J. Math. Kyoto Univ. (JMKYAZ)
20-2 (1980) 315-325

The comodule structure of $K_*(\Omega Sp(n))$

By

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(Received Feb. 5, 1979)

§0. Introduction

Let $Sp(n)$ be the n-th symplectic group, $\Omega Sp(n)$ its loop space. F. Clarke determined the Hopf algebra $K_*(\Omega Sp(n))$ for $n \leqq 3$ where $K_*(\ )$ is the Z_2-graded K-homology theory, following to the corresponding result of the ordinary homology by R. Bott ([6], [9]). Recently, the Hopf algebra $H^*(\Omega Sp(n))$ was determined by Kono-Kozima [11]. Though our method used in [11] is not applicable for K-theory directly, we can determine the Hopf algebra $K_*(\Omega Sp(n))$ by some algebraic devices ([12]).

Using these results, we can fix the generators of $K_*(\Omega Sp(n))$. The purpose of this paper is to determine the $K_*(K)$-comodule structures of $K_*(\Omega Sp(n))$, and to write down the formulas for the above generators.

Under some algebraic notations, the main result of this paper is

Theorem 3.10.

$$\Psi_K Z(x) = \frac{\left(\frac{d}{dx} Q(x)\right) \cdot Z(Q(x))}{1 \otimes 1 + S(x) \cdot Z(Q(x))}.$$

For details, see §3.

This paper is organized as follows:

In §1, we recall some notations in [11], [12].

The results of [12] are written by the words of Z_2-graded K-theory, but we can easily obtain the corresponding ones in Z-graded K-theory. So in this paper, we use only Z-graded theory.

In §2, we quote the results of [2], [3] and [15] for the structures of $K_*(K)$, $KO_*(KO)$ and the comodule structure of $K_*(BU)$ where BU is the classifying space of the infinite unitary group U.

We also summarize the result of [12] for the structure of $K_*(\Omega Sp(n))$ where $K_*(\ )$ is Z-graded K-homology theory.

In §3, we introduce some algebraic notations which are needed to state the main result, and prove it.

Notice that the Euler class in K-theory which is used in this paper is different from the ordinary one in [2], [15]. So, β_i used in §2, §3 is $t^i\beta_i$ in [2], or [15] where $t\in\pi_2(K)$ is the usual generator.

Throughout the paper, the ring of integers is denoted by Z, the ring of integers modulo n by Z_n, and the rational numbers by Q.

If R is a ring with unit, then the formal power series ring over R is denoted by $R[[x]]$. If $f(x)=\sum_i f_i x^i\in R[[x]]$ then the coefficient of x^n in $f(x)$ is denoted by $[f(x)]_n$.

Then the binomial coefficient $\binom{n}{m}$ is equal to $[(1+x)^n]_m$.

§1. Notations

First, recall some notations (see [11], [12]).

Let $U(n)$, $Sp(n)$ be the n-th unitary and symplectic group, and U, Sp the infinite unitary and symplectic group, respectively.

Let $\mathbf{C}$ (resp. $\mathbf{H}$) be the field of complex (resp. quaternion) numbers.

Let $a_{ij}\in\mathbf{H}$ and $a_{ij}=b_{ij}+jc_{ij}$ for b_{ij}, $c_{ij}\in\mathbf{C}$ and define a map $c\colon Sp(n)\to U(2n)$ by

$$c((a_{ij}))=\begin{pmatrix} N_{11}\cdots N_{1n}\\ \vdots \quad \vdots\\ N_{n1}\cdots N_{nn}\end{pmatrix},$$

where

$$N_{ij}=\begin{pmatrix} b_{ij} & -c_{ij}\\ \bar{c}_{ij} & \bar{b}_{ij}\end{pmatrix}.$$

Let BG be the classifying space of a topological group G and $Bf\colon BG\to BH$ the map induced by a continuous homomorphism $f\colon G\to H$.

Let ΩX be the space of loops on a space X and $\Omega f\colon \Omega X\to\Omega Y$ the map induced by a map $f\colon X\to Y$.

Let $SU(n)$, SU be the n-th and the infinite special unitary group.

Let $g\colon BU\to\Omega SU$ be the Bott map. Since $\Omega SU=\Omega_0^2BU$, we may regard g as a map to Ω_0^2BU.

Let $i_n\colon Sp(n)\to Sp$ be the natural inclusion (see [11]).

Let $\gamma\to BU(1)$ be the Hopf line bundle. Though we usuary use $[\gamma-1]\in K^0(BU(1))$ as the Euler class in K-theory ([2], [15]), we use $t^{-1}[\gamma-1]\in K^2(BU(1))$ as the Euler class in K-theory where $t\in\pi_2(K)=K^{-2}(pt)$ throughout this paper.

§2. $K_*(K)$-comodules

In this section, we quote some theorems in [2], [15], and state the main results.

Let K be BU-spectrum and KO BO-spectrum. In general, we have next theorem [15].

Theorem 2.1. *Suppose* E *is a commutative ring spectrum such that* $E_*(E)=C$ *is flat as a right module over* $E_*(pt)=R$. *Then there are homomorphisms*

$$\phi:\ C\otimes_R C\longrightarrow C \qquad \varepsilon:\ C\longrightarrow R$$

$$\eta_L:\ R\longrightarrow C \qquad \eta_R:\ R\longrightarrow C$$

$$c:\ C\longrightarrow C \qquad \Psi_E:\ C\longrightarrow C\otimes_R C$$

and Ψ_X: $E_*(X)\to C\otimes_R E_*(X)$ *for all spectra* X, *with the following properties*:

i) C *is a commutative Hopf algebra with product* ϕ *having left and right units* η_L, η_R *and associative coproduct* Ψ_E *having augmentation* ε;

ii) *if* $\lambda\in R$ *and* $x\in C$, *then* $\lambda x=\phi(\eta_L(\lambda)\otimes x)$, $x\lambda=\phi(x\otimes\eta_R(\lambda))$;

iii) $\varepsilon\eta_L=1=\varepsilon\eta_R$, $c\eta_L=\eta_R$, $c\eta_R=\eta_L$, $\varepsilon c=\varepsilon$, $c^2=1$;

iv) $\Psi_E(1)=1\otimes 1$ *and hence* $\Psi_E\eta_L(\lambda)=\eta_L(\lambda)\otimes 1$ *and* $\Psi_E\eta_R(\lambda)=1\otimes\eta_R(\lambda)$ *for all* $\lambda\in R$;

v) Ψ_X *is natural with respect to maps of* X;

vi) Ψ_X *is a coaction map*;

vii) *if* $\Psi_X(x)=\sum_i e_i'\otimes x_i$ *and* $\Psi_Y(y)=\sum_j e_j''\otimes y_j$ *for* $x, x_i\in E_*(X)$, $y, y_j\in E_*(Y)$, $e_i', e_j''\in C$ *then* $\Psi_{X\wedge Y}(x\wedge y)=\sum_{i,j}(-1)^{|x_i|\cdot|e''_j|}e_i'e_j''\otimes(x_i\wedge y_j)$;

viii) Ψ_{S^0}: $R\to C\otimes_R R=C$ *is just* η_L.

For details, see [15]. Notice that $K_*(K)$ and $KO_*(KO)$ are right flat over $K^*(pt)$ and $KO_*(pt)$, respectively. Then (2.1) is applicable to the cases $E=K$, KO.

We prefer to write Ψ_K in place of Ψ_X in (2.1) because we want to clarify that we work in K-theory.

Let $t\in K_2(pt)$ be a generator and $h_i\in K_{2i}(BU(1))$ be the dual element of i-th power of the Euler class in K-theory. Let $\beta_i\in K_{2i}(BU)$ be the image of h_i by the homomorphism induced by the natural map Bi: $BU(1)\to BU$ (see [2], [11]).

Let $u=\eta_L(t)$, $v=\eta_R(t)$.

If we consider BU as the $2n$-th term of K-spectrum, then we have a homomorphism

$$\iota_n:\ K_{2q}(BU)\longrightarrow K_{2q-2n}(K).$$

Let c: $KO\to K$ be the complexification map. Then we have

$$(c\wedge c)^*:\ KO_*(KO)\longrightarrow K_*(K).$$

Proposition 2.2.

i) $K_*(K)\to K_*(K)\otimes Q=Q[u, u^{-1}, v, v^{-1}]$ *is monic.*

ii) *the composition*

$$\oplus_n KO_{4n}(KO)\xrightarrow{(c\wedge c)_*}K_*(K)\longrightarrow K_*(K)\otimes Q=Q[u, u^{-1}, v, v^{-1}]$$

is monic.

Theorem 2.3. *In* $K_*(K)\otimes Q$

$$\iota_1(\beta_n)=p_n(u, v)=\frac{1}{n!}(v-u)(v-2u)\cdots(v-(n-1)u), \qquad \text{for } n>1.$$

Theorem 2.4.

i) *$K_*(K)$ is spanned by $p_n(u, v)$ over $Z[u, u^{-1}, v^{-1}]$ in $K_*(K)\otimes Q$.*

ii) *The image of $\oplus_n KO_{4n}(KO)$ in $K_*(K)\otimes Q$ is spanned by*

$$q_n(u, v)=\frac{2}{(2n+2)!}(v^2-u^2)(v^2-2^2u^2)\cdots(v^2-n^2u^2),$$

for $n>0$ over $Z[u^4, 2u^2, u^{-4}, v^{-4}]$.

In [15], Switzer also determined $\Psi_K(p_n(u, v))$ and $\Psi_K(q_n(u, v))$.

We abbreviate $p_n(u, v)$, $q_n(u, v)$ to p_n, q_n.

Put $p_0=q_0=1$.

Put

$$P(x)=\sum_{i\geq 0} p_i x^{i+1},$$

$$Q(x)=\sum_{i\geq 0} q_i x^{i+1} \in K_*(K)[[x]].$$

Theorem 2.5.

i) $\Psi_K(u)=u\otimes 1$, $\Psi_K(v)=1\otimes v$ *and*

$$\Psi_K(p_n)=\sum_{j\geq 0}[(P(x))^{j+1}]_{n+1}\otimes p_j, \quad n\geq 0,$$

ii) $\Psi_K(q_n)=\sum_{j\geq 0}[(Q(x))^{j+1}]_{n+1}\otimes q_j, \quad n\geq 0.$

For the proofs of (2.2)–(2.5), see [2], [3] and [15]. The formulas of (2.5) are slightly different from the original ones in [15], but one can easily show that they are essentially equivalent.

Let B: $S^2BU\rightarrow BU$ be an adjoint map of the Bott map g: $BU\rightarrow\Omega_0^2BU$. Define $\underline{B}$: $\tilde{K}_m(BU)\longrightarrow\tilde{K}_{m+2}(BU)$ by the composition

$$\tilde{K}_m(BU)\cong\tilde{K}_{m+2}(S^2BU)\xrightarrow{B_*}\tilde{K}_{m+2}(BU).$$

Then the diagram

$$\begin{array}{ccc}\tilde{K}_m(BU) & \xrightarrow{\iota_n} & K_{m-2n}(K)\\ \downarrow{\underline{B}} & & \downarrow{v\cdot}\\ \tilde{K}_{m+2}(BU) & \xrightarrow{\iota_n} & K_{m-2n+2}(K)\end{array}$$

commutes (see [15]).

Since $v\cdot$ is an isomorphism and $\underline{B}$ kills the decomposable elements in $K_*(BU)$, ι_n also vanishes on the decomposable elements. As a corollary of (2.2) and (2.3), we have

Proposition 2.6. *ι_n: $Q(K_*(BU))\rightarrow K_{*-2n}(K)$ is a monomorphism where Q is indecomposable functor.*

Proof. One can easily show that $\{v^{-n+1}p_i(=\iota_n(\beta_i))\}_{i>0}$ are linearly independent over $Z[u, u^{-1}]$. Since $Q(K_*(BU))$ is a free $K_*(pt)$-module generated by $\{\beta_i\}_{i>0}$, (2.6) is clear.

Let I be the composition

$$\tilde{K}_m(BU(1)) \longrightarrow \tilde{K}_m(BU) \xrightarrow{\iota_1} K_{m-2}(K).$$

Then (2.6) says

Proposition 2.7. *I is a monomorphism.*

Next we rewrite the results of [12] in the words of Z-graded K-theory. We will use the same notation both in Z_2- and Z-graded theories.

So, put $b_{2n-1}=\sum_{i\geqq 0}\binom{n-1}{i}t^i\beta_{2n-1-i}\in K_{4n-2}(BU)$ and

$$b_{2n}=\sum_{i\geqq 0}\binom{n-1}{t}t^i\beta_{2n-1}\in K_{4n}(BU) \quad \text{for} \quad n>0.$$

We define also $b_{od}(x)$ and $b_{ev}(x)\in K_*(BU)[[x]]$ to be $\sum_{i>0}b_{2i-1}x^{2i-1}$ and $1+\sum_{i>0}b_{2i}x^{2i}$. Put $r(x)=\sum_{i>0}r_{2i-1}x^{2i-1}=b_{od}(x)/b_{ev}(x)$. Let g, c and i_n be the maps as in § 1.

Theorem 2.8. *There are $z_{2k-1}\in K_{4k-2}(\Omega Sp)$ such that*

i) $K_*(\Omega Sp)=K_*(pt)[z_1, z_3,\dots, z_{2k-1},\dots]$ *as an algebra*

ii) $g_*^{-1}\circ(\Omega c)_*\colon K_*(\Omega Sp)\to K_*(BU)$ *is monic. Moreover* $g_*^{-1}\circ(\Omega c)_* z_{2k-1}=r_{2k-1}$, *for* $k>0$,

iii) $(\Omega i_n)_*\colon K_*(\Omega Sp(n))\to K_*(\Omega Sp)$ *is monic, and* $\operatorname{Im}(\Omega i_n)_*$ *is generated by* $z_1, z_3,\dots, z_{2n-1}$ *as a subalgebra of* $K_*(\Omega Sp)$.

For the proofs of i)–iii), one has only to modify those of the corresponding theorems of [12].

Since the coactions are natural, and since $\Psi_K(\beta_n)$ is known, we can know the comodule structure of $K_*(\Omega Sp(n))$ or $K_*(\Omega Sp)$ by virtue of the above theorem.

In the next section, we will obtain 'internal' formulas, that is, we will write down $\Psi_K(z_n)$ by z_i, q_j and the new element $s_k\in K_*(K)$.

§ 3. Comodule structure of $K_*(\Omega Sp(n))$

First, we determine $\Psi_K(b_{2n-1})$ and $\Psi_K(b_{2n})$.

Let $H_{2n-1}=\sum_{i\geqq 0}\binom{n-1}{i}t^i h_{2n-1-i}$ and

$$H_{2n}=\sum_{i\geqq 0}\binom{n-1}{i}t^i h_{2n-i}\in K_*(BU(1)).$$

By the naturarity of Ψ_K, we have only to determine $\Psi_K(H_{2n-1})$ and $\Psi_K(H_{2n})$. We need the following lemma.

Lemma 3.1.

i) $I(H_{2n-1})=\iota_1(b_{2n-1})=nq_{n-1}$ and,

ii) $I(H_{2n})=\iota_1(b_{2n})=q_{n-1}\left(\frac{v-nu}{2}\right)$.

Proof. For i), we need to show that

$$\sum_{i\geq 0}\binom{n-1}{i}u^i p_{2n-1-i}=nq_{n-1}.$$

Put
$$\tilde{p}_k(x)=\frac{1}{(k+1)!}(x-1)(x-2)\cdots(x-k) \quad \text{and}$$

$$\tilde{q}_k(x)=\frac{2}{(2k+2)!}(x^2-1)(x^2-2^2)\cdots(x^2-k^2)\,.$$

Then $p_{k+1}=\tilde{p}_k\left(\frac{v}{u}\right)u^k$ and $q_k=\tilde{q}_k\left(\frac{v}{u}\right)u^{2k}$.

So we have only to prove the equation

$$(3.2) \qquad \sum_{i\geq 0}\binom{n-1}{i}\tilde{p}_{2n-2-i}(x)=n\tilde{q}_{n-1}(x)\,.$$

For ii), we need to prove

$$(3.3) \qquad 2\cdot\sum_{i\geq 0}\binom{n-1}{i}\tilde{p}_{2n-1-i}(x)=(x-n)\tilde{q}_{n-1}(x)\,.$$

Clearly the both sides of (3.2) (respectively (3.3)) are the polynomials of degree $2n-2$ (respectively $2n-1$), and have the same coefficients at the maximal degree. Since $\binom{n-1}{i}=0$ for $i>n-1$, the both sides of (3.2) (respectively (3.3)) have common roots $1, 2,\dots, n-1$ (respectively $1, 2,\dots, n$). So we may show that

$$(3.4) \qquad \sum_{i\geq 0}\binom{n-1}{i}\tilde{p}_{2n-2-i+\varepsilon}(-k)=0 \qquad \text{for} \quad k=1, 2,\dots, n-1$$

where $\varepsilon=1$ or 0.

If $k>0$, then we have

$$\tilde{p}_m(-k)=\frac{1}{(m+1)!}(-k-1)(-k-2)\cdots(-k-m)$$
$$=(-1)^m\cdot\frac{1}{k}\cdot\binom{m+k}{m+1}.$$

Then

$$k\cdot\sum_{i\geq 0}\binom{n-1}{i}\tilde{p}_{2n-2-i+\varepsilon}(-k)$$

$$=\sum_{i\geqq 0}\binom{n-1}{i}(-1)^i[(1+x)^{2n-2+\varepsilon-i+k}]_{2n-1-i+\varepsilon}$$

Since

$$\sum_{i\geqq 0}\binom{n-1}{i}(-1)^i[(1+x)^{2n-2+\varepsilon-i+k}]_{2n-i-1+\varepsilon}$$
$$=\sum_{i\geqq 0}\binom{n-1}{i}(-1)^i[x^i(1+x)^{2n-2+\varepsilon-i+k}]_{2n-1+\varepsilon}$$
$$=\left[\left\{\sum_{i\geqq 0}\binom{n-1}{i}(-x)^i(1+x)^{n-i-1}\right\}(1+x)^{n+k-1+\varepsilon}\right]_{2n-1+\varepsilon}$$
$$=[\{(1+x)-x\}^{n-1}(1+x)^{n+k-1+\varepsilon}]_{2n-1+\varepsilon}$$
$$=[(1+x)^{n+k-1+\varepsilon}]_{2n-1+\varepsilon}=0$$ for $k=1, 2,\ldots, n-1$, (3.4) holds and (3.1) does.

Q.E.D.

Now we consider the following commutative diagram

$$\begin{array}{ccc} K_*(BU(1)) & \xrightarrow{\Psi_K} & K_*(K)\otimes_{K_*(pt)}K_*(BU(1)) \\ \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle id\otimes I} \\ K_{*-2}(K) & \xrightarrow{\Psi_K} & K_*(K)\otimes_{K_*(pt)}K_*(K)\,. \end{array}$$

As remarked in §2, $K_*(K)$ is a right flat module over $K_*(pt)$. So (2.7) says also that $id\otimes I$ is monic. Thus, to determine $\Psi_K(H_n)$, we have only to calculate $\Psi_K(nq_{n-1})$ and $\Psi_K\left(q_{n-1}\left(\frac{v-nu}{2}\right)\right)$.

proposition 3.5.

$$\Psi_K(nq_{n-1})=\sum_{j\geqq 0}\left[\left(\frac{d}{dx}Q(x)\right)\cdot(Q(x))^j\right]_{n-1}\otimes(j+1)q_j.$$

Proof. By (2.5) ii),

$$\Psi_K n(q_{n-1})=n\sum_{j\geqq 0}[(Q(x))^{j+1}]_n\otimes q_j.$$

So we have to prove that

$$n[(Q(x))^{j+1}]_n=\left[(j+1)(Q(x))^j\left(\frac{d}{dx}Q(x)\right)\right]_{n-1}. \tag{3.6}$$

This is clear, because $(j+1)(Q(x))^j\left(\frac{d}{dx}Q(x)\right)=\frac{d}{dx}\{(Q(x))^{j+1}\}$.

For brevity, we put $s_n=q_{n-1}\left(\frac{v-nu}{2}\right)$, $n>0$. Also we put $S(x)=\sum_{i>0}s_ix^i \in K_*(K)[[x]]$. Then

Proposition 3.7.

$$\Psi_Ks_n=\sum_{j>0}[(Q(x))^j]_n\otimes s_j+\sum_{j\geqq 0}[S(x)\cdot(Q(x))^j]_n\otimes(j+1)q_j.$$

Proof. As in (3.5), we have

$$\Psi_K\left(q_{n-1}\cdot\left(\frac{v-nu}{2}\right)\right)=(\textstyle\sum_{j>0}[(Q(x))^j]_n\otimes q_{j-1})\cdot\left(\frac{1\otimes v-nu\otimes 1}{2}\right)$$

by (2.5). On the other hand, since $v\otimes 1=1\otimes u$ in $K_*(K)\otimes_{K_*(pt)}K_*(K)$, we have

$$\begin{aligned}&\textstyle\sum_{j>0}(Q(x))^j\otimes q_{j-1}(v-ju)\\ &\quad+((v-u)x+q_1(v-2u)x^2+\cdots)\otimes 1\cdot\textstyle\sum_{j\geqq 0}(Q(x))^j\otimes(j+1)q_j\\ &=\textstyle\sum_{j>0}(Q(x))^j\otimes q_{j-1}\cdot v-\sum_{j>0}(Q(x))^j\cdot jv\otimes q_{j-1}\\ &\quad+\textstyle\sum_{j\geqq 0}vQ(x)\cdot(Q(x))^j\otimes(j+1)q_j-\sum_{j\geqq 0}u(x+2q_1x^2+\cdots)\cdot(Q(x))^j\otimes(j+1)q_j\\ &=\textstyle\sum_{j>0}(Q(x))^j\otimes q_{j-1}v-\sum_{j\geqq 0}u(x+2q_1x^2+\cdots)(Q(x))^j\otimes(j+1)q_j.\end{aligned}$$

So we have to prove that

$$\begin{aligned}&\textstyle\sum_{j>0}nu[(Q(x))^j]_n\otimes q_{j-1}\\ &\quad=\textstyle\sum_{j\geqq 0}u[(x+2q_1x^2+\cdots)\cdot(Q(x))^j]_n\otimes(j+1)q_j.\end{aligned}$$

Thus we have only to show that

$$n[(Q(x))^{j+1}]_n=[(j+1)(Q(x))^j(x+2q_1x^2+\cdots)]_n. \tag{3.8}$$

Since the right side of (3.8) is $\left[(j+1)(Q(x))^j\left(\frac{d}{dx}Q(x)\right)\right]_{n-1}$, (3.8) is equivalent to (3.6). Q.E.D.

As a collorary, we have

Theorem 3.9.

i) $\Psi_K b_{2n-1}=\sum_{j\geqq 0}\left[\left(\frac{d}{dx}Q(x)\right)\cdot(Q(X))^j\right]_{n-1}\otimes b_{2j+1}$ and

ii) $\Psi_K b_{2n}=\sum_{j>0}[(Q(x))^j]_n\otimes b_{2j}+\sum_{j\geqq 0}[S(x)\cdot(Q(x))^j]_n\otimes b_{2j+1}$

To determine $\Psi_K z_{2n-1}$ or $\Psi_K r_{2n-1}$, some algebraic notations are necessary.

Let R be a ring with unit.

If A and B are R-algebras, then we define i_A: $A\to A\otimes_R B$ and i_B: $B\to A\otimes_R B$ by

$$i_A(a)=a\otimes 1 \quad\text{and}$$
$$i_B(b)=1\otimes b \quad\text{where}\quad a\in A,\ b\in B.$$

Thus we regard A and B as the subalgebras of $A\otimes_R B$.

If f: $A\to B$ is an algebra homomorphism, we can define an algebra homomorphism f: $A[[x]]\to B[[x]]$ by

$$f(\textstyle\sum_i a_ix^i)=\sum_i f(a_i)x^i \quad\text{where}\quad a_i\in A.$$

So, we can regard $A[[x]]$ and $B[[x]]$ as the subalgebras of $(A\otimes_R B)[[x]]$.

Now we can state our main result. Let $Q(x)$, $S(x)$ be as before, and put $Z(x) = \sum_{i\geq 0} z_{2i+1}x^i$ where z_{2i+1} is the element in (2.8).

To determine $\Psi_K z_{2n-1}$, we have only to determine $\Psi_K Z(x)$.

Teorem 3.10.

$$\Psi_K Z(x) = \frac{\left(\frac{d}{dx}Q(x)\right)\cdot Z(Q(x))}{1\otimes 1 + S(x)\cdot Z(Q(x))}.$$

Proof. By virtue of (2.8), we may identify $Z(x)$ and

$\bar{r}(x) = \sum_{i\geq 0} r_{2i+1}x^i$. Put

$\overline{b_{od}}(x) = \sum_{i\geq 0} b_{2i+1}x^i$ and

$\overline{b_{ev}}(x) = 1 + \sum_{i>0} b_{2i}x^i$.

Then formally

$$\overline{b_{od}}(x)/\overline{b_{ev}}(x) = \frac{1}{\sqrt{x}}\cdot b_{od}(\sqrt{x})/b_{ev}(\sqrt{x}) = \frac{1}{\sqrt{x}}\cdot r(\sqrt{x}) = \bar{r}(x).$$

Thus

$$\begin{aligned}\Psi_K\bar{r}(x) &= \Psi_K(\overline{b_{od}}(x)/\overline{b_{ev}}(x)) \\ &= \Psi_K\overline{b_{od}}(x)/\Psi_K\overline{b_{ev}}(x).\end{aligned}$$

On the other hand, by (3.9), we have

$$\Psi_K\overline{b_{od}}(x) = \left(\frac{d}{dx}Q(x)\right)\cdot\overline{b_{od}}(Q(x)) \quad \text{and}$$

$$\Psi_K\overline{b_{ev}}(x) = \overline{b_{ev}}(Q(x)) + S(x)\cdot\overline{b_{od}}(Q(x)).$$

Then

$$\Psi_K\bar{r}(x) = \frac{\left(\frac{d}{dx}Q(x)\right)\cdot\overline{b_{od}}(Q(x))}{\overline{b_{ev}}(Q(x)) + S(x)\cdot\overline{b_{od}}(Q(x))}.$$

So, we have the equation

$$\Psi_K\bar{r}(x) = \frac{\left(\frac{d}{dx}Q(x)\right)\cdot\bar{r}(Q(x))}{1\otimes 1 + S(x)\cdot\bar{r}(Q(x))}.$$

Q. E. D.

As a corollary, we can obtain the comodule structure of $K_*(\Omega Sp(n))$ by (2.8).

As another corollary, we can obtain the comodule structures of $K_*(Sp(n))$ and $K_*(Sp)$.

By the result of [4], the Atiyah-Hirzebruch spectral sequence

$$H_*(Sp(n); K_*(pt)) \Rightarrow K_*(Sp(n)) \quad \text{collapses.}$$

So, if $\sigma: QK_m(\Omega Sp(n)) \to PK_{m+1}(Sp(n))$ is the homomorphism induced by the K-homology suspension where P and Q denote the primitive and indecomposable modules, we have an isomorphism

$$K_*(Sp(n)) = \Lambda_K(\sigma z_1, \sigma z_3, \dots, \sigma z_{2n-1})$$

where Λ_K represents an exterior algebra over $K_*(pt)$ (see [9] Proposition (6.6)).

Put $w_{4k-1} = \sigma z_{2k-1}$, for $k>0$. Since the homology suspension kills decomposable elements and commutes with coaction, that is, if $\Psi_K x = \sum_i a_i \otimes x_i$, then

$$\Psi_K(\sigma x) = \sum_i (-1)^{|a_i|} a_i \otimes \sigma x_i$$

where x, $x_i \in K_*(X)$, and $a_i \in K_*(K)$,
we have

Theorem 3.11. *There are $w_{4k-1} \in K_{4k-1}(Sp(n))$ for $0<k\leqq n$ such that*

$$K_*(Sp(n)) = \Lambda_K(w_3, w_7, \dots, w_{4n-1}) \quad \text{and}$$

$$\Psi_K(w_{4k-1}) = \sum_{j\geqq 0}\left[\left(\frac{d}{dx}Q(x)\right)\cdot(q(x))^j\right]_{k-1} \otimes w_{4j+3}.$$

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