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A note on stable G-cohomotopy Groups

By

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Let G be a finite group, X and Y pointed G-spaces and α be an element of the real representation ring $RO(G)$ of G. By $\{X, Y\}_G^\alpha$ we denote the abelian group of stable G-homotopy classes of pointed G-maps of degree α from X to Y. Let $\mathcal{C}$ be a class of abelian groups. It is natural to conjecture that "if $\{X^H, Y^H\}^k \in \mathcal{C}$ for all subgroups H of G and all integers k, then $\{X, Y\}_G^\alpha \in \mathcal{C}$ for all $\alpha \in RO(G)$".

The purpose of this paper is to prove the above conjecture under a suitable condition (cf. Theorem 2.2).

1. Stable G-cohomotopy groups

Let G be a finite group. By a G-module V we mean a real representation space of G. Let $\{V_1, \cdots, V_{n(G)}\}$ be the set of representatives of irreducible G-modules and fix it throughout this paper. Identify a G-module V with the G-module

$$k_1V_1 \oplus \cdots \oplus k_{n(G)}V_{n(G)}$$

if the former is isomorphic to the latter.

For a G-module V we denote by $S(V)$ and $B(V)$ the unit sphere and the unit ball in V with respect to a G-invariant inner product, respectively. Put $\Sigma^V = B(V)/S(V)$ and $\Sigma^V X = \Sigma^V \wedge X$ for a pointed G-space X. An effective G-module V is a G-module such that $S(V)$ is an effective G-space which is equivalent to $V^G = \{0\}$.

In this section V and W denote G-modules, $\alpha \in RO(G)$ and X and Y denote pointed G-spaces. If $[\ ,\ ]_G$ denotes the set of G-homotopy classes of pointed G-maps, then define the α-th stable G-cohomotopy group $\{\ ,\ \}_G^\alpha$ by

$$\{X, Y\}_G^\alpha = \mathrm{Colim}[\Sigma^{2n\omega - \alpha} X, \Sigma^{2n\omega} Y]_G$$

where $\omega = \omega(G)$ is the real regular representation of G. When G is a trivial group, we write simply $\{\ ,\ \}^n$ for $\{\ ,\ \}_G^n$. The suspension isomorphism for G-module V

$$\sigma^V : \{X, Y\}_G^\alpha \to \{\Sigma^V X, Y\}_G^{\alpha + V}$$

is defined in the obvious way.

Let H be a subgroup of G. By Top_0^G we denote the category of pointed G-spaces and pointed G-maps. Since any pointed G-space may be considered as a pointed H-space, there is a functor $\psi_H: Top_0^G \to Top_0^H$ and we have a natural transformation (up to sign)

$$\psi_H: \{X, Y\}_G^\alpha \to \{\psi_H(X), \psi_H(Y)\}_H^{\alpha|H}, \tag{1.1}$$

where $\alpha|H$ denotes $i_H^*(\alpha)$ where $i_H^*: RO(G) \to RO(H)$ is the restriction homomorphism. The restriction (1.1) may be considered as follows. It is easy to check that G-maps $(G/H)_+ \wedge X \to Y$ correspond precisely to H-maps $\psi_H(X) \to \psi_H(Y)$ and G-homotopic maps to H-homotopic maps. Thus we have an isomorphism

$$\{(G/H)_+ \wedge X, Y\}_G^\alpha \approx \{\psi_H(X), \psi_H(Y)\}_H^{\alpha|H}. \tag{1.2}$$

Let

$$\tilde{\beta}_H: \{X, Y\}_G^\alpha \to \{(G/H)_+ \wedge X, Y\}_G^\alpha \tag{1.3}$$

be the homomorphism induced by the map $(G/H)_+ \to \Sigma^0 = \{0\}_+$ which collapses G/H to a point 0. Then the restriction (1.1) is identified with $\tilde{\beta}_H$ through the isomorphism (1.2).

Put $W(H) = N(H)/H$, where $N(H)$ is the normalizer of H in G. For a pointed G-space X, X^H (the fixed point set of H) may be considered as a pointed $W(H)$-space. Thus there is a fixed-point functor $\phi_H: Top_0^G \to Top_0^{W(H)}$. Since $(X \wedge Y)^H = X^H \wedge Y^H$ and $\omega(G)^H = |G/N(H)|\omega(W(H))$ as $W(H)$-module, we have a natural transformation (up to sign)

$$\phi_H: \{X, Y\}_G^\alpha \to \{X^H, Y^H\}_{W(H)}^{\alpha^H} \tag{1.4}$$

where $\alpha^H = V^H - W^H$ for $\alpha = V - W \in RO(G)$.

If H is a normal subgroup of G, we may regard the fixed-point homomorphism (1.4) as follows. Let

$$\chi_V: \{X, Y\}_G^\alpha \to \{X, Y\}_G^{\alpha+V}$$

be the composition of the following sequence:

$$\{X, Y\}_G^\alpha \xrightarrow{\sigma^V} \{\Sigma^V X, Y\}_G^{\alpha+V} \xrightarrow{(i_V \wedge 1)^*} \{X, Y\}_G^{\alpha+V}$$

where $i_V: \Sigma^0 \subset \Sigma^V$ is the inclusion. Obviously $\chi_V \chi_W = \chi_W \chi_V = \chi_{V \oplus W}$. By $I(H)$ we denote the set of G-modules such that $V^H = \{0\}$. Since $(V \oplus W)^H = V^H \oplus W^H$, $I(H)$ is an abelian semi-group. For fixed $\alpha \in RO(G)$, $\{\{X, Y\}_G^{\alpha+V}, \chi_W \mid V, W \in I(H)\}$ forms a direct system of abelian groups. Write its colimit as

$$\operatorname*{Colim}_{I(H)} \{X, Y\}_G^\alpha$$

and denote the canonical homomorphism by

$$l_H: \{X, Y\}_G^\alpha \longrightarrow \operatorname*{Colim}_{I(H)} \{X, Y\}_G^\alpha.$$

For an irreducible G-module V, $V^H=\{0\}$ if and only if H does not act trivially. If H acts on X trivially, there are following isomorphisms:

$$\begin{aligned}\operatorname*{Colim}_{I(H)}\{X, Y\}_G^{\alpha} &= \operatorname*{Colim}_{V\in I(H), n}[\Sigma^{2n\omega-\alpha-V}X, \Sigma^{2n\omega}Y]_G \\ &\approx \operatorname*{Colim}_{n}[\Sigma^{2n\omega^H-\alpha^H}X, \Sigma^{2n\omega}Y]_G \\ &\approx \operatorname*{Colim}_{n}[\Sigma^{2n\omega(G/H)-\alpha^H}X, \Sigma^{2n\omega(G/H)}Y]_G \\ &\approx \{X, Y^H\}_{G/H}^{\alpha^H}.\end{aligned}$$

Let $j^H : X^H \subset X$ be the inclusion, then the composition of a sequence

$$\{X, Y\}_G^{\alpha} \xrightarrow{j_H^*} \{X^H, Y\}_G^{\alpha} \xrightarrow{l_H} \operatorname*{Colim}_{I(H)}\{X^H, Y\}_G^{\alpha} = \{X^H, Y^H\}_{G/H}^{\alpha^H}$$

equals ϕ_H defined by (1.4).

2. Fixed-point isomorphism

For a pointed G-space Y we denote by $N(Y)\geqq -1$ an integer such that $\{\Sigma^0, \phi(Y)\}^k=0$ for $k\geqq -N(Y)$.

We will prove the following theorems by induction on the order of G.

Theorem 2.1. *Let $\alpha\in RO(G)$, X be a finite dimensional pointed G-complex* ([2], [4]) *and Y be a pointed G-space. If $|\alpha^H|\geqq \dim X^H - N(Y^H)$ for all subgroups H of G then*

$$\{X, Y\}_G^{\alpha}=0.$$

Theorem 2.2. *Let $\alpha\in RO(G)$, X be a finite dimensional pointed G-complex, Y be a pointed G-space and $\mathcal{C}$ be a class of abelian groups. Denote by $\{k_H\}_{H\subset G}$ the given set of integers. If $\{X^H, Y^H\}^k\in\mathcal{C}$, $k\geqq k_H$, and $|\alpha^H|\geqq k_H$ for all subgroups H of G then*

$$\{X, Y\}_G^{\alpha}\in\mathcal{C}.$$

Theorem 2.1 is a corollary of Theorem 2.2.

First we prepare three lemmas and a theorem by using the assumption of the induction. Hence, fix a finite group G and assume that Theorem 2.1 and 2.2 are valid for all proper subgroups of G.

Lemma 2.3. *Let V be an effective G-module. If $\{X^H, Y^H\}^k\in\mathcal{C}$, $k\geqq k_H$, and $|\alpha^H|\geqq k_H+(|V^H|-1)$ for all proper subgroups H of G then*

$$\{S(V)_+\wedge X, Y\}_G^{\alpha}\in\mathcal{C}.$$

Proof. Since $C=S(V)_+$ is a $(|V|-1)$-dimensional G-complex, by C_k we denote the k-skeleton of C (C_{-1}=base point).

We prove $\{C_k\wedge X, Y\}_G^\alpha\in\mathcal{C}$ by induction on $k\leqq|V|-1$. Since $C_{-1}=$base point, it is clear. Next suppose that $0\leqq k\leqq|V|-1$, then

$$C_k/C_{k-1}\approx\vee_\beta(G/H_\beta)_+\wedge S^k,\qquad H_\beta\neq G,\qquad k\leqq|V^{H_\beta}|-1,$$

and

$$\{C_k/C_{k-1}\wedge X, Y\}_G^\alpha\approx\Pi_\beta\{(G/H_\beta)_+\wedge S^k\wedge X, Y\}_G^\alpha$$
$$\approx\Pi_\beta\{\psi_{H_\beta}(X), \psi_{H_\beta}(Y)\}_{H_\beta}^{\alpha|H_\beta-k}\in\mathcal{C}.$$

Assume $\{C_{k-1}\wedge X, Y\}_G^\alpha\in\mathcal{C}$, then by the exact sequence

$$\{C_k/C_{k-1}\wedge X, Y\}_G^\alpha\longrightarrow\{C_k\wedge X, Y\}_G^\alpha\longrightarrow\{C_{k-1}\wedge X, Y\}_G^\alpha,$$

we see that $\{C_k\wedge X, Y\}_G^\alpha\in\mathcal{C}$, which completes the proof. Q. E. D.

G-cofibration $S(V)_+\to B(V)_+\to\Sigma^V$ induces an exact sequence

$$\longrightarrow\{S(V)_+\wedge X, Y\}_G^{\alpha+V-1}\longrightarrow\{X, Y\}_G^\alpha\xrightarrow{\chi_V}\{X, Y\}_G^{\alpha+V}$$
$$\longrightarrow\{S(V)_+\wedge X, Y\}_G^{\alpha+V}\longrightarrow\cdots.$$

This exact sequence and Lemma 2.3 imply

Lemma 2.4. *Let V be an effective G-module. If $\{X^H, Y^H\}^k\in\mathcal{C}$, $k\geqq k_H$, and $|\alpha^H|\geqq k_H$ for all proper subgroups H of G then*

$$\chi_V:\{X, Y\}_G^\alpha\longrightarrow\{X, Y\}_G^{\alpha+V}$$

is a $\mathcal{C}$-isomorphism.

Lemma 2.5. *Let X be a finite dimensional pointed G-complex such that X^G* =base point. *If $|\alpha^H|\geqq\dim X^H-N(Y^H)$ for all proper subgroups H of G then*

$$\{X, Y\}_G^\alpha=0.$$

Proof. By induction on $\dim X$. If $\dim X=-1$, then $X=$base point and it is clear. Next, suppose $\dim X=n\geqq0$ and put $A=X_{n-1}$. Note that $A^G=$base point. Then

$$X/A\approx\vee_\beta(G/H_\beta)_+\wedge S^n,\qquad H_\beta\neq G,$$

and so $\dim X^{H_\beta}\geqq n$. Hence $|\alpha^{H_\beta}|\geqq n-N(Y^{H_\beta})$ and so

$$\{X/A, Y\}_G^\alpha\approx\Pi_\beta\{S^n, \psi_{H_\beta}(Y)\}_{H_\beta}^{\alpha|H_\beta}=0$$

since Theorem 2.1 is valid for $H_\beta\neq G$. Assume the lemma holds for A. By the exact sequence

$$\{X/A, Y\}_G^\alpha\longrightarrow\{X, Y\}_G^\alpha\longrightarrow\{A, Y\}_G^\alpha$$

we see the lemma holds for X. Q. E. D.

Theorem 2.6. *Let X be a finite dimensional pointed G-complex. If $|\alpha^H|\geqq\dim X^H-N(Y^H)$ for all proper subgroups H of G then the homomorphism*

$$\phi_G : \{X, Y\}_G^\alpha \longrightarrow \{X^G, Y^G\}^{\alpha^G}$$

is isomorphic.

Proof. Observe the exact sequence of the pair (X, X^G): $\{X/X^G, Y\}_G^\alpha \longrightarrow \{X, Y\}_G^\alpha \xrightarrow{j_G^*} \{X^G, Y\}_G^\alpha \longrightarrow \{X/X^G, Y\}_G^{\alpha+1}$, where $j_G : X^G \subset X$ is the inclusion. By Lemma 2.5 we have $\{X/X^G, Y\}_G^\alpha = 0$ and $\{X/X^G, Y\}_G^{\alpha+1} = 0$, hence

$$j_G^* : \{X, Y\}_G^\alpha \longrightarrow \{X^G, Y\}_G^\alpha$$

is isomorphic.

Let V be a non-trivial irreducible G-module. Then by Lemma 2.4 we know that $\chi_V : \{X^G, Y\}_G^\alpha \to \{X^G, Y\}_G^{\alpha+V}$ is isomorphic, which in turn implies that the canonical homomorphism

$$l_G : \{X^G, Y\}_G^\alpha \longrightarrow \mathrm{Colim}\{X^G, Y\}_G^\alpha \approx \{X^G, Y^G\}^{\alpha^G}$$

is isomorphic.

Since $\phi_G = l_G \circ j_G^*$, we get the theorem. Q. E. D.

Proof of Theorem 2.2. Since $|\omega(G)^H| - |\omega(G)^G| = |G/H| - 1$, there is a G-module V such that $V^H \neq \{0\}$ for all proper subgroups H of G and $V^G = \{0\}$. By Lemma 2.4 $\chi_{rV} : \{X, Y\}_G^\alpha \to \{X, Y\}_G^{\alpha+rV}$ is $\mathcal{C}$-isomorphic for any positive integer r. Choose r so that $|(\alpha+rV)^H| \geqq \dim X^H - N(Y^H)$ for all proper subgroups H of G. Then by Theorem 2.6 $\phi_G : \{X, Y\}_G^{\alpha+rV} \to \{X^G, Y^G\}^{\alpha^G}$ is isomorphic. Hence we see that $\{X, Y\}_G^\alpha$ is $\mathcal{C}$-isomorphic to $\{X^G, Y^G\}^{\alpha^G}$, which completes the proof.

Q. E. D.

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