \rho_2 = 1\,.$$

Assume that (a) and (b) are proven. Since all fibres of f are irreducible, this (b) implies that $\deg \nu = 1$. It is now easy to check that, for a suitable modification $\mu: X' \to X$ from a complex manifold X', we have morphisms $g := \pi_\alpha \circ j$ and $h := j^{-1} \circ \rho_1 \circ \rho_2^{-1} \circ \mu$ with the required properties (i), (ii), and (iii) above.

Step 2: Since $(\pi_\alpha \circ j)^{-1}(o)$ contains $j^{-1}(i(V))$ $(\cong V)$ as its subset, there exists a point o' of $\nu^{-1}(o)$ such that an irreducible component (denoted by T) of $\lambda^{-1}(o')_{red}$ is bimeromorphic to $B(X_o)$. Let Δ $(\subseteq S')$ be a small open disc centered at o' such that λ is smooth over $\Delta - \{o'\}$. Then by a standard argument (cf. Ashikaga-Ueno [1]), one easily obtains

$$\kappa(\lambda^{-1}(s')) \geqq \kappa(T) = \kappa(B(X_o)) \geqq 0\,, \qquad \text{for every} \quad s' \in \Delta - \{o'\}\,,$$

which in particular proves (a) of Step 1.

Step 3: General elements of D_α are, as curves on X, isomorphic to $\boldsymbol{P}^1$. Hence every element of D_α represents a connected curve whose support is a union of rational curves on X. Let $W = \{\gamma \in \pi_\alpha^{-1}(o)\,;\, \dim b_{X_o}(V' \cap \rho_2(\rho_1^{-1}(\gamma))) = 1\}$. Since $B(X_o)$ is not uniruled, the subset $U := V - b_{X_o}(V' \cap \rho_2(\rho_1^{-1}(W)))$ of V is nonempty. We then pick a point u of U, and fix a point x_1 on the rational curve $i(u)$, where for each $\gamma \in D_\alpha$, we write $\rho_2(\rho_1^{-1}(\gamma))$ simply as γ if no confusion seems likely to result. We now claim that $\rho_2^{-1}(x_1)$ consists of a single point. Assume the contrary. Then there exists an element γ_1 of D_α such that (1) γ_1 as a curve on X passes through x_1, and that (2) $\gamma_1 \neq i(u)$. Since γ_1 is a connected curve whose restriction to V' is mapped to the point u by b_{X_o}, we have $(\gamma_1)_{red} = i(u) \cong \boldsymbol{P}^1$. By $c_1(X_o)[\gamma_1] = c_1(X_o)[i(u)] = 2$, γ_1 is generically reduced. In view of $\chi(\gamma_1, \mathcal{O}) = \chi(i(u), \mathcal{O}) = \chi((\gamma_1)_{red}, \mathcal{O})$, it would now follow that $\gamma_1 = i(u)$ in contradiction. Thus $\rho_2^{-1}(x_1)$ consists of the single point $z := (i(u), x_1) \in D_\alpha \times_S X$. Next by Step 1, D_α is nonsingular at $i(u)$, and so is Z_α at z. Since $N_{i(u)/X}$ is trivial, (cf. Step 1), we have the natural isomorphisms

$$T_{i(u)}(D_\alpha) \cong H^0(i(u), N_{i(u)/X}) \cong (N_{i(u)/X})_{x_1} \cong T_{x_1}(X)/T_{x_1}(i(u))\,.$$

This then shows that ρ_2 is unramified at z, and hence we obtain (b) of Step 1. The proof of 4.3.1 is now complete. Q. E. D.

Proposition 4.3.2. *Let X and S be complex manifolds with* $\dim X=n$ *and* $\dim S=1$, *where n is an integer with $n\geqq 2$. Let $f: X\to S$ be a proper surjective smooth morphism of class C_{loc} with irreducible fibres. Assume that there exists a dense subset S^0 of S such that* (1) *$S-S^0$ is a countable union of analytic subsets of S and* (2) *$\beta(X_s)=n-2$ for all $s\in S^0$. Then $\beta(X_s)=n-2$ for all $s\in S$.*

Proof. Since the assertion is obvious for $n=2$, we may assume that $n\geqq 3$. For each $m\in \boldsymbol{Z}_+$, we denote by $\mathcal{F}_m$ the locally free sheaf $S^m(\Omega^{n-2}_{X/S})$ on X, and let D_m $(\subseteqq S)$ be the support of the sheaf $f_*(\mathcal{F}_m)$. Note that, by $\dim S=1$, the torsion free sheaf $f_*(\mathcal{F}_m)$ is locally free. Since f is smooth, the natural homomorphism

$$\rho_{m,s}: f_*(\mathcal{F}_m)_s\otimes_{\mathcal{O}_{S,s}}\boldsymbol{C} \longrightarrow H^0(X_s, \mathcal{F}_{m|X_s})(=H^0(X_s, S^m(\Omega^{n-2}_{X_s})))$$

is injective everywhere on S, and is isomorphic for every point s of S outside a nowhere dense analytic subset K_m of S. Now, for every $s\in S^0$, we have $\beta(X_s)=n-2$ and therefore $h^0(X_s, S^m(\Omega^{n-2}_{X_s}))\neq 0$. In particular,

$$\bigcup_{m=1}^{\infty} D_m \supseteqq S^0-\bigcup_{m=1}^{\infty} K_m\,.$$

Since each D_m is a closed analytic subset of S, it then follows that $S=D_{m_0}$ for some $m_0\in \boldsymbol{Z}_+$. We now fix an arbitrary point t of $S-S^0$, and let $s_n\in S^0$, $n=1, 2, \cdots$, be such that $\lim_{n\to\infty} s_n=t$. On a small open neighbourhood U of t in S, we choose a local section $\xi\in H^0(U, f_*(\mathcal{F}_{m_0}))$ such that $\xi(t)\neq 0$. Take a large enough $N\in \boldsymbol{Z}_+$ so that one obtains $s_n\in U$ and $\xi(s_n)\neq 0$ for all $n\geqq N$. Then by Theorem 2.3.1 applied to $p=n-2$ and $m=m_0$, it follows that $\rho_{m_0,s_n}(\xi(s_n))$ is (very neatly foliated and hence) of purely multiple type for all $n\geqq N$, (cf. 2.4.1 and 2.4.2). Here, letting $n\to\infty$, we see that $\rho_{m_0,t}(\xi(t))$ $(\neq 0)$ is also of purely multiple type, (cf. (ii-a) of 2.4.1). On the other hand, by a theorem of Fujiki [5], X_t is uniruled. Theorem 2.4.3 now says that $\beta(X_t)=n-2$, as required.

Q. E. D.

Proof of II-a) *of* 4.1.1. Connecting $o\in S$ and $s\in S$ by a chain of nonsingular holomorphic curves, we may assume that $\dim S=1$ without loss of generality. If either $\beta(X_o)$ or $\beta(X_s)$ is 2, then Propositions 4.3.1 and 4.3.2 immediately imply that $\beta(X_o)=\beta(X_s)$. Therefore we may further assume $\beta(X_o)=1$ and $\beta(X_s)\neq 2$. Since X_o is of class $\mathcal{C}$, it follows that X_o is uniruled with $0<b_1(X_o)=2\cdot\dim \mathrm{Alb}(X_o)$. By the deformation invariance of uniruledness and also of the first Betti number b_1, we see that X_s is uniruled with $0<b_1(X_s)=2\cdot\dim \mathrm{Alb}(X_s)$. Since $\beta(X_s)\neq 2$, we now conclude that $\beta(X_s)=1=\beta(X_o)$. Q. E. D.

Proof of II-b) *of* 4.1.1. In view of II-a) above, we may assume that $\beta(X_o)=0$. Since X_o is uniruled, so is every X_s, and hence $\kappa(X_s)=-\infty$. Then II-c), which we shall prove below, completes the proof. Q. E. D.

Proof of II-c) *of* 4.1.1. By II-a) above, we may assume that $\beta(X_s)$ is neither 1 nor 2. On the other hand, $\kappa(X_s)=-\infty$, i. e., $\beta(X_s)\leqq 2$. Thus $\beta(X_s)=0$.

Q. E. D.

§ 5. Semipositivity.

For later purposes, using the standard technique of Fujita [7], we shall give some delicate analysis of semiposivity of the direct image sheaves of relative differential forms. First we fix our notation: For a proper surjective morphism $g: W\to S$ of normal complex varieties, we denote by $\Omega^1_{W/S}$ the sheaf of germs of holomorphic S-differentials on W in the sense of Grothendieck. Then we put $\Omega^q_{W/S}=\wedge^q\Omega^1_{W/S}$ for each $q\in \boldsymbol{Z}_+$, and let $(\Omega^q_{W/S})^{**}$ be the double dual of $\Omega^q_{W/S}$. Note that $(\Omega^q_{W/S})^{**}$ is a torsion free sheaf on W which coincides with $\Omega^q_{W/S}$ modulo torsion outside an analytic subset of W of codimension$\geqq 2$.

Definition 5.1.1 (cf. Fujita [7]). A vector bundle (or equivalently a locally free sheaf) E over a nonsingular projective curve S is said to be *pseudo-semipositive* if either rank $E=0$ or one has $\deg_S Q\geqq 0$ for any quotient line bundle Q of E.

Theorem 5.1.2. *Let S (resp. W) be a nonsingular projective curve (resp. an n-dimensional compact complex normal variety of class $\mathcal{C}$), and $g: W\to S$ be a surjective morphism only with generically reduced connected fibres. Then for every $q\in\{1, 2, \cdots, n-1\}$, the locally free sheaf $g_*((\Omega^q_{W/S})^{**})$ is pseudo-semipositive.*

Proof. Step 1. Put $E=g_*((\Omega^q_{W/S})^{**})$ and fix an arbitrary quotient line bundle Q of E with an exact sequence $E\xrightarrow{\pi}Q\to 0$. Since W is of class $\mathcal{C}$, there exists a surjective morphism $h: Z\to W$ from a compact Kähler manifold Z. Let U be the Zariski open subset $\{s\in S;$ the fibre $Z_s(=(g\circ h)^{-1}(s))$ is smooth$\}$ of S. Put $m=\dim Z-\dim W$ and we define a Hermitian metric $(\ ,\ )_E$ of the vector bundle $E_{|U}$ by setting

$$(\varphi, \psi)_E=(\sqrt{-1})^{q^2}\int_{Z_s}\omega^{m+n-q-1}\wedge h^*(\varphi)\wedge\overline{h^*(\psi)}$$

for all $\varphi, \psi\in E_s(=H^0(Z_s, \Omega^q_{Z_s}))$ at each $s\in U$, where ω denotes the Kähler form on Z. Since this Hermitian metric on $E_{|U}$ naturally extends to an indefinite flat Hermitian metric on $R^q g_*(\boldsymbol{C})_{|U}$, the standard argument of second fundamental forms (cf. Griffiths [8], Schmid [25]) shows that the curvature form Θ_E of $E_{|U}$ is positive semidefinite. Let $(\ ,\ )_Q$ be the canonical metric on $Q_{|U}$ obtained by identifying $Q_{|U}$ with the orthogonal complement of Ker π in $E_{|U}$. Then again by the argument of second fundamental forms, the curvature form Θ_Q of $Q_{|U}$ is positive semidefinite.

Step 2. Fix an arbitrary point s_0 of $S-U$. We choose an open neighbourhood $T=\{|t|<1\}$ of s_0 in S with a local coordinate t such that $T\cap(S-U)=$

$\{s_0\}=\{t=0\}$. Take a local base $\{\psi_0, \psi_1, \cdots, \psi_r\}$ over T for the vector bundle E so that (i) $\pi(\psi_\alpha)=0$ in $H^0(T, E)$ for $\alpha=1, 2, \cdots, r$, and that (ii) $\pi(\psi_0)$ is a local base of Q over T. Write the fibre $g^{-1}(s_0)$ as a union $\cup_{j=1}^m G_j$ of its irreducible components, and we choose a sufficiently small open neighbourhood $W_j=\{|w_{j;\nu}|<\varepsilon$ for all $\nu\}$ $(0<\varepsilon\ll 1)$ in W of a general point of G_j with a system of local coordinates $(w_{j;1}, w_{j;2}, \cdots, w_{j;n})$ such that $g^*(t)=w_{j;1}$. We next express each $h^*(G_j)\in \mathrm{Div}(Z)$ as $\sum_{k=1}^{m_j} d_{j;k}G_{j;k}$ with multiplicities $d_{j;k}\in \boldsymbol{Z}_+$ and prime divisors $G_{j;k}$ on Z. Furthermore, let $Z_j=\{|z_{j;\mu}|\leqq\delta$ for all $\mu\}$ $(0<\delta\ll\varepsilon)$ be a neighbourhood $(\subseteqq Z)$ of a general point of $G_{j;1}$ with a system of local coordinates $(z_{j;1}, z_{j;2}, \cdots, z_{j;n+m})$ such that (i) $h^*(w_{j;1})=(z_{j;1})^{d_{j;1}}$ and (ii) $h^*(w_{j;\nu})=z_{j;\nu}$ for $\nu=2, 3, \cdots, n$. Choosing δ small enough, we may assume that Z_j, $j=1, 2, \cdots, m$, are mutually disjoint. Now, for every $c\in\boldsymbol{C}$ with $|c|<\delta$, we put:

$$F_{c;j}=\{p\in W_j;\ w_{j;1}(p)=c\}, \quad F'_{c;j}=h^{-1}(F_{c;j})\cap Z_j, \qquad j=1, 2, \cdots, m,$$

$$F_c=\bigcup_{j=1}^m F_{c;j}, \qquad F'_c=\bigcup_{j=1}^m F'_{c;j}.$$

We then define $A_{\alpha\beta}(c)\in\boldsymbol{C}$ ($\alpha, \beta\in\{0, 1, \cdots, r\}$) by

(1) if $c\neq 0$, $\quad A_{\alpha\beta}(c)=\int_{F'_c}\omega^{m+n-q-1}\wedge h^*(\psi_{\alpha|F_c})\wedge\overline{h^*(\psi_{\beta|F_c})}$,

(2) if $c=0$, $\quad A_{\alpha\beta}(0)=\sum_{j=1}^m d_{j;1}\int_{F'_{0;j}}\omega^{m+n-q-1}\wedge h^*(\psi_{\alpha|F_0})\wedge\overline{h^*(\psi_{\beta|F_0})}$,

where in both cases, ψ_α and ψ_β being regarded as elements of $H^0(g^{-1}(T), (\Omega^q_{W/S})^{**})$, their restrictions $\psi_{\alpha|F_c}$ and $\psi_{\beta|F_c}$ are naturally in $H^0(F_c, \Omega^q_{F_c})$ via the isomorphism $(\Omega^q_{W/S})^{**}{}_{|F_c}\cong\Omega^q_{F_c}$. Here, in view of $h^*(w_{j;1})=(z_{j;1})^{d_{j;1}}$, every $A_{\alpha\beta}(c)$ is a continuous function of c on $\{c\in\boldsymbol{C};\ |c|<\delta\}$. Let $\lambda(c)$ be the least eigen value of the $(r+1)\times(r+1)$ Hermitian matrix $(A_{\alpha\beta}(c))_{0\leqq\alpha,\beta\leqq r}$ and $\sigma(2r+1)$ be the sphere $\{\boldsymbol{a}=(a_0, a_1, \cdots, a_r)\in\boldsymbol{C}^{r+1};\ \sum_{\alpha=0}^r|a_\alpha|^2=1\}$. Pick an arbitrary $\boldsymbol{a}=(a_0, a_1, \cdots a_r)\in\sigma(2r+1)$. Since $\{\psi_0, \psi_1, \cdots, \psi_r\}$ is a local base for E, and since $g^{-1}(s_0)$ is generically reduced, $\sum_{\alpha=0}^r a_\alpha\psi_{\alpha|F_0}$ regarded as an element of $H^0(F_0, \Omega^q_{F_0})$ does not vanish identically on F_0, and hence $\sum_{\alpha=0}^r a_\alpha h^*(\psi_{\alpha|F_0})\neq 0\in H^0(F'_0, \Omega^q_{F'_0})$. Thus $(A_{\alpha\beta}(0))_{0\leqq\alpha,\beta\leqq r}$ is a positive definite Hermitian matrix, i.e., $\lambda(0)>0$. On the other hand, for $c\neq 0$,

$$\left(\sum_{\alpha=0}^r a_\alpha\psi_\alpha,\ \sum_{\alpha=0}^r a_\alpha\psi_\alpha\right)_{E|t=c}\geqq\sum_{\alpha,\beta=0}^r a_\alpha\bar{a}_\beta A_{\alpha\beta}(c)\geqq\lambda(c).$$

Since $\lambda(c)$ is a continuous function of c, in view of $\delta\ll 1$, there exists a positive constant K such that:

$$\operatorname*{Inf}_{t,\boldsymbol{a}}\left\{\left(\sum_{\alpha=0}^r a_\alpha\psi_\alpha,\ \sum_{\alpha=0}^r a_\alpha\psi_\alpha\right)_E;\ 0\neq|t|<\delta,\ \boldsymbol{a}\in\sigma(2r+1)\right\}\geqq K.$$

Then by a lemma of Fujita [7; (1.13)], it follows that:

$$(\pi(\psi_0),\ \pi(\psi_0))_Q \geqq K, \qquad \text{for all } t\ (\in T) \text{ with } 0 \neq |t| < \delta.$$

Step 3. Finally by the formula of Fujita [7; (1.16)], the above two steps give us $\deg_S Q \geqq \int_U \Theta_Q \geqq 0$, as required. Q. E. D.

§ 6. Some criteria of positiveness of Kodaira dimension.

Throughout this section, we use the following notation: Fix positive integers p, n such that $p<n$, and we denote by c and c' the binomial coefficients ${}_nC_p$ and ${}_{n-1}C_{p-1}$ respectively. X is an n-dimensional compact complex manifold of class $\mathcal{C}$ such that $h^{p,0}(X)\neq 0$, and $f: X\to Y$ is a surjective morphism of X onto a nonsingular projective curve Y with connected fibres. For every smooth fibre X_y $(=f^{-1}(y))$ $(y\in Y)$, we denote by $i_y^*: H^0(X, \Omega_X^p)\to H^0(X_y, \Omega_{X_y}^p)$ the natural pull-back of p-forms induced by the inclusion $i_y: X_y \subsetneq X$. For every coherent sheaf $\mathcal{F}$ on X, the natural isomorphism of $H^0(X, \mathcal{F})$ and $H^0(Y, f_*(\mathcal{F}))$ is written as $\iota: H^0(X, \mathcal{F})\cong H^0(Y, f_*(\mathcal{F}))$ by using a common letter ι.

(6.1) The purpose of this section is to prove the following:

Theorem 6.1.1. *Let L_1 be a nonzero $\boldsymbol{C}$-linear subspace of $H^0(X, \Omega_X^p)$ satisfying the following conditions:*

(1) *$L_1 \subseteqq \operatorname{Ker} i_y^*$ for every smooth fibre X_y of f.*

(2) *$\iota(L_1)$ generates an invertible subsheaf $\mathcal{L}_1$ of $f_*(\Omega_X^p)$.*

(3) *$\dim L_1 \geqq 2$ (or we may replace this by the weaker condition that $\mathcal{L}_1$ is ample). Furthermore, let $\lambda_1: f_*((\Omega_{X/Y}^{n-p})^{**})\to \operatorname{Hom}(\mathcal{L}_1, f_*(\omega_X))$ be the sheaf homomorphism on Y naturally induced by the wedge product $(\psi, \varphi)\in \Omega_{X/Y,x}^{n-p}\times L_1 \to \psi\wedge\varphi \in \Omega_{X,x}^n = \omega_{X,x}$ $(x\in X)$, and we assume that λ_1 is not trivial. Then $\kappa(X)>0$.*

Theorem 6.1.2. *Assume that there exist $\gamma_1, \gamma_2, \cdots, \gamma_{c-1}\in H^0(X, \Omega_X^p)$ satisfying the following conditions:*

(1) *$i_y^*(\gamma_1)=i_y^*(\gamma_2)=\cdots=i_y^*(\gamma_{c'-1})=i_y^*(\gamma_{c'})=0$ for every smooth fibre X_y of f.*

(2) *Let $\mathcal{E}$ denote Ω_X^p regarded only as a locally free sheaf on X. Then $\gamma_1\wedge\gamma_2\wedge\cdots\wedge\gamma_{c-1}\neq 0$ as an element of $H^0(X, \wedge^{c-1}\mathcal{E})$.*

Let L_2 be a $\boldsymbol{C}$-linear subspace of $\{g\cdot\gamma_1\wedge\gamma_2\wedge\cdots\wedge\gamma_{c-1};\ g\in \boldsymbol{C}(X)\}\cap H^0(X, \wedge^{c-1}\mathcal{E})$ with $\dim L_2\geqq 2$ such that $\iota(L_2)$ generates an invertible subsheaf $\mathcal{L}_2$ of $f_*(\wedge^{c-1}\mathcal{E})$. Furthermore, let $\lambda_2: f_*((\Omega_{X/Y}^p)^{**})\to \operatorname{Hom}(\mathcal{L}_2, f_*(\omega_X^{\otimes c'}))$ be the sheaf homomorphism on Y naturally induced by the pairing $(\psi, \eta)\in \Omega_{X/Y,x}^p\times L_2\to \psi\wedge\eta\in \det(\mathcal{E})_x=(\omega_X^{\otimes c'})_x$ $(x\in X)$, and we assume that λ_2 is not trivial. Then $\kappa(X)>0$.

(6.2) We shall first show that the above λ_1 and λ_2 are well-defined:

(i) Fix a smooth fibre X_{y_0} and its point x_0 arbitrarily. In view of the exact sequence of coherent sheaves on Y,

$$0 \longrightarrow f^*(\Omega_Y^1) \longrightarrow \Omega_X^1 \longrightarrow \Omega_{X/Y} \longrightarrow 0,$$

we have $\Omega_{X/Y}^{n-p}=\wedge^{n-p}\Omega_{X/Y}^1=\Omega_X^{n-p}/(f^*(\Omega_Y^1)\wedge\Omega_X^{n-p-1})$. Let t be a local coordinate of Y centered at y_0. Then by (1) of 6.1.1, every $\varphi \in L_1$ is locally written as $f^*(dt) \wedge$(holomorphic $(p-1)$-form) around x_0, and hence $\psi \wedge \varphi = 0$ for all $\psi \in (f^*(\Omega_Y^1) \wedge \Omega_X^{n-p-1})_{x_0}$. This shows that the pairing

$$(\psi, \varphi) \in (f^*(\Omega_Y^1) \wedge \Omega_X^{n-p-1})_x \times L_1 \longrightarrow \psi \wedge \varphi \in \Omega_{X,x}^n = \omega_{X,x}$$

is trivial for every x on a Zariski open dense subset of X. Moreover, ω_X being a torsion free sheaf, this pairing is trivial everywhere on X. Then we have a natural sheaf homomorphism: $f_*(\Omega_{X/Y}^{n-p}) \to \mathrm{Hom}(\mathcal{L}_1, f_*(\omega_X))$, which induces a well-defined λ_1.

(ii) The conditions (1) and (2) of 6.1.2 show that, on a Zariski open dense subset U of X, the sheaf $f^*(\Omega_Y^1) \wedge \Omega_X^{p-1}$ is locally free with a base $\{\gamma_1, \gamma_2, \cdots, \gamma_{c'}\}$. Hence for every $\psi \in (f^*(\Omega_Y^1) \wedge \Omega_X^{p-1})_x$ with $x \in U$, we have $\psi \wedge \gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_{c'} = 0 \in (\wedge^{c'+1} \mathcal{E})_x$. In particular, the pairing

$$(\psi, \eta) \in (f^*(\Omega_Y^1) \wedge \Omega_X^{p-1})_x \times L_2 \longrightarrow \psi \wedge \eta \in \det(\mathcal{E})_x = (\omega_X^{\otimes c'})_x$$

is trivial for every point x on U. Then by the same argument as in (i), the sheaf homomorphism λ_2 is well-defined.

(6.3) Let $\{y_1, y_2, \cdots y_\rho\}$ be the set of all those points of Y over which f has singular fibres, and we express each $f^*(y_i) \in \mathrm{Div}(X)$ as $\sum_{j=0}^{m_i} e_{ij} D_{ij}$ with multiplicities $e_{ij} \in \boldsymbol{Z}_+$ and prime divisors D_{ij} on X. Fixing a point y_0 on $Y - \{y_1, \cdots, y_\rho\}$, we denote the prime divisor $f^*(y_0)$ by D_{01}. Put $e_{01}=1$ and $m_0=1$. We then define positive integers d and e_i, $0 \leqq i \leqq \rho$, by

$$e_i = \mathrm{l.c.m.}(e_{i1}, e_{i2}, \cdots, e_{im_i}) \quad i=1, 2, \cdots, \rho,$$

$$e_0 = \mathrm{l.c.m.}(e_1, e_2, \cdots, e_\rho), \qquad d = e_1 e_2 \cdots e_\rho .$$

We now have a d-fold abelian covering $\pi: \tilde{Y} \to Y$ which is unramified over $Y - \{y_0, y_1, \cdots y_\rho\}$ and has d/e_i points $\tilde{y}_{i\alpha} \in \tilde{Y}$, $1 \leqq \alpha \leqq d/e_i$, of ramification index e_i over each y_i, $0 \leqq i \leqq \rho$, (see, for instance, Kodaira [20]). Let $\nu: \tilde{X} \to X \times_Y \tilde{Y}$ be the normalization of $X \times_Y \tilde{Y}$ $(=\{(x, \tilde{y}) \in X \times \tilde{Y}; f(x) = \pi(\tilde{y})\}$, and $\tilde{\pi}: \tilde{X} \to X$ (resp. $\tilde{f}: \tilde{X} \to \tilde{Y}$) be the composite of ν with the projection $pr_1: X \times_Y \tilde{Y} \to X$ (resp. $pr_2: X \times_Y \tilde{Y} \to \tilde{Y}$) to the first (resp. second) factor. Then for each i, j $(0 \leqq i \leqq \rho, 1 \leqq j \leqq m_i)$, the divisor $\tilde{\pi}^*(D_{ij})$ is written in the form

$$\tilde{\pi}^*(D_{ij}) = \sum_{\alpha=1}^{d/e_i} \sum_{k=1}^{e_{ij}} (e_i/e_{ij}) \tilde{D}_{ijk}^\alpha$$

with prime divisors $\tilde{D}_{ijk}^\alpha$ $(1 \leqq \alpha \leqq d/e_i, 1 \leqq k \leqq e_{ij})$ on $\tilde{X}$. Note that every fibre of $\tilde{f}$ is reduced and connected. By $X \in \mathcal{C}$, we have $\tilde{X} \in \mathcal{C}$, and hence by 5.1.2,

(a) $f_*((\Omega_{\tilde{X}/\tilde{Y}}^q)^{**})$, $1 \leqq q \leqq n-1$, are pseudo-semipositive locally free sheaves on $\tilde{Y}$.

Let $\tilde{X}^0$ be the Zariski open nonsingular subset $\tilde{X} - \bigcup_{i=1}^{\rho} \tilde{\pi}^{-1}(\mathrm{Sing}(f^{-1}(y_i)_{red}))$ of $\tilde{X}$, where $\mathrm{Sing}(f^{-1}(y_i)_{red})$ denotes the singular locus of $f^{-1}(y_i)_{red}$. Clearly $\mathrm{codim}_{\tilde{X}}(\tilde{X} - \tilde{X}^0) \geqq 2$. We now define a Cartier divisor R on $\tilde{X}^0$ by

$$R=\sum_{i=0}^{\rho}\sum_{j=1}^{m_i}\sum_{\alpha=1}^{d/e_i}\sum_{k=1}^{e_{ij}}((e_i/e_{ij})-1)D^{\alpha}_{ijk|\tilde{X}^0}.$$

Then by a straightforward computation, we obtain

(b) $(\tilde{\pi}^*\omega_X)_{|\tilde{X}^0}=\omega_{\tilde{X}^0}(-R)$.

Let $\tilde{f}^0$ be the restriction $\tilde{f}_{|\tilde{X}^0}$ of $\tilde{f}$ to $\tilde{X}^0$, and for every sheaf $\mathcal{F}$ on $\tilde{X}^0$, the natural isomorphism of $H^0(\tilde{X}^0, \mathcal{F})$ and $H^0(\tilde{Y}, (\tilde{f}^0)_*(\mathcal{F}))$ will be written as $\iota^0: H^0(\tilde{X}^0, \mathcal{F})\cong H^0(\tilde{Y}, (\tilde{f}^0)_*(\mathcal{F}))$ by using the common ι^0. Let $\tilde{\mathcal{E}}$ denote $\Omega^p_{\tilde{X}}$ regarded only as a coherent sheaf on $\tilde{X}$. Since $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) is generated by $\iota(L_1)$ (resp. $\iota(L_2)$), the mapping which sends $\pi^*(\iota(\varphi))\in H^0(\tilde{Y}, \pi^*\mathcal{L}_1)$ (resp. $\pi^*(\iota(\eta))\in H^0(Y, \pi^*\mathcal{L}_2)$) to $\iota^0(\tilde{\pi}^*(\varphi)_{|\tilde{X}^0})\in H^0(\tilde{Y}, (\tilde{f}^0)_*(\Omega^p_{\tilde{X}}))$ (resp. $\iota^0(\tilde{\pi}^*(\eta)_{|\tilde{X}^0})\in H^0(\tilde{Y}, (\tilde{f}^0)_*(\wedge^{c-1}\tilde{\mathcal{E}})))$ for each $\varphi\in L_1$ (resp. $\eta\in L_2$) naturally induces a sheaf homomorphism

$$j_1: \pi^*\mathcal{L}_1 \hookrightarrow (\tilde{f}^0)_*(\Omega^p_{\tilde{X}}) \qquad (\text{resp. } j_2: \pi^*\mathcal{L}_2 \hookrightarrow (\tilde{f}^0)_*(\wedge^{c-1}\tilde{\mathcal{E}})).$$

Then $\pi^*\mathcal{L}_1$ (resp. $\pi^*\mathcal{L}_2$), as a subsheaf of $(\tilde{f}^0)_*(\Omega^p_{\tilde{X}})$ (resp. $(\tilde{f}^0)_*(\wedge^{c-1}\tilde{\mathcal{E}})$), satisfies the following:

Lemma 6.3.1. (i) $\pi^*\mathcal{L}_1\subseteq(\tilde{f}^0)_*(\Omega^p_{\tilde{X}}(-R))$.
(ii) $\pi^*\mathcal{L}_2\subseteq(\tilde{f}^0)_*((\wedge^{c-1}\tilde{\mathcal{E}})(-c'R))$.

Proof. Let $\varphi\in L_1$ and $\eta\in L_2$. It then suffices to show $\tilde{\pi}^*(\varphi)_{|\tilde{X}^0}\in H^0(\tilde{X}^0, \Omega^p_{\tilde{X}}(-R))$ and $\tilde{\pi}^*(\eta)_{|\tilde{X}^0}\in H^0(\tilde{X}^0, (\wedge^{c-1}\tilde{\mathcal{E}})(-c'R))$. Fix an arbitrary $\tilde{D}^{\alpha}_{ijk}$ and let $\tilde{u}$ be its general point. We choose a sufficiently small open neighbourhood $\tilde{U}$ of $\tilde{u}$ (resp. U of $\tilde{\pi}(\tilde{u})$) with a system of local coordinates $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ (resp. $(x_1, x_2, \cdots, x_n)$) such that (1) $\tilde{\pi}(\tilde{U})\subseteq U$, (2) $\tilde{x}_1=0$ locally defines $\tilde{D}^{\alpha}_{ijk}$, (3) $\tilde{\pi}^*(x_1)=\tilde{x}_1^{e_i/e_{ij}}$, and (4) $\tilde{\pi}^*(x_\gamma)=\tilde{x}_\gamma$ for $2\leqq\gamma\leqq n$. Let $\mathcal{A}$ be the set of all those subsets of $\{2, 3, \cdots n\}$ whose cardinality is $p-1$, and for each $A=\{\alpha_2, \alpha_3, \cdots, \alpha_p\}\in\mathcal{A}$ with $\alpha_2<\alpha_3<\cdots<\alpha_p$, we put $s_A=dx_1\wedge dx_{\alpha_2}\wedge dx_{\alpha_3}\wedge\cdots\wedge dx_{\alpha_p}$ and $\tilde{s}_A=d\tilde{x}_1\wedge d\tilde{x}_{\alpha_2}\wedge d\tilde{x}_{\alpha_3}\wedge\cdots\wedge d\tilde{x}_{\alpha_p}$. Then, in view of (1) of 6.1.1 and (1), (2) of 6.1.2, the restrictions $\varphi_{|U}$ and $\eta_{|U}$ are written in the form $\varphi_{|U}=dx_1\wedge\zeta$ and $\eta_{|U}=(\bigwedge_{A\in\mathcal{A}} s_A)\wedge\psi$ for some $\zeta\in H^0(U, \Omega^{p-1}_X)$ and $\psi\in H^0(U, \wedge^{c-c'-1}\mathcal{E})$. Hence

$$\tilde{\pi}^*(\varphi)_{|\tilde{U}}=(e_i/e_{ij})\tilde{x}_1^{(e_i/e_{ij})-1}d\tilde{x}_1\wedge\tilde{\pi}^*(\zeta),$$

$$\tilde{\pi}^*(\eta)_{|\tilde{U}}=(e_i/e_{ij})^{c'}\tilde{x}_1^{c'((e_i/e_{ij})-1)}(\bigwedge_{A\in\mathcal{A}}\tilde{s}_A)\wedge\tilde{\pi}^*(\psi).$$

Thus, when restricted to $\tilde{X}^0$, our $\tilde{\pi}^*(\varphi)$ (resp. $\tilde{\pi}^*(\eta)$) has a zero along each $\tilde{D}^{\alpha}_{ijk}$, of order at least $(e_i/e_{ij})-1$ (resp. $c'((e_i/e_{ij})-1)$), as required. Q. E. D.

(6.4) By the same argument as in (6.2), to every $(\psi, \varphi)\in((\Omega^{n-p}_{\tilde{X}/\tilde{Y}})^{**})_{\tilde{x}}\times L_1$ (resp. $(\psi, \eta)\in((\Omega^p_{\tilde{X}/\tilde{Y}})^{**})_{\tilde{x}}\times L_2$), we can associate the wedge product

$$(\psi, \pi^*(\varphi))\longmapsto\psi\wedge\pi^*(\varphi)\in((\Omega^n_{\tilde{X}})^{**})_{\tilde{x}},$$

$$(\text{resp. } (\psi, \pi^*(\eta))\longmapsto\psi\wedge\pi^*(\eta)\in((\wedge^c\tilde{\mathcal{E}})^{**})_{\tilde{x}}),$$

which naturally induces the sheaf homomorphism

$$\mu_1 : \tilde{f}_*((\Omega^{n-p}_{\tilde{X}/\tilde{Y}})^{**}) \otimes_{\mathcal{O}_{\tilde{Y}}} \tilde{\pi}^*\mathcal{L}_1 \longrightarrow f_*((\Omega^n_{\tilde{X}})^{**}),$$

$$(\text{resp. } \mu_2 : \tilde{f}_*((\Omega^p_{\tilde{X}/\tilde{Y}})^{**}) \otimes_{\mathcal{O}_{\tilde{Y}}} \tilde{\pi}^*\mathcal{L}_2 \longrightarrow f_*((\wedge^c \tilde{\mathcal{E}})^{**})).$$

Now by 6.3.1, $\pi^*\mathcal{L}_1 \subseteqq (\tilde{f}^0)_*(\Omega^p_{\tilde{X}}(-R))$ (resp. $\pi^*\mathcal{L}_2 \subseteqq (\tilde{f}^0)_*((\wedge^{c-1}\tilde{\mathcal{E}})(-c'R))$), and hence

$$\begin{aligned} \text{Image } \mu_1 &\subseteqq (\tilde{f}^0)_*((\Omega^n_{\tilde{X}})^{**}(-R)) = (\tilde{f}^0)_*(\omega_{\tilde{X}^0}(-R)) \\ &= (\tilde{f}^0)_*((\tilde{\pi}^*\omega_X)_{|\tilde{X}^0}), \quad (\text{cf. (b) of (6.3)}), \\ &= \tilde{f}_*(\tilde{\pi}^*\omega_X), \quad (\text{because } \operatorname{codim}_{\tilde{X}}(\tilde{X}-\tilde{X}^0) \geqq 2), \end{aligned}$$

$$\left(\begin{aligned} \text{resp. } \quad \text{Image } \mu_2 &\subseteqq (\tilde{f}^0)_*((\wedge^c\tilde{\mathcal{E}})^{**}(-c'R)) = (\tilde{f}^0)_*((\omega_{\tilde{X}^0})^{\otimes c'}(-c'R)) \\ &= (\tilde{f}^0)_*((\tilde{\pi}^*\omega_X)^{\otimes c'}{}_{|\tilde{X}^0}) = \tilde{f}_*((\tilde{\pi}^*\omega_X)^{\otimes c'}) \end{aligned}\right).$$

Thus we have the natural homomorphism

$$\tilde{\lambda}_1 : \tilde{f}_*((\Omega^{n-p}_{\tilde{X}/\tilde{Y}})^{**}) \longrightarrow \operatorname{Hom}(\pi^*\mathcal{L}_1, \tilde{f}_*(\tilde{\pi}^*\omega_X))$$

$$(\text{resp. } \tilde{\lambda}_2 : \tilde{f}_*((\Omega^p_{\tilde{X}/\tilde{Y}})^{**}) \longrightarrow \operatorname{Hom}(\pi^*\mathcal{L}_2, \tilde{f}_*((\tilde{\pi}^*\omega_X)^{\otimes c'})))$$

of locally free sheaves on $\tilde{Y}$.

Proof of 6.1.1 (*resp.* 6.1.2). Note that λ_1 (resp. λ_2) is not trivial. Hence neither is $\tilde{\lambda}_1$ (resp. $\tilde{\lambda}_2$). Now by 5.1.2, $f_*((\Omega^{n-p}_{\tilde{X}/\tilde{Y}})^{**})$ (resp. $f_*((\Omega^p_{\tilde{X}/\tilde{Y}})^{**})$) is pseudo-semipositive, and therefore its image $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) under the map $\tilde{\lambda}_1$ (resp. $\tilde{\lambda}_2$) is again a pseudo-semipositive locally free sheaf on $\tilde{Y}$. Since $\dim L_1 \geqq 2$ (resp. $\dim L_2 \geqq 2$), $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) is ample, and so is $\pi^*\mathcal{L}_1$ (resp. $\pi^*\mathcal{L}_2$). Then every quotient line bundle of $\mathcal{S}_1 \otimes \pi^*\mathcal{L}_1$ (resp. $\mathcal{S}_2 \otimes \pi^*\mathcal{L}_2$) has positive degree, and hence by a theorem of Hartshorne [9], $\mathcal{S}_1 \otimes \pi^*\mathcal{L}_1$ (resp. $\mathcal{S}_2 \otimes \pi^*\mathcal{L}_2$) is ample. Since $\mathcal{S}_1 \subseteqq \operatorname{Hom}(\pi^*\mathcal{L}_1, \tilde{f}_*(\tilde{\pi}^*\omega_X))$ (resp. $\mathcal{S}_2 \subseteqq \operatorname{Hom}(\pi^*\mathcal{L}_2, \tilde{f}_*((\tilde{\pi}^*\omega_X)^{\otimes c'}))$), we naturally have an inclusion of sheaves

$$\mathcal{F} \subseteqq \tilde{f}_*((\tilde{\pi}^*\omega_X)^{\otimes e})$$

by putting $e=1$ (resp, $e=c'$) and $\mathcal{F} = \mathcal{S}_1 \otimes \tilde{\pi}^*\mathcal{L}_1$ (resp. $\mathcal{F} = \mathcal{S}_2 \otimes \tilde{\pi}^*\mathcal{L}_2$), where $\mathcal{F}$ is an ample locally free sheaf. Note that, for every $d \in \boldsymbol{Z}_+$, the image of the sub-sheaf $S^d(\mathcal{F})$ of $S^d(\tilde{f}_*((\tilde{\pi}^*\omega_X)^{\otimes e}))$ under the natural sheaf homomorphism

$$\zeta_d : S^d(\tilde{f}_*((\tilde{\pi}^*\omega_X)^{\otimes e})) \longrightarrow \tilde{f}_*((\tilde{\pi}^*\omega_X)^{\otimes de})$$

is nontrivial. We first choose a large enough $d' \in \boldsymbol{Z}_+$ such that $S^{d'}(\mathcal{F})$ is generated by global sections. Then there exists a section $\sigma \in H^0(\tilde{Y}, S^{d'}(\mathcal{F}))$ which satisfies $\zeta_{d'}(\sigma(\tilde{y}_0)) \neq 0$ for some point $\tilde{y}_0$ on $\tilde{Y}$. Next, let $d'' \in \boldsymbol{Z}_+$ be such that $\mathcal{J}_{\tilde{y}_0} \cdot S^{d''}(\mathcal{F})$ is generated by global sections, where $\mathcal{J}_{\tilde{y}_0}$ is the ideal sheaf of $\tilde{y}_0$ in $\tilde{Y}$. We can then find a section $\tau \in H^0(\tilde{Y}, S^{d''}(\mathcal{F}))$ with $\zeta_{d''}(\tau) \neq 0$ and $\tau(\tilde{y}_0) = 0$. Now, $(\zeta_{d'}(\sigma))^{\otimes d''}$ and $(\zeta_{d''}(\tau))^{\otimes d'}$ are $\boldsymbol{C}$-linearly independent in $H^0(\tilde{Y}, \tilde{f}_*((\tilde{\pi}^*\omega_X)^{\otimes d'd''e}))$. Thus

$$0 < \kappa(\tilde{\pi}^*\omega_X, \tilde{X}) = \kappa(\omega_X, X) = \kappa(X),$$

which completes the proof. Q. E. D.

Remark 6.4.1. (a) The arguments in (6.3) and (6.4) are still valid even if we replace $\tilde{f}:\tilde{X}\to\tilde{Y}$ by the "semi-stable reduction" (cf. Mumford et al. [24]) of $f:X\to Y$, although such replacement doesn't so much simplify the proofs of 6.1.1 and 6.1.2.
(b) Theorems 6.1.1 and 6.1.2 can be generalized in various ways. For instance, it is easy to extend them to asymptotic cases. We shall discuss such a topic in a separate paper [22].

§ 7. L-fibrations.

In this section, we shall define the concept of L-fibrations and give the basic properties.

Definition 7.1.1. Fix positive integers p, n with $p\leqq n$. Let X be an n-dimensional compact complex manifold such that $h^{p,0}(X)>0$, and L be an arbitrary $\boldsymbol{C}$-linear subspace of $H^0(X, \Omega_X^p)$ with $l=\dim L>0$. We denote by r the rank of the subsheaf of Ω_X^p generated by L, and let $Gr(l, l-r)$ be the complex Grassmann variety of $(l-r)$-planes in L $(=\boldsymbol{C}^l)$. We consider the meromorphic map $\Psi: X\to Gr(l, l-r)$ defined generically by

$$\begin{aligned}\Psi: X &\longrightarrow Gr(l, l-r)\\ x &\longmapsto \{\omega\in L\,;\,\omega(x)=0\}\,.\end{aligned}$$

1) The closed subvariety $\operatorname{Im}\Psi$ of $Gr(l, l-r)$ denotes the meromorphic image of X under the meromorphic map Ψ.
2) The *fundamental subspace* L_0 *of* L is the linear subspace of L spanned by $\bigcup_{z\in\operatorname{Im}\Psi} P(z)$, where $P(z)$ denotes the $(l-r)$-plane in L corresponding to z.
3) A surjective morphism $f: X'\to Y$ of compact complex manifolds with connected fibres is called an *L-fibration of* X, if there exist a modification $j: X'\to X$ and a generically finite surjective morphisn $\nu: Y\to\operatorname{Im}\Psi$ such that the following diagram commutes:

$$\begin{array}{ccc} X' & \xrightarrow{\ j\ } & X \\ {\scriptstyle f}\downarrow & \circlearrowright & \downarrow{\scriptstyle \Psi} \\ Y & \xrightarrow{\ \nu\ } & \operatorname{Im}\Psi. \end{array}$$

Note that, given X and L, an L-fibration of X always exists as follows: Let K be the algebraic closure of $\boldsymbol{C}(\operatorname{Im}\Psi)$ in $\boldsymbol{C}(X)$, and we take a nonsingular projective variety Y such that (i) $\boldsymbol{C}(Y)=K$, and that (ii) the meromorphic map $\nu: Y\to\operatorname{Im}\Psi$ induced by $\boldsymbol{C}(\operatorname{Im}\Psi)\subseteqq\boldsymbol{C}(Y)$ is a morphism. Since $\boldsymbol{C}(Y)\subseteqq\boldsymbol{C}(X)$, we have a meromorphic map $g: X\to Y$. Choose a modification $j: X'\to X$ from a compact complex manifold X' so that $f\overset{defn}{=\!=}g\circ j: X'\to Y$ is a morphism. This f now defines an L-fibration of X. Moreover, given X and L, all possible L-fibrations of X are mutually bimeromorphically equivalent, because Y is a Moishezon manifold characterized bimeromorphically by $C(Y)=K$.

4) Let X'_y $(=f^{-1}(y))$ $(y\in Y)$ be an arbitrary smooth fibre of f. Then, in terms of the notation above, $j^*(\omega)_{|X'_y}=0\in H^0(X'_y, (\Omega^p_{X'})_{|X'_y})$ for all $\omega\in P(\nu(y))$.

5) Let $i_y: X'_y\hookrightarrow X'$ be the natural inclusion. Then we define the *horizontal subspace* L_h *of* L by

$$L_h=\{\omega\in L\,;\,(i_y^*\circ j^*)(\omega)=0 \text{ for every smooth fibre } X'_y \text{ of } f\}\,,$$

where $i_y^*\circ j^*: H^0(X, \Omega^p_X)\to H^0(X'_y, \Omega^p_{X'_y})$ denotes the natural pullback of p-forms by $j\circ i_y$. For fixed X and L, this L_h is independent of the choice of L-fibrations of X.

The following theorem is of crucial importance in our later study of holomorphic 2-forms on compact complex threefolds.

Theorem 7.1.2. *Let $p, n\in \boldsymbol{Z}_+$ with $p\leqq n$, and let X be an n-dimensional compact complex manifold of class $\mathcal{C}$ such that $h^{p,0}(X)>0$. Fix an arbitrary nonzero $\boldsymbol{C}$-linear subspace L of $H^0(X, \Omega^p_X)$. Then, between the fundamental subspace L_0 of L and the horizontal one L_h of L, we have the inclusion $L_0\subseteqq L_h$.*

Proof. Let Y^0 be the Zariski open subset $\{y\in Y\,;\,X'_y \text{ is smooth}\}$ of Y. Since $\bigcup_{y\in Y^0} P(\nu(y))$ spans L_0 in L, the proof is reduced to showing $(i_y^*\circ j^*)(P(\nu(y')))=\{0\}$ for all $y, y'\in Y^0$. Fix an arbitrary $\omega\in P(\nu(y'))$. By 4) of 7.1.1, we have $j^*(\omega)_{|X'_{y'}}=0\in H^0(X'_{y'}, (\Omega^p_{X'})_{|X'_{y'}})$, and in particular $(i_{y'}^*\circ j^*)(\omega)=0$. Since $\Psi_{|\Psi^{-1}(Y^0)}: \Psi^{-1}(Y^0)\to Y^0$ is locally trivial as a C^∞-fibration, every p-cycle γ on X'_y can be deformed to a p-cycle γ' on $X'_{y'}$ over a piecewise-smooth path $\subseteqq Y^0$. Hence $(i_y)_*(\gamma)-(i_{y'})_*(\gamma')=\partial\tau$ for some $(p+1)$-chain τ on X', and we obtain

$$\int_\gamma (i_y^*\circ j^*)(\omega)=\int_\gamma (i_y^*\circ j^*)(\omega)-\int_{\gamma'} (i_{y'}^*\circ j^*)(\omega)$$

$$=\int_{(i_y)_*(\gamma)-(i_{y'})_*(\gamma')}\omega=\int_{\partial\tau}\omega=\int_\tau d\omega=0\,,$$

where $d\omega=0$ follows from $X\in\mathcal{C}$. Thus $(i_y^*\circ j^*)(\omega)$ is cohomologous to 0. In view of $X_y\in\mathcal{C}$ (which follows from $X\in\mathcal{C}$), the holomorphic p-form $(i_y^*\circ j^*)(\omega)$ vanishes identically. We now conclude that $(i_y^*\circ j^*)(P(\nu(y')))=\{0\}$. Q. E. D.

Remark 7.1.3. If $p=n-1$ and $\dim Y=1$, then Theorem 7.1.2 is valid even if we get rid of the assumption that X is of class $\mathcal{C}$. This is an immediate consequence of the following facts:

Let M be an m-dimensional compact complex manifold. Then

(1) Every holomorphic $(m-1)$-form ω on M is d-closed, because $\int_M d\omega\wedge\overline{d\omega}=\int_M d(\omega\wedge d\bar{\omega})=0$ implies $d\omega=0$.

(2) Every holomorphic m-form η on M which is cohomologous to 0 is identically

0, because η is written as $d\xi$ for some C^∞ $(m-1)$-form ξ on M and hence $\int_M \eta\wedge\bar{\eta} = \int_M d(\xi\wedge d\bar{\xi})=0$ implies $\eta=0$.

(7.2) We shall here make a little more study of L-fibrations. First we give the following technical lemma.

Lemma 7.2.1. *Let X (resp. Z) be an n-dimensional (resp. p-dimensional) compact complex manifold, and $g: X\to Z$ be a surjective morphism with connected fibres. Assume that $h^{p,0}(X)>0$, and suppose $\omega\in H^0(X, \Omega_X^p)$ satisfies the following condition:*

(♯) *There exists a Zariski open dense subset U of X such that, for every $x\in U$, ω is locally written as $s\cdot g^*(\xi)$ for some $s\in\mathcal{O}_{X,x}$ and $\xi\in\Omega^p_{Z,g(x)}$, where $g^*(\xi)\in\Omega^p_{X,x}$ denotes the natural pullback of ξ by g.*

Then there exists $\eta\in H^0(Z, \Omega_Z^p)$ such that $\omega=g^*(\eta)$.

Proof. Let V be the Zariski open dense subset $\{z\in Z; g^{-1}(z)$ is smooth$\}$ of Z. Fix a point v on V and a point w on $g^{-1}(v)$ arbitrarily. We choose a sufficiently small open neighbourhood N_v of v in Z (resp. M_w of w in X) with a system of local coordinates $(z_1, z_2, \cdots, z_p)$ (resp. $(g^*(z_1), g^*(z_2), \cdots, g^*(z_p), x_1, x_2, \cdots, x_{n-p})$). Then around w, we can express ω as a sum $s\cdot g^*(dz_1\wedge dz_2\wedge\cdots\wedge dz_p)+\omega'$, where $s\in\mathcal{O}_{X,w}$ and $\omega'=\sum_{i=1}^{n-p}\psi_i\wedge dx_i$ with $\psi_i\in\Omega^{p-1}_{X,w}$. Now, the condition (♯) above shows that $\omega'=0$ (even if $w\notin U$). Thus $\omega/g^*(dz_1\wedge\cdots\wedge dz_p)$ is a holomorphic function on $g^{-1}(N_v)$. Since $g_*\mathcal{O}_X=\mathcal{O}_Z$, it then follows that $\omega_{|g^{-1}(N_v)}=g^*(\xi)$ for some $\xi\in H^0(N_v, \Omega_Z^p)$. Varying v in V, we obtain $\eta\in H^0(V, \Omega_Z^p)$ such that $\omega_{|g^{-1}(V)}=g^*(\eta)$, and the assertion of our lemma is now straightforward from 2.2.3. Q.E.D.

The following fact on horizontal subspaces associated with L-fibrations will be needed in §8.

Proposition 7.2.2. *Let $p, n\in \mathbf{Z}_+$ with $p\leqq n$, and let X be an n-dimensional compact complex manifold such that $h^{p,0}(X)>0$. Fix an arbitrary nonzero $\mathbf{C}$-linear subspace L of $H^0(X, \Omega_X^p)$, and let $f: X'\to Y$ be an L-fibration of X. We assume that* dim $Y=1$ *and that the horizontal subspace L_h of L has positive dimension. Then*

(a) *The locally free sheaf $f_*((\Omega^p_{X'/Y})^{**})$ on Y is not a zero sheaf.*

(b) *Assume furthermore that there exists a quadruple (Z, g, g', Y^0) satisfying the following conditions:*

i) *$g: X'\to Z$ is a surjective morphism of X' onto a p-dimensional compact complex manifold Z with connected fibres.*

ii) *$g': Z\to Y$ is a surjective morphism such that $f=g'\circ g$.*

iii) *Y^0 is a Zariski open dense subset of Y such that, for every $y\in Y^0$, the re-*

striction $g_{|X'_y}: X'_y(=f^{-1}(y))\to Z_y(=g'^{-1}(y))$ is a morphism of compact complex manifolds inducing an isomorphism $(g_{|X'_y})^*: H^0(Z_y, \Omega^{p-1}_{Z_y})\cong H^0(X'_y, \Omega^{p-1}_{X'_y})$.

Then every $\omega\in L_h$ is expressible as $g^*(\eta)$ for some $\eta\in H^0(Z, \Omega^p_Z)$ $(=H^0(Z, K_Z))$. In particular $p_g(Z)\geqq\dim L_h$.

Proof. Note that, without loss of generality, we may assume $X'=X$.

(a) Let $0\neq\omega\in L_h$. Then for a general smooth fibre X'_y of f, we have $\omega_{|X'_y}\neq 0$ in $H^0(X'_y, (\Omega^p_X)_{|X'_y})$. Let $i_y: X'_y\hookrightarrow X'$ $(=X)$ be the natural inclusion as usual. In view of $i^*_y(\omega)=0$, to each tangent vector $0\neq\theta\in T_y(Y)$, we can associate a well-defined $(p-1)$-form $0\neq\omega_\theta\in H^0(X'_y, \Omega^p_{X'_y})$ by the following equality. Namely, we put

$$\omega_\theta(\theta_2, \theta_3, \cdots, \theta_p)=\omega(\theta_1, \theta_2, \cdots, \theta_p), \quad \text{for all } \theta_2, \theta_3, \cdots, \theta_p\in T_{x'}(X'_y),$$

at each point x' of X'_y, where $\theta_1\in T_{x'}(X')$ is a tangent vector satisfying $f_*(\theta_1)=\theta$. Thus $f_*((\Omega^p_{X'/Y})^{**})_y\otimes_{\mathcal{O}_{Y,y}}\boldsymbol{C}\cong H^0(X'_y, \Omega^p_{X'_y})\neq\{0\}$ for general smooth fibres X'_y of f, i.e., $f^*((\Omega^p_{X'/Y})^{**})$ is not a zero sheaf.

(b) Let $0\neq\omega\in L_h$. Fix a point x' on $f^{-1}(Y^0)$ arbitrarily, and we put $y=f(x')$ and $z=g(x')$. Choose a $\boldsymbol{C}$-basis $\{\theta_1, \theta_2, \cdots, \theta_n\}$ for the tangent space $T_{x'}(X')$ such that $\theta_i\in T_{x'}(g^{-1}(z))$ for $1\leqq i\leqq n-p$ and that $\theta_i\in T_{x'}(X'_y)$ for $1\leqq i\leqq n-1$. Let $\mathcal{A}$ be the set of all those subsets A of $\{1, 2, \cdots, n\}$ which satisfies $A\cap\{1, 2, \cdots, n-p\}\neq\phi$. For each $A=\{\alpha_1, \alpha_2, \cdots, \alpha_p\}\in\mathcal{A}$ (where $\alpha_1<\alpha_2<\cdots<\alpha_p$), we have two possibilities:

Case 1: $n\notin A$. Then by $i^*_y(\omega)=0$, it follows that $\omega(\theta_{\alpha_1}, \theta_{\alpha_2}, \cdots, \theta_{\alpha_p})=0$.
Case 2: $n\in A$. In this case, $\alpha_p=n$ and we put $\theta=f_*(\theta_n)\in T_y(Y)$. Then, using the notation in (a) above, we have

$$\omega(\theta_{\alpha_1}, \theta_{\alpha_2}, \cdots, \theta_{\alpha_p})=(-1)^{p-1}\omega_\theta(\theta_{\alpha_1}, \theta_{\alpha_2}, \cdots, \theta_{\alpha_{p-1}})=0,$$

where in the last equality, we use the condition iii) above.

Thus, in both cases, we obtain $\omega(\theta_{\alpha_1}, \theta_{\alpha_2}, \cdots, \theta_{\alpha_p})=0$. This now shows that, around each point x' of $f^{-1}(Y^0)$, ω is written as $s\cdot g^*(\xi)$ for some $s\in\mathcal{O}_{X',x'}$ and $\xi\in\Omega^p_{Z,g(x')}$. Then by 7.2.1, we can find $\eta\in H^0(Z, \Omega^p_Z)$ such that $\omega=g^*(\eta)$.

Q. E. D.

§ 8. Holomorphic 2-forms.

The main purpose of this section is to give a partial affirmative answer (cf. 8.4.1) to the following conjecture:

Conjecture 8.1.1. *Let X be a 3-dimensional compact complex manifold of class $\mathcal{C}$, and let r be the rank of the subsheaf of Ω^2_X generated by the global sections $H^0(X, \Omega^2_X)$. Then*

(i) *If $\kappa(X)=-\infty$ and $h^{2,0}(X)>r$, then $\beta(X)=2$.*
(ii) (Ueno [32]). *If $\kappa(X)=0$, then $h^{2,0}(X)=r$ (and in particular $h^{2,0}(X)\leqq 3$).*

Remark 8.1.2. *If Conjecture* 2.1.3 *is true, then so is* (i) *of* 8.1.1.

Proof of 8.1.2. Since $\kappa(X)=-\infty$, one has three possibilities: $\beta(X)=0, 1, 2$. First, if $\beta(X)=0$, then 2.1.3 says that $h^{2,0}(X)=0$. Secondly, if $\beta(X)=1$, Theorem 2.1.1 shows that $h^{2,0}(X)=h^{2,0}(B(X))=0$. Thus, in both cases, we have a contradiction to $h^{2,0}(X)>r$. Hence $\beta(X)=2$. Q.E.D.

(8.2) The following observation due to Ueno and myself gives an affirmative answer to the case $r=3$ of (ii) of 8.1.1.

Proposition 8.2.1. *Fix positive integers p ond n arbitrarily with $p\leqq n$. Let X be an n-dimensional compact complex manifold with $\kappa(X)\leqq 0$, and r be the rank of the subsheaf of Ω_X^p generated by the global sections $H^0(X, \Omega_X^p)$. Suppose r coincides with the binomial coefficient ${}_nC_p$. Then $\kappa(X)=0$ and $h^{p,0}(X)=r$.*

Proof. Since the rank of the locally free sheaf Ω_X^p coincides with r, there exist sections $\gamma_1, \gamma_2, \cdots \gamma_r\in H^0(X, \Omega_X^p)$ which form a local base for Ω_X^p over a Zariski open dense subset of X. Then $\gamma_1\wedge\gamma_2\wedge\cdots\wedge\gamma_r$, regarded as an element of $H^0(X, \det(\Omega_X^p))$, is nonzero. On the other hand, putting $e={}_{n-1}C_{p-1}$, we have $\det(\Omega_X^p)=\omega_X^{\otimes e}$. It now follows that $\kappa(X)\geqq 0$, and hence $\kappa(X)=0$. Assume, for contradiction, that $h^{p,0}(X)>r$. Completing $\{\gamma_1, \cdots, \gamma_r\}$ to a $\boldsymbol{C}$-basis $\{\gamma_1, \cdots, \gamma_r, \gamma_{r+1}, \cdots\}$ for $H^0(X, \Omega_X^p)$, we express γ_{r+1} as $\sum_{i=1}^r f_i\gamma_i$ with meromorphic functions f_i on X. Since all of $f_1, f_2, \cdots f_r$ cannot be constant, we may assume that f_1 is nonconstant. Then $\gamma_{r+1}\wedge\gamma_2\wedge\cdots\wedge\gamma_r$ $(=f_1\gamma_1\wedge\gamma_2\wedge\cdots\wedge\gamma_r)$ and $\gamma_1\wedge\gamma_2\wedge\cdots\wedge\gamma_r$ are $\boldsymbol{C}$-linearly independent global sections of $\omega_X^{\otimes e}$ on X, which contradicts the equality $\kappa(X)=0$. Q.E.D.

(8.3) We next give a couple of results which brought us a definite progress in the study of the case $r\leqq 2$ of Conjecture 8.1.1.

Theorem 8.3.1. *Let X be a compact complex 3-dimensional manifold of class $\mathcal{C}$ with $\kappa(X)\leqq 0$. Assume that there exists a 2-dimensional $\boldsymbol{C}$-linear subspace L of $H^0(X, \Omega_X^2)$ which generates a subsheaf of rank 1 in Ω_X^2 Then $\beta(X)=2$.*

Theorem 8.3.2. *Let X be a compact complex 3-dimensional manifold of class $\mathcal{C}$ with $\kappa(X)\leqq 0$. Assume that there exists a 3-dimensional $\boldsymbol{C}$-linear subspace L of $H^0(X, \Omega_X^2)$ which generates a subsheaf of rank 2 in Ω_X^2. Let $\Psi: X\to\boldsymbol{P}^2$ be the meromorphic map defined generically by*

$$\begin{aligned}\Psi: X &\longrightarrow Gr(3, 2)(\cong\boldsymbol{P}^2)\\ x &\longmapsto \{\omega\in L\,;\,\omega(x)=0\}\,.\end{aligned}$$

Then either $\beta(X)=2$ or Ψ is generically surjective.

Proof of 8.3.1. *Step* 1. Let $\{\omega_0, \omega_1\}$ be a $\boldsymbol{C}$-basis for L, and consider the generically surjective meromorphic map $\Psi: X\to\boldsymbol{P}^1$ defined generically by

$$\Psi : X \longrightarrow \boldsymbol{P}^1 = \{(z_0 : z_1)\}$$
$$x \longmapsto (\omega_0(x) : \omega_1(x)) .$$

Fixing an L-fibration $f: X' \to Y$ (cf. 7.1.1) of X, we have the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\ j\ } & X \\ {\scriptstyle f}\downarrow & \circlearrowright & \downarrow{\scriptstyle \Psi} \\ Y & \xrightarrow{\ \nu\ } & \boldsymbol{P}^1 , \end{array}$$

where $j: X' \to X$ is a modification and $\nu: Y \to \boldsymbol{P}^1$ is a finite morphism. Since Ψ is generically surjective, the fundamental subspace L_0 of L must be L itself, (cf. 7.1.1). Hence by 7.1.2, for every smooth fibre X'_y of f, the subspace $j^*(L) = \{j^*(\omega) ; \omega \in L\}$ of $H^0(X', \Omega^2_{X'})$ is contained in the kernel of $i^*_y : H^0(X', \Omega^2_{X'}) \to H^0(X'_y, \Omega^2_{X'_y})$, where $i_y : X'_y \hookrightarrow X'$ denotes the natural inclusion. Moreover, since $j^*(\omega_1) = j^*(\Psi^*(z_1/z_0)) \cdot j^*(\omega_0) = f^*(\nu^*(z_1/z_0)) \cdot j^*(\omega_0)$, we see that $\iota(j^*(L))$ generates an invertible subsheaf $\mathcal{L}$ in $f_*(\Omega^2_{X'})$, where $\iota : H^0(X', \Omega^2_{X'}) \cong H^0(Y, f_*(\Omega^2_{X'}))$ is the canonical isomorphism. Let $\lambda : f_*((\Omega^1_{X'/Y})^{**}) \to \mathrm{Hom}(\mathcal{L}, f_*(\omega_{X'}))$ be the natural sheaf homomorphism induced by the wedge product $(\phi, \varphi) \in \Omega^1_{X'/Y, x'} \times L \mapsto \phi \wedge \varphi \in \omega_{X', x'} (x' \in X')$. Since $\kappa(X') = \kappa(X) \leqq 0$, Theorem 6.1.1 then asserts that λ is trivial. We now fix a general smooth fibre X'_y of f, and let $\alpha_y : X'_y \to \mathrm{Alb}(X'_y)$ be the Albanese map. First, by (a) of 7.2.2, $\dim \alpha_y(X'_y) \geqq 1$. It then follows that $\dim \alpha_y(X'_y) = 1$, because otherwise the pairing

$$f_*((\Omega^1_{X'/Y})^{**})_y \otimes_{\mathcal{O}_{Y,y}} \boldsymbol{C} (= H^0(X'_y, \Omega^1_{X'_y})) \times L \longrightarrow f_*(\omega_{X'})_y \otimes_{\mathcal{O}_{Y,y}} \boldsymbol{C} (= H^0(X'_y, \omega_{X'_y}))$$
$$(\eta, \quad \varphi) \longmapsto \eta \wedge \varphi$$

would be nontrivial in contradiction to the fact that λ is a zero homomorphism.

Step 2. We put $U = \{y \in Y ; X'_y \text{ is smooth}\}$. Since a general fibre of f is a nonsingular surface, a theorem of Fujiki [4] states that, choosing a suitable bimeromorphic model X'' of X' with $f^{-1}(U) \subseteqq X''$, one has a compact complex variety T and a surjective morphism $f'' : X'' \to Y$ with the next two properties: (1) f'' coincides with f when restricted to the Zariski open subset $f^{-1}(U)$ of X''; (2) $f'' : X'' \to Y$ factors through T, where α and γ below are morphisms such that for each $y \in U$, i) $\gamma^{-1}(y)$ is a complex torus, and ii) the morphisms $\alpha_{|X'_y} : X'_y \to \alpha(X'_y)$ and $\alpha_y : X'_y \to \alpha_y(X'_y)$ coincide via an identification of $\alpha(X'_y)$ with $\alpha_y(X'_y)$.

$$\begin{array}{ccc} X'' & \xrightarrow{\ f''\ } & Y \\ & {\scriptstyle \alpha}\searrow \ \circlearrowright \ \nearrow{\scriptstyle \gamma} & \\ & T & \end{array}$$

Note that the image $\alpha(X'')$ of X'' is a (possibly singular) surface. Taking a suitable nonsingular bimeromorphic model $\tilde{X}$ of X'' (resp. Z of $\alpha(X'')$) with $f^{-1}(U) \subseteqq \tilde{X}$ (resp. $\gamma^{-1}(U) \subseteqq Z$), we have the following commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & Y \\ & {\scriptstyle\tilde{\alpha}}\searrow \quad \circlearrowright \quad \nearrow{\scriptstyle\tilde{\gamma}} & \\ & Z, & \end{array}$$

where over U, the morphisms $\tilde{f}$, $\tilde{\alpha}$, $\tilde{\gamma}$ coincide with f'', α, γ respectively. Now by (b) of 7.2.2, we have $p_g(Z) \geqq \dim L_h = \dim L = 2$, and in particular $\kappa(Z) > 0$. Then from

$$0 \geqq \kappa(\tilde{X}) \geqq \kappa(\text{general fibre of } \tilde{\alpha}) + \kappa(Z), \qquad (\text{cf. Viehweg [27]}),$$

it follows that $\kappa(\text{general fibre of } \tilde{\alpha}) = -\infty$. Hence $\kappa(\tilde{X}) \leqq \kappa(\text{general fibre of } \tilde{\alpha}) + \dim Z = -\infty$. Thus $2 \geqq \beta(\tilde{X}) \geqq \dim Z = 2$, and we conclude that $\beta(X) = \beta(\tilde{X}) = 2$.

Q. E. D.

Proof of 8.3.2. Once for all, we assume that Ψ is not generically surjective. Let $\{\omega_0, \omega_1, \omega_2\}$ be a $\boldsymbol{C}$-basis for L, and we write Ω_X^2 as $\mathcal{E}$, regarding it only as a locally free sheaf on X. Then, from our assumption, $\omega_0 \wedge \omega_1, \omega_1 \wedge \omega_2, \omega_0 \wedge \omega_2 \in H^0(X, \wedge^2 \mathcal{E})$ generate a subsheaf of rank 1 in $\wedge^2 \mathcal{E}$. Note that Ψ is generically given by

$$\Psi : X \longrightarrow \boldsymbol{P}^2 = \{(z_0 : z_1 : z_2)\}$$
$$x \longmapsto ((\omega_1 \wedge \omega_2)(x) : (\omega_0 \wedge \omega_1)(x) : (\omega_0 \wedge \omega_2)(x)).$$

Because of Theorem 8.3.1, we may assume that any two distinct elements of L are linearly independent over $\boldsymbol{C}(X)$. Hence no hyperplanes of $\boldsymbol{P}^2$ can contain the meromorphic image $\operatorname{Im} \Psi$, and therefore

i) $S := \boldsymbol{C}\omega_1 \wedge \omega_2 + \boldsymbol{C}\omega_0 \wedge \omega_1 + \boldsymbol{C}\omega_0 \wedge \omega_2$ is a 3-dimensional subspace of $H^0(X, \wedge^2 \mathcal{E})$;
ii) the fundamental subspace L_0 of L coincides with L.

Now, fixing an L-fibration $f : X' \to Y$ (cf. 7.1.1) of X, we have the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\;j\;} & X \\ {\scriptstyle f}\downarrow & \circlearrowright & \downarrow{\scriptstyle \Psi} \\ Y & \xrightarrow{\;\nu\;} & \operatorname{Im} \Psi, \end{array}$$

where j is a modification and ν is a finite morphism. In view of ii) above, Theorem 7.1.2 shows that, for every smooth fibre X'_y of f, all $j^*(\omega_\alpha)$ $(\alpha = 0, 1, 2)$ are contained in the kernel of $i_y^* : H^0(X', \Omega_{X'}^2) \to H^0(X'_y, \Omega_{X'_y}^2)$, where $i_y : X'_y \hookrightarrow X'$ denotes the natural inclusion. We now write $\Omega_{X'}^2$ as $\tilde{\mathcal{E}}$, regarding it only as a locally free sheaf on X'. From the equalities $f^*(\omega_0 \wedge \omega_\alpha) = f^*(\nu^*(z_\alpha / z_0)) \cdot j^*(\omega_1 \wedge \omega_2)$, $\alpha = 1, 2$, it then follows that $\iota(j^*(S))$ $(:= \{\iota(j^*(\sigma)) ; \sigma \in S\})$ generates an invertible subsheaf $\mathcal{S}$ of $f_*(\wedge^2 \tilde{\mathcal{E}})$, where $\iota : H^0(X', \wedge^2 \tilde{\mathcal{E}}) \cong H^0(Y, f_*(\wedge^2 \tilde{\mathcal{E}}))$ denotes the canonical isomorphism. We now apply Theorem 6.1.2 to the fibration $f : X' \to Y$ with

$n=3$ and $p=2$: Let $\lambda: f_*((\Omega^2_{X'/Y})^{**}) \to \mathrm{Hom}(\mathcal{S}, f_*(\omega_{X'}^{\otimes 2}))$ be the sheaf homomorphism on Y naturally induced by the pairing $(\eta, j^*(\sigma)) \in \Omega^2_{X'/Y, x'} \times j^*(S) \mapsto \eta \wedge j^*(\sigma) \in \det(\Omega^2_{X'})_{x'} = (\omega_{X'}^{\otimes 2})_{x'} (x' \in X')$. Since $\kappa(X') \leqq 0$, Theorem 6.1.2 then asserts that λ is trivial. We now fix a general smooth fibre X'_y of f, and let $\alpha_y: X'_y \to \mathrm{Alb}(X'_y)$ be the Albanese map. First, by (a) of 7.2.2, $\dim \alpha_y(X'_y) \geqq 1$. It then follows that $\dim \alpha_y(X'_y)=1$, because otherwise we should have a nonzero locally free sheaf $f_*((\Omega^2_{X'/Y})^{**})$ on Y in contradiction to the fact that λ is a zero homomorphism. Thus, just by the same argument as in Step 2 of the proof of 8.3.1, we now conclude that $\beta(X)=2$. Q.E.D.

Remark 8.3.3. Without the assumption that X is of class $\mathcal{C}$, we still have the following statement[*)].

Let X be a compact complex manifold of dimension $n \geqq 2$. Assume that there exists a nonzero $\boldsymbol{C}$-linear subspace L of $H^0(X, \Omega^{n-1}_X)$ which generates a subsheaf of rank 1 *in Ω^{n-1}_X. Let $\{\omega_1, \omega_2, \cdots, \omega_l\}$ be a $\boldsymbol{C}$-basis for L, and $\Psi: X \to \boldsymbol{P}^{l-1}$ be the meromorphic map defined generically by*

$$\Psi: X \longrightarrow \boldsymbol{P}^{l-1} = \{(z_1 : z_2 : \cdots : z_l)\}$$
$$x \longmapsto (\omega_1(x) : \omega_2(x) : \cdots : \omega_l(x)) .$$

We furthermore assume that the meromorphic image $\mathrm{Im}\,\Psi$ *has dimension at least $n-1$. Then*

i) $\dim \mathrm{Im}\,\Psi = n-1$, (cf. Bogomolov [2]).
ii) *Fixing an arbitrary L-fibration $f: X' \to Y$ of X with its associated modification $j: X' \to X$, we can express each element ω in L as $(f \circ j^{-1})^*(\eta)$ for some $\eta \in H^0(Y, \Omega^{n-1}_Y)$.*
iii) *If $\kappa(X) \leqq 0$, then general fibres of f in* ii) *above are $\boldsymbol{P}^1$, (and in particular $\kappa(X) = -\infty$).*

Proof of 8.3.3. i) is straightforward from the inequality $\dim \mathrm{Im}\,\Psi \leqq n-1$ which is a consequence of a theorem of Bogomolov [2; (12.2)].
ii) Since $f: X' \to Y$ is an L-fibration of X, there exists a generically finite morphism $\nu: Y \to \mathrm{Im}\,\Psi$ such that the following diagram commutes:

$$\begin{array}{ccc} X' & \xrightarrow{\ j\ } & X \\ {\scriptstyle f}\downarrow & \circlearrowright & \downarrow{\scriptstyle \Psi} \\ Y & \xrightarrow[\ \nu\ ]{} & \mathrm{Im}\,\Psi \subseteq \boldsymbol{P}^{l-1} . \end{array}$$

For each $\alpha, \beta \in \boldsymbol{Z}$ with $1 \leqq \alpha < \beta \leqq l$, let $\pi_{\alpha\beta}: \boldsymbol{P}^{l-1} \to \boldsymbol{P}^1$ denote the meromorphic projection $(z_1 : z_2 : \cdots : z_l) \mapsto (z_\alpha : z_\beta)$ to the α-th and β-th factors. Replacing X' and Y by their suitable bimeromorphic models respectively, we may assume that every $\pi_{\alpha\beta} \circ \nu: Y \to \boldsymbol{P}^1$ $(1 \leqq \alpha < \beta \leqq l)$ is a morphism. We now take a general

*) When $n=3$, this reduces to a special case of 8.3.1.

point $\theta=(\theta_1:\theta_2:\cdots:\theta_l)$ of $\mathrm{Im}\,\Psi$ such that the following conditions are all satisfied:

(a) θ is a nonsingular point of $\mathrm{Im}\,\Psi$.

(b) $\theta_1\neq 0$, and $F_k:=(\pi_{1\beta_k}\circ\nu\circ f)^{-1}((\theta_1:\theta_{\beta_k}))$ is smooth for any $k\in\{1,2,\cdots,n-1\}$, where $\{\beta_1,\cdots,\beta_{n-1}\}$ $(\subseteqq\{1,\cdots,l\})$ is such that $(\pi_{1\beta_k}\circ\nu\circ f)^*(z_{\beta_k}/z_1)$, $k=1,2,\cdots,n-1$, form a system of local parameters of $\mathrm{Im}\,\Psi$ at θ.

(c) $(\nu\circ f)^{-1}(\theta)$ is smooth (and in particular reduced).

Then by Remark 7.1.3, $i_k^*(j^*(\omega_\gamma))=0$ for all $k\in\{1,2,\cdots,n-1\}$ and $\gamma\in\{1,2,\cdots,l\}$, where $i_k: F_k\hookrightarrow X'$ denotes the natural inclusion. Now, one easily sees that there exists a Zariski open dense subset U of X' such that, for any $x'\in U$, each $j^*(\omega_\gamma)$ $(1\leqq\gamma\leqq l)$ is locally written as $s\cdot f^*(\xi)$ for some $s\in\mathcal{O}_{X',x'}$ and $\xi\in\Omega^{n-1}_{Y,f(x')}$. Applying Lemma 7.2.1, we can finally express each $j^*(\omega)$ $(\omega\in L)$ as $f^*(\eta)$ for some $\eta\in H^0(Y,\Omega_Y^{n-1})$.

iii) From ii) above, we obtain $p_g(Y)\geqq l>1$, and in particular $\kappa(Y)>0$. Hence in view of the inequality $0\geqq\kappa(X')\geqq\kappa(Y)+\kappa(\text{general fibre of } f)$, (cf. Viehweg [27]), we now conclude that $\kappa(\text{general fibre of } f)=-\infty$, i.e., general fibres of f are $\boldsymbol{P}^1$. Q.E.D.

(8.4) Combining 8.2.1, 8.3.1, and 8.3.2, we finally obtain:

Theorem 8.4.1. *Let X be a compact complex 3-dimensional manifold of class C with $\kappa(X)\leqq 0$. Put $l=h^{2,0}(X)$. Let r be the rank of the subsheaf of Ω_X^2 generated by the global sections $H^0(X,\Omega_X^2)$, and we denote by $\Phi: X\to Gr(l,l-r)$ the meromorphic map defined generically by*

$$\Phi: X\longrightarrow Gr(l,l-r)$$
$$x\longmapsto\{\omega\in H^0(X,\Omega_X^2);\ \omega(x)=0\}\,.$$

Then we have at least one of the following:

(1) *$\beta(X)=2$ (and hence $\kappa(X)=-\infty$).*

(2) *$r=l=3$ and $\kappa(X)=0$.*

(3) *$r=l\leqq 2$.*

(4) *X is not uniruled with $r=2<l$ and the meromorphic image $\mathrm{Im}\,\Phi$ of Φ has dimension at least 2.*

Proof. If $r=3$, then by 8.2.1, we obtain (2) above. On the other hand, if $r=0$, it follows that $l=0$, and hence (3) is the case. In view of $r\leqq l$, there are three remaining possibilities:

Case (a). $1\leqq r=l\leqq 2$: Then we have (3) above.

Case (b). $r=1<l$: Then, applying Theorem 8.3.1 to an arbitrary 2-dimensional $\boldsymbol{C}$-linear subspace L of $H^0(X,\Omega_X^2)$, we obtain (1) above.

Case (c). $r=2<l$: Choose a 3-dimensional $\boldsymbol{C}$-linear subspace L of $H^0(X,\Omega_X^2)$ which generates a subsheaf of rank 2 in Ω_X^2. Let $\Psi: X\to Gr(3,2)$ be the meromorphic map defined generically by

$$\Psi : X \longrightarrow Gr(3, 2)(=\boldsymbol{P}^2)$$
$$x \longmapsto \{\omega \in L\,;\, \omega(x)=0\}\,.$$

Then by Theorem 8.3.2, either we have (1) above or $2=\dim \operatorname{Im} \Psi \leqq \dim \operatorname{Im} \Phi$. Thus, in view of (4) above, it suffices to consider the following subcase of (c):

Subcase: X is uniruled with $r=2<l$.

But then Corollary 2.4.4 asserts that (1) above is true in this subcase, and now the proof of 8.4.1 is complete.

Remark 8.4.2. Conjecture 8.1.1 and Theorem 8.4.1 will be reconsidered in a separate paper [22] from a different viewpoint.

§ 9. Appendix (I).

In this appendix, we shall give a rough sketch of how one can divide the compact complex threefolds of negative Kodaira dimension into several interesting classes. The theorems given below are somewhat of expository nature, heavily depending on the recent results of Fujiki [4], [6], and Ueno [31]; we also use the standard techniques of Kawai [14], [15], and Ueno [30].

Definition 9.1.1. Let X be a compact complex variety.

(a) (cf. Ueno [30]). An *algebraic reduction* $alg_X : X^* \to X_{alg}$ of X is a natural morphism of a nonsingular bimeromorphic model X^* of X onto a projective algebraic manifold X_{alg} with $\boldsymbol{C}(X)=\boldsymbol{C}(X_{alg})$. We then put $a(X):=\dim X_{alg}$.
(b) (cf. Fujiki [6]). A *$\mathcal{C}$-reduction* $c_X : X^* \to X_c$ of X is a morphism of a nonsingular bimeromorphic model X^* of X onto "the largest" compact complex manifold X_c of class $\mathcal{C}$ which is dominated by X. We put $c(X):=\dim X_c$.
(c) (cf. Fujiki [6]). X is called *simple*, if there are no pair of surjective morphisms of compact complex varieties $(\pi : Z \to X,\ \rho : Z \to Y)$ such that $0<\dim \pi(\rho^{-1}(y))<\dim X$ for a general point y of Y.

We need the following results of Fujiki:

Theorem 9.1.2. (Fujiki [4]). *Let X be a compact complex variety with $\dim X - a(X)=2$ and $X \in \mathcal{C}$. Then we have an algebraic reduction $alg_X : X^* \to X_{alg}$ of X such that, if U is the Zariski open dense subset $\{u \in X_{alg} \mid X_u^* := alg_X^{-1}(u)$ is smooth$\}$ of X_{alg}, then one of the following holds:*

(a) *Each X_u^* $(u \in U)$ is a complex torus.*
(b) *Each X_u^* is a $K3$ surface, and $a(X_u^*)=0$ if u is a general point of U.*
(c) *Each X_u^* is an almost homogeneous relatively minimal ruled surface of genus* 1.

Theorem 9.1.3. (Fujiki [6]). *Let X be a compact Kähler 3-dimensional manifold with $a(X)=1$. Assume that general fibres of an algebraic reduction of X*

are either $K3$ surfaces or non-algebraic complex tori. Then there exists a generically surjective meromorphic map from X to a non-algebraic $K3$ surface.

(9.2) We now consider compact complex threefolds of negative Kodaira dimension. The cases $X\in\mathcal{C}$ and $X\notin\mathcal{C}$ will be treated separately.

Theorem 9.2.1. *Any compact complex 3-dimensional manifold X with $X\in\mathcal{C}$ and $\kappa(X)=-\infty$ belongs to one of the following seven types of compact complex manifolds:*

Type	$a(X)$	$\beta(X)$	general fibre of an algebraic reduction of X	general fibre of $b_X: X\to B(X)$	other structure
I	0	0	X	X	simple
II	1	0	complex torus	X	$X_{alg}=\boldsymbol{P}^1$
III	1	0	$K3$ surface	X	$X_{alg}=\boldsymbol{P}^1$
IV	2	0	elliptic curve	X	a suitable X_{alg} is $\boldsymbol{P}^2$
V	3	0	single point	X	?
VI	3	1	single point	rational surface	uniruled
VII		2		$\boldsymbol{P}^1$	uniruled

Proof. Since $\kappa(X)=-\infty$, we have $\beta(X)=0$, 1, or 2. Then the following six cases are possible:

Case 1. $a(X)=\beta(X)=0$: In this case, by Fujiki's theory of W^*-reduction [6], X is easily shown to be simple as follows. For contradiction, we assume that X is not simple. Then, letting $f: X\to Y$ be an W^*-reduction of X, one sees that Y is a compact complex surface of class $\mathcal{C}$ with $a(Y)=0$. Y would now be bimeromorphic to either a complex torus or a $K3$ surface in contradiction to $0\leqq\beta(Y)\leqq\beta(X)=0$. Thus X is simple, and is of type I above.

Case 2. $a(X)=1$ and $\beta(X)=0$: First, in view of $0\leqq\beta(X_{alg})\leqq\beta(X)=0$, we obtain $X_{alg}=\boldsymbol{P}^1$. Secondly, by 9.1.2, we can find an algebraic reduction $alg_X: X^*\to X_{alg}$ of X whose general fibre is one of the following: (a) a complex torus, (b) a $K3$ surface, (c) a ruled surface of genus 1. We shall eliminate the last case (c). If (c) is the case, then denoting by $\mathrm{Alb}(X/X_{alg})$ the relative Albanese variety of X over X_{alg}, (cf. Fujiki [4]), we have a generically surjective meromorphic map of X to the surface $\mathrm{Alb}(X/X_{alg})$. Since $\mathrm{Alb}(X/X_{alg})$ is of class $\mathcal{C}$, the inequality $0\leqq\beta(\mathrm{Alb}(X/X_{alg}))\leqq\beta(X)=0$ implies that $\mathrm{Alb}(X/X_{alg})$ is rational. Then $a(X)$

$\geqq \dim \mathrm{Alb}(X/X_{alg})=2$ which contradicts our assumption $a(X)=1$. Thus (c) cannot occur. Hence in our Case 2, X is of type either II or III.

Case 3. $a(X)=2$ and $\beta(X)=0$: Since $a(X)=\dim X-1$, a general fibre of alg_X: $X^*\to X_{alg}$ is an elliptic curve. On the other hand, $0\leqq\beta(X_{alg})\leqq\beta(X)\leqq 0$, and therefore X_{alg} is a rational surface. Thus, in this case, X is of type IV.

Case 4. $a(X)=3$ and $\beta(X)=0$: Then X is clearly of type V.

Case 5. $\beta(X)=1$: By Theorem 1.3.4, a general fibre of b_X is a rational surface, and in particular X is uniruled. Furthermore, since $B(X)$ is algebraic and since a general fibre of $b_X: X\to B(X)$ is Moishezon with irregularity 0, it follows that X is also Moishezon, i.e., $a(X)=3$. Hence X is of type VI.

Case 6. $\beta(X)=2$: By Theorem 1.3.4, general fibres of b_X are isomorphic to $\boldsymbol{P}^1$, and in particular X is uniruled. Thus X is of type VII. Q.E.D.

Remark 9.2.2. In Theorem 9.2.1, we further assume that X is Kähler. Then from Theorem 9.1.3, one immediately obtains:

1) Type III cannot occur;
2) if X is of Type II, general fibres of an algebraic reduction of X are abelian varieties.

Theorem 9.2.3. *Let X be a compact complex 3-dimensional manifold with $X\notin\mathcal{C}$ and $\kappa(X)=-\infty$. Assume that X cannot dominate any non-Kähler K3 surface by a generically surjective meromorphic map. Then alg_X and c_X are bimeromorphically equivalent, and we have one of the following:*

(1) $a(X)=\beta(X)=c(X)=0$: *Then two cases are possible.*

(a) *X is simple.*

(b) (cf. Fujiki [6]). *There exists a generically surjective meromorphic map $\sigma: X\to S$ of X to a compact complex surface S of class* VII_0 *with $\kappa(S)=-\infty$ and $a(S)=0$ such that* i) *for some Zariski open dense subset U of X, $\sigma_{|U}: U\to\sigma(U)$ is a proper morphism having a general fibre isomorphic to either $\boldsymbol{P}^1$ or an elliptic curve, and that* ii) *for any generically surjective meromorphic map $f: X\to Y$ of X to a compact complex variety Y with $0<\dim Y<\dim X$, there exists a generically finite meromorphic map $f': S\to Y$ satisfying $f=f'\circ\sigma$.*

(2) $a(X)=c(X)=1$ *and* $\beta(X)=0$: *Then a general fibre of alg_X is bimeromorphic to one of the following surfaces,* (cf. Kawai [14], [15], Ueno [30]):

(a) *K3 surface;*
(b) *hyperelliptic surface;*
(c) *Enriques surface;*
(d) *complex torus;*
(e) *elliptic surface with trivial canonical bundle;*
(f) *surface of class* VII_0*;*
(g) *rational surface;*
(h) *ruled surface of genus* 1.

(3) $a(X)=c(X)=1$ *and* $\beta(X)=1$: *Then a general fibre of alg_X is bimeromorphic to one of the following surfaces,* (cf. Ueno [30], [31]).

(a) *non-algebraic K3 surface;*
(b) *non-algebraic complex torus;*
(c) *surface of class* VII_0 *with* $\kappa=-\infty$*;*
(d) *rational surface;*
(e) *ruled surface of genus* 1.

(4) $a(X)=c(X)=2$ *and* $\beta(X)\leqq 1$: *Then* X_{alg} *is an either rational or ruled surface, and a general fibre of* alg_X *is an elliptic curve.*

(5) $a(X)=c(X)=1$ *and* $\beta(X)=2$: *Then* $B(X)$ *is an elliptic surface with odd first Betti number fibred over the curve* $B(X)_{alg}=X_{alg}$, *and a general fibre of* b_X *is isomorphic to* $\boldsymbol{P}^1$.

(6) $a(X)=\beta(X)=c(X)=2$: *We choose a suitable* X_{alg}. *Then* X_{alg} *(resp.* $B(X)$*) is a ruled surface (resp. an elliptic surface with odd first Betti number) over the curve* $C:=B(X)_{alg}$, *and a general fibre of* alg_X *(resp.* b_X*) is an elliptic (resp. rational) curve. Furthermore, replacing* X *by its suitable bimeromorphic model, we have a generically finite surjective morphism* $b_X\times alg_X: X\to B(X)\times_C X_{alg}$ *which sends each* $x\in X$ *to* $(b_X(x), alg_X(x))\in B(X)\times_C X_{alg}$.

Remark 9.2.4. Since there is a gap in the paper of Todorov [26] who claims that every $K3$ surface is Kähler, we are unable to eliminate the above cumbersome assumption that X cannot dominate any non-Kähler $K3$ surface by a generically surjective meromorphic map.

Proof of 9.2.3. Replacing X by its suitable bimeromorphic model, we may assume that $b_X: X\to B(X)$ is holomorphic and that a $\mathcal{C}$-reduction of X is given by a morphism $c_X: X\to X_c$.

Step 1. First we consider the case $\beta(X)=2$: Then by a theorem of Viehweg [27], general fibres of b_X are isomorphic to $\boldsymbol{P}^1$. In particular, $X\notin\mathcal{C}$ implies $B(X)\notin\mathcal{C}$. Thus $B(X)$ is a non-Kähler surface with $\kappa(B(X))\geqq 0$. Being unable to be a $K3$ surface, $B(X)$ is now an elliptic surface with odd first Betti number, (cf. Kodaira [21], Miyaoka [23]).

Step 2. Next we consider the case $c(X)=2$: In this case, a general fibre of c_X is an elliptic curve, (cf. Fujiki [6]). Then $\kappa(X_c)=-\infty$, because otherwise $\kappa(X)=-\infty$ would imply κ(general fibre of c_X)$=-\infty$, which is a contradiction. Thus X_c is an either rational or ruled surface. In particular, c_X and alg_X are bimeromorphically equivalent.

Step 3. Now we come back to the general situation: In view of $\kappa(X)=-\infty$ and $X\notin\mathcal{C}$, we have $\beta(X)\leqq 2$ and $c(X)\leqq 2$. Since $X_c=X_{alg}$ for $c(X)\leqq 1$, Step 2 above implies that c_X and alg_X are always bimeromorphically equivalent. Hence $a(X)=c(X)$, and we may assume $c_X=alg_X$. Since $(a(X), \beta(X))\neq(0, 1)$, and since by Step 1, $(a(X), \beta(X))\neq(0, 2)$, we have one of the following: (i) $a(X)=\beta(X)=c(X)=0$, (ii) $a(X)=c(X)=1$ and $\beta(X)=0$, (iii) $a(X)=c(X)=1$ and $\beta(X)=1$, (iv) $a(X)=c(X)=2$ and $\beta(X)\leqq 1$, (v) $a(X)=c(X)=1$ and $\beta(X)=2$, (vi) $a(X)=\beta(X)=c(X)=2$.

Step 4. We consider each of six cases of Step 3:

Case (i): Assume that X is not simple. Choose a W*-reduction $f: X\to Y$ of

X, (cf. Fujiki [6]), where in our case Y can be taken as a relatively minimal surface with $a(Y)=0$. Then a general fibre of f is a curve which is either elliptic or rational. If $\kappa(Y)=-\infty$, then Y is a surface of class VII_0, and hence according to Fujiki [6], we have the situation (b) of (1) above. If $\kappa(Y)\geqq 0$, then Y would be either a $K3$ surface or a complex torus in contradiction to $0\leqq\beta(Y)\leqq\beta(X)=0$. It now follows that, in our Case (i), we have (1) above.

Case (ii): In this case, it is well-known that (2) above holds, (cf. Ueno [30]).

Case (iii): Then a recent result of Ueno [31] together with standard facts (cf. Ueno [30]) immediately implies (3) above.

Case (iv): Since $X_{alg}=X_c$, Step 2 shows that (4) above holds.

Case (v): Then by Step 1, $B(X)$ is an elliptic surface with odd first Betti number. Since $alg_{B(X)}\circ b_X: X\to B(X)_{alg}$ is a morphism with connected fibres, we infer from $a(X)=1$ that the curve $B(X)_{alg}$ coincides with X_{alg}. Hence (5) immediately follows.

Case (vi): Then by Step 1, $B(X)$ is an elliptic surface with odd first Betti number (naturally fibred over the curve $C:=B(X)_{alg}$). Now, $b_X: X\to B(X)$ naturally induces $\bar{b}_X: X_{alg}\to C$ (we may assume that $\bar{b}_X$ is a morphism by replacing X and X_{alg} by their suitable bimeromorphic models if necessary), and we obtain the following commutative diagram of surjective morphisms:

$$\begin{array}{ccc} X & \xrightarrow{\quad b_X \quad} & B(X) \\ {\scriptstyle alg_X}\downarrow & \circlearrowright & \downarrow{\scriptstyle alg_{B(X)}} \\ X_{alg} & \xrightarrow[\quad \bar{b}_X \quad]{} & C=B(X)_{alg}. \end{array}$$

Since $alg_{B(X)}\circ b_X$ is a morphism with connected fibres, so is $\bar{b}_X$. Now choose a general point z of $B(X)$. Then $b_X^{-1}(z)\cong \boldsymbol{P}^1$, and $alg_X(b_X^{-1}(z))$ sits in a fibre of $\bar{b}_X$. Here $alg_X(b_X^{-1}(z))$ is not a point, because otherwise alg_X would factor through $B(X)$ so that $alg_X=g\circ b_X$ for some generically finite surjective meromorphic map $g: B(X)\to X_{alg}$ in contradiction to $a(B(X))=1$. Thus X_{alg} is a ruled surface over C, and $b_X\times alg_X: X\to B(X)\times_C X_{alg}$ is a generically finite surjective morphism. We now obtain (6). Q. E. D.

§ 10. Appendix (II).

Finally, we shall generalize Theorems 1.3.3 and 1.3.10 as follows: Let X be a nonsingular S-variety of class $\mathcal{C}_{loc}$ with $\dim(X/S)=3$, (cf. § 1). Considering the meromorphic maps $b_{X/S}: X\to B(X/S)$ and $b'_{X/S}: X\to B'(X/S)$, (cf. 1.1.3), we put $X_s=\pi_X^{-1}(s)$, $B(X/S)_s=(\pi_{B(X/S)})^{-1}(s)$, $B'(X/S)_s=(\pi_{B'(X/S)})^{-1}(s)$ for each $s\in S$, where the morphisms π_X, $\pi_{B(X/S)}$, $\pi_{B'(X/S)}$ are as in (a) of 1.1.1.

Theorem 10.1.1. *If X_s is a general fibre of π_X, then there is a natural bimeromorphic identification of $B(X/S)_s$ with $B(X_s)$ such that the restriction*

$(b_{X/S})_{|X_s}: X_s \to B(X/S)_s$ coincides with $b_{X_s}: X_s \to B(X_s)$ bimeromorphically. In particular, $\beta(X_s)=\beta(X/S)$ for general points s of S.

Theorem 10.1.2. *If X_s is a general fibre of π_X, then there is a natural bimeromorphic identification of $B'(X/S)_s$ with $B'(X_s)$ such that the restriction $(b'_{X/S})_{|X_s}: X_s \to B'(X/S)_s$ coincides with $b'_{X_s}: X_s \to B'(X_s)$ bimeromorphically. In particular, $\beta'(X_s)=\beta'(X/S)$ for general points s of S.*

Outline of the proof of 10.1.1. If κ (general fibre of π_X)$\geqq 0$, then one has $B(X/S)=X$ together with $B(X_s)=X_s$, where $s \in S$ is general, and therefore the assertion is obvious. Thus we may assume that κ(general fibre of π_X)$=-\infty$. Now in view of Theorem 4.1.1, the following three cases are possible:

Case 1. β (general fibre of π_X)$=0$: Then $\beta(X/S)=0$ and if $s \in S$ is general, $\beta(X_s)=0$. Hence this case is clear.

Case 2. β (every smooth fibre of π_X)$=1$: Note that, in this case, every smooth fibre of π_X is Moishezon (see the proof of 4.3.1). Then we may take $B(X/S)$ as the meromorphic image of the relative Albanese map $\alpha_{X/S}: X \to \mathrm{Alb}(X/S)$, and $b_{X/S}$ is naturally identified with $\alpha_{X/S}$. Now if $s \in S$ is general, $b_{X_s}: X_s \to B(X_s)$ is regarded as $\alpha_s: X_s \to \alpha_s(X_s)$, where $\alpha_s: X_s \to \mathrm{Alb}(X_s)$ is the Albanese map. The assertion is then straightforward.

Case 3. β (every smooth fibre of π_X)$=2$: In this case, the assertion follows from the same arguments as in the proof of 4.3.1, (we don't go into details).

Q. E. D.

Proof of 10.1.2. In view of 10.1.1 and (i) of 1.3.5, this is an easy consequence of 1.3.7.

Remark 10.2.1. Let X be a 4-dimensional Moishezon manifold, and we conjecture the following:

(a) *Choose a modification $\mu: X^* \to X$ from a compact complex manifold X^* so that $b_X \circ \mu: X^* \to B(X)$ is a morphism. Then β(general fibre of $b_X \circ \mu$)$=0$.*

(b) *Choose a modification $\nu: X^\# \to X$ from a compact complex manifold $X^\#$ so that $b'_X \circ \nu: X^\# \to B'(X)$ is a morphism. Then β'(general fibre of $b'_X \circ \nu$)$=0$.*

By 10.1.1 and 10.1.2 above, one easily sees that these conjectures are true if 1) of (b) of 1.1.4 is so under the condition $\dim V=4$.

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Added in proof: By recent results of Kawamata-Viehweg, 1) of (b) of 1.1.4 is true for dim $V=4$. Hence conjectures in 10.2.1 are both affirmative.