J. Math. Kyoto Univ. (JMKYAZ)
21-2 (1981) 301-317

A characterization of the identity operator on L^{∞}-spaces and its application to locally compact groups

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(Received and communicated by Prof. Yoshizawa, December 27, 1979)

Introduction.

The purpose of the present paper is twofold. The first purpose is to establish a theorem of the following type:

Let X be a locally compact space, and ν an inner regular Borel measure on X. If an isometric linear operator T on $L^{\infty}(X, \nu)$ fixes every continuous function on X vanishing at infinity, then T is the identity.

In Part I, we prove this as Theorem 1 under somewhat more general formulation. In fact, we formulate an L^{∞}-space relative to a Boolean algebra of sets, where we need neither measure nor σ-completeness.

Part II is devoted to the second purpose, an application of the above result to a characterization of left translations on $L^{\infty}(G)$, where G is a locally compact group. The principal result in Part II is Theorem 6, which states that an isometric linear operator on $L^{\infty}(G)$ commuting with every right translation is a scalar multiple of a left translation. Similar results on $L^{p}(G)$ were obtained by Wendel [12] for $p=1$ and generally by Strichartz [9] and Parrott [7] for $1 \leqq p < \infty$, $p \neq 2$.

Let us explain the contents in more detail. Part I consists of six sections. In §§ 1 & 2, we give the formulation of L^{∞}-spaces and some basic properties. In § 3, we introduce a key notion, "tracing a function by its perturbation", and then present two important propositions that extract information of a function from its perturbations. After these preparations, we prove our main theorem, Theorem 1, in § 4. A modification of the main theorem, Theorem 3, which characterizes the identity operator as a bipositive operator, is treated in § 5. In § 6, as a supplement to Part I, we discuss the change of the base space of an L^{∞}-space and describe its spectrum space in terms of the Boolean algebra that defines the L^{∞}-space.

Part II consists of three sections. We treat there the problem of determining the isometric linear operator on $L^{\infty}(G)$ commuting with all right translations. In § 7, we apply the main theorem of Part I to this problem and obtain Theorem 6. For a weaker theorem with an additional assumption of the surjectivity of the operator, we present another proof in § 8. We also mention there some related facts and their relations. In the last section, we give an alternative proof for the theorem of Takesaki-Tatsuuma, which is closely related to the consideration in § 8.

The author wishes to express his thanks to Professor H. Yoshizawa for valuable suggestions in this presentation. Thanks also go deeply to Professor N. Tatsuuma and to Professor T. Hirai for their encouragement and various advices.

Notations.

$\boldsymbol{R}$: the set of real numbers,
$\boldsymbol{C}$: the set of complex numbers,
$\boldsymbol{B}_\varepsilon=\{a\in\boldsymbol{C};\ |a|\leqq\varepsilon\}$, the closed disk of radius ε in $\boldsymbol{C}$.

Let X be a set. For subsets M and M' of X, we use the following notations throughout this paper.

M^C : the complement of M in X,
$M\triangle M'$: the symmetric difference of M and M',
χ_M : the characteristic function of M.

For $a\in\boldsymbol{C}$, we denote the constant function $a\chi_X$ simply by a.

Part I. Characterization on the identity operator on L^∞-spaces.

§ 1. Formulation of L^∞-spaces.

In this section, we formulate an L^∞-space for a Boolean algebra of sets.

1.1. Let X be a set. We consider a pair $(\mathfrak{B}, \mathfrak{N})$ of a Boolean lattice $\mathfrak{B}$ of subsets of X and an ideal $\mathfrak{N}$ of $\mathfrak{B}$: $B, B'\in\mathfrak{B}\Rightarrow B^C, B\cup B'\in\mathfrak{B}$, and $N, N'\in\mathfrak{N}, B\in\mathfrak{B}\Rightarrow N\cup N', N\cap B\in\mathfrak{N}$. We call a set null if it belongs to $\mathfrak{N}$. We exclude the trivial case $\mathfrak{N}=\mathfrak{B}$. Let $\mathcal{S}_{\mathfrak{B}}$ and $\mathcal{S}_{\mathfrak{N}}$ be the space of step functions and that of null step functions (i.e., of the linear combinations of χ_B's, $B\in\mathfrak{B}$, and of those of χ_N's, $N\in\mathfrak{N}$ respectively). We define seminorms $\|\cdot\|_E$ on $\mathcal{S}_{\mathfrak{B}}$ for $E\in\mathfrak{B}$ by

$$(*)_E \qquad \|f\|_E=\inf\{k\geqq 0;\ \{x;\ |f(x)|>k\}\cap E\in\mathfrak{N}\}\ .$$

It is clear that $\|\cdot\|_E\leqq\|\cdot\|_{E'}$ if $E\subset E'$, and that $\|f\|_X=0$ if and only if $f\in\mathcal{S}_{\mathfrak{N}}$. Now, we define the space $L^\infty(X, \mathfrak{B}, \mathfrak{N})$ as the completion of $\mathcal{S}_{\mathfrak{B}}/\mathcal{S}_{\mathfrak{N}}$ by the norm $\|\cdot\|_X$. Note that the seminorms $\|\cdot\|_E$, $E\in\mathfrak{B}$, are well defined on $L^\infty(X, \mathfrak{B}, \mathfrak{N})$.

Remark 1. Note that the space $\mathcal{S}_{\mathfrak{B}}$ forms a ring under the pointwise multiplication and that $\mathcal{S}_{\mathfrak{N}}$ is an ideal of $\mathcal{S}_{\mathfrak{B}}$. Further the involution $\sum a_i\chi_{B_i}\mapsto\sum\bar{a}_i\chi_{B_i}$ is naturally extended to $L^\infty(X, \mathfrak{B}, \mathfrak{N})$. Thus $L^\infty(X, \mathfrak{B}, \mathfrak{N})$ forms a commutative C^*-algebra. Note also that $\|f\|_E$ is equal to $\|f\chi_E\|_X$. The structure of $L^\infty(X, \mathfrak{B}, \mathfrak{N})$ as a C^*-algebra depends only on the quotient Boolean lattice $\mathfrak{B}/\mathfrak{N}$. For details, see § 6.

1.2. Realization of $L^{\infty}(X, \mathfrak{B}, \mathfrak{N})$ as a function space on X.

The space $L^{\infty}(X, \mathfrak{B}, \mathfrak{N})$ can be realized as the space of (equivalence classes of) functions on X. First, when $\mathfrak{B}$ is σ-complete (i.e., $\mathfrak{B}$ is a σ-field on X), put

$$\mathcal{L}^{\infty}(X, \mathfrak{B}, \mathfrak{N})=\{f\,;\, \mathfrak{B}\text{-measurable},\ \|f\|_X<\infty\}\ ,$$

$$\mathcal{N}(X, \mathfrak{B}, \mathfrak{N})=\{f\,;\, \mathfrak{B}\text{-measurable},\ \|f\|_X=0\}\ .$$

Here a function f on X is called $\mathfrak{B}$-measurable if $f^{-1}(U)\in\mathfrak{B}$ for every open set U in $\boldsymbol{C}$, and $\|f\|_X$ is defined by the formula $(*)_X$. Note that the σ-completeness of $\mathfrak{N}$ is not required. As is easily seen, $\mathcal{L}^{\infty}(X, \mathfrak{B}, \mathfrak{N})$ and $\mathcal{N}(X, \mathfrak{B}, \mathfrak{N})$ are linear spaces. Then $L^{\infty}(X, \mathfrak{B}, \mathfrak{N})$ is realized as the quotient space $\mathcal{L}^{\infty}(X, \mathfrak{B}, \mathfrak{N})/\mathcal{N}(X, \mathfrak{B}, \mathfrak{N})$.

Now, for a general $\mathfrak{B}$, take a σ-complete Boolean lattice $\tilde{\mathfrak{B}}$ of subsets of X including $\mathfrak{B}$, and let $\tilde{\mathfrak{N}}$ be the ideal generated by $\mathfrak{N}$ in $\tilde{\mathfrak{B}}$. Then $L^{\infty}(X, \mathfrak{B}, \mathfrak{N})$ is realized as the closure of the image of $\mathcal{S}_{\mathfrak{B}}$ in $L^{\infty}(X, \tilde{\mathfrak{B}}, \tilde{\mathfrak{N}})=\mathcal{L}^{\infty}(X, \tilde{\mathfrak{B}}, \tilde{\mathfrak{N}})/\mathcal{N}(X, \tilde{\mathfrak{B}}, \tilde{\mathfrak{N}})$. Here the image of $\mathcal{S}_{\mathfrak{B}}$ in $L^{\infty}(X, \tilde{\mathfrak{B}}, \tilde{\mathfrak{N}})$ is isomorphic to $\mathcal{S}_{\mathfrak{B}}/\mathcal{S}_{\mathfrak{N}}$, because $\mathcal{S}_{\mathfrak{N}}=\mathcal{S}_{\mathfrak{B}}\cap\mathcal{N}(X, \tilde{\mathfrak{B}}, \tilde{\mathfrak{N}})$.

Thus a function f on X represents an element of $L^{\infty}(X, \mathfrak{B}, \mathfrak{N})$ if it can be approximated by the step functions in $\mathcal{S}_{\mathfrak{B}}$ in the sense of $\|\cdot\|_X$. In this case we say that the function f is admitted by the system $(X, \mathfrak{B}, \mathfrak{N})$.

Hereafter we often employ the following conventions on notation. (1) $\|\cdot\|=\|\cdot\|_X$. (2) A subset of X without any notice will be understood to be in $\mathfrak{B}$. (3) For a function and for the element of $L^{\infty}(X, \mathfrak{B}, \mathfrak{N})$ represented by it, we use the same notation.

1.3. The essential image of a set in $\mathfrak{B}$.

Definition 1. For $f\in L^{\infty}(X, \mathfrak{B}, \mathfrak{N})$ and $E\in\mathfrak{B}$, we define the *essential image* of E by f, denote by $f[E]$, as the set of $a\in\boldsymbol{C}$ satisfying the following: for every $\varepsilon>0$, there exists a non-null set $B\in\mathfrak{B}$ such that $B\subset E$ and $\|f-a\|_B<\varepsilon$.

It is clear by definition that for $a\in f[E]$ and for every $\varepsilon>0$, there exists a non-null set $B\in\mathfrak{B}$ such that $B\subset E$ and $f[B]\subset a+\boldsymbol{B}_{\varepsilon}$. Note that $f[E]$ is closed and bounded, hence compact in $\boldsymbol{C}$. In fact, for $a\notin f[E]$, there exists a $\delta>0$ such that $\|f-a\|_B\geqq\delta$ for any non-null set $B\subset E$, whence $\|f-b\|_B\geqq\frac{\delta}{2}$ for $|b-a|\leqq\frac{\delta}{2}$. The boundedness of $f[E]$ follows from $f[E]\subset\mathfrak{B}_{\|f\|_E}$.

For $E\in\mathfrak{N}$, $f[E]$ is the empty set. The converse of this fact is fundamental in what follows.

Proposition 1. *The essential image $f[E]$ is not empty for $E\notin\mathfrak{N}$.*

Proof. Suppose $f[E]$ is empty. Then for every $a\in\boldsymbol{C}$, there exists a $\delta_a>0$ such that $\|f-b\|_B\geqq\delta_a$ for any non-null $B\subset E$ and for $|b-a|<\delta_a$. Moreover, if $|a|>2\|f\|_E$, then $\|f-a\|_B>\|f\|_E$. Therefore by the compactness of the closed disk $\boldsymbol{B}_{2\|f\|_E}$, there exists a $\delta>0$ such that $\|f-a\|_B\geqq\delta$ for any non-null $B\subset E$ and for any $a\in\boldsymbol{C}$. But this is impossible because f can be approximated by step functions. Q.E.D.

Proposition 2. $(f+g)[E]\subset f[E]+g[E]$.

Proof. Let $c\in(f+g)[E]$. For $\varepsilon>0$, there exists a non-null $B\subset E$ such that $\|f+g-c\|_B<\varepsilon$. Take an $a\in f[B]$ and a non-null $B'\subset B$ such that $\|f-a\|_{B'}<\varepsilon$. Further take a $b\in g[B']$ and a non-null $B''\subset B'$ such that $\|g-b\|_{B''}<\varepsilon$. Then we have $|c-a-b|=\|c-a-b\|_{B''}<3\varepsilon$, or $c\in f[E]+g[E]+\boldsymbol{B}_{3\varepsilon}$. Since $f[E]+g[E]$ is compact and $\varepsilon>0$ is arbitrary, we see that $c\in f[E]+g[E]$. Q. E. D.

Corollary 1. *If* $\|f-g\|_E\leqq\varepsilon$, *then* $f[E]\subset g[E]+\boldsymbol{B}_\varepsilon$.

Proof. $f[E]\subset g[E]+(f-g)[E]$ Q. E. D.

Corollary 2. $\|f\|_E=\sup\{|a|\,;\,a\in f[E]\}$ *for* $E\notin\mathfrak{N}$.

Proof. This relation is clear for a step function. For a general $f\in L^\infty(X, \mathfrak{B}, \mathfrak{N})$, we can see this from the corollary above, because f is approximated by step functions. Q. E. D.

We have also the following useful relations:

$$f[E\cup E']=f[E]\cup f[E'],$$

$$f[E]=f[E'] \quad \text{if} \quad E\triangle E'\in\mathfrak{N}.$$

The assertions can be obtained directly from the definitions.

§2. Support of a set in $\mathfrak{B}$.

From now on, the base space X is assumed to be a Hausdorff topological space, and we put a natural topological condition on $(\mathfrak{B}, \mathfrak{N})$:

(O) $\mathfrak{B}$ contains a basis of open sets in X.

We make further an essential assumption, the inner regularity of $(\mathfrak{B}, \mathfrak{N})$:

(I) For every non-null set $B\in\mathfrak{B}$, there exists a non-null compact set $K\in\mathfrak{B}$ such that $K\subset B$.

Definition 2. For $E\in\mathfrak{B}$, we define the *support* $[E]$ of E as the set of $x\in X$ satisfying the condition that $V\cap E$ is not null for every open neighbourhood $V\in\mathfrak{B}$ of x.

We see easily the following properties.

(1) $[E]$ is closed.

(2) $[E]\subset E^-=$the closure of E.

(3) $[E\cup E']=[E]\cup[E']$.

(4) If E is null, then $[E]$ is empty.

The converse of (4) holds under the assumption of the inner regularity.

Proposition 3. *If E is not null, then $[E]$ is not empty.*

Proof. By the inner regularity, there exists a non-null compact set $K\in\mathfrak{B}$ such that $K\subset E$. Therefore if $[E]$ is empty, so is $[K]$. Then for every $x\in X$, there exists an open neighbourhood $V_x\in\mathfrak{B}$ of x such that $K\cap V_x$ is null. Since K is compact, we can find a finitely many V_x's that cover K. So we have

$$K=K\cap(\bigcup_{\text{finite}} V_x)=\bigcup_{\text{finite}}(K\cap V_x)\in\mathfrak{N},$$

a contradiction. Q. E. D.

Remark 2. Under the inner regularity, we have the following for the essential image $f[E]$:

Let $a\in f[E]$. Then for any $\varepsilon>0$, there exists a non-null compact set $K\in\mathfrak{B}$ such that $K\subset E$ and $f[K]\subset a+\boldsymbol{B}_\varepsilon$.

Remark 3. Let us assume the condition (O^*) stronger than (O):

(O^*) $\mathfrak{B}$ contains every open set in X.

In that case, $[E]\in\mathfrak{B}$ for $E\in\mathfrak{B}$. Then we have the following (✡) quite similarly as Proposition 3:

(✡) $E\cap[E]^C\in\mathfrak{N}$.

From this we can deduce the following.

(5) If F is closed, then $F\triangle[F]\in\mathfrak{N}$.

(6) $[[E]]=[E]$.

(7) If E is not null, neither is $[E]$.

Note further that under the condition (O^*) every bounded continuous function on X is admitted by $(X, \mathfrak{B}, \mathfrak{N})$.

§ 3. Tracing of f by its perturbations.

Let us begin with a definition.

Definition 3. A set $\mathcal{F}$ of continuous functions on X is called *fundamental* if it satisfies the following: (i) For every $h\in\mathcal{F}$ and $x\in X$, $0\leqq h(x)\leqq 1$. (ii) For every point $a\in X$ and a closed set $F\not\ni a$, there exists an $h\in\mathcal{F}$ such that $h(a)=1$ and $h(x)=0$ on F.

For a fundamental set $\mathcal{F}$ and $a\in X$, put $\mathcal{F}_a=\{h\in\mathcal{F}\,;\,h(a)=1\}$.

We assume that there exists a fundamental set $\mathcal{F}$ admitted by $(X, \mathfrak{B}, \mathfrak{N})$. Under this assumption, the base space X must be completely regular.

For $f\in L^\infty(X, \mathfrak{B}, \mathfrak{N})$, $c\in\boldsymbol{C}$, and $k\geqq 0$, we define

$$M(f\,;\,c)(x)=\inf\{\|f+ch\|\,;\,h\in\mathcal{F}_x\}\,,$$

$$m(f\,;\,k)(x)=\sup\{M(f\,;\,c)(x)\,;\,|c|\leqq k\}\,.$$

These functions play an important role for our main theorem, Theorem 1, through the following propositions.

Proposition 4. *If there exists an* $x_0 \in X$ *such that* $M(f; 1)(x_0)=\|f\|+1$, *then* $f[X] \ni \|f\|$.

Proof. For any $h \in \mathcal{F}_{x_0}$, we have

$$(f+h)[X] \subset f[X]+h[X] \subset f[X]+\{a \in \boldsymbol{R};\, 0 \leqq a \leqq 1\}.$$

This shows that the absolute value of any $b \in (f+h)[X]$ cannot attain the value $\|f\|+1$ when $f[X]$ does not contain the value $\|f\|$.

Proposition 5. *If* $m(f; k)(x) \leqq k$ *holds on a non-null set* E, *then* $f[E]=\{0\}$.

Proof. Assume that $f[E] \ni a \neq 0$. Then for any $\varepsilon>0$, there exists a non-null compact set $K \in \mathfrak{B}$ such that $K \subset E$ and $f[K] \subset a+\boldsymbol{B}_\varepsilon$. Put $c=\frac{a}{|a|}k$. Let $s \in [K]$, then $s \in E$. For every $h \in \mathcal{F}_s$, we can take an open neighbourhood $V \in \mathfrak{B}$ of s such that $h[V] \subset 1+\boldsymbol{B}_\varepsilon$. Therefore

$$\begin{aligned}(f+ch)[K \cap V] &\subset f[K]+ch[V]\\ &\subset a+\boldsymbol{B}_\varepsilon+\frac{a}{|a|}k+\boldsymbol{B}_{\varepsilon k}\\ &=\frac{a}{|a|}(|a|+k)+\boldsymbol{B}_{\varepsilon(1+k)}.\end{aligned}$$

Hence $m(f; k)(s) \geqq |a|+k$, a contradiction. Q.E.D.

From the definitions, we can easily deduce the following.

Lemma 1. *For* $E \in \mathfrak{B}$,

$$M(\chi_E; 1)(x)=m(\chi_E; 1)(x)=\chi_{[X]}(x)+\chi_{[E]}(x) \qquad (x \in X).$$

(Needless to say, this equality holds exactly at each point $x \in X$, not in the sense of $L^\infty(X, \mathfrak{B}, \mathfrak{N})$.)

Lemma 2. *Let* $k \geqq \|f\|$. *If* $f[V]=\{0\}$ *for an open set* $V \in \mathfrak{B}$, *then* $m(f; k)(x) \leqq k$ *on* V.

Remark 4. When we replace the condition of the inner regularity (I) with the following (I_*), Proposition 5 does not hold.

(I_*) For every non-null set $B \in \mathfrak{B}$, there exist a null set N and a non-null compact set $K \in \mathfrak{B}$ such that $K \subset B \cup N$.

For example, let X be the interval $[0, 1]$, and put $\mathfrak{B}=2^X$, and $\mathfrak{N}=$the totality of finite subsets of X. Then the Bolzano-Weierstrass theorem shows that $(\mathfrak{B}, \mathfrak{N})$ satisfies (I_*). Put $A=\left\{\frac{1}{n};\, n \geqq 1, \text{ integer}\right\}$. Then $m(\chi_A; 1)(x)=1$ for $x \in (0, 1)$, but $\chi_A[(0, 1)] \ni 1$.

§ 4. Characterization of the identity operator on $L^\infty(X, \mathfrak{B}, \mathfrak{N})$.

In this section, we prove the main theorem of Part I. Our assumptions are the following.

(A) The system $(\mathfrak{B}, \mathfrak{N})$ satisfies the condition (O) and the inner regularity (I).

(B) There exists a fundamental set $\mathfrak{F}$ admitted by $(X, \mathfrak{B}, \mathfrak{N})$.

Theorem 1. *Let T be an isometric linear operator on $L^\infty(X, \mathfrak{B}, \mathfrak{N})$. Assume that $Th=h$ for $h\in\mathfrak{F}$. Then T is the identity operator on the whole $L^\infty(X, \mathfrak{B}, \mathfrak{N})$.*

Let us recall that according to our convention we use the same notation for a function on X and the element in $L^\infty(X, \mathfrak{B}, \mathfrak{N})$ represented by it.

Before the proof of Theorem 1, we make some preparations. The next lemma is quite obvious.

Lemma 3. *Let T be a linear operator on $L^\infty(X, \mathfrak{B}, \mathfrak{N})$. Assume that T fixes each element in $\mathfrak{F}$. Then the following hold.*

(i) *$M(Tf\,;c)\leqq\|T\|M(f\,;c)$, and $m(Tf\,;k)\leqq\|T\|m(f\,;k)$.*

(ii) *If $\|Tf\|\geqq r\|f\|$ holds for all $f\in L^\infty(X, \mathfrak{B}, \mathfrak{N})$, then*

$$M(Tf\,;c)\geqq r\cdot M(f\,;c)\,,\qquad \text{and}\quad m(Tf\,;k)\geqq r\cdot m(f\,;k)\,.$$

Corollary. *Let T be an isometric linear operator on $L^\infty(X, \mathfrak{B}, \mathfrak{N})$. Assume that T fixes each element in $\mathfrak{F}$. Then the following hold.*

(i) *For a non-hull set $E\in\mathfrak{B}$, $(T\chi_E)[X]\ni 1$.*

(ii) *If $f[V]=\{0\}$ for an open set $V\in\mathfrak{B}$, then $(Tf)[V]=\{0\}$.*

Proof. By Lemma 3, we have $M(Tf\,;c)=M(f\,;c)$ and $m(Tf\,;k)=m(f\,;k)$. Then (i) follows from Proposition 4 and Lemma 1, and (ii) from Proposition 5 and Lemma 2.

Proposition 6. *Let T be a linear operator with norm 1. Assume that $(T\chi_K)[K]\ni 1$ for every non-null compact set $K\in\mathfrak{B}$. Then T is the identity.*

Proof. It suffices to show that $T\chi_E=\chi_E$ for all $E\in\mathfrak{B}$. For this, we have only to show $(T\chi_E)[E]\subset\{1\}$ and $(T\chi_E)[E^C]\subset\{0\}$. But the former implies the latter. In fact, putting $E=X$, we see that $T1=1$ and then obtain $(T\chi_E)[E^C]=(1-T\chi_{E^C})[E^C]\subset\{0\}$. Hence it remains to prove that $(T\chi_E)[E]\subset\{1\}$.

Suppose $(T\chi_E)[E]\ni a\neq 1$. For $\varepsilon>0$, take a non-null compact set $K\in\mathfrak{B}$ such that $K\subset E$ and $(T\chi_E)[K]\subset a+\boldsymbol{B}_\varepsilon$. Put $f=\chi_E-\chi_K+\frac{b}{|b|}\chi_K$, with $b=a-1$. Then $\|f\|\leqq 1$. On the other hand, since $(T\chi_K)[K]\ni 1$, there exists a non-null set $B\in\mathfrak{B}$ such that $B\subset K$ and $(T\chi_K)[B]\subset 1+\boldsymbol{B}_\varepsilon$. Then

$$\begin{aligned}(Tf)[B]&\subset(T\chi_E)[B]-(T\chi_K)[B]+\frac{b}{|b|}(T\chi_K)[B]\\&\subset a+\boldsymbol{B}_\varepsilon-1+\boldsymbol{B}_\varepsilon+\frac{b}{|b|}+\boldsymbol{B}_\varepsilon\end{aligned}$$

$$=\frac{b}{|b|}(1+|b|)+\boldsymbol{B}_{3\varepsilon}.$$

Hence $\|Tf\|\geqq 1+|b|$, a contradiction. Q.E.D.

Proof of Theorem 1. In view of Proposition 6, we have only to show that $(T\chi_K)[K]\ni 1$ for every non-null compact set $K\in\mathfrak{B}$. This follows from Corollary of Lemma 3. In fact, we have

$$(T\chi_K)[X]\ni 1 \quad \text{and} \quad (T\chi_K)[K^C]\subset\{0\}. \qquad \text{Q.E.D.}$$

In case where X is locally compact, we have a typical situation: $\mathfrak{B}$ contains all open subsets of X, and $\mathfrak{F}$ is taken from $C_0(X)$, the space of all continuous functions on X vanishing at infinity. In this case, Theorem 1 is rewritten in the following form.

Theorem 2. *Let X be locally compact, and assume (O^*) and (I) on $(\mathfrak{B}, \mathfrak{N})$. Let T be an isometric linear operator on $L^\infty(X, \mathfrak{B}, \mathfrak{N})$. If T is the identity on $C_0(X)$, then T is also the identity on the whole $L^\infty(X, \mathfrak{B}, \mathfrak{N})$.*

Remark 5. Let X be a totally disconnected locally compact space, $\mathfrak{B}$ the totality of closed open subsets of X, and $\mathfrak{N}=\{\varnothing\}$. In this case $(\mathfrak{B}, \mathfrak{N})$ satisfies (O) and (I). By Proposition 6, for such a system $(X, \mathfrak{B}, \mathfrak{N})$, we find that a continuous linear operator on $L^\infty(X, \mathfrak{B}, \mathfrak{N})$ with norm 1 is the identity if it fixes each χ_K, for K compact. In particular, in the case that X is discrete, we see that a continuous linear operator on $l^\infty(X)$ with norm 1 is the identity if it fixes every element which vanishes except on a finite subset of X.

§5. A variant of the main theorem.

In this section, we characterize the identity operator as a bipositive operator that fixes every element of a fundamental set.

Referring to its essential range, we call an element f of $L^\infty(X, \mathfrak{B}, \mathfrak{N})$ real or positive according as $f[X]\subset\boldsymbol{R}$ or $f[X]\subset\{a\in\boldsymbol{R}; a\geqq 0\}$ respectively. We denote by $L^\infty_{\boldsymbol{R}}(X, \mathfrak{B}, \mathfrak{N})$ or $L^\infty_+(X, \mathfrak{B}, \mathfrak{A})$ the set of all real elements or positive elements respectively. Then $L^\infty_{\boldsymbol{R}}(X, \mathfrak{B}, \mathfrak{N})$ is an ordered vector space with positive cone $L^\infty_+(X, \mathfrak{B}, \mathfrak{N})$. We use the notation $\geqq$ for the order.

As usual we call an operator on $L^\infty_{\boldsymbol{R}}(X, \mathfrak{B}, \mathfrak{N})$ positive or bipositive if it satisfies that $f\geqq 0\Rightarrow Tf\geqq 0$ or $f\geqq 0\Leftrightarrow Tf\geqq 0$ respectively.

We keep to the same assumptions on $(X, \mathfrak{B}, \mathfrak{N})$ as in §4:

(A) The system $(\mathfrak{B}, \mathfrak{N})$ satisfies the condition (O) and the inner regularity (I).

(B) There exists a fundamental set $\mathfrak{F}$ admitted by $(X, \mathfrak{B}, \mathfrak{N})$.

Theorem 3. *Let T be a bipositive linear operator on $L^\infty_{\boldsymbol{R}}(X, \mathfrak{B}, \mathfrak{N})$. Assume that $Th=h$ for $h\in\mathfrak{F}$. Then T is the identity operator on $L^\infty_{\boldsymbol{R}}(X, \mathfrak{B}, \mathfrak{N})$.*

For the proof we need a lemma.

Lemma 4. *Let T be a positive linear operator on $L_R^\infty(X, \mathfrak{B}, \mathfrak{N})$. Assume that T fixes each element of $\mathcal{F}$. Then we have the following.*
(i) *For every open set $V \in \mathfrak{B}$, $T\chi_V \geqq \chi_V$.*
(ii) *For every compact set $K \in \mathfrak{B}$, $(T\chi_K)[K^C] \subset \{0\}$.*
(iii) *For every non-null compact set $K \in \mathfrak{B}$ and $r>1$, there exists a non-null compact set $K_1 \in \mathfrak{B}$ such that $K_1 \subset K$ and $T\chi_{K_1} \leqq r\chi_{K_1}$.*

Proof. For short, we write $A \geqq a$ for $A \subset \{b \in \boldsymbol{R}; b \geqq a\}$, and similarly for $A \leqq a$, $A>a$, and $A<a$.

For (i). Since T is positive, $T\chi_V \geqq 0$. Therefore it suffices to show that $(T\chi_V)[V] \geqq 1$. Suppose not, then there exist an $\varepsilon>0$ and a non-null compact set $K \in \mathfrak{B}$ such that $K \subset V$ and $(T\chi_V)[K] < 1-\varepsilon$. Let $s \in [K]$ and take an $h \in \mathcal{F}_s$ satisfying $h \leqq \chi_V$. We can choose an open neighbourhood $W \in \mathfrak{B}$ of s such that $h[W] \geqq 1-\varepsilon$, whence $h[K \cap W] \geqq 1-\varepsilon$. But this contradicts the fact that $(T\chi_V)[K] < 1-\varepsilon$.

For (ii). Let F be a compact set disjoint from K. For every $a \in K$, take an $h_a \in \mathcal{F}$ such that $h_a(a)=1$ and $h_a(F)=\{0\}$. Let V_a be an open neighbourhood of a such that $h_a(V_a)>1/2$. Since K is compact, we can choose a finitely many $a_1, \cdots, a_n$ such that $\bigcup_{i=1}^n V_{a_i} \supset K$. Put $h=\sum_{i=1}^n 2h_{a_i}$. Then $h \geqq \chi_K$, hence $h=Th \geqq T\chi_K$. This and the inner regularity show that $(T\chi_K)[K^C] \subset \{0\}$.

For (iii). As is shown in (ii), we have an h such that $h \geqq \chi_K$ and that h is a linear combination of elements in $\mathcal{F}$. Let $s \in [K]$, and take an open neighbourhood $W \in \mathfrak{B}$ of s such that $h(W) \subset h(s)+\boldsymbol{B}_\varepsilon$, with $r-1 \geqq 2\varepsilon > 0$. Put $f=\dfrac{1}{1-\varepsilon} \cdot \dfrac{1}{h(s)} \cdot h$. Then $f[W \cap K] \subset 1+\varepsilon+\boldsymbol{B}_\varepsilon$. Let K_1 be a non-null compact set included in $W \cap K$. Then $f \geqq \chi_{K_1}$, so that $f=Tf \geqq T\chi_{K_1}$. Note that $f[K_1] \leqq 1+2\varepsilon \leqq r$. On the other hand, we know by (ii) that $(T\chi_{K_1})[K_1{}^C] \subset \{0\}$. Hence $T\chi_{K_1} \leqq r\chi_{K_1}$. Q.E.D.

Proof of Theorem 3. Let us first show that $T1=1$. By (i) of Lemma 3, we have $T1 \geqq 1$. Therefore if $T1 \neq 1$, then there exist a non-null compact set $K \in \mathfrak{B}$ and $r>1$ such that $T1 \geqq r\chi_K$. By (iii) of Lemma 4, we can find a non-null compact set K_1 such that $K_1 \subset K$ and $T\chi_{K_1} \leqq r^{1/2}\chi_{K_1}$. Therefore $T1 \geqq r\chi_K \geqq r\chi_{K_1} \geqq r^{1/2}T\chi_{K_1}$. But this implies by the bipositivity of T a contradiction that $1 \geqq r^{1/2}\chi_{K_1}$. Hence $T1=1$.

Now, for $f \in L_R^\infty(X, \mathfrak{B}, \mathfrak{N})$ and $r \geqq 0$, we see by the bipositivity of T that $-r \leqq f \leqq r \Leftrightarrow -r \leqq Tf \leqq r$. This shows that T is isometric with respect to the norm of $L_R^\infty(X, \mathfrak{B}, \mathfrak{N})$. Hence by Theorem 1, T is the identity operator on $L_R^\infty(X, \mathfrak{B}, \mathfrak{N})$. We note here that Theorem 1 does not depend on the scalar field $\boldsymbol{C}$ or $\boldsymbol{R}$. Q.E.D.

Similarly as Theorem 2, this theorem is rewritten in the following form when

X is locally compact.

Theorem 4. *Let X be locally compact, and assume (O^*) and (I) on $(\mathfrak{B}, \mathfrak{N})$. Let T be a bipositive linear operator on $L_R^\infty(X, \mathfrak{B}, \mathfrak{N})$. If T is the identity on $C_0(X)$, then T is also the identity on the whole $L_R^\infty(X, \mathfrak{B}, \mathfrak{N})$.*

§ 6. Remarks on the change of the base space X.

In this section we make some remarks on the change of the base space X and the structure of $L^\infty(X, \mathfrak{B}, \mathfrak{N})$.

6.1. Let us consider two systems $(X_1, \mathfrak{B}_1, \mathfrak{N}_1)$ and $(X_2, \mathfrak{B}_2, \mathfrak{N}_2)$. Let φ be a Boolean lattice homomorphism of $\mathfrak{B}_1$ to $\mathfrak{B}_2$ with $\varphi(\mathfrak{N}_1)\subset\mathfrak{N}_2$. Then it is not difficult to see that $\tilde{\varphi}(\chi_E)=\chi_{\varphi(E)}$ defines a linear map $\tilde{\varphi}$ of $\mathcal{S}_{\mathfrak{B}_1}$ to $\mathcal{S}_{\mathfrak{B}_2}$ with $\tilde{\varphi}(\mathcal{S}_{\mathfrak{N}_1})\subset\mathcal{S}_{\mathfrak{N}_2}$. Further since $\|f\|_E\geqq\|\tilde{\varphi}(f)\|_{\varphi(E)}$, $\tilde{\varphi}$ is continuous with respect to the norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$, so that $\tilde{\varphi}$ can be extended to a linear map of $L^\infty(X_1, \mathfrak{B}_1, \mathfrak{N}_1)$ to $L^\infty(X_2, \mathfrak{B}_2, \mathfrak{N}_2)$. We can easily see that if $\varphi^{-1}(\mathfrak{N}_2)=\mathfrak{N}_1$, then $\tilde{\varphi}$ is isometric, and that if φ is surjective, then $\tilde{\varphi}$ is also surjective.

It should be noted that $\tilde{\varphi}$ is an algebra homomorphism when we consider the natural algebra structures on L^∞'s.

6.2. Change of the base space X by a map $\psi: X\to Y$.

Let us consider $(X, \mathfrak{B}, \mathfrak{N})$. Let Y be a set and $\psi: X\to Y$ be a map. We define $(\psi_*\mathfrak{B}, \psi_*\mathfrak{N})$ on Y by $\psi_*\mathfrak{B}=\{B'\subset Y; \psi^{-1}(B')\in\mathfrak{B}\}$ and $\psi_*\mathfrak{N}=\{N'\subset Y; \psi^{-1}(N')\in\mathfrak{N}\}$. Further define $\psi^*(B')=\psi^{-1}(B')$ for $B'\in\psi_*\mathfrak{B}$. Then ψ^* is a Boolean lattice homomorphism of $\psi_*\mathfrak{B}$ to $\mathfrak{B}$ such that $\psi^*(\psi_*\mathfrak{N})\subset\mathfrak{N}$. Thus we get a situation in 6.1, so that we have a continuous linear map $\psi^{*\sim}$ of $L^\infty(Y, \psi_*\mathfrak{B}, \psi_*\mathfrak{N})$ to $L^\infty(X, \mathfrak{B}, \mathfrak{N})$. Note that $\psi^{*\sim}$ is automatically isometric because $\psi^{*-1}(\mathfrak{N})=\psi_*\mathfrak{N}$. Moreover if ψ is injective, then ψ^* is surjective. Hence $\psi^{*\sim}$ gives an isometric isomorphism between $L^\infty(X, \mathfrak{B}, \mathfrak{N})$ and $L^\infty(Y, \psi_*\mathfrak{B}, \psi_*\mathfrak{N})$ when ψ is injective.

Next let us consider topological conditions. Assume that X and Y are Hausdorff topological spaces and ψ is continuous. Then the system $(Y, \psi_*\mathfrak{B}, \psi_*\mathfrak{N})$ satisfies the conditions (O^*) and (I) when so does the system $(X, \mathfrak{B}, \mathfrak{N})$. Moreover if ψ is a homeomorphism onto a subspace of Y and $(X, \mathfrak{B}, \mathfrak{N})$ satisfies (O) and (I), then $(Y, \psi_*\mathfrak{B}, \psi_*\mathfrak{N})$ satisfies the same conditions.

Thus we can change the base space X by Y preserving the topological conditions (O) and (I) of the systems. One may therefore expect further that keeping the conditions (A) and (B) we can arrive at a locally compact space Y from any $(X, \mathfrak{B}, \mathfrak{N})$ in a certain way; for example using the spectrum space of the algebra generated by the fundamental set $\mathcal{F}$. But the auther does not know whether it is possible or not.

6.3. The spectrum space of $L^\infty(X, \mathfrak{B}, \mathfrak{N})$. Let us recall the Stone's representation theorem for a Boolean lattice: *For a Boolean lattice $\mathcal{B}$, let $\mathfrak{M}$ be the set of all maximal ideals of $\mathcal{B}$. Then the map $\mathcal{B}\ni E\mapsto\hat{E}=\{M\in\mathfrak{M}; M\not\ni E\}$*

defines a Boolean lattice isomorphism onto a sublattice of $2^{\mathfrak{M}}$. *Being equipped with the topology generated by* $\hat{\mathscr{B}}=\{\hat{E}\,;\,E,\ \mathscr{B}\}$, $\mathfrak{M}$ *is found to be compact. Further* $\hat{\mathscr{B}}$ *is characterized as the set of all closed open subsets of* $\mathfrak{M}$.

By the definition of the topology on $\mathfrak{M}$, $\chi_{\hat{E}}$ for $E\in\mathscr{B}$ is continuous. Further the algebra of step functions $\mathcal{S}_{\hat{\mathscr{B}}}$ separates the points of $\mathfrak{M}$. Therefore by the Stone-Weierstrass theorem, $\mathcal{S}_{\hat{\mathscr{B}}}$ is dense in the space $C(\mathfrak{M})$ of all continuous functions on $\mathfrak{M}$ with respect to the supremum norm. So we see that $L^\infty(\mathfrak{M},\ \hat{\mathscr{B}},\ \{\varnothing\})=C(\mathfrak{M})$.

Applying the argument in 6.1 to the lattice isomorphism $\mathscr{B}\rightarrow\hat{\mathscr{B}}$ for $\mathscr{B}=\mathfrak{B}/\mathfrak{N}$, we obtain the following.

Theorem 5. *Let* $\mathfrak{M}$ *be the maximal ideal space of the quotient Boolean lattice* $\mathfrak{B}/\mathfrak{N}$. *Then* $L^\infty(X,\ \mathfrak{B},\ \mathfrak{N})$ *is isomorphic to* $C(\mathfrak{M})$ *as a C*-algebra. The isomorphism* $f\mapsto\hat{f}$ *is given by* $f[E]=\hat{f}(\hat{E})$ *for* $E\in\mathfrak{B}$.

By abuse of notation, we denote the class of E in $\mathfrak{B}/\mathfrak{N}$ by the same notation E.

Proof. The fact $\hat{f}(\hat{E})=f[E]$ follows from $\hat{\chi}_E=\chi_{\hat{E}}$. Conversely from $\hat{\chi}_E(\hat{B})=\chi_E[B]$ for $B,\ E\in\mathfrak{B}$, we see $\hat{\chi}_E=\chi_{\hat{E}}$. And the isomorphism is uniquely determined by $\hat{\chi}_E=\chi_{\hat{E}}$. Q.E.D.

Corollary. *Every character, i.e., algebra homorphism onto* $\boldsymbol{C}$, *of* $L^\infty(X,\ \mathfrak{B},\ \mathfrak{N})$ *is given in the form* $\lim_{E\in F} f[E]$, *where* F *is a maximal filtre of* $\mathfrak{B}/\mathfrak{N}$.

Remark 6. From the above theorem, the properties on essential images stated in § 1 are readily seen.

PART II. LINEAR ISOMETRIES ON $L^\infty(G)$ COMMUTING WITH TRANSLATIONS.

Throughout Part II, G will always denote a locally compact group, and e its neutral element. For $1\leqq p<\infty$, $p\neq 2$, Strichartz [9] and Parrott [7] proved that isometric linear operators on $L^p(G)$ commuting with every translation are scalar multiples of left translations. Here $L^p(G)$ is the space of p-th power integrable functions relative to the Haar measure. In Part II, we shall apply our main theorem of Part I, Theorem 2, to the case of $L^\infty(G)$ and obtain a similar result as of Strichartz and Parrott.

§ 7. Linear isometries on $L^\infty(G)$ commuting with translations.

7.1. In this paper we understand the Haar measure as the left invariant regular Borel measure defined on $\mathfrak{B}_0$, the σ-ring generated by all compact subsets of G. Let $\mathfrak{B}$ be the totality of locally measurable sets in G and $\mathfrak{N}$ that of locally null sets in G. Here a subset B is called *locally measurable* (resp. *locally null*) if $B\cap K$ belongs to $\mathfrak{B}_0$ (resp. belongs to $\mathfrak{B}_0$ and measure zero) for an arbitrary compact set K. We define $L^\infty(G)$ as $L^\infty(G, \mathfrak{B}, \mathfrak{N})$ given in § 1. From the definition, we see that the system $(\mathfrak{B}, \mathfrak{N})$ satisfies the condition (O^*) and the inner regularity (I) in § 2.

The left and right translation operators $L(t)$ and $R(t)$ for $t\in G$ are defined by $L(t)f(x)=f(t^{-1}x)$ and $R(t)f(x)=f(xt)$ $(x\in G)$.

The main theorem in Part II is the following.

Theorem 6. *Let T be an isometric linear operator on $L^\infty(G)$ commuting with every right translation. Then T is of the form $\alpha L(s)$, where $s\in G$ and α is a scalar of modulus* 1.

Note that we do not assume the surjectivity of T.

Before the proof, we deduce from Theorem 6 the case that T preserves the pointwise multiplication of $L^\infty(G)$.

Corollary. *Let S be an injective algebra endomorphism of $L^\infty(G)$ commuting with every right translation. Then S is a left translation.*

Proof. Let us prove that an injective endomorphism S is isometric. Then by Theorem 6, $S=\alpha L(s)$ for some $s\in G$ and $|\alpha|=1$. And letting S act on $1=1^2$, we get $\alpha=\alpha^2$, whence $\alpha=1$.

Now, since $\chi_E=\chi_E^2$, we have $S\chi_E=(S\chi_E)^2$. So $S\chi_E$ is of the form $\chi_{\tilde{E}}$. Here $\tilde{E}$ is not in $\mathfrak{N}$ if E is not, because S is injective. Thus S transfers a step function isometrically to a step function. Since the space of step functions is dense, S is isometric on the whole $L^\infty(G)$. Q.E.D.

By virtue of Theorem 2, the proof of Theorem 6 is reduced to the following Proposition 7. We note here that to Corollary of Theorem 6, we can take a shortcut not through Proposition 7 but through Proposition 8 in § 9.

Proposition 7. *Let T be an isometric linear operator on $L^\infty(G)$ commuting with every right translation. Then $C_0(G)$ is stable under T and the restriction of T to $C_0(G)$ is of the form $\alpha L(s)$, where $s\in G$ and α is a scalar of modulus* 1.

7.2. Lemmas on continuous linear forms on $C_0(G)$. For the proof of Proposition 7, we make some preparations.

Let μ be a continuous linear form on $C_0(G)$ with respect to the supremum norm. As usual we define supp μ (the support of μ) to be the set of $x\in G$ satisfying the following: for every neighbourhood W of x, there exists some $f\in C_0(G)$ such that supp $f\subset W$ and $\mu f\neq 0$. Note that supp μ is closed, and that $\mu f=0$ if supp $f\cap$ supp $\mu=\varnothing$. For a point a of G, δ_a denotes the Dirac measure at a : $\delta_a f=f(a)$ $(f\in C_0(G))$.

We need the following three lemmas. The first one is well known.

Lemma 5. *Let μ be a continuous linear form on $C_0(G)$. Assume that* supp μ *is a one point set $\{a\}$. Then μ is a constant multiple of the Dirac measure at a.*

Lemma 6. *Let μ be a continuous linear form on $C_0(G)$, and let $f\in C_0(G)$ be non-zero. If* $\mathrm{supp}\,\mu\not\subset\mathrm{supp}\,f$, *then* $|\mu f|<\|\mu\|\|f\|$, *where* $\|f\|=\sup_{x\in G}|f(x)|$.

Proof. Since $\mathrm{supp}\,\mu\not\subset\mathrm{supp}\,f$, we can find an $h\in C_0(G)$ satisfying $\mathrm{supp}\,h\cap\mathrm{supp}\,f=\emptyset$ and $\mu h\neq 0$. We may assume in addition that $\|h\|\leqq\|f\|$. Then $\|cf+c'h\|=\|f\|$ for c, c', complex numbers with modulus 1. Taking such an h, and choosing c, c' in such a way that $|\mu f|=c\cdot\mu f$, $|\mu h|=c'\cdot\mu h$, we have

$$\begin{aligned}|\mu f|<|\mu f|+|\mu h|&=c\cdot\mu f+c'\cdot\mu h\\&=\mu(cf+c'h)\leqq\|\mu\|\|cf+c'h\|=\|\mu\|\|f\|.\end{aligned}$$

This completes the proof. Q.E.D.

Lemma 7. *Let μ be a continuous linear form on $C_0(G)$, and $f\in C_0(G)$. Put $\varphi(x)=\mu(R(x)f)$. Then φ is in $C_0(G)$.*

Proof. Clearly φ is continuous, because f is uniformly continuous. Moreover since $|\varphi(x)|\leqq\|\mu\|\|f\|$, we have only to prove the assertion for f with compact support.

For $\varepsilon>0$, take an $h\in C_0(G)$ with compact support such that $|\mu h|\geqq\|\mu\|-\varepsilon$ and $\|h\|\leqq 1$. Then we have the following (♯) for $g\in C_0(G)$ with $\mathrm{supp}\,g\cap\mathrm{supp}\,h=\emptyset$:

$$(\sharp)\qquad |\mu g|\leqq\varepsilon\|g\|.$$

In fact, put $g'=c\|g\|h+c'g$, where c, c' are such complex numbers with modulus 1 that $|\mu h|=c\cdot\mu h$, $|\mu g|=c'\cdot\mu g$. Then we see that $\|g\|=\|g'\|$ whence $\|\mu\|\|g\|\geqq|\mu g'|$, and that

$$g'=\|g\||\mu h|+|\mu g|\geqq\|g\|(\|\mu\|-\varepsilon)+|\mu g|.$$

So we obtain (♯).

Now, let f be with compact support. If $x\notin(\mathrm{supp}\,h)^{-1}(\mathrm{supp}\,f)$, then $\mathrm{supp}\,h\cap\mathrm{supp}\,R(x)f=\emptyset$. Therefore by (♯) we get

$$|\varphi(x)|=|\mu(R(x)f)|\leqq\varepsilon\|R(x)f\|=\varepsilon\|f\|$$

for any x outside the compact set $(\mathrm{supp}\,h)^{-1}(\mathrm{supp}\,f)$. Q.E.D.

7.3. Proof of Proposition 7. Note first that the space $B_r(G)$ of all right uniformly continuous bounded functions is stable under T. In fact, T is isometric and commutes with every right translation, whence

$$\|R(t)Tf-Tf\|=\|TR(t)f-Tf\|=\|R(t)f-f\|.$$

Although this norm is not the supremum norm but the essential supremum norm, one can see without difficulty that $Tf\in B_r(G)$ for $f\in B_r(G)$ using the convolution by an approximate identity (see e.g. Parrott [7, Lemma 2]). Then since $|Tf(e)|\leqq\|Tf\|=\|f\|$, the linear form $\mu: f\mapsto Tf(e)$ is continuous and with norm $\leqq 1$ on

$C_0(G)$. Since $Tf(x)=R(x)Tf(e)=TR(x)f(e)$, we have

$$Tf(x)=\mu(R(x)f) \quad (f\in C_0(G)).$$

So by Lemma 7, we see that $C_0(G)$ is stable under T.

We show next that supp μ is of one point. Suppose that supp μ contains distinct two points a and b. Let V be a neighbourhood of e such that $VV^{-1}\not\ni ab^{-1}$. Then for any $x\in G$, $Vx\not\supset\{a, b\}$, hence $Vx\not\supset \text{supp}\,\mu$. Therefore for non-zero $f\in C_0(G)$ satisfying supp $f\subset V$, we have supp $R(x)f=(\text{supp}\,f)x^{-1}\not\supset\text{supp}\,\mu$. For such an f, we see from Lemma 6 that $|Tf(x)|=|\mu(R(x)f)|<\|f\|$. On the other hand, the function $|Tf(x)|$ attains its maximum, because $Tf\in C_0(G)$. So we have $\|Tf\|<\|f\|$, a contradiction. It is clear that supp $\mu\neq\emptyset$, therefore supp μ is a one point set.

By lemma 5, μ is of the form $\alpha\delta_{s^{-1}}$ for some $s\in G$ and α a scalar. Consequently $Tf(x)=\alpha f(s^{-1}x)=\alpha L(s)f(x)$ holds for $f\in C_0(G)$. Note here that $|\alpha|=1$ because T is isometric. This completes the proof of Proposition 7. Q. E. D.

Thus Theorem 6 is now proved.

§ 8. Remarks on related facts.

In Theorem 6, we do not assume that the surjectivity of the operator T. When we add this assumption on T in Theorem 6, we can prove it in another way. We shall explain this and give some remarks on related facts.

8.1. Relation between isometries and algebra automorphisms.

Theorem BS. *Let A be a commutative C^*-algebra with the unit element* 1, *and T an isometric linear operator on A onto itself. Then there exists an algebra automorphism S of A such that*

$$Tf=T1\cdot Sf \quad (f\in A).$$

This follows from the following Banach-Stone theorem via the Gelfand-Naimark representation theorem.

Theorem (Banach-Stone). *Let X and X' be compact spaces, and T an isometric linear isomorphism of $C(X')$ onto $C(X)$. Then there exist a homeomorphism τ of X onto X' and a continuous function α on X with values of modulus* 1 *such that*

$$Tf(x)=\alpha(x)f(\tau(x)) \quad (x\in X,\ f\in C(X')).$$

Here for a compact space X, $C(X)$ denotes the space of all continuous complex-valued functions on X with the supremum norm.

The Banach-Stone theorem is due to Banach [1, p. 173] and to Stone [8, p. 469]. They consider real-valued function spaces. For the case of complex-valued functions, see for example Dunford-Schwartz [4, p. 442].

8.2. The following theorem is obtained in Takesaki-Tatsuuma [10, Theorem 1]. We shall give an elementary proof of it in the next section.

Theorem (Takesaki-Tatsuuma). *Let S be an algebra automorphism of $L^\infty(G)$ commuting with every right translation. Tnen S is a left translation.*

Combining Theorem BS and the above theorem, we get the following.

Theorem 7. *Let T be a surjective isometric linear operator on $L^\infty(G)$ commuting with every right translation. Then T is of the form $\alpha L(s)$, where $s \in G$ and α is a scalar of modulus* 1.

Proof. By Theorem BS, there exists an algebra automorphism S of $L^\infty(G)$ such that $Tf = T1 \cdot Sf$ $(f \in L^\infty(G))$. Observe that $T1$ is a scalar of modulus 1. In fact, $T1$ is invariant under every right translation, because so is the constant 1 and T commutes with these translations. Since T is isometric, $T1$ is of modulus 1. Thus Theorem 7 is reduced to the theorem of Takesaki-Tatsuuma.

Q. E. D.

8.3. Implications among our results and related facts. It is known that algebra automorphisms of a W^*-algebra are automatically weak* continuous. By virtue of this fact and Theorem BS, Theorem 7 can be rewritten in its dual form:

Theorem 7*. *Let T be a surjective isometric linear operator on $L^1(G)$ commuting with every right translation. Then T is of the form $\alpha L(s)$, where $s \in G$ and α is a scalar of modulus* 1.

This in turn is a special case of Wendel's theorem [12, Theorem 3] that requires no surjectivity of T.

We illustrate in the following diagram how these theorems imply each other.

Th. of Wendel		Th. 6	$\Longrightarrow$	Cor. of Th. 6
$\Downarrow$		$\Downarrow$		$\Downarrow$
Th. 7*	$\Longleftrightarrow$ (by duality)	Th. 7	$\Longleftrightarrow$ (by Th. BS)	Th. of Takesaki-Tatsuuma

§ 9. A proof of the theorem of Takesaki-Tatsuuma.

In this section, we give another proof of the theorem of Takesaki-Tatsuuma, which is essentially on the same line as the original one in [10], but somewhat direct.

By $C(G)$ or $\mathcal{K}(G)$, we denote the space of all continuous functions or that of all continuous functions with compact support on G respectively.

Proposition 8. *Let A be a subalgebra of $C(G)$ including $\mathcal{K}(G)$, and S an algebra homomorphism of A to $C(G)$. Assume that $SR(x)h = R(x)Sh$ for $h \in \mathcal{K}(G)$,*

$x\in G$, and that S does not vanish identically on $\mathcal{K}(G)$. Then S is a left translation.

Remark. For A and S, we assume no topological property.

Proof. It is not difficult to see that every character of $\mathcal{K}(G)$, i.e., algebra homomorphism of $\mathcal{K}(G)$ to $\boldsymbol{C}$, is of the form $h\mapsto h(t)$ for some $t\in G$ if it is not identically zero. Now, let us consider a character ξ of $\mathcal{K}(G)$ defined by $\xi h=Sh(e)$. Then we have $\xi(R(x)h)=Sh(x)$ because S commutes with $R(x)$ on $\mathcal{K}(G)$. Since S is not identically zero on $\mathcal{K}(G)$, neither is ξ. Therefore from the fact mentioned above, we see that $\xi h=h(s^{-1})$ for some $s\in G$. Then for every $h\in\mathcal{K}(G)$, $Sh(x)=\xi(R(x)h)=h(s^{-1}x)=L(s)h(x)$. Since $fh\in\mathcal{K}(G)$ for $f\in A$ and $h\in\mathcal{K}(G)$, we have

$$(L(s)^{-1}Sf)h=L(s)^{-1}S(fh)=fh\,.$$

Here h is an arbitrary function in $\mathcal{K}(G)$, so we obtain $L(s)^{-1}Sf=f$ for $f\in A$. Hence $S=L(s)$ on A. Q.E.D.

Proof of the theorem of Takesaki-Tatsuuma. As is shown in the proof of Proposition 7, the space $B_r(G)$ of all right uniformly continuous bounded functions is stable under S. Applying Proposition 8 for $A=B_r(G)$, we see that $S=L(s)$ on $B_r(G)$ for some $s\in G$. Since both S and $L(s)$ are continuous in the weak* topology $\sigma(L^\infty, L^1)$ and coincide on a weak* dense subspace $B_r(G)$, they coincide on the whole $L^\infty(G)$. Q.E.D.

Remark 7. Corollary of Theorem 6, which is stronger result than the theorem above, can also be proved directly by Theorem 2 and Proposition 8 independently of Theorem 6.

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