J. Math. Kyoto Univ. (JMKYAZ)
20-1 (1980) 67-104

The initial value problem for the equations of motion of viscous and heat-conductive gases

By

Akitaka MATSUMURA and Takaaki NISHIDA

(Received, Sept. 30, 1978)

§ 1. Introduction.

The motion for a compressible viscous, heat-conductive, isotropic Newtonian fluid is described by the system of equations ([10], [11])

(1.1) $$\rho_t = -(\rho u^j)_{x_j}$$

(1.2) $$u_t^i = -u^j u_{x_j}^i + \rho^{-1}(\mu(u_{x_j}^i + u_{x_i}^j))_{x_j} + \rho^{-1}(\mu' u_{x_j}^j)_{x_i} - \rho^{-1} p_{x_i} + f^i$$

(1.3) $$\theta_t = (c\rho)^{-1}(\kappa\theta_{x_j})_{x_j} - u^j\theta_{x_j} - (c\rho)^{-1}\theta\frac{\partial p}{\partial\theta}u_{x_j}^j + \mu(2c\rho)^{-1}(u_{x_j}^i + u_{x_i}^j)(u_{x_j}^i + u_{x_i}^j) + \mu'(c\rho)^{-1}(u_{x_j}^j)^2,$$

where $t \geqq 0$, $x = (x_1, x_2, x_3) \in \boldsymbol{R}^3$,
ρ: density ($\rho > 0$), $u = (u^1, u^2, u^3)$: velocity,
θ: absolute temperature ($\theta > 0$), p: pressure, $f = (f^1, f^2, f^3)$: outer force,
μ: coefficient of viscosity ($\mu > 0$),
μ': second coefficient of viscosity ($\mu' + 2\mu/3 \geqq 0$),
κ: coefficient of heat conduction ($\kappa > 0$),
c: heat capacity at constant volume, and the subscripts t and x_i, $i = 1, 2, 3$, denote the partial differentiation with respect to the corresponding variable. We use the summation convention here and in what follows.

The existence theorems of unique solution local in time for (1.1)-(1.3) are obtained by Nash [14], Itaya [2, 3] for the initial value problem, and by Tani [15] for the first initial-boundary value problem. On the other hand the existence theorem of global solutions in time for (1.1)-(1.3) is not known in general. Recently some one-dimensional equations are investigated on the existence of global (in time) solutions by Kanel' [6], Itaya [3, 4], Kazhikhov and Shelukhin [8, 9]. More precisely, Kanel' obtained the global solution for the model equation

(1.4) $$\begin{cases} v_t = u_x \\ u_t = -p(v)_x + \mu\left(\dfrac{u_x}{v}\right)_x, \qquad t \geqq 0, \qquad x \in \boldsymbol{R}, \end{cases}$$

with the initial data

$$(v-v_0,\ u)(0,\ x)\in H^1(\boldsymbol{R}) \tag{1.5}$$

for v_0=constant>0. (See § 2 for the notations of the function spaces.) It contains the barotropic model in the Lagrangian coordinates: $p=a^2/v^\gamma$, γ=constant $\geqq 1$ and a, μ: constant>0. Itaya obtained the global solution for the isothermal gas model equation

$$\begin{cases} \rho_t=-(\rho u)_x \\ u_t=\mu\rho^{-1}u_{xx}-a^2\rho^{-1}\rho_x-uu_x\,, \qquad t\geqq 0\,, \qquad x\in\boldsymbol{R}\,, \end{cases} \tag{1.6}$$

μ, a: constant>0, with the general initial data

$$\begin{cases} \rho(0,\ x)\in\mathscr{B}^{1+\sigma}\,, \\ u(0,\ x)\in\mathscr{B}^{2+\sigma}\,, \end{cases} \tag{1.7}$$

where $0<\rho_1\leqq\rho(0,\ x)\leqq\rho_2<\infty$, ρ_1, ρ_2 are constants, $\sigma\in(0,\ 1)$ and $u(0,\ x)_x\in L_1(\boldsymbol{R})$. Moreover Kazhikhov and Shelukhin considered the initial-boundary value problem of (1.1)-(1.3) in the one space-dimension. They obtained the global solution in $t\geqq 0$, $x\in[0,\ 1]$ for (1.1)-(1.3) under the same assumptions on gases as ours (see the conditions (i) (ii) (iii) below) with the initial data

$$\begin{cases} \rho(0,\ x)\in\mathscr{B}^{1+\sigma}\,, \qquad 0<\rho_1\leqq\rho(0,\ x)\leqq\rho_2<\infty \\ u(0,\ x)\,, \qquad \theta(0,\ x)\in\mathscr{B}^{2+\sigma}\,, \qquad x\in[0,\ 1]\,, \end{cases} \tag{1.8}$$

where ρ_1, ρ_2 are constants, $\sigma\in(0,\ 1)$, and with the boundary data

$$u(t,\ 0)=u(t,\ 1)=\theta_x(t,\ 0)=\theta_x(t,\ 1)=0,\ t\geqq 0\,. \tag{1.9}$$

For the initial value problem of (1.1)-(1.3) the global (in time) solution is not known even in the one space-dimension. In the present paper we consider the initial value problem (1.1)-(1.3) in the three space-dimensions. We assume the following conditions on (1.1)-(1.3).

(i) The gas is ideal: $p=R\rho\theta$, R=constant>0.

(ii) The gas is polytropic: $e=c\theta$, where e denotes the internal energy and c is a constant which denotes the specific heat at constant volume. Then the equation of state for the gas is

$$e=\frac{a^2}{\gamma-1}\Big(\rho^{\gamma-1}\exp\frac{\gamma-1}{R}S\Big)+\text{constant}\,, \tag{1.10}$$

where a is a constant >0, γ is the ratio of specific heats $\geqq 1$ and S is the entropy. We have $\theta=e_S$, $p=\rho^2 e_\rho$ and $c=\dfrac{R}{\gamma-1}$.

(iii) μ, μ' and κ are positive constants and $f^i\equiv 0$, $i=1,\ 2,\ 3$. We treat more general fluids in the subsequent paper. We suppose that the initial data

$$(\rho,\ u,\ \theta)(0,\ x)\,, \qquad x\in\boldsymbol{R}^3 \tag{1.11}$$

are given as

(1.12) $$\rho(0, x)-\rho_0, \quad u(0, x), \quad \theta(0, x)-\theta_0 \in H^3(\boldsymbol{R}^3)$$

and that their norms are small in the space $H^3(\boldsymbol{R}^3)$, where ρ_0, θ_0 are positive constants. Then we show that the solution of the initial value problem (1.1)-(1.3) with (1.11) exists uniquely globally in time and that it is a classical solution for $t>0$ and decays:

(1.13) $$(\rho, u, \theta)(t, x) \longrightarrow (\rho_0, 0, \theta_0) \quad \text{as} \quad t \longrightarrow \infty .$$

We use the energy method for the proof. cf. [6], [12]. Last we remark that our method also applies to the compressible Navier-Stokes equation for the ideal isentropic fluid, where the equation of state of gas is the following:

(1.14) $$p=p(\rho), \quad p'(\rho)>0 \quad \text{for} \quad \rho>0 \quad \text{and} \quad S=\text{constant}.$$

In particular, the case $\gamma=1$ of (1.10) can be treated in this way. The first energy form for this model gas may be the following:

(1.15) $$E^0(\rho, u) \equiv (1+\rho)\int_0^\rho \frac{p(1+\rho)}{(1+\rho)^2} d\rho - \rho p(1) + \frac{1}{2}(1+\rho)u^i u^i$$

in particular, for $\gamma=1$

(1.16) $$E^0(\rho, u) \equiv a^2\{(1+\rho)\log(1+\rho)-\rho\} + \frac{1}{2}(1+\rho)u^i u^i .$$

Compare these with $E^0(\rho, u, s)$ in (2.19). We omit the argument for these cases.

§ 2. Notations and Basic Lemmas.

If we change the unknown variables $\rho \to \rho_0(1+\rho)$, $\theta \to \theta_0(1+\theta)$ in (1.1)-(1.3) under assumptions (i)-(iii) and regard $\mu\rho_0^{-1}$, $\mu'\rho_0^{-1}$, $R\theta_0$, $\kappa\theta_0\rho_0^{-1}$ as μ, μ', R, κ respectively, we may consider the following system of equations:

(2.1) $$\rho_t + u^j \rho_{x_j} + (1+\rho)u^j_{x_j} = 0$$

(2.2) $$u^i_t - \frac{\mu}{1+\rho} u^i_{x_j x_j} - \frac{\mu+\mu'}{1+\rho} u^j_{x_i x_j} = -u^j u^i_{x_j} - R\theta_{x_i} - R\frac{1+\theta}{1+\rho}\rho_{x_i} \equiv g^i , \qquad i=1, 2, 3$$

(2.3) $$\theta_t - \frac{\kappa(\gamma-1)}{R(1+\rho)}\theta_{x_j x_j} = -u^j \theta_{x_j} - (\gamma-1)(1+\theta)u^j_{x_j} + \frac{\gamma-1}{R(1+\rho)}\left\{\mu'(u^j_{x_j})^2 + \frac{\mu}{2}(u^j_{x_i}+u^i_{x_j})(u^j_{x_i}+u^i_{x_j})\right\} \equiv h .$$

Our problem is to seek the solution (2.1)-(2.3) such that $|\rho|, |u|, |\theta|<1$ globally in time, when the initial data are given in a small neighbourhood of $(\rho, u, \theta)=(0, 0, 0)$.

Definition 2.1.

(i) $L_p(1\leqq p<\infty)$: the Lebesgue space of measurable functions on $\boldsymbol{R}^3$ whose p-th powers are integrable with the norm

$$\|f\|_{L_p}\equiv\left(\int_{R^3}|f(x)|^p dx\right)^{1/p}. \tag{2.4}$$

For $p=2$, we simply write $\|\cdot\|$.

(ii) $\mathscr{B}^k(k=0, 1, 2, \cdots)$: the Banach space of bounded continuous functions on $\boldsymbol{R}^3$ such that all their partial derivatives of order $\leqq k$ exist and are bounded continuous with the norm

$$\|f\|_{\mathscr{B}^k}\equiv\sum_{|\alpha|\leq k}\sup_{R^3}\left|\left(\frac{\partial}{\partial x}\right)^\alpha f(x)\right|, \tag{2.5}$$

where $\alpha=(\alpha_1, \alpha_2, \alpha_3)$, $|\alpha|=\sum_{i=1}^{3}\alpha_i$ and

$$\left(\frac{\partial}{\partial x}\right)^\alpha f=\partial^{|\alpha|}f/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}.$$

(iii) $\mathscr{B}^{k+\sigma}(k=0, 1, 2, \cdots, 0<\sigma<1)$: the Hölder space of $\mathscr{B}^k$-functions such that their partial derivatives of order k are uniformly Hölder continuous (exponent σ) with the norm

$$\|f\|_{\mathscr{B}^{k+\sigma}}=\|f\|_{\mathscr{B}^k}+\sum_{|\alpha|=k}\sup_{R^3}\frac{\left|\left(\frac{\partial}{\partial x}\right)^\alpha f(x)-\left(\frac{\partial}{\partial y}\right)^\alpha f(y)\right|}{|x-y|^\sigma} \tag{2.6}$$

(iv) Let $f=(f^1(x), f^2(x), \cdots, f^n(x))$.

Denote

$$D^k f=\left(\left(\frac{\partial}{\partial x}\right)^\alpha f^i, \quad |\alpha|=k, \quad i=1, 2, \cdots, n\right), \tag{2.7}$$

which is a vector composed of all k-th partial derivatives. $D^k f\cdot D^k g$ denotes the usual inner product of $D^k f$ and $D^k g$.

$$|D^k f|=(D^k f\cdot D^k f)^{1/2}. \tag{2.8}$$

(v) $H^l(l=0, 1, 2, \cdots)$: the Sobolev space on $\boldsymbol{R}^3$ of L_2-functions whose partial derivatives up to l-th order are also L_2-functions, with the norm

$$\|f\|_l\equiv\Big(\sum_{0\leq k\leq l}\int|D^k f|^2 dx\Big)^{1/2}. \tag{2.9}$$

We note two lemmas playing an important role in the paper.

Lemma 2.2. *Let* $x\in\boldsymbol{R}^3$.

(i) *If* $f(x)\in H^2$, *then* $f\in\mathscr{B}^\sigma$ *for any* $\sigma\in(0, 1/2)$ *and*

$$\|f\|_{\mathscr{B}^\sigma}\leqq C\|f\|_2 \tag{2.10}$$

(ii) *If* $f(x)\in H^1$, *then* $f\in L_p$ *for any* $p\in[2, 6]$ *and*

$$\|f\|_{L_p}\leqq C\|f\|_1. \tag{2.11}$$

Lemma 2.3. *Suppose* $g(x)\in\mathscr{B}^1$ *and* $f(x)\in L_2$. *Set for* $i=1, 2, 3$

$$C_\delta^i\equiv\phi_\delta*(gf_{x_i})-g(\phi_\delta*f_{x_i}), \tag{2.12}$$

where $\phi_\delta*$ *denotes the Friedrichs' mollifier with respect to* x. *We have*

$$\|C_\delta^i\|\leqq C\|f\|, \tag{2.13}$$

$$\|C_\delta^i\|\longrightarrow 0 \quad as \quad \delta\longrightarrow 0. \tag{2.14}$$

As for the proofs, see [13] for example. By Lemma 2.2 we obtain estimates for the terms on the right hand side of equations (2.2) (2.3).

Lemma 2.4. *Suppose* $(\rho, u, \theta)\in H^l$ *for* $l=2$ *or* 3 *and* $-1<\rho_1\leqq\rho(x)$ *for some constant* ρ_1. *Consider the functions*

$$\begin{cases} \tilde{g}^i(\rho, u, \theta)\equiv -u^j u_{x_j}^i-R\theta_{x_i}, & i=1, 2, 3 \\ g^i\equiv\tilde{g}^i-R\dfrac{1+\theta}{1+\rho}\rho_{x_i}, & i=1, 2, 3 \\ h(\rho, u, \theta)\equiv -u^j\theta_{x_j}-(\gamma-1)(1+\theta)u_{x_j}^j & \\ \qquad +\dfrac{R(\gamma-1)}{1+\rho}\left\{\mu'(u_{x_j}^j)^2+\dfrac{\mu}{2}(u_{x_i}^j+u_{x_j}^i)(u_{x_i}^j+u_{x_j}^i)\right\}. & \end{cases} \tag{2.15}$$

Then it holds for $m=0, 1, \cdots, l-1$

$$\begin{cases} \|\tilde{g}\|_m\leqq C(1+\|u\|_3)\|D(u, \theta)\|_{l-1} \\ \left\|R\dfrac{1+\theta}{1+\rho}\rho_{x_i}\right\|_m\leqq C(1+\|\rho, u, \theta\|_3)^2\|D(\rho, u, \theta)\|_{l-1} \\ \|h\|_m\leqq C(1+\|\rho, u, \theta\|_3)^2\|D(\rho, u, \theta)\|_{l-1}, \end{cases} \tag{2.16}$$

where C *is a constant depending on* ρ_1. *Furthermore suppose* (ρ, u, θ), (ρ', u', θ') $\in H^3$. *Then it holds*

$$\begin{aligned} &\|g^i(\rho, u, \theta)-g^i(\rho', u', \theta')\|_1, \ \|h(\rho, u, \theta)-h(\rho', u', \theta')\|_1 \\ &\qquad \leqq C(1+\|\rho, u, \theta\|_3+\|\rho', u', \theta'\|_3)^2\cdot\|\rho-\rho', u-u', \theta-\theta'\|_2. \end{aligned} \tag{2.17}$$

Set

$$s=(1+\theta)/(1+\rho)^{\gamma-1}-1, \tag{2.18}$$

which is a function of the entropy by (1.10). Define a function $E^0(\rho, u, s)$ for ρ, $u=(u^1, u^2, u^3)$ and s by

$$\begin{aligned} E^0(\rho, u, s)\equiv &\frac{R}{\gamma-1}((1+\rho)^\gamma-1-\gamma\rho)(1+s) \\ &+\frac{1}{2}(1+\rho)u^iu^i+Rs\rho+\frac{R}{2(\gamma-1)}(1+\rho)s^2, \end{aligned} \tag{2.19}$$

which will be used in the first energy form in §6 on the base of following lemma.

Lemma 2.5. *There exist constants* $\rho_2>0$, $(\rho_2\leqq 1/2)$ *and* C_1, $C_2(0<C_1\leqq C_2<\infty)$ *such that* E^0 *is positive definite, i.e.,*

$$\rho^2+u^2+\theta^2\leqq C_1E^0\leqq C_2(\rho^2+u^2+\theta^2) \quad \text{for} \quad |\rho|\leqq\rho_2, \tag{2.20}$$

where ρ_2, C_1, C_2 *depend on* $\gamma>1$.

Proof First we note that for $|\rho|\leqq 1/2$

$$\theta^2\leqq 8\left(\frac{3}{2}\right)^{2(\gamma-1)}(s^2+(\gamma-1)^2\rho^2). \tag{2.21}$$

Next by the mean value theorem we have

$$\frac{R}{\gamma-1}((1+\rho)^\gamma-1-\gamma\rho)=\frac{\gamma R}{2}\rho^2+\frac{\gamma R(\gamma-2)}{3!}\rho^3(1+\xi\rho)^{\gamma-3}$$

for some $\xi\in(0, 1)$.

Therefore when $|\rho|\leqq 1/2$, (2.19) has the estimate

$$\begin{aligned}
E^0 &\geqq R\left\{\frac{\gamma}{2}\rho^2+\rho s+\frac{s^2}{2(\gamma-1)}\right.\\
&\quad -\frac{\gamma|\gamma-2|}{6}2^{|\gamma-3|}|\rho|^3-\left(\frac{\gamma}{2}\rho^2+\frac{\gamma|\gamma-2|}{6}2^{|\gamma-3|}|\rho|^3\right)|s|\\
&\quad \left.-\frac{1}{2(\gamma-1)}|\rho|s^2\right\}+\frac{1}{4}u^iu^i\\
&\geqq \frac{R}{2}\left\{\frac{\rho^2}{2}\left(1-\frac{2\gamma|\gamma-2|}{3}2^{|\gamma-3|}|\rho|-(\gamma-1)(\gamma-1/2)\right.\right.\\
&\quad \left.\cdot\left(\gamma+\frac{\gamma|\gamma-2|}{3}2^{|\gamma-2|}|\rho|\right)\rho^2\right)\\
&\quad \left.+\frac{s^2}{2(\gamma-1)(\gamma-1/2)}\left(1-\frac{2|\rho|}{(\gamma-1/2)}\right)\right\}+\frac{1}{4}u^iu^i\\
&\geqq \frac{R}{8}\left(\rho^2+\frac{s^2}{(\gamma-1)(\gamma-1/2)}\right)+\frac{1}{4}u^iu^i,
\end{aligned} \tag{2.22}$$

provided $|\rho|<\rho_2=\rho_2(\gamma)$.

Inequalities (2.21) (2.22) give the first one in (2.20). The second inequality of (2.20) is trivial for $|\rho|\leqq 1/2$. This completes the proof of Lemma 2.5.

The spaces of the solution are the following.

Definition 2.6. Let B be a Banach space, k be a non-negative integer and T be some positive constant. $C^k(0, T; B)$ (respectively $L^k_\infty(0, T; B)$): the Banach space of functions $f(t)$ on $[0, T]$ which have the values in B for every fixed $t\in[0, T]$ and are k-times continuously (respectively boundedly) differentiable with respect to $t\in[0, T]$ in B-topology. $L_2(0, T; B)$: the Banach space of functions $f(t)$ on $[0, T]$ which have the values in B for $t\in[0, T]$ and square summable with respect to $t\in[0, T]$ in B-topology.

Definition 2.7. For $k=2, 3$ and $l=0, 1, 2, 3$

(2.23)
$$\mathcal{E}(0, T; H^k)\equiv\{(\rho, u, \theta):$$
$$\rho(t, x)\in C^0(0, T; H^k)\cap C^1(0, T; H^{k-1}),$$
$$u^i(t, x), \theta(t, x)\in C^0(0, T; H^k)\cap C^1(0, T; H^{k-2}),$$
$$i=1, 2, 3\}$$

(2.24)
$$\mathcal{L}_\infty(0, T; H^k)\equiv\{(\rho, u, \theta):$$
$$\rho(t, x)\in L^0_\infty(0, T; H^k)\cap L^1_\infty(0, T; H^{k-1}),$$
$$u^i(t, x), \theta(t, x)\in L^0_\infty(0, T; H^k)\cap L^1_\infty(0, T; H^{k-2}),$$
$$i=1, 2, 3\}$$

Denote

$$\|(\rho, u, \theta)(t)\|_l^2\equiv\|\rho(t)\|_l^2+\|u(t)\|_l^2+\|\theta(t)\|_l^2,$$

where $\|u\|_l^2=\sum_{i=1}^3\|u^i\|_l^2$.

Last we note that the solution for (2.1)-(2.3) global in time is sought in $\mathcal{E}(0, \infty; H^3)$ so small that it satisfies for $t\geqq 0$

(2.25)
$$\|(\rho, u, \theta)(t)\|_3\leqq\varepsilon,$$

where $\varepsilon>0$ is a constant such that (2.25) implies $|\rho(t, x)|\leqq\rho_2$, where ρ_2 is defined in Lemma 2.5. This is because we will use the fcnction E^0 in the first energy form for a priori estimate. The existence of $\varepsilon>0$ is a trivial consequence of Lemma 2.2. Although the local solution [14] [2] is obtained in the Hölder-continuous space, for our purpose we reconstruct it belonging to $\mathcal{E}(0, T; H^3)$ for some $T>0$ in § 3-§ 5.

§ 3. Energy Estimates for Linearized Equation.

We establish some basic energy estimates for the linearized equation of (2.1)-(2.3) at (η, v):

(3.1)
$$L^0_{\eta,v}(\rho, u)\equiv\rho_t+v^j\rho_{x_j}+(1+\eta)u^j_{x_j}=f,$$

(3.2)
$$L^i_\eta(u)\equiv u^i_t-\frac{\mu}{1+\eta}u^i_{x_jx_j}-\frac{\mu+\mu'}{1+\eta}u^j_{x_ix_j}=g^i, \qquad i=1, 2, 3,$$

(3.3)
$$L^4_\eta(\theta)\equiv\theta_t-\frac{\kappa(\gamma-1)}{R(1+\eta)}\theta_{x_jx_j}=h,$$

where $\eta=\eta(t, x)$, $v=(v^1(t, x), v^2(t, x), v^3(t, x))$, $f=f(t, x)$, $g=(g^1(t, x), g^2(t, x), g^3(t, x))$ and $h=h(t, x)$ are given functions. When we regard the term $(1+\eta)u^j_{x_j}$ in (3.1) as known by use of the solution of (3.2)-(3.3), we consider the following simple equation

$$L_v(\rho)\equiv\rho_t+v^j\rho_{x_j}=f^0, \tag{3.4}$$

where $f^0=f^0(t, x)$ is a known function.

First we prepare some estimates for the commutators of the operators (3.1), $\cdots$, (3, 4), D^m, $m=1, 2, \cdots$, and mollifiers, which will be used later.

Lemma 3.1. *Let* $(\eta, v)(t)\in L^0_\infty(0, T; H^3)$ *for some* $T>0$ *and* $-1<\rho_1\leqq\eta(t, x)$ *for some constant* ρ_1. *Put*

$$E=\max\{\sup_{0\leq t\leq T}\|\eta(t)\|_3, \sup_{0\leq t\leq T}\|v(t)\|_3\}.$$

Suppose $(\rho, u, \theta)(t)\in\mathcal{L}_\infty(0, T; H^l)$ *for* $l=2$ *or* 3. *Then we have the following estimates* (i)-(iii) *for every* $t\in[0, T]$ *and for any* $m\leqq l-1$, *where* $C=C(\rho_1, E)<\infty$ *is a constant.*

(i) *For any* m, $0\leqq m\leqq l-1$

$$\begin{cases} \|D^m L_v(\rho)-L_v(D^m\rho)\|+\|D^{m+1}L_v(\rho)-L_v(D^{m+1}\rho)\| \\ \qquad\leqq CE\|D\rho\|_m \\ \|D^m L^0_{\eta,v}(\rho, u)-L^0_{\eta,v}(D^m\rho, D^m u)\| \\ \qquad +\|D^{m+1}L^0_{\eta,v}(\rho, u)-L^0_{\eta,v}(D^{m+1}\rho, D^{m+1}u)\| \\ \qquad\leqq CE\|D(\rho, u)\|_m. \end{cases} \tag{3.5}$$

For any m, $1\leqq m\leqq l-1$

$$\begin{cases} \|D^m L^i_\eta(u)-L^i_\eta(D^m u)\|\leqq CE\|Du\|_m, \qquad i=1, 2, 3, \\ \|D^m L^4_\eta(\theta)-L^4_\eta(D^m\theta)\|\leqq CE\|D\theta\|_m \end{cases} \tag{3.6}$$

(ii) *For any* m, $0\leqq m\leqq l-1$

$$\begin{cases} \|\phi_\delta*L_v(\rho)-L_v(\phi_\delta*\rho)\|_{m+1}\leqq C\|\rho\|_{m+1}, \\ \|\phi_\delta*L_v(\rho)-L_v(\phi_\delta*\rho)\|_{m+1}\longrightarrow 0 \quad as \quad \delta\longrightarrow 0, \\ \|\phi_\delta*L^0_{\eta,v}(\rho, u)-L^0_{\eta,v}(\phi_\delta*\rho, \phi_\delta*u)\|_{m+1}\leqq C\|\rho, u\|_{m+1}, \\ \|\phi_\delta*L^0_{\eta,v}(\rho, u)-L^0_{\eta,v}(\phi_\delta*\rho, \phi_\delta*u)\|_{m+1}\longrightarrow 0 \quad as \quad \delta\longrightarrow 0, \\ \|\phi_\delta*L^i_\eta(u)-L^i_\eta(\phi_\delta*u)\|_m\leqq C\|Du\|_m, \\ \|\phi_\delta*L^i_\eta(u)-L^i_\eta(\phi_\delta*u)\|_m\longrightarrow 0 \quad as \quad \delta\longrightarrow 0, \qquad i=1, 2, 3, \\ \|\phi_\delta*L^4_\eta(\theta)-L^4_\eta(\phi_\delta*\theta)\|_m\leqq C\|D\theta\|_m, \\ \|\phi_\delta*L^4_\eta(\theta)-L^4_\eta(\phi_\delta*\theta)\|_m\longrightarrow 0 \quad as \quad \delta\longrightarrow 0. \end{cases} \tag{3.7}$$

(iii) *Further we suppose that* $(\rho, u, \theta)\in\mathcal{L}_\infty(0, T; H^3)$ *and* $(\eta', v')\in L^0_\infty(0, T; H^3)$ *with* $-1<\rho_1\leqq\eta'(t, x)$. *Then we have for a constant* $C=C(\rho_1)$

$$(3.8)\quad \begin{cases} \|L^0_{\eta,v}(\rho,\ u)-L^0_{\eta',v'}(\rho,\ u)\|_2 \\ \qquad \leqq C(\sup_{0\leqq t\leqq T}\|(\rho,\ u)(t)\|_3)\|\eta-\eta',\ v-v'\|_2 \\ \|L^i_\eta(u)-L^i_{\eta'}(u)\|_1 \\ \qquad \leqq C(\sup_{0\leqq t\leqq T}\|u(t)\|_3)\|\eta-\eta'\|_2, \qquad i=1,\ 2,\ 3, \\ \|L^4_\eta(\theta)-L^4_{\eta'}(\theta)\|_1 \\ \qquad \leqq C(\sup_{0\leqq t\leqq T}\|\theta(t)\|_3)\|\eta-\eta'\|_2. \end{cases}$$

Proof. (i) We show the inequality only for L^i_η with $l=3,\ m=2$. The others are proved in the same way. Compute

$$\begin{aligned} &\{L^i_\eta(u)\}_{x_kx_n}-L^i_\eta(u_{x_kx_n}) \\ &=(1+\eta)^{-2}\eta_{x_kx_n}(\mu u^i_{x_jx_j}+(\mu+\mu')u^j_{x_ix_j}) \\ &\quad -2(1+\eta)^{-3}\eta_{x_k}\eta_{x_n}(\mu u^i_{x_jx_j}+(\mu+\mu')u^j_{x_ix_j}) \\ &\quad +(1+\eta)^{-2}\eta_{x_n}(\mu u^i_{x_jx_jx_k}+(\mu+\mu')u^j_{x_ix_jx_k}) \\ &\quad +(1+\eta)^{-2}\eta_{x_k}(\mu u^i_{x_jx_jx_n}+(\mu+\mu')u^j_{x_ix_jx_n}) \\ &\equiv A^1-2A^2+A^3_{nk}+A^3_{kn}. \end{aligned}$$

By Lemma 2.2 we can estimate each term as follows:

$$\begin{aligned} \|A^1\| &\leqq \frac{2\mu+\mu'}{(1+\rho_1)^2}\|\eta_{x_kx_n}\|_1(\|u^i_{x_jx_j}\|_1+\|u^j_{x_ix_j}\|_1) \\ &\leqq CE\|Du\|_2, \\ \|A^2\| &\leqq \frac{2\mu+\mu'}{(1+\rho_1)^3}\|\eta_{x_k}\|_2\|\eta_{x_l}\|_2(\|u^i_{x_jx_j}\|+\|u^j_{x_ix_j}\|) \\ &\leqq CE^2\|Du\|_1 \\ \|A^3_{kn}\| &\leqq \frac{2\mu+\mu'}{(1+\rho_1)^2}\|\eta_{x_k}\|_2(\|u^i_{x_jx_jx_n}\|+\|u^j_{x_ix_jx_n}\|) \\ &\leqq CE\|Du\|_2. \end{aligned}$$

Therefore noting $D^2f=(\partial^2f/\partial x_k\partial x_n,\ 1\leqq k,\ n\leqq 3)$, we arrive at

$$\|D^2L^i_\eta(u)-L^i_\eta(D^2u)\|\leqq CE\|Du\|_2,$$

where $C=C(\rho_1,\ E,\ \mu,\ \mu')<\infty$ is a constant.

(ii) We prove it only for L^i_η with $l=3,\ m=2$.

$$\begin{aligned} &\{\phi_\delta*L^i_\eta(u)-L^i_\eta(\phi_\delta*u)\}_{x_kx_n} \\ &=[\phi_\delta*L^i_\eta(u_{x_kx_n})-L^i_\eta(\phi_\delta*u_{x_kx_n})] \\ &\quad +[\{L^i_\eta(u-\phi_\delta*u)\}_{x_kx_n}-L^i_\eta(\{u-\phi_\delta*u\}_{x_kx_n})] \\ &\quad +[\phi_\delta*\{(L^i_\eta(u))_{x_kx_n}-L^i_\eta(u_{x_kx_n})\}-\{(L^i_\eta(u))_{x_kx_n}-L^i_\eta(u_{x_kx_n})\}] \end{aligned}$$

$$\equiv B_\delta^1+B_\delta^2+B_\delta^3.$$

Noting that $\frac{1}{1+\eta}\in\mathscr{B}^1$ and $u_{x_k x_n x_j}\in L_2$, we have by use of Lemma 2.3

$$\|B_\delta^1\|\leqq\mu\Big\|\phi_\delta*\frac{1}{1+\eta}(u^i_{x_k x_n x_j})_{x_j}-\frac{1}{1+\eta}\phi_\delta*(u^i_{x_k x_n x_j})_{x_j}\Big\|$$

$$+(\mu+\mu')\Big\|\phi_\delta*\frac{1}{1+\eta}(u^j_{x_k x_n x_i})_{x_j}-\frac{1}{1+\eta}\phi_\delta*(u^j_{x_k x_n x_i})_{x_j}\Big\|$$

$$\leqq C\|u\|_3,$$

$$\|B_\delta^1\|\longrightarrow 0\quad\text{as}\quad\delta\longrightarrow 0.$$

By (i) above and by the property of mollifier we have

$$\|B_\delta^2\|\leqq C\|u-\phi_\delta*u\|_3\leqq 2C\|u\|_3,$$

$$\|B_\delta^2\|\longrightarrow 0\quad\text{as}\quad\delta\longrightarrow 0$$

In the same way

$$\|B_\delta^3\|\leqq C\|\{L_\eta^i(u)\}_{x_k x_n}-L_\eta^i(u_{x_k x_n})\|$$

$$\leqq C\|u\|_3,$$

$$\|B_\delta^3\|\longrightarrow 0\quad\text{as}\quad\delta\longrightarrow 0.$$

(iii) We prove it for $L_\eta^i(u)$ only.

$$\{L_\eta^i(u)-L_{\eta'}^i(u)\}_{x_k}$$

$$=-\mu\Big(\frac{1}{1+\eta}-\frac{1}{1+\eta'}\Big)_{x_k}u^i_{x_j x_j}-\mu\Big(\frac{1}{1+\eta}-\frac{1}{1+\eta'}\Big)u^i_{x_k x_j x_j}$$

$$-(\mu+\mu')\Big(\frac{1}{1+\eta}-\frac{1}{1+\eta'}\Big)_{x_k}u^j_{x_i x_j}-(\mu+\mu')\Big(\frac{1}{1+\eta}-\frac{1}{1+\eta'}\Big)u^j_{x_k x_i x_j}$$

$$\equiv\mu A^1+\mu A^2+(\mu+\mu')A^3+(\mu+\mu')A^4.$$

We can estimate each term as follows:

$$\|A^1\|=\Big\|\Big(\frac{1}{(1+\eta)^2}\eta_{x_k}-\frac{1}{(1+\eta')^2}\eta'_{x_k}\Big)u^i_{x_j x_j}\Big\|$$

$$\leqq\Big\|\frac{1}{(1+\eta)^2}(\eta_{x_k}-\eta'_{x_k})\,u^i_{x_j x_j}\Big\|$$

$$+\Big\|\eta'_{x_k}\Big(\frac{1}{(1+\eta)^2}-\frac{1}{(1+\eta')^2}\Big)u^i_{x_j x_j}\Big\|$$

$$\leqq C\|D(\eta-\eta')\|_1\|D^2u\|_1+C\|\eta-\eta'\|_{\mathscr{B}^0}\|D^2u\|$$

$$\leqq C\|\eta-\eta'\|_2\|u\|_3.$$

$$\|A^2\|\leqq C\|\eta-\eta'\|_{\mathscr{B}^0}\|u\|_3\leqq C\|\eta-\eta'\|_2\|u\|_3.$$

In the same way we have

$$\|A^3\|+\|A^4\|\leqq C\|\eta-\eta'\|_2\|u\|_3 .$$

This completes the proof of Lemma 3.1.

Lemma 3.2. *Suppose* $v(t)\in L^0_\infty(0, T; H^3)$ *for some* $T>0$. *Let* $\rho(t)\in L^0_\infty(0, T; H^l)\cap L^1_\infty(0, T; H^{l-1})$ *and* $f^0(t)\in L^0_\infty(0, T; H^l)$ *satisfy the equation* (3.4) *for* $l=1, 2$ *or* 3. *Then there exists a constant* C*(independent of* t*)*$<\infty$ *such that the following energy estimates hold for* $l=1, 2$ *or* 3 *respectively.*

$$\|\rho(t)\|_l\leqq e^{CEt}(\|\rho(0)\|_l+\int_0^t e^{-CE\tau}\|f^0(\tau)\|_l d\tau), \tag{3.9}$$

where $E=\sup_{0\leq t\leq T}\|v(t)\|_3$.

Proof. Consider $\rho_\delta(t)=\phi_\delta*\rho(t)$, where $\phi_\delta*$ is the Friedrichs mollifier with respect to x, which belongs to $C^0(0, T; H^{l+1})\cap C^1(0, T; H^l)$. Applying $\phi_\delta*$ to $L_v(\rho)$, we have

$$L_v(\rho_\delta)=\phi_\delta*f^0+C_\delta , \tag{3.10}$$

where

$$C_\delta=L_v(\phi_\delta*\rho)-\phi_\delta*L_v(\rho) .$$

Apply D to (3.10). Then we have

$$L_v(D\rho_\delta)=L_v(D\rho_\delta)-DL_v(\rho_\delta)+\phi_\delta*Df^0+DC_\delta . \tag{3.11}$$

Multiply ρ_δ to (3.10). Take the inner product of (3.11) and $D\rho_\delta$. Adding them and integrating it with respect to x, we have

$$(\|\rho_\delta(t)\|_1^2)_t\leqq 2\{CE\|\rho_\delta(t)\|_1+\|f^0_\delta\|_1+\|C_\delta\|_1\}\|\rho_\delta(t)\|_1 . \tag{3.12}$$

Integration of (3.12) in $t\in[0, T]$ gives

$$\|\rho_\delta(t)\|_1\leqq e^{CEt}\{\|\rho_\delta(0)\|_1+\int_0^t e^{-CE\tau}(\|f^0_\delta(\tau)\|_1+\|C_\delta(\tau)\|_1)d\tau\} .$$

By Lemma 3.1 as $\delta\to 0$, we obtain

$$\|\rho(t)\|_1\leqq e^{CEt}\{\|\rho(0)\|_1+\int_0^t e^{-CE\tau}\|f^0(\tau)\|_1 d\tau\} . \tag{3.13}$$

For $l=2$ apply D^2 to (3.10)

$$L_v(D^2\rho_\delta)=\phi_\delta*D^2f^0+D^2C_\delta+L_v(D^2\rho_\delta)-D^2L_v(\rho_\delta) . \tag{3.14}$$

Taking the inner product of (3.14) and $D^2\rho_\delta$, and integrating it in x we obtain by Lemma 3.1

$$\begin{aligned}\|D^2\rho_\delta(t)\|_t^2\leqq 2\{&CE\|D^2\rho_\delta(t)\|+\|D^2f^0_\delta\|+\|D^2C_\delta\| \\ &+CE\|D\rho_\delta(t)\|_1\}\|D^2\rho_\delta(t)\| .\end{aligned} \tag{3.15}$$

Addition of (3.15) to (3.12) gives

$$(\|\rho_\delta(t)\|_2^2)_t \leqq 2(CE\|\rho_\delta(t)\|_2+\|f_\delta^0\|_2+\|C_\delta\|_2)\|\rho_\delta(t)\|_2 . \tag{3.16}$$

Thus we have by use of Lemma 3.1 as $\delta \to 0$

$$\|\rho(t)\|_2 \leqq e^{CEt}(\|\rho(0)\|_2+\int_0^t e^{-CE\tau}\|f^0(\tau)\|_2 d\tau) . \tag{3.17}$$

For $l=3$ apply D^3 to (3.10).

$$L_v(D^3\rho_\delta)=\phi_\delta * D^3 f^0+D^3C_\delta+L_v(D^3\rho_\delta)-D^3L_v(\rho_\delta) .$$

By Lemma 3.1 in the same way as above we have

$$(\|D^3\rho_\delta(t)\|^2)_t \leqq 2(CE\|D^3\rho_\delta(t)\|+\|D^3 f_\delta^0\|+\|D^3C_\delta\| +CE\|D\rho_\delta\|_2)\|D^3\rho_\delta\| .$$

Add this to (3.16), integrate it in $t\in[0, T]$. We obtain as $\delta\to 0$

$$\|\rho(t)\|_3 \leqq e^{CEt}(\|\rho(0)\|_3+\int_0^t e^{-CE\tau}\|f^0(\tau)\|_3 d\tau) ,$$

where C is a constant independent of E, t.

This completes the proof of Lemma 3.2.

Proposition 3.3. *Let* $\eta(t)\in L_\infty^0(0, T ; H^3)$ *for some* $T>0$ *and* $-1<\rho_1\leqq\eta(t, x)$ *for some constant* ρ_1. *Put* $E=\sup_{0\leq t\leq T}\|\eta(t)\|_3$. *If for* $l=2$ *or* 3, $(u, \theta)(t)\in L_\infty^0(0, T ; H^l)\cap L_\infty^1(0, T ; H^{l-2})$ *and* $g(t)$, $h(t)\in L_\infty^0(0, T ; H^{l-1})$ *satisfy the equation* (3.2) (3.3), *then there exist constants* $\nu>0$, $C<\infty$ (*independent of* t) *such that* $(u, \theta)(t)\in L_2(0, T ; H^{l+1})$ *and has the estimates*:

$$\|(u, \theta)(t)\|^2+\nu\int_0^t\|D(u, \theta)(\tau)\|^2 d\tau \leqq e^{C(1+E^2)t}(\|(u, \theta)(0)\|^2+\int_0^t\|(g, h)(\tau)\|^2 d\tau) , \tag{3.18}$$

$$\|D^k(u, \theta)(t)\|^2+\nu\int_0^t\|D^{k+1}(u, \theta)(\tau)\|^2 d\tau \leqq \|D^k(u, \theta)(0)\|^2+C\int_0^t\|(g, h)(\tau)\|_{k-1}^2 ds +CE^2\int_0^t\|D(u, \theta)(\tau)\|_{k-1}^2 d\tau \tag{3.19}$$

$$\leqq e^{CE^2t}(\|D^k(u, \theta)(0)\|^2+C\int_0^t\|(g, h)(\tau)\|_{k-1}^2 d\tau) \tag{3.20}$$

for any k, $1\leqq k\leqq l$.

Proof. (3.18) is easy to see. Multiplying u, θ to (3.2) (3.3) and integrating it in $x\in\boldsymbol{R}^3$, we have

$$\frac{1}{2}(\|(u,\theta)(t)\|^2)_t+\int\Big(\frac{\mu}{1+\eta}u^i_{x_j}u^i_{x_j}+\frac{\mu+\mu'}{1+\eta}u^j_{x_j}u^i_{x_i}+\frac{\kappa(\gamma-1)}{R(1+\eta)}\theta_{x_j}\theta_{x_j}\Big)dx$$

$$=\int\Big(\frac{\mu}{(1+\eta)^2}\eta_{x_j}u^i_{x_j}u^i+\frac{\mu+\mu'}{(1+\eta)^2}\eta_{x_i}u^j_{x_j}u^i$$

$$+\frac{\kappa(\gamma-1)}{R(1+\eta)^2}\eta_{x_j}\theta_{x_j}\theta+g^iu^i+h\theta\Big)dx$$

$$=\alpha\int\frac{\mu}{1+\eta}u^i_{x_j}u^i_{x_j}+\frac{\kappa(\gamma-1)}{R(1+\eta)}\theta_{x_j}\theta_{x_j}$$

$$+C_\alpha(1+E^2)\|(u,\theta)(t)\|^2+\|g(t)\|^2+\|h(t)\|^2$$

for any $\alpha>0$. Thus after integration in $t\in[0, T]$, we get

$$\|(u,\theta)(t)\|^2+\nu\int_0^t\|D(u,\theta)(\tau)\|^2d\tau$$

$$\leqq e^{C(1+E^2)t}(\|(u,\theta)(0)\|^2+\int_0^t(\|g(\tau)\|^2+\|h(\tau)\|^2)d\tau),$$

where ν depends on μ, κ, γ and ρ_1 but not on t.

In order to obtain (3.19) (3.20), we first prove them for $k=1$ under which $(u,\theta)\in L^0_\infty(0, T; H^{k+2})$ and $g, h\in C(0, T; H^k)$. Multiply $-\Delta u^i$ to (3.2) and $-\Delta\theta$ to (3.3). We compute each term as follows:

$$\int(\Delta u^i\cdot g^i+\Delta\theta\cdot h)dx\leqq\alpha(\sum\|\Delta u^i\|^2+\|\Delta\theta\|^2)+\frac{1}{4\alpha}(\|g\|^2+\|h\|^2)$$

for any $\alpha>0$.

$$-\int\Delta u^i\cdot L^i(u)+\Delta\theta\cdot L^4(\theta)dx$$

$$=\frac{1}{2}(\|D(u,\theta)(t)\|^2)_t+\int\frac{\mu}{1+\eta}\Delta u^i\cdot u^i_{x_jx_j}$$

$$+\frac{\mu+\mu'}{1+\eta}\Delta u^i\cdot u^j_{x_ix_j}+\frac{\kappa(\gamma-1)}{R(1+\eta)}\Delta\theta\cdot\theta_{x_jx_j}dx$$

$$\geqq\frac{1}{2}(\|D(u,\theta)(t)\|^2)_t+2\nu_0(\|\Delta u(t)\|^2+\|\Delta\theta(t)\|^2)$$

$$+\int\frac{\mu+\mu'}{1+\eta}\Delta u^i\cdot u^j_{x_ix_j}dx\,.$$

The last term has the estimate

$$\int\frac{1}{1+\eta}\Delta u^i\cdot u^j_{x_ix_j}dx=\int\Big(\frac{1}{(1+\eta)^2}\eta_{x_i}u^i_{x_kx_k}u^j_{x_j}$$

$$-\frac{1}{1+\eta}u^i_{x_kx_kx_i}u^j_{x_j}\Big)dx$$

$$=\int\Big(\frac{1}{(1+\eta)^2}\eta_{x_i}u^j_{x_j}u^i_{x_kx_k}-\frac{1}{(1+\eta)^2}\eta_{x_k}u^j_{x_j}u^i_{x_kx_i}$$

$$+\frac{1}{1+\eta}u^i_{x_k x_i}u^j_{x_j x_k}\Big)dx$$

$$\geqq -\alpha\|\Delta u\|^2 - C_\alpha E^2\|Du\|^2$$

for any $\alpha>0$.

Thus after integration in t we get the estimate for $(u,\theta)\in L^0_\infty(0,T;H^3)\cap L^1_\infty(0,T;H^1)$ satisfying (3.2) (3.3)

$$\begin{aligned}&\|D(u,\theta)(t)\|^2+\nu\int_0^t\|D^2(u,\theta)(\tau)\|^2d\tau\\ &\leqq\|D(u,\theta)(0)\|^2+CE^2\int_0^t\|Du(\tau)\|^2d\tau\\ &\quad+C\int_0^t\|g(\tau)\|^2+\|h(\tau)\|^2d\tau\\ &\leqq e^{CE^2t}(\|D(u,\theta)(0)\|^2+C\int_0^t\|g(\tau)\|^2+\|h(\tau)\|^2d\tau\end{aligned}\tag{3.21}$$

This holds also for $(u,\theta)\in L^0_\infty(0,T;H^2)\cap L^1_\infty(0,T;H^1)$ satisfying (3.2) (3.3). In fact it is proved by use of Friedrichs mollifier. Let $(u_\delta,\theta_\delta)=(\phi_\delta*u,\phi_\delta*\theta)$ for $(u,\theta)\in L^0_\infty(0,T;H^l)\cap L^1_\infty(0,T;H^{l-2})$. Apply $\phi_\delta*$ to (3.2) (3.3)

$$\begin{cases}L^i_\eta(u_\delta)=\phi_\delta*g^i+C^i_\delta, & i=1,2,3\\ L^4_\eta(\theta_\delta)=\phi_\delta*h+C^4_\delta,\end{cases}\tag{3.22}$$

where

$$\begin{cases}C^i_\delta\equiv L^i_\eta(u_\delta)-\phi_\delta*L^i_\eta(u)\\ C^4_\delta\equiv L^4_\eta(\theta_\delta)-\phi_\delta*L^4_\eta(\theta).\end{cases}$$

Moreover apply D^m, $1\leqq m\leqq l$, to (3.22). We obtain the system for m, $1\leqq m\leqq l$

$$\begin{cases}L^i_\eta(D^m u_\delta)=\phi_\delta*D^m g^i+D^m C^i_\delta+M^{i,m}_\delta, & i=1,2,3,\\ L^4_\eta(D^m\theta_\delta)=\phi_\delta*D^m h+D^m C^4_\delta+M^{4,m}_\delta,\end{cases}\tag{3.23}$$

where

$$\begin{cases}M^{i,m}_\delta\equiv L^i_\eta(D^m u_\delta)-D^m L^i_\eta(u_\delta)\\ M^{4,m}_\delta\equiv L^4_\eta(D^m\theta_\delta)-D^m L^4_\eta(\theta_\delta).\end{cases}$$

Here we can apply the inequality (3.21) to $D^m(u_\delta,\theta_\delta)$ in (3.22), (3.23)($m=0,1,\cdots,l-1$ recursively) to obtain (3.19) (3.20) for $k=m+1$. In fact for $k=1$ we get from (3.22) (3.21)

$$\begin{aligned}&\|D(u_\delta,\theta_\delta)(t)\|^2+\nu\int_0^t\|D^2(u_\delta,\theta_\delta)(\tau)\|^2d\tau\\ &\leqq\|D(u_\delta,\theta_\delta)(0)\|^2+CE^2\int_0^t\|Du_\delta(\tau)\|^2d\tau\\ &\quad+C\int_0^t(\|(g_\delta,h_\delta)(\tau)\|^2+\sum_{i=1}^4\|C^i_\delta\|^2)d\tau.\end{aligned}$$

As $\delta\to 0$, by Lemma 3.1 we get

$$\|D(u,\theta)(t)\|^2+\nu\int_0^t\|D^2(u,\theta)(\tau)\|^2d\tau$$

$$\leqq\|D(u,\theta)(0)\|^2+CE^2\int_0^t\|Du(\tau)\|^2d\tau+C\int_0^t\|(g,h)(\tau)\|^2d\tau \tag{3.24}$$

$$\leqq e^{CE^2t}(\|D(u,\theta)(0)\|^2+C\int_0^t\|(g,h)(\tau)\|^2d\tau).$$

For $k=2$ we get from (3.21) and (3.23) $m=1$

$$\|D^2(u_\delta,\theta_\delta)(t)\|^2+\nu\int_0^t\|D^3(u_\delta,\theta_\delta)(\tau)\|^2d\tau$$

$$\leqq\|D^2(u_\delta,\theta_\delta)(0)\|^2+CE^2\int_0^t\|D^2u_\delta(\tau)\|^2d\tau$$

$$+C\int_0^t\|D(g_\delta,h_\delta)(\tau)\|^2+\sum_{i=1}^4\|DC_\delta^i\|^2+\|M_\delta^{i,1}\|^2d\tau.$$

Noting that $\|M_\delta^{i,m}\|\leqq CE\|D(u_\delta,\theta_\delta)\|_m$ by Lemma 3.1, we have by Lemma 3.1 as the limit $\delta\to 0$

$$\|D^2(u,\theta)(t)\|^2+\nu\int_0^t\|D^3(u,\theta)(\tau)\|^2d\tau$$

$$\leqq\|D^2(u,\theta)(0)\|^2+CE^2\int_0^t\|D^2(u,\theta)(\tau)\|^2 \tag{3.25}$$

$$+\|D(u,\theta)(\tau)\|^2d\tau+C\int_0^t\|D(g,h)(\tau)\|^2d\tau$$

Estimates (3.19) (3.20) for $k=2$ follow from (3.24) (3.25). For $k=3$ we have from (3.21) and (3.23) $m=2$ in the same way

$$\|D^3(u_\delta,\theta_\delta)(t)\|^2+\nu\int_0^t\|D^4(u_\delta,\theta_\delta)(\tau)\|^2d\tau$$

$$\leqq\|D^3(u_\delta,\theta_\delta)(0)\|^2+CE^2\int_0^t\|D^3(u_\delta,\theta_\delta)(\tau)\|^2d\tau$$

$$+C\int_0^t\|D^2(g_\delta,h_\delta)(\tau)\|^2+\sum_{i=1}^4\|D^2C_\delta^i\|^2+\|M_\delta^{i,2}(\tau)\|^2d\tau.$$

As $\delta\to 0$ by Lemma 3.1 we obtain

$$\|D^3(u,\theta)(t)\|^2+\nu\int_0^t\|D^4(u,\theta)(\tau)\|^2d\tau$$

$$\leqq\|D^3(u,\theta)(0)\|^2+CE^2\int_0^t\|D^3(u,\theta)(\tau)\|^2 \tag{3.26}$$

$$+\|D(u, \theta)(\tau)\|_l^2 d\tau + C\int_0^t \|D^2(g, h)(\tau)\|^2 d\tau .$$

Thus (3.19) (3.20) for $k=2$ and (3.26) give (3.19) (3.20) for $k=3$.

This completes the proof of Proposition 3.3.

Proposition 3.4. *Suppose* $(\eta, v)(t)\in L_\infty^0(0, T; H^3)$ *for some* $T>0$ *and* $-1<\rho_1 \leqq \eta(t, x)$ *for some constant* ρ_1. *Let* $f(t)\in L_\infty^0(0, T; H^l)$ *and* $g(t), h(t)\in L_\infty^0(0, T; H^{l-1})$ *for* $l=2$ *or* 3. *Let* $(\rho, u, \theta)(t)\in \mathcal{L}_\infty(0, T; H^l)$ *satisfy the system* (3.1)-(3.3). *Then there exist constant* $\nu>0$, $C<\infty$ *such that* $(u, \theta)(t)\in L_2(0, T; H^{l+1})$ *and for any* $t\in[0, T]$

$$\|(\rho, u, \theta)(t)\|_l, \left(\nu\int_0^t \|D(u, \theta)(\tau)\|_l^2 d\tau\right)^{1/2}$$

$$\leqq e^{C(1+E)^2 t}\left\{\|(\rho, u, \theta)(0)\|_l + \int_0^t \|f(\tau)\|_l d\tau \right. \tag{3.27}$$

$$\left. + \left(C\int_0^t \|(g, h)(\tau)\|_{l-1}^2 d\tau\right)^{1/2}\right\}.$$

Proof. It follows from Lemma 3.2 and Proposition 3.3. In fact we note for $k=0, 1, 2, 3$

$$\int_0^t \|f^0(\tau)\|_k d\tau \leqq \int_0^t \|f(\tau)\|_k + \|(1+\eta)u_{x_j}^j\|_k d\tau$$

$$\leqq \int_0^t \|f(\tau)\|_k + C(1+E)\|Du\|_k d\tau$$

$$\leqq \int_0^t \|f(\tau)\|_k d\tau + C(1+E)t^{1/2}\left(\int_0^t \|Du\|_k^2 d\tau\right)^{1/2}.$$

By use of (3.9) and (3.18), (3.20), $k=1, \cdots, l$, we obtain

$$\|(\rho, u, \theta)(t)\|_l \leqq e^{C(1+E)^2 t}\left\{\|\rho(0)\|_l + \int_0^t \|f(\tau)\|_l d\tau\right.$$

$$+C(1+E)t^{1/2}e^{CE^2 t}(\|(u, \theta)(0)\|_l^2 + C\int_0^t \|(g, h)(\tau)\|_{l-1}^2 d\tau)^{1/2}$$

$$\left. + \|(u, \theta)(0)\|_l + \left(C\int_0^t \|(g, h)(\tau)\|_{l-1}^2 d\tau\right)^{1/2}\right\}$$

$$\leqq e^{C(1+E)^2 t}\left\{\|(\rho, u, \theta)(0)\|_l + \int_0^t \|f(\tau)\|_l d\tau\right.$$

$$\left. + \left(C\int_0^t \|(g, h)(\tau)\|_l^2 d\tau\right)^{1/2}\right\}.$$

Also by (3.18) (3.20) we have

$$\nu\int_0^t \|D(u, \theta)(\tau)\|_l^2 d\tau$$

$$\leqq e^{CE^2 t}\left(\|(u, \theta)(0)\|_l^2 + C\int_0^t \|(g, h)(\tau)\|_{l-1}^2 d\tau\right).$$

This completes the proof of Proposition 3.4.

§ 4. Solution of Linearized Equation.

We solve the initial value problem for the linearized equation (3.1)-(3.3).

(4.1) $$L^0(\rho, u)\equiv \rho_t+v^j\rho_{x_j}+(1+\eta)u^j_{x_j}=f,$$

(4.2) $$L^i(u)\equiv u^i_t-\frac{\mu}{1+\eta}u^i_{x_jx_j}-\frac{\mu+\mu'}{1+\eta}u^j_{x_ix_j}=g^i, \qquad i=1, 2, 3,$$

(4.3) $$L^4(\theta)\equiv\theta_t-\frac{\kappa(\gamma-1)}{R(1+\eta)}\theta_{x_jx_j}=h,$$

where it is supposed that for some $T>0$

$$(\eta, v)(t)\in C^0(0, T; H^3), \ -1<\rho_1\leqq\eta \text{ for some constant } \rho_1,$$

(4.4) $$f(t)\in C^0(0, T; H^3)$$

$$g(t)=(g^1, g^2, g^3)(t),\ h(t)\in C^0(0, T; H^2).$$

The initial data are given by

(4.5) $$(\rho, u, \theta)(0)\in H^3.$$

First instead of (4.1) we solve the initial value problem for the simple hyperbolic equation

(4.6) $$L(\rho)\equiv\rho_t+v^j\rho_{x_j}=f^0,$$

where v and $f^0=f-(1+\eta)u^j_{x_j}$ are considered as known functions of t and x. The initial data are given by

(4.7) $$\rho(0)\in H^l \quad \text{for } l=1 \text{ or } 2.$$

Proposition 4.1. *Let $v(t)\in C^0(0, T; H^3)$, $f^0(t)\in C^0(0, T; H^l)$ and $\rho(0)\in H^l$ for $l=1$ or 2 and some $T>0$. Then the initial value problem* (4.6) (4.7) *has a unique solution*

(4.8) $$\rho(t)\in C^0(0, T; H^l)\cap C^1(0, T; H^{l-1}),$$

which satisfies the energy inequality:

(4.9) $$\|\rho(t)\|_l\leqq e^{CEt}\Big(\|\rho(0)\|_l+\int_0^t e^{-CE\tau}\|f^0(\tau)\|_l d\tau\Big),$$

where

$$E=\sup_{0\leq t\leq T}\|v(t)\|_3$$

and C is a constant independent of t.

Proof. If we note $v(t)\in C^0(0, T; H^3)\subset C^0(0, T; \mathscr{B}^{1+\sigma})$, $0<\sigma<1/2$, the theorem for $l=1$ is well known by the theory of hyperbolic equation. (cf. for example [7], [13]) For $l=2$ we differentiate the equation (4.6) with respect to x.

$$(4.10)\qquad \begin{cases} L(\rho_{x_i})=f^0_{x_i}+v^j_{x_i}\rho_{x_j} \\ \rho_{x_i}(0,\ x)=\rho(0)_{x_i}, \qquad i=1,\ 2,\ 3\,. \end{cases}$$

If we note the inequality

$$(4.11)\qquad \|f^0_{x_i}+v^j_{x_i}\rho_{x_j}\|_1\leqq\|Df\|_1+C\|v\|_3\|D\rho\|_1,$$

the initial value problem (4.10) is solved by the iteration:

$$\rho^{(0)}_{x_i}(t,\ x)\equiv\rho(0)_{x_i}, \qquad i=1,\ 2,\ 3$$

and $\rho^{(m)}_{x_i}(t,\ x)$, $i=1,\ 2,\ 3$, $m=1,\ 2,\ 3,\ \cdots$, is the solution, belonging to $C^0(0,\ T\ ;\ H^1)$, of the problem

$$\begin{cases} L(\rho^{(m+1)}_{x_i})=f^0_{x_i}+v^j_{x_i}\rho^{(m)}_{x_j} \\ \rho^{(m+1)}_{x_i}(0,\ x)=\rho(0)_{x_i}, \qquad i=1,\ 2,\ 3\,. \end{cases}$$

We have the estimates for the approximation $D\rho^{(m)}$.

$$\begin{aligned} &\|D\rho^{(0)}(t)\|_1\leqq\|D\rho(0)\|_1, \\ &\|D\rho^{(1)}(t)\|_1\leqq e^{CEt}(\|D\rho(0)\|_1 \\ &\qquad +\int_0^t(e^{-CE\tau}\|Df^0(\tau)\|_1+CEe^{-CE\tau}\|D\rho(0)\|_1)d\tau \\ &\leqq e^{CEt}(\|D\rho(0)\|_1+\int_0^t e^{-CE\tau}\|Df^0(\tau)\|_1 d\tau)\,. \end{aligned}$$

Noting for $m=1,\ 2,\ 3,\ \cdots$.

$$L(\rho^{(m+1)}_{x_i}-\rho^{(m)}_{x_i})=v^j_{x_i}(\rho^{(m)}_{x_j}-\rho^{(m-1)}_{x_j}), \qquad i=1,\ 2,\ 3\,,$$

we have by Lemma 3.2

$$\begin{aligned} &\|D\rho^{(m+1)}(t)-D\rho^{(m)}(t)\|_1\leqq e^{CEt}\int_0^t e^{-CE\tau_1}CE\|D\rho^{(m)}(\tau_1)-D\rho^{(m-1)}(\tau_1)\|_1 d\tau_1 \\ &\leqq\cdots\leqq e^{CEt}(CE)^m\int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{m-1}}e^{-CE\tau_m}\|D\rho^{(1)}(\tau_m)-D\rho^{(0)}(\tau_m)\|_1 d\tau_m\cdots d\tau_1 \\ &\leqq e^{CEt}\frac{(CEt)^m}{m\,!}(2\|D\rho(0)\|_1+\int_0^t e^{-CE\tau}\|Df^0(\tau)\|_1 d\tau)\,. \end{aligned}$$

Hence there exists for $i=1,\ 2,\ 3$

$$\lim\,\rho^{(m)}_{x_i}(t)=\rho_{x_i}(t)\in C^0(0,\ T\ ;\ H^1)\,,$$

which is the unique solution of (4.10), and so the solution $\rho(t)\in C^0(0,\ T\ ;\ H^2)\cap C^1(0,\ T\ ;\ H^1)$ for the initial value problem (4.6) (4.7) is obtained. By Lemma 3.2 the solution $\rho(t)$ satisfies the energy estimate (4.9), $l=2$. This completes the proof of Proposition 4.1.

Next we solve the initial value problem for (4.2) (4.3).

Proposition 4.2. *Suppose $\eta(t)\in C(0,\ T\ ;\ H^3)$ for some $T>0$ and $-1<\rho_1\leqq\eta$ for some constant ρ_1. Let $g(t)$, $h(t)\in C^0(0,\ T\ ;\ H^{l-1})$ and the initial data $(u,\ \theta)(0)$*

$\in H^l$ *for* $l=2$ *or* 3. *Then the initial value problem* (4.2) (4.3) *has a unique solution* $(u, \theta)(t)\in C^0(0, T; H^l)\cap C^1(0, T; H^{l-2})$, *which satisfies the energy estimates*:

$$\|(u, \theta)(t)\|_l^2+\nu\int_0^t\|D(u, \theta)(\tau)\|_l^2 d\tau \tag{4.12}$$
$$\leqq e^{C(1+E^2)t}(\|(u, \theta)(0)\|_l^2+C\int_0^t\|(g, h)(\tau)\|_{l-1}^2 d\tau),$$

where $\nu>0$ *and* $C=C(\rho_1)<\infty$ *are constants independent of* t, E, *and* $E=\sup_{0\leq t\leq T}\|\eta(t)\|_3$.

Proof. By the assumption on $\eta(t)$ in Proposition 4.2, Equation (4.2) (4.3) may be considered an evolution equation. If $g(t), h(t)\in C^0(0, T; H^2)$ and the initial data $(u, \theta)(0)\in H^2$, then the initial value problem for (4.2) (4.3) is solved by the abstract theory of linear evolution equation (cf. [7] for example). The solution $(u, \theta)(t)$ belongs to $C^0(0, T; H^2)\cap C^1(0, T; L^2)$. In order to weaken the hypothesis on $(g, h)(t)$ to $(g, h)(t)\in C^0(0, T; H^1)$, we use the parabolicity of Equation (4.2) (4.3) and the energy estimates in Proposition 3.3. Let us consider the solution $(u_\delta, \theta_\delta)(t)$ for the initial value problem

$$\begin{cases} L^i(u_\delta)=\phi_\delta * g^i\equiv g_\delta^i\in C^0(0, T; H^\infty), & i=1, 2, 3, \\ L^4(\theta_\delta)=\phi_\delta * h\equiv h_\delta\in C^0(0, T; H^\infty), \\ (u_\delta, \theta_\delta)(0)=(u, \theta)(0)\in H^2, \end{cases} \tag{4.13}$$

where $\phi_\delta *$ denotes the Friedrichs mollifier with respect to x. By Proposition 4.2 already proved, (4.13) has a unigue solution

$$(u_\delta, \theta_\delta)(t)\in C^0(0, T; H^2)\cap C^1(0, T; L^2).$$

Further, using Proposition 3.3 for $(u_\delta-u_{\delta'}, \theta_\delta-\theta_{\delta'})$, the following energy inequality holds for any $\delta, \delta'>0$:

$$\|(u_\delta-u_{\delta'}, \theta_\delta-\theta_{\delta'})(t)\|_2^2+\nu\int_0^t\|D(u_\delta-u_{\delta'}, \theta_\delta-\theta_{\delta'})(\tau)\|_2^2 d\tau$$
$$\leqq Ce^{C(1+E^2)t}\int_0^t\|(g_\delta-g_{\delta'}, h_\delta-h_{\delta'})(\tau)\|_1^2 d\tau.$$

Tending δ to zero, we have a solution $(u, \theta)(t)$ of (4.2) (4.3) for $(g, h)(t)\in C^0(0, T; H^1)$ such that by Proposition 3.3.

$$\begin{cases} (u, \theta)(t)\in C^0(0, T; H^2)\cap C^1(0, T; L_2)\cap L_2(0, T; H^3), \\ \|(u, \theta)(t)\|_2^2+\nu\int_0^t\|D(u, \theta)(\tau)\|_2^2 d\tau \\ \leqq e^{C(1+E^2)t}(\|(u, \theta)(0)\|_2^2+C\int_0^t\|(g, h)(\tau)\|_1^2 d\tau). \end{cases} \tag{4.14}$$

The uniqueness follows from the energy inequality (4.14) immediately.

Last we can get Proposition 4.2 for $l=3$ as follows: Differentiate (4.2) (4.3) with respect to x_k, $k=1, 2, 3$.

$$
(4.15)\quad \begin{cases} L^i(u_{x_k}) = g^i_{x_k} + \dfrac{\mu}{(1+\eta)^2}\eta_{x_k}(u^i_{x_j})_{x_j} \\ \qquad + \dfrac{\mu+\mu'}{(1+\eta)^2}\eta_{x_k}(u^j_{x_j})_{x_i} \equiv g^{ik}, \qquad i=1, 2, 3. \\ L^4(\theta_{x_k}) = h_{x_k} + \dfrac{\kappa(\gamma-1)}{R(1+\eta)^2}\eta_{x_k}(\theta_{x_j})_{x_j} \equiv h^k, \end{cases}
$$

$$
(u^i_{x_k}, \theta_{x_k})(0) \in H^2, \qquad i, k=1, 2, 3.
$$

We note that if $u_{x_k}(t) \in C^0(0, T; H^2)$, then the right hand side of (4.15)

$$
g^{ik}(t),\ h^k(t) \in C^0(0, T; H^1), \qquad i, k=1, 2, 3.
$$

Therefore we can solve (4.15) by the iteration as in the proof of the latter part of Proposition 4.1, when each approximate solution of the iteration is given by the former part of Proposition 4.2 for $l=2$. Hence as the limit of iteration we have

$$
u_{x_k}(t) \in C^0(0, T; H^2) \cap C^1(0, T; L^2), \qquad k=1, 2, 3,
$$

which implies

$$
u(t) \in C^0(0, T; H^3) \cap C^1(0, T; H^1).
$$

Proposition 3.3 gives the energy inequality (4.12) for $l=3$ and the fact

$$
(u, \theta)(t) \in L_2(0, T; H^4).
$$

This completes the proof of Proposition 4.2.

Now we can combine Proposition 4.1 and 4.2 to get the solution of the initial value problem for the linearized equation (4.1)-(4.3).

Proposition 4.3. *Suppose* $(\eta, v)(t) \in C^0(0, T; H^3)$ *for some* $T>0$ *and* $-1<\rho_1 \leqq \eta$ *with some constant* ρ_1. *Let* $f(t) \in C^0(0, T; H^2)$ *and* $g(t), h(t) \in C^0(0, T; H^1)$. *Consider the initial value problem* (4.1)-(4.3) *with the initial data*

$$
(4.16)\qquad (\rho, u, \theta)(0) \in H^2.
$$

Then the problem (4.1)-(4.3), (4.16) *has a unique solution*

$$
(4.17)\qquad (\rho, u, \theta)(t) \in \mathcal{E}(0, T; H^2)
$$

which satisfies

$$
(u, \theta)(t) \in L_2(0, T; H^3)
$$

and for any $t \in [0, T]$

$$
\begin{aligned} &\|(\rho, u, \theta)(t)\|_2,\ (\nu\int_0^t \|D(u, \theta)(\tau)\|_2^2 d\tau)^{1/2} \\ (4.18)\qquad &\leqq e^{C(1+E)^2 t}\{\|(\rho, u, \theta)(0)\|_2 + \int_0^t \|f(\tau)\|_2 d\tau \\ &\qquad + (C\int_0^t \|(g, h)(\tau)\|_1^2 d\tau)^{1/2}\}, \end{aligned}
$$

where $\nu>0$ *and* $C<\infty$ *are constants independent of* t *and*

$$E=\max\{\sup_{0\leq t\leq T}\|\eta(t)\|_3, \sup_{0\leq t\leq T}\|v(t)\|_3\}.$$

Proof. For any $\delta>0$, let $(\rho_\delta, u_\delta, \theta_\delta)(t)$ be the solution of

$$(4.19)\quad \begin{cases} L^0(\rho_\delta, u_\delta)=f \\ L^i(u_\delta)=\phi_\delta * g^i \equiv g_\delta^i, \qquad i=1, 2, 3 \\ L^4(\theta_\delta)=\phi_\delta * h \equiv h_\delta, \end{cases}$$

$$(4.20)\quad \begin{cases} \rho_\delta(0)=\rho(0) \\ (u_\delta, \theta_\delta)(0)=(\phi_\delta * u(0), \phi_\delta * \theta(0)) \end{cases}$$

Since $g_\delta, h_\delta \in C^0(0, T; H^3)$ and $(u_\delta, \theta_\delta)(0) \in H^3$, Proposition 4.2 implies that

$$(u_\delta, \theta_\delta)(t) \in C^0(0, T; H^3) \cap C^1(0, T; H^1)$$

and

$$f(t)-(1+\eta)u^j_{\delta, x_j}(t) \in C^0(0, T; H^2).$$

Thus Proposition 4.1 gives

$$\rho_\delta(t) \in C^0(0, T; H^2) \cap C^1(0, T; H^1).$$

Further by Proposition 3.4 for the difference of solutions for any $\delta, \delta'>0$, we have the estimate for any $t\in[0, T]$

$$(4.21)\quad \begin{aligned} &\|(\rho_\delta-\rho_{\delta'}, u_\delta-u_{\delta'}, \theta_\delta-\theta_{\delta'})(t)\|_2, \\ &(\nu\int_0^t \|D(u_\delta-u_{\delta'}, \theta_\delta-\theta_{\delta'})(\tau)\|_2^2 d\tau)^{1/2} \\ \leqq\ & e^{C(1+E)^2 t}\{\|(u_\delta-u_{\delta'}, \theta_\delta-\theta_{\delta'})(0)\|_2 \\ &\quad +(C\int_0^t \|(g_\delta-g_{\delta'}, h_\delta-h_{\delta'})(\tau)\|_1^2 d\tau)^{1/2}\}. \end{aligned}$$

Since

$$\begin{cases} \|(u_\delta-u_{\delta'}, \theta_\delta-\theta_{\delta'})(0)\|_2 \longrightarrow 0 \\ \sup_{0\leq t\leq T}\|(g_\delta-g_{\delta'}, h_\delta-h_{\delta'})(t)\|_1 \longrightarrow 0 \end{cases}$$

as $\delta, \delta' \to 0$ in (4.21), we have the solution $(\rho, u, \theta)(t)$ for (4.1)-(4.3), (4.16) as the limit $\delta\to 0$, which satisfies (4.17). The estimate (4.18) follows from Proposition 3.4. The uniqueness follows from (4.18). This completes the proof of Proposition 4.3.

We are ready to prove the existence of solution in $\mathcal{E}(0, T; H^3)$ for any $T>0$ of the initial value problem (4.1)-(4.3), (4.5).

Theorem 4.4. *Suppose* $(\eta, v)(t) \in C^0(0, T; H^3)$ *for some* $T>0$ *and* $-1<\rho_1\leqq\eta$ *for some constant* ρ_1, $(g, h)(t)\in C^0(0, T; H^2)$ *and the initial data* $(\rho, u, \theta)(0)\in H^3$. *Then the initial value problem* (4.1)-(4.3), (4.5) *has a uniqu solution*

$$(\rho, u, \theta)(t)\in\mathcal{E}(0, T; H^3)$$

such that

$$(u, \theta)(t)\in L_2(0, T; H^4)$$

and it satisfies the energy estimate:

For any $t\in[0, T]$

$$\|(\rho, u, \theta)(t)\|_3, \ (\nu\int_0^t\|D(u, \theta)(\tau)\|_3^2d\tau)^{1/2}$$
$$=e^{C(1+E)^2t}\{\|(\rho, u, \theta)(0)\|_3+(C\int_0^t\|(g, h)(\tau)\|_2^2d\tau)^{1/2}\}, \tag{4.22}$$

where $\nu>0$, $C<\infty$ *are constants independent of* t *and* $E=\sup_{0\leq t\leq T}\|(\eta, v)(t)\|_3$.

Proof. Differentiate (4.1)-(4.3) with respect to x_k, $k=1, 2, 3$ and put $f\equiv 0$.

$$\begin{cases} L^0(\rho_{x_k}, u_{x_k})=v^i_{x_k}\rho_{x_j}+\eta_{x_k}u^j_{x_j}\equiv f^k, \\ L^i(u_{x_k})=g^i_{x_k}+\dfrac{\mu}{(1+\eta)^2}\eta_{x_k}(u^i_{x_j})_{x_j} \\ \qquad +\dfrac{\mu+\mu'}{(1+\eta)^2}\eta_{x_k}(u^j_{x_j})_{x_i}\equiv g^{ik}, \qquad i=1, 2, 3, \\ L^4(\theta_{x_k})=h_{x_k}+\dfrac{\kappa(\gamma-1)}{R(1+\eta)^2}\eta_{x_k}(\theta_{x_j})_{x_j}\equiv h^k, \\ (\rho_{x_k}, u_{x_k}, \theta_{x_k})(0)\in H^2, \qquad k=1, 2, 3. \end{cases} \tag{4.23}$$

If we note

$$\|f^k\|_2\leqq CE\|D(\rho, u)\|_2, \ \|g^{ik}\|_1, \ \|h^k\|_1\leqq CE\|D(u, \theta)\|_2+\|g, h\|_2,$$

we can solve this problem in $\mathcal{E}(0, T; H^2)$ by the iteration using Proposition 4.3 in the same way as the latter part of Proposition 4.1. Thus we have a solution $(\rho, u, \theta)\in\mathcal{E}(0, T; H^3)$ such that $(u, \theta)\in L_2(0, T; H^4)$. The energy estimate and the uniqueness are consequences of Proposition 3.4. This completes the proof of Theorem 4.4.

§ 5. Local Solution for Nonlinear Equation.

In order to obtain the solution of the initial value problem for the nonlinear equation (2.1)-(2.3) belonging to $\mathcal{E}(0, T; H^3)$ for some $T>0$ and satisfying $-1<\rho(t, x)$, we construct the approximate sequence

$$\{(\rho, u, \theta)_n(t)=(\rho_n, u_n, \theta_n)(t, x)\}_{n=3}^{\infty}$$

as follows:

$$(\rho, u, \theta)_3\equiv(\rho, u, \theta)(0)\in H^3, \qquad n=3, \tag{5.1}$$

$$
(5.2)\quad \begin{cases} L^0_{\rho_{n-1},u_{n-1}}(\rho_n, u_n)=0 \\ L^i_{\rho_{n-1}}(u_n)=-u^j_{n-1}u^i_{n-1,x_j}-R\theta_{n-1,x_i} \\ \qquad -R\dfrac{1+\theta_{n-1}}{1+\rho_{n-1}}\rho_{n-1,x_i}\equiv g^i_{n-1}, \qquad i=1, 2, 3 \\ L^4_{\rho_{n-1}}(\theta_n)=-u^j_{n-1}\theta_{n-1,x_j}-(\gamma-1)(1+\theta_{n-1})u^j_{n-1,x_j} \\ \qquad +\dfrac{\gamma-1}{R(1+\rho_{n-1})}\Big\{\mu'(u^j_{n-1,x_j})^2+\dfrac{\mu}{2}(u^j_{n-1,x_i}+u^i_{n-1,x_j}) \\ \qquad \cdot(u^j_{n-1,x_i}+u^i_{n-1,x_j})\Big\}\equiv h_{n-1}, \qquad n=4, 5, 6, \cdots. \end{cases}
$$

with the initial data

$$
(5.3)\qquad (\rho, u, \theta)_n(0)=(\rho, u, \theta)(0)\in H^3,
$$

which satisfies

$$
(5.4)\qquad -1<\inf_{x\in R^3}\rho(0, x).
$$

Define

$$
(5.5)\qquad \begin{cases} E=2\|(\rho, u, \theta)(0)\|_3<\infty \\ \rho_1=(-1+\inf\rho(0, x))/2>-1. \end{cases}
$$

Lemma 5.1. *If T is suitably small, we have for all $n\geqq 3$*

$$
(5.6)\qquad (\rho, u, \theta)_n(t)\in\mathcal{E}(0, T; H^3),
$$

which satisfy for any $t\in[0, T]$

$$
(5.7)\qquad \|(\rho, u, \theta)_n(t)\|_3,\ (\nu\int_0^t\|D(u_n, \theta_n)(\tau)\|_3^2d\tau)^{1/2}\leqq E,
$$

and

$$
(5.8)\qquad -1<\rho_1\leqq\rho_n(t, x).
$$

Proof. It is trivial for $n=3$ by (5.5). Suppose $(\rho, u, \theta)_k(t)$, $k=3, 4, \cdots, n-1$, satisfy (5.6) (5.7) (5.8). Then by Theorem 4.4 we have

$$
(\rho, u, \theta)_n(t)\in\mathcal{E}(0, T; H^3)
$$

and for E defined by (5.5) and appeared in (5.7) it satisfies for any $t\in[0, T]$

$$
\begin{aligned}
&\|(\rho, u, \theta)_n(t)\|_3,\ (\nu\int_0^t\|D(u, \theta)_n(\tau)\|_3^2d\tau)^{1/2} \\
&\leqq e^{C(1+E)^2t}\Big\{\|(\rho, u, \theta)(0)\|_3+(C\int_0^t\|(g_{n-1}h_{n-1})(\tau)\|_2^2d\tau)^{1/2}\Big\} \\
&\leqq e^{C(1+E)^2t}\Big\{\frac{E}{2}+(\int_0^t C(E)E^2d\tau)^{1/2}\Big\} \\
&\leqq E,
\end{aligned}
$$

provided that $t\in[0, T_1]$ for some small $T_1>0$. Because $C(E)$ is a constant independent of t. Furthermore (5.8) is true for $k=n$ in the time interval $[0, T_2]$, where T_2 is determined as follows: The equation

$$L^0_{\rho_{n-1}, u_{n-1}}(\rho_n, u_n)=L_{u_{n-1}}(\rho_n)+(1+\rho_{n-1})u^j_{n,x_j}=0$$

is a single hyperbolic equation for $\rho_n(t, x)$, if ρ_{n-1}, u_{u-1}, u_n are considered known. Therefore along the characteristic curve $y=y(\tau; t, x)$ we can estimate

$$\begin{aligned}\rho_n(t, x)&=\rho_n(0, y(0; t, x))+\int_0^t(1+\rho_{n-1})u^j_{n,x_j}(\tau, y(\tau; t, x))d\tau\\ &\geqq\inf\rho(0, y)+C(E)t\geqq\rho_1\end{aligned}$$

for $0\leqq t\leqq T_2(E)$.

Therefore for $t\in[0, T]$, $T=\min(T_1, T_2)$, we obtain (5.6)-(5.8) for any $n=3, 4, 5, \cdots$. This completes the proof of Lemma 5.1.

Theorem 5.2. *Suppose* $(\rho, u, \theta)(0)\in H^3$ *and* $-1<\inf\rho(0, x)$. *Then there exists a positive constant* T *such that the initial value problem* (2.1)-(2.3) *with the initials* $(\rho, u, \theta)(0)$ *has a unique solution*

$$(5.9)\qquad \begin{aligned}&(\rho, u, \theta)(t)\in\mathcal{E}(0, T; H^3),\\ &(u, \theta)(t)\in L_2(0, T; H^4)\end{aligned}$$

which satisfies for $t\in[0, T]$ *and for some* $\nu>0$

$$(5.10)\qquad \begin{cases}-1<\frac{1}{2}(-1+\inf\rho(0, x))\leqq\rho(t, x),\\ \|(\rho, u, \theta)(t)\|_3, (\nu\int_0^t\|D(u, \theta)(\tau)\|_3^2d\tau)^{1/2}\\ \qquad\leqq 2\|(\rho, u, \theta)(0)\|_3.\end{cases}$$

Proof. We show a convergence of the approximate sequence $(\rho, u, \theta)_n$ constructed in Lemma 5.1. Subtract the equation (5.2), $n=m\geqq 4$ from that for $n=m+1$. We have

$$(5.11)\qquad \begin{aligned}&L^0_{\rho_m, u_m}(\rho_{m+1}-\rho_m, u_{m+1}-u_m)\\ &\qquad=-(L^0_{\rho_m, u_m}(\rho_m, u_m)-L^0_{\rho_{m-1}, u_{m-1}}(\rho_m, u_m)),\\ &L^i_{\rho_m}(u_{m+1}-u_m)\\ &\qquad=-(L^i_{\rho_m}(u_m)-L^i_{\rho_{m-1}}(u_m))-(g^i_m-g^i_{m-1}),\qquad i=1, 2, 3,\\ &L^4_{\rho_m}(\theta_{m+1}-\theta_m)=-(L^4_{\rho_m}(\theta_m)-L^4_{\rho_{m-1}}(\theta_m))-(h_m-h_{m-1}).\end{aligned}$$

$$(5.12)\qquad (\rho_{m+1}-\rho_m, u_{m+1}-u_m, \theta_{m+1}-\theta_m)(0)=0,\qquad m=4, 5, 6, \cdots.$$

It follows from Proposition 3.4, Lemmas 2.4, 3.1 and 5.1 that the difference satisfies the energy estimate:

$$\|(\rho_{m+1}-\rho_m,\ u_{m+1}-u_m,\ \theta_{m+1}-\theta_m)(t)\|_2$$
$$\leqq e^{C(1+E)^2 t}C(E)\Big(\int_0^t \|(\rho_m-\rho_{m-1},\ u_m-u_{m-1},\ \theta_m-\theta_{m-1})(\tau)\|_2^2 d\tau\Big)^{1/2}. \tag{5.13}$$

Thus as $n\to\infty$ the limit exists such as

$$(\rho,\ u,\ \theta)_n(t) \longrightarrow (\rho,\ u,\ \theta)(t)$$

strongly in $C^0(0,\ T;\ H^2)$ and as $n'\to\infty$, where $\{n'\}$ is a subsequence of $\{n\}$, we have

$$D(u,\ \theta)_{n'}(t) \longrightarrow D(u,\ \theta)(t)$$

weakly in $L_2(0,\ T;\ H^3)$ by Lemma 5.1. Also by Lemma 5.1 we know

$$(\rho,\ u,\ \theta)_{n''}(t) \longrightarrow (\rho,\ u,\ \theta)(t)$$

weakly in H^3 for every fixed $t\in[0,\ T]$, where $n''=n''(t)$ is a subsequence of $\{n'\}$, depending on t. Thus we have a solution $(\rho,\ u,\ \theta)(t)\in\mathcal{L}_\infty(0,\ T;\ H^3)$ for the problem (2.1)-(2.3), which satisfies

$$-1<\frac{1}{2}(-1+\inf\rho(0,\ x))\leqq\rho(t,\ x).$$

Moreover we can show that the solution belongs also to $C^0(0,\ T;\ H^3)$ as follows: Consider $(\rho_\delta,\ u_\delta,\ \theta_\delta)(t)=(\phi_\delta*\rho,\ \phi_\delta*u,\ \phi_\delta*\theta)(t)$ for the solution $(\rho,\ u,\ \theta)(t)\in\mathcal{L}_\infty(0,\ T;\ H^3)$. It follows from $(\rho,\ u,\ \theta)(t)\in C^0(0,\ T;\ H^2)$ that $(\rho_\delta,\ u_\delta,\ \theta_\delta)(t)\in C^0(0,\ T;\ H^\infty)$. Apply $\phi_\delta*$ to Equation (2.1)-(2.3). We have

$$\begin{cases} L^0_{\rho,u}(\rho_\delta,\ u_\delta)=C^0_\delta, \\ L^i_\rho(u_\delta)=g^i_\delta+C^i_\delta, \qquad i=1,\ 2,\ 3, \\ L^4_\rho(\theta_\delta)=h_\delta+C^4_\delta, \\ (\rho_\delta,\ u_\delta,\ \theta_\delta)(0)=(\phi_\delta*\rho,\ \phi_\delta*u,\ \phi_\delta*\theta)(0), \end{cases} \tag{5.14}$$

where $g_\delta=\phi_\delta*g,\ h_\delta=\phi_\delta*h$,

$$\begin{cases} C^0_\delta\equiv L^0_{\rho,u}(\rho_\delta,\ u_\delta)-\phi_\delta*L^0_{\rho,u}(\rho,\ u), \\ C^i_\delta\equiv L^i_\rho(u_\delta)-\phi_\delta*L^i_\rho(u), \qquad i=1,\ 2,\ 3, \\ C^4_\delta\equiv L^4_\rho(\theta_\delta)-\phi_\delta*L^4_\rho(\theta). \end{cases}$$

Using Proposition 3.4 and Lemma 3.1, we can estimate the difference for any $\delta,\ \delta'>0$

$$\sup_{0\leq t\leq T}\|(\rho_\delta-\rho_{\delta'},\ u_\delta-u_{\delta'},\ \theta_\delta-\theta_{\delta'})(t)\|_3$$
$$\leqq e^{C(1+E)^2 T}\Big\{\|(\rho_\delta-\rho_{\delta'},\ u_\delta-u_{\delta'},\ \theta_\delta-\theta_{\delta'})(0)\|_3$$
$$+\Big(C\int_0^T\|(g_\delta-g_{\delta'},\ h_\delta-h_{\delta'})(\tau)\|_2^2 \tag{5.15}$$

$$+\sum_{i=1}^{4}(\|C_\delta^i(\tau)\|_2^2+\|C_{\delta'}^i(\tau)\|_2^2)$$

$$+\|C_\delta^0(\tau)\|_3^2+\|C_{\delta'}^0(\tau)\|_3^2 d\tau\Big)^{1/2}\Big\}.$$

By Lemma 3.1, (5.15) implies

$$\sup_{0\leq t\leq T}\|(\rho_\delta-\rho_{\delta'},\ u_\delta-u_{\delta'},\ \theta_\delta-\theta_{\delta'})(t)\|_3 \longrightarrow 0$$

as $\delta,\ \delta'\to 0$. Therefore as the uniform limit we have

$$(\rho,\ u,\ \theta)(t)\in\mathcal{E}(0,\ T\ ;\ H^3).$$

The uniqueness of solution follows from the energy estimate for the difference of solutions in the same way as (5.13). This completes the proof of Theorem 5.2.

§6. A Priori Estimates for Nonlinear Equation.

For some fixed positive number T, we suppose that $(\rho,\ u,\ \theta)(t)\in\mathcal{E}(0,\ T\ ;\ H^3)$ satisfies (2.1)-(2.3) i. e.,

$$L_{\rho,u}^0(\rho,\ u)=0\,, \tag{6.1}$$

$$L_\rho^i(u)+R\frac{1+\theta}{1+\rho}\rho_{x_i}=\tilde{g}^i\,,\qquad i=1,\ 2,\ 3\,, \tag{6.2}$$

$$L_\rho^4(\theta)=h\,, \tag{6.3}$$

where $\tilde{g}_i=-u^j u_{x_j}^i-R\theta_{x_i}$, and

$$E\equiv\max_{0\leq t\leq T}\|(\rho,\ u,\ \theta)(t)\|_3\leqq\varepsilon\,, \tag{6.4}$$

where ε is defined in Lemma 2.5 and (2.25). We first show an essential energy estimate for the solution.

Lemma 6.1. *There exists a constant* $\varepsilon_0>0(\varepsilon_0\leqq\varepsilon)$ *such that if the solution is so small that* $E<\varepsilon_0$, *then the following a priori estimate holds for* $t\in[0,\ T]$.

$$\begin{aligned}&\|(\rho,\ u,\ \theta)(t)\|^2+\|D\rho(t)\|^2\\&+\nu_0\int_0^t\|D(\rho,\ u,\ \theta)(\tau)\|^2d\tau\leqq C\|(\rho,\ u,\ \theta)(0)\|_3^2\,,\end{aligned} \tag{6.5}$$

where $\nu_0>0$, $C=C(\varepsilon_0)$ *are independent of* t.

Proof. Since $E<\varepsilon$, by Lemmas 2.4, 2.5 the norm $\|\rho,\ u,\ \theta\|$ is equivalent to $E^0(\rho,\ u,\ s)$. Hence we will estimate E^0 at first. If $\rho,\ u,\ \theta$ satisfy (6.1), then $\rho,\ u,\ s$ (See (2.18) for the definition of s) satisfy the system of equations

$$
(6.6)\quad \begin{cases} \rho_t+\{(1+\rho)u^j\}_{x_j}=0 \\ u_t^i+\dfrac{R\{(1+\rho)^\gamma(1+s)\}_{x_i}}{1+\rho}+u^j u^i_{x_j}-\dfrac{\mu}{1+\rho}u^i_{x_j x_j} \\ \qquad -\dfrac{\mu+\mu'}{1+\rho}u^j_{x_i x_j}=0 \\ s_t+u^j s_{x_j}-\dfrac{\kappa(\gamma-1)}{R}\Big\{\dfrac{s_{x_j}}{1+\rho}+\dfrac{(\gamma-1)(1+s)}{(1+\rho)^2}\rho_{x_j}\Big\}_{x_j} \\ \qquad -\dfrac{\kappa(\gamma-1)\gamma}{R}\Big(\dfrac{s_{x_j}}{(1+\rho)^2}+\dfrac{(\gamma-1)(1+s)}{(1+\rho)^3}\rho_{x_j}\Big)\rho_{x_j} \\ \qquad =\dfrac{\gamma-1}{R(1+\rho)^\gamma}\Big\{\mu'(u^j_{x_j})^2+\dfrac{\mu}{2}(u^j_{x_i}+u^i_{x_j})(u^j_{x_i}+u^i_{x_j})\Big\}. \end{cases}
$$

We compute

$$
\frac{\partial}{\partial t}E^0(\rho,u,s)=(1+\rho)u^i u_t^i+\Big\{\frac{u^i u^i}{2}+\frac{\gamma R}{\gamma-1}(1+s)((1+\rho)^{\gamma-1}-1)
$$

$$
+Rs+\frac{R}{2(\gamma-1)}s^2\Big\}\rho_t+\Big\{\frac{R}{\gamma-1}((1+\rho)^\gamma-(1+\rho))+\frac{R}{\gamma-1}(1+\rho)s\Big\}s_t
$$

=(by use of (6.6))

$$
=\sum_j\{\ \}_{x_j}-\mu u^i_{x_j}u^i_{x_j}-(\mu+\mu')(u^i_{x_i})^2-\frac{\kappa(\gamma(1+\rho)^{\gamma-1}-1)}{1+\rho}s_{x_j}\rho_{x_j}
$$

$$
-\frac{\kappa(\gamma-1)(1+s)(\gamma(1+\rho)^{\gamma-1}-1)}{(1+\rho)^2}\rho_{x_j}\rho_{x_j}
$$

$$
-\kappa s_{x_j}s_{x_j}-\frac{\kappa(\gamma-1)(1+s)}{1+\rho}\rho_{x_j}s_{x_j}+O(E)|D(\rho,u,s)|^2
$$

$$
=\sum_j\{\ \}_{x_j}-\mu u^i_{x_j}u^i_{x_j}-(\mu+\mu')(u^i_{x_i})^2
$$

$$
(6.7)\qquad -2\kappa(\gamma-1)s_{x_j}\rho_{x_j}-\kappa(\gamma-1)^2\rho_{x_j}\rho_{x_j}-\kappa s_{x_j}s_{x_j}+O(E)|D(\rho,u,s)|^2,
$$

where $\sum_j\{\ \}_{x_j}$ means the terms in divergent form of functions of ρ, u, s and their derivatives which will disappear after the integration in x, and $O(E)$ means the same order as $E=\|\rho,u,\theta\|_3$ when E tends to zero. In addition to (6.7) we calculate

$$
\Big\{\frac{1}{2}\rho_{x_i}\rho_{x_i}+\frac{(1+\rho)^2}{2\mu+\mu'}\rho_{x_i}u^i\Big\}_t
$$

$$
=\Big\{\rho_{x_i}+\frac{(1+\rho)^2}{2\mu+\mu'}u^i\Big\}\rho_{x_i t}+\frac{(1+\rho)^2}{2\mu+\mu'}\rho_{x_i}u_t^i+\frac{2(1+\rho)}{2\mu+\mu'}u^i\rho_{x_i}\rho_t
$$

=(by use of (6.6) after differentiation in x_i)

$$= \sum_j \{ \ \}_{x_j} - \frac{R(1+\rho)^{\gamma+1}}{2\mu+\mu'} \rho_{x_i} s_{x_i} - \frac{R\gamma(1+\rho)^{\gamma}(1+s)}{2\mu+\mu'} \rho_{x_i}\rho_{x_i}$$

$$+ \frac{(1+\rho)^3}{2\mu+\mu'} (u^i_{x_i})^2 + O(E)|D(\rho, u, s)|^2$$

$$= \sum_j \{ \ \}_{x_j} - \frac{R}{2\mu+\mu'} \rho_{x_i} s_{x_i} - \frac{R\gamma}{2\mu+\mu'} \rho_{x_i}\rho_{x_i} + \frac{1}{2\mu+\mu'} (u^i_{x_i})^2 + O(E)|D(\rho, u, s)|^2 \tag{6.8}$$

Add (6.7) to $\beta(0<\beta<1)$ times (6.8) and integrate it in $x \in \boldsymbol{R}^3$. We obtain the following energy inequality

$$\frac{\partial}{\partial t}\int E^0(\rho, u, s) + \beta\left(\frac{1}{2}\rho_{x_i}\rho_{x_i} + \frac{(1+\rho)^2}{2\mu+\mu'}\rho_{x_i}u^i\right)dx$$

$$+\int \mu u^i_{x_j}u^i_{x_j} + (\mu+\mu')(u^i_{x_i})^2 + 2\kappa(\gamma-1)s_{x_j}\rho_{x_j}$$

$$+\kappa(\gamma-1)^2\rho_{x_j}\rho_{x_j} + \kappa s_{x_j}s_{x_j} + \beta\left(\frac{R}{2\mu+\mu'}\rho_{x_i}s_{x_i}\right.$$

$$\left. + \frac{R\gamma}{2\mu+\mu'}\rho_{x_i}\rho_{x_i} - \frac{(u^i_{x_i})^2}{2\mu+\mu'}\right)dx$$

$$\leqq O(E)\int |D(\rho, u, s)|^2 dx$$

Therefore if we take β such that

$$\beta < \min\left\{\frac{(2\mu+\mu')^2}{8(1+\rho_2)^4}, (\mu+\mu')(2\mu+\mu'), \frac{4\kappa(2\mu+\mu')}{R}\right\},$$

where ρ_2 is that given in Lemma 2.5, then

$$\frac{R}{8}\left(\rho^2 + \frac{s^2}{(\gamma-1)\left(\gamma-\frac{1}{2}\right)}\right) + \frac{1}{8}u^iu^i + \frac{\beta}{4}\rho_{x_i}\rho_{x_i}$$

$$\leqq E^0(\rho, u, s) + \beta\left(\frac{1}{2}\rho_{x_i}\rho_{x_i} + \frac{(1+\rho)^2}{2\mu+\mu'}\rho_{x_i}u^i\right),$$

$$\mu u^i_{x_j}u^i_{x_j} + \kappa\gamma\left(\frac{s_{x_j}s_{x_j}}{(\gamma+1)^2+\gamma} + \rho_{x_j}\rho_{x_j}\right)$$

$$\leqq \mu u^i_{x_j}u^i_{x_j} + \left(\mu+\mu' - \frac{\beta}{2\mu+\mu'}\right)(u^i_{x_i})^2 + \kappa s_{x_j}s_{x_j}$$

$$+\left(2\kappa(\gamma-1) + \frac{\beta R}{2\mu+\mu'}\right)s_{x_j}\rho_{x_j} + \left(\kappa(\gamma-1)^2 + \frac{\beta R\gamma}{2\mu+\mu'}\right)\rho_{x_j}\rho_{x_j}.$$

Thus after integration in t we obtain

$$\int\frac{R}{8}\Big(\rho^2+\frac{s^2}{(\gamma-1)\big(\gamma-\frac{1}{2}\big)}+\frac{u^iu^i}{8}+\frac{\beta}{4}\rho_{x_i}\rho_{x_i}dx$$

$$(6.9)\qquad +\int_0^t\int\Big\{\mu u^i_{x_j}u^i_{x_j}+\kappa\gamma\Big(\rho_{x_j}\rho_{x_j}+\frac{1}{(\gamma+1)^2+\gamma}s_{x_i}s_{x_i}\Big)$$

$$+O(E)|D(\rho,\ u,\ s)|^2\Big\}dxd\tau$$

$$\leqq\int E^0(\rho,\ u,\ s)+\beta\Big(\frac{1}{2}\rho_{x_i}\rho_{x_i}+\frac{(1+\rho)^2}{2\mu+\mu'}\rho_{x_i}u^i\Big)dx\,|_{t=0}$$

$$\leqq C\|(\rho,\ u,\ \theta)(0)\|_3^2$$

This is the desired estimate provided that $E<\varepsilon_0$ for some $\varepsilon_0=\varepsilon_0(\mu,\ \kappa,\ \gamma)>0$. In fact the left hand side of (6.9) is equivalent to the norm

$$\|(\rho,\ u,\ s)(t)\|^2+\|D\rho(t)\|^2+\nu_0\int_0^t\|D(\rho,\ u,\ s)(\tau)\|^2d\tau\,.$$

Furthermore it is equivalent to

$$\|(\rho,\ u,\ \theta)(t)\|^2+\|D\rho(t)\|^2+\nu_0\int_0^t\|D(\rho,\ u,\ \theta)(\tau)\|^2d\tau\,,$$

by (2.18) (2.20) and by $E<\varepsilon_0\leqq\varepsilon$. This completes the proof of Lemma 6.1.

Next we proceed to estimate the derivatives of solution. Proposition 3.3 (3.19) gives those for u, θ, i.e., for $k=1, 2, 3$

$$(6.10)\qquad \begin{aligned}&\|D^k(u,\ \theta)(t)\|^2+\nu\int_0^t\|D^{k+1}(u,\ \theta)(\tau)\|^2d\tau\\ &\leqq\|D^k(u,\ \theta)(0)\|^2+C\int_0^t\|(g,\ h)(\tau)\|_{k-1}^2d\tau\\ &\qquad+CE^2\int_0^t\|D(u,\ \theta)(\tau)\|_{k-1}^2d\tau\,,\end{aligned}$$

where $\nu=\nu(\varepsilon)>0$, $C<\infty$ are constants independent of t, and g, h have the estimates (2.16)

$$\|(g,\ h)\|_{k-1}\leqq C(1+E)^2\|D(\rho,\ u,\ \theta)\|_{k-1}\,.$$

Substituting this into (6.10) we get

Lemma 6.2. *If $E<\varepsilon$, then it holds for $k=1, 2, 3$*

$$(6.11)_k\qquad \begin{aligned}&\|D^k(u,\ \theta)(t)\|^2+\nu\int_0^t\|D^{k+1}(u,\ \theta)(\tau)\|^2d\tau\\ &\leqq\|D^k(u,\ \theta)(0)\|^2+C\int_0^t\|D(\rho,\ u,\ \theta)(\tau)\|_{k-1}^2d\tau\,,\end{aligned}$$

where $\nu=\nu(\varepsilon)>0$ and $C=C(\varepsilon,\ \nu)<\infty$ are constants independent of t.

Add (6.11), $k=1$, to $\Big(\dfrac{C}{\nu_0}+1\Big)$ times (6.5). We have the estimate

$$\begin{aligned}(6.12)\qquad &\|(\rho,u,\theta)(t)\|_1^2+\nu_0\int_0^t\|D(\rho,u,\theta)(\tau)\|^2\\ &+\|D^2(u,\theta)(\tau)\|^2d\tau\leqq C\|(\rho,u,\theta)(0)\|_3^2,\end{aligned}$$

where $\nu_0>0(\nu_0\leqq\nu$ and $C=C(\varepsilon_0,\nu_0)$ are independent of t.

Now we need the estimates for the derivatives of ρ.

Lemma 6.3. *Suppose $(\rho,u,\theta)\in\mathcal{E}(0,T;H^3)$ for some $T>0$ is a solution of* (6.1)-(6.3) *satisfying* (6.4). *Then there exist $\varepsilon_0>0(\varepsilon_0\leqq\varepsilon)$, $\nu_0>0$ and $C=C(\varepsilon_0,\nu_0)<\infty$ such that if $E<\varepsilon_0$, then it holds:*

$$\begin{aligned}&\|D^k\rho(t)\|^2-C\|D^{k-1}u(t)\|^2+\nu_0\int_0^t\|D^k\rho(\tau)\|^2d\tau\\ (6.13)\qquad &\leqq C\|(\rho,u)(0)\|_3^2+C\int_0^t(\|D^k(u,\theta)(\tau)\|^2\\ &+\sum_{j=1}^{k-1}\|D^j(\rho,u,\theta)(\tau)\|^2)d\tau \qquad for \quad k=1,2,3.\end{aligned}$$

Proof. Consider $(\rho_\delta,u_\delta)=(\phi_\delta*\rho,\phi_\delta*u)\in\mathcal{E}(0,T;H^\infty)$ and apply $\phi_\delta*$ to equations (6.1) (6.2). We have

$$(6.14)\qquad L^0_{\rho,u}(\rho_\delta,u_\delta)=L^0_{\rho,u}(\rho_\delta,u_\delta)-\phi_\delta*L^0_{\rho,u}(\rho,u)$$

$$\begin{aligned}&L^i_\rho(u_\delta)+R\frac{1+\theta}{1+\rho}\rho_{\delta x_i}=\phi_\delta*\tilde{g}^i\\ (6.15)\qquad &+\Big(L^i_\rho(u_\delta)+R\frac{1+\theta}{1+\rho}\rho_{\delta x_i}\Big)-\phi_\delta*\Big(L^i_\rho(u)-R\frac{1+\theta}{1+\rho}\rho_{x_i}\Big)\\ &\equiv\tilde{g}^i_\delta+C^i_\delta,\qquad i=1,2,3.\end{aligned}$$

Differentiate (6.14) with respect to x_i, $i=1,2,3$.

$$\begin{aligned}&L^0_{\rho,u}(\rho_{\delta x_i},u_{\delta x_i})=\{L^0_{\rho,u}(\rho_\delta,u_\delta)-\phi_\delta*L^0_{\rho,u}(\rho,u)\}_{x_i}\\ (6.16)\qquad &+L^0_{\rho,u}(\rho_{\delta x_i},u_{\delta x_i})-L^0_{\rho,u}(\rho_\delta,u_\delta)_{x_i}\\ &=C^{0i}_\delta+f^i_\delta,\qquad i=1,2,3.\end{aligned}$$

For $k=1$ we calculate the following in the same way as (6.8):

$$\begin{aligned}&\int_0^t\int\rho_{\delta x_i}\Big\{L^0_{\rho,u}(\rho_{\delta x_i},u_{\delta x_i})+\frac{(1+\rho)^2}{2\mu+\mu'}\Big(L^i_\rho(u_\delta)\\ (6.17)\qquad &+R\frac{1+\theta}{1+\rho}\rho_{\delta x_i}\Big)\Big\}dxd\tau\\ &=\int_0^t\int\rho_{\delta x_i}\Big\{f^i_\delta+C^{0i}_\delta+\frac{(1+\rho)^2}{2\mu+\mu'}(\tilde{g}^i_\delta+C^i_\delta)\Big\}dxd\tau.\end{aligned}$$

We have for the left hand side

$$\int\frac{1}{2}\rho_{\delta x_i}\rho_{\delta x_i}+\frac{(1+\rho)^2}{2\mu+\mu'}\rho_{\delta x_i}u^i_\delta dx\Big|_{t=0}^{t=t}$$

$$+\int_0^t\int\Big[(1+\rho)\rho_{\delta x_i}u^j_{\delta x_i x_j}-\frac{1+\rho}{2\mu+\mu'}\{\mu\rho_{\delta x_i}u^i_{\delta x_j x_j}$$

$$+(\mu+\mu')\rho_{\delta x_i}u^j_{\delta x_i x_j}\}+\frac{R}{2\mu+\mu'}(1+\rho)(1+\theta)\rho_{\delta x_i}\rho_{\delta x_i}$$

$$-\frac{1}{2}u^j_{x_j}\rho_{\delta x_i}\rho_{\delta x_i}+\frac{1}{2\mu+\mu'}\{(1+\rho)^2u^i_\delta\}_{x_i}\rho_{\delta t}\Big]dxd\tau$$

$$(6.18)\qquad =\int\frac{1}{2}\rho_{\delta x_i}\rho_{\delta x_i}+\frac{(1+\rho)^2}{2\mu+\mu'}u^i_\delta\rho_{\delta x_i}dx\Big|_{t=0}^{t=t}$$

$$+\int_0^t\int\Big(\frac{R}{2\mu+\mu'}(1+\rho)(1+\theta)-\frac{1}{2}u^j_{x_j}\Big)\rho_{\delta x_i}\rho_{\delta x_i}dxd\tau$$

$$+\int_0^t\int\frac{\mu}{2\mu+\mu'}(\rho_{x_j}\rho_{\delta x_i}u^i_{\delta x_j}-\rho_{x_i}\rho_{\delta x_j}u^i_{\delta x_j})dxd\tau$$

$$-\int_0^t\int\frac{1}{2\mu+\mu'}(1+\rho)(2u^i_\delta\rho_{x_i}+(1+\rho)u^i_{\delta x_i})$$

$$\cdot\{\phi_\delta*(u^j\rho_{x_j}+(1+\rho)u^j_{x_j})\}\,dxd\tau\,.$$

As $\delta\to 0$, (6.18) has the expression and estimate.

$$\int\Big(\frac{1}{2}\rho_{x_i}\rho_{x_i}+\frac{(1+\rho)^2}{2\mu+\mu'}u^i\rho_{x_i}\Big)dx\Big|_{t=0}^{t=t}$$

$$+\int_0^t\int\Big(\frac{R}{2\mu+\mu'}(1+\rho)(1+\theta)-\frac{1}{2}u^j_{x_j}\Big)\rho_{x_i}\rho_{x_i}dxd\tau$$

$$-\int_0^t\int\frac{1}{2\mu+\mu'}(1+\rho)(2u^i\rho_{x_i}+(1+\rho)u^i_{x_i})$$

$$(6.19)\qquad \cdot(u^j\rho_{x_j}+(1+\rho)u^j_{x_j})dxd\tau$$

$$\geqq\int\Big(\frac{1}{2}\rho_{x_i}\rho_{x_i}+\frac{(1+\rho)^2}{2\mu+\mu'}u^i\rho_{x_i}\Big)dx\Big|_{t=0}^{t=t}$$

$$+\int_0^t\int\Big(\frac{R}{4(2\mu+\mu')}-CE\Big)\rho_{x_i}\rho_{x_i}dxd\tau-C\int_0^t\|Du(\tau)\|^2d\tau\,.$$

The right hand side of (6.17) has the estimate by use of Lemmas 2.4, 3.1 as $\delta\to 0$.

$$\lim_{\delta\to 0}\int_0^t\int\rho_{\delta x_i}\Big\{f^i_\delta+C^{0i}_\delta+\frac{(1+\rho)^2}{2\mu+\mu'}(\tilde g^i_\delta+C^i_\delta)\Big\}dxd\tau$$

$$\leqq\lim_{\delta\to 0}\int_0^t\|D\rho(\tau)\|(CE\|D(\rho,\,u)(\tau)\|+C\|D\theta(\tau)\|$$

$$(6.20)\qquad +\Sigma\|C^{0i}_\delta\|+\|C^i_\delta\|)d\tau$$

$$\leqq\int_0^t(CE+\alpha)\|D\rho(\tau)\|^2+C\|D(u,\,\theta)(\tau)\|^2d\tau$$

for any $\alpha>0$.

Therefore from (6.19) and (6.20) we get

$$
\begin{aligned}
&\int \frac{1}{2}\rho_{x_i}\rho_{x_i}+\frac{(1+\rho)^2}{2\mu+\mu'}u^i\rho_{x_i}dx\Big|_{t=0}^{t=t} \\
(6.21)\qquad &+\int_0^t\int\Big(\frac{R}{4(2\mu+\mu')}-2CE-\alpha\Big)\rho_{x_i}\rho_{x_i}dxd\tau \\
&\leqq C\int_0^t\|D(u,\theta)(\tau)\|^2d\tau \\
&\qquad\qquad \text{for any}\quad \alpha>0.
\end{aligned}
$$

Thus there exist $\varepsilon_0>0$, $\nu_0>0$ and $C=C(\varepsilon_0, \nu_0)$ such that for $E<\varepsilon_0$ we have

$$
\begin{aligned}
&\|D\rho(t)\|^2-C\|u(t)\|^2+\nu_0\int_0^t\|D\rho(\tau)\|^2d\tau \\
(6.22)\qquad &\leqq C\|(\rho, u)(0)\|_3^2+C\int_0^t\|D(u,\theta)(\tau)\|^2d\tau.
\end{aligned}
$$

To get the estimate (6.13), $k=2$ or 3 we apply $D^m(m=1$ or $2)$ to (6.15) (6.16) and get, $i=1, 2, 3$,

$$
(6.23)\qquad \begin{cases}
L^0_{\rho,u}(D^m\rho_{\delta x_i}, D^m u_{\delta x_i})=D^m C^{0i}_\delta+f^{im}_\delta \\
L^i_\rho(D^m u_\delta)+R\dfrac{1+\theta}{1+\rho}D^m\rho_{\delta x_i}=D^m(\tilde g^i_\delta+C^i_\delta)+M^{im}_\delta,
\end{cases}
$$

where

$$
\begin{cases}
f^{im}_\delta\equiv L^0_{\rho,u}(D^m\rho_{\delta x_i}, D^m u_{\delta x_i})-D^m\{L^0_{\rho,u}(\rho_\delta, u_\delta)_{x_i}\} \\
M^{im}_\delta\equiv L^i_\rho(D^m u_\delta)-D^m L^i_\rho(u_\delta) \\
\qquad +R\Big(D^m\dfrac{1+\theta}{1+\rho}\rho_{\delta x_i}-\dfrac{1+\theta}{1+\rho}D^m\rho_{\delta x_i}\Big).
\end{cases}
$$

Compute the equality for $m=1$ or 2

$$
\begin{aligned}
&\int_0^t\int D^m\rho_{\delta x_i}\Big\{L^0_{\rho,u}(D^m\rho_{\delta x_i}, D^m u_{\delta x_i}) \\
&\qquad +\frac{(1+\rho)^2}{2\mu+\mu'}\Big(L^i_\rho(D^m u_\delta)+R\frac{1+\theta}{1+\rho}D^m\rho_{\delta x_i}\Big)\Big\}dxd\tau \\
&=\int_0^t\int D^m\rho_{\delta x_i}\Big\{D^m C^{0i}_\delta+f^{im}_\delta+\frac{(1+\rho)^2}{2\mu+\mu'}(D^m(\tilde g^i_\delta+C^i_\delta)+M^{im}_\delta)\Big\}dxd\tau
\end{aligned}
$$

in the same way as (6.17)-(6.22). By Lemmas 2.3, 2.4 and 3.1 we have the estimates.

$$
\begin{aligned}
&\|f^{im}_\delta\|+\|D^m\tilde g^i_\delta\|+\|M^{im}_\delta\| \\
&\leqq CE\|D(\rho_\delta, u_\delta)\|_m+C\|D(u,\theta)\|_m \\
(6.24)\qquad &\quad +CE\|Du_\delta\|_m+C\sum_{j=1}^m\|D^j(\rho_\delta,\theta)\| \\
&\leqq CE\|D\rho\|_m+C\|D\rho\|_{m-1}+C\|D(u,\theta)\|_m
\end{aligned}
$$

and as $\delta\to 0$

$$\|D^m C_\delta^{0i}\| + \|D^m C_\delta^i\| \longrightarrow 0\,. \tag{6.25}$$

The same argument as (6.17)-(6.22) using (6.24) (6.25) gives the desired estimates (6.13) for $k=2$ and 3. This completes the proof of Lemma 6.3.

Now we are ready to obtain a priori estimate in H^3 for the solution of (6.1)-(6.3)

Theorem 6.4. *Suppose that for some* $T>0$ $(\rho, u, \theta)(t) \in \mathcal{E}(0, T; H^3)$ *is a solution of* (6.1)-(6.3) *satisfying* (6.4). *Then there exist* $\varepsilon_0 > 0 (\varepsilon_0 \leqq \varepsilon)$, $\nu_0 = \nu_0(\varepsilon_0) > 0$ *and* $C_0 = C_0(\varepsilon_0, \nu_0)$ *such that the following a priori estimate holds for any* $(\rho, u, \theta)(t)$ *satisfying* $E < \varepsilon_0$.

$$\begin{aligned} &\|(\rho, u, \theta)(t)\|_3^2 + \nu_0 \int_0^t (\|D\rho(\tau)\|_2^2 + \|D(u, \theta)(\tau)\|_3^2) d\tau \\ &\leqq C_0 \|(\rho, u, \theta)(0)\|_3^2\,. \end{aligned} \tag{6.26}$$

Proof. We combine the inequalities (6.11) (6.12) and (6.13) as follows: Add (6.13), $k=2$, to $\left(C + \dfrac{C}{\nu_0} + 1\right)$ times (6.12). We have

$$\begin{aligned} &\|(\rho, u, \theta)(t)\|_1^2 + \|D^2 \rho(t)\|^2 + \nu_0 \int_0^t \|D(\rho, u, \theta)(\tau)\|_1^2 d\tau \\ &\leqq C \|(\rho, u, \theta)(0)\|_3^2\,. \end{aligned} \tag{6.27}$$

Add (6.11), $k=2$, to $\left(\dfrac{C}{\nu_0} + 1\right)$ times (6.27). We have

$$\begin{aligned} &\|(\rho, u, \theta)(t)\|_2^2 + \nu_0 \int_0^t \|D(\rho, u, \theta)(\tau)\|_1^2 + \|D^3(u, \theta)(\tau)\|^2 d\tau \\ &\leqq C \|(\rho, u, \theta)(0)\|_3^2\,. \end{aligned} \tag{6.28}$$

Add (6.13), $k=3$, to $\left(C + \dfrac{C}{\nu_0} + 1\right)$ times (6.28). We have

$$\begin{aligned} &\|(\rho, u, \theta)(t)\|_2^2 + \|D^3 \rho(t)\|^2 + \nu_0 \int_0^t \|D(\rho, u, \theta)(\tau)\|_2^2 d\tau \\ &\leqq C \|(\rho, u, \theta)(0)\|_3^2\,. \end{aligned} \tag{6.29}$$

At last add (6.11), $k=3$, to $\left(\dfrac{C}{\nu_0} + 1\right)$ times (6.29). We have

$$\begin{aligned} &\|(\rho, u, \theta)(t)\|_3^2 + \nu_0 \int_0^t \|D(\rho, u, \theta)(\tau)\|_2^2 + \|D^4(u, \theta)(\tau)\|^2 d\tau \\ &\leqq C \|(\rho, u, \theta)(0)\|_3^2\,, \end{aligned}$$

which is the desired estimate. This completes the proof of Theorem 6.4.

§ 7. Global Solutions in Time.

We get the global (in time) solution $(\rho, u, \theta)(t) \in \mathcal{E}(0, T; H^3)$ for (2.1)-(2.3) by combining local existence (Theorem 5.2) and a priori estimate (Theorem 6.4) at length.

Theorem 7.1. *Suppose the initial data*

(7.1) $$(\rho, u, \theta)(0) \in H^3$$

and set $E_0 \equiv \|(\rho, u, \theta)(0)\|_3 < \infty$. *Then there exists* $\varepsilon_0 > 0$, $\nu_0 > 0$ *and* $C_0 < \infty$, *which are defined in Theorem 6.4, such that if* $E_0 < \min(\varepsilon_0/2, \varepsilon_0/2\sqrt{C_0})$, *then the initial value problem* (2.1)-(2.3) *with* (7.1) *has a unique solution* $(\rho, u, \theta)(t)$ *globally in time such as* $(\rho, u, \theta)(t) \in \mathcal{E}(0, \infty; H^3)$, $D\rho \in L_2(0, \infty; H^2)$ *and* $D(u, \theta) \in L_2(0, \infty; H^3)$ *which is a classical one for* $t > 0$, *and has the estimate for any* $t \geqq 0$

(7.2) $$\|(\rho, u, \theta)(t)\|_3^2 + \nu_0 \int_0^t \|D(\rho, u, \theta)(\tau)\|_2^2 + \|D(u, \theta)(\tau)\|_3^2 d\tau \leqq C_0 E_0^2 .$$

Furthermore the solution has the decay:

(7.3) $$\|D(\rho, u, \theta)(t)\|_1 \longrightarrow 0 \qquad as \quad t \longrightarrow \infty .$$

Proof. If the initial data satisfies $E_0 < \varepsilon_0/2$, then the local solution of (2.1)-(2.3) for the data exists in $\mathcal{E}(0, T_1; H^3)$ and has the estimate by Theorem 5.2.

(7.4) $$E_1 \equiv \sup_{0 \leq t \leq T_1} \|(\rho, u, \theta)(t)\|_3 \leqq 2E_0 < \varepsilon_0 .$$

Therefore by Theorem 6.4 the solution satisfies the a priori estimate (6.26):

(7.5) $$E_1 \leqq \sqrt{C_0}\, E_0 < \varepsilon_0/2$$

provided $E_0 < \varepsilon_0/2\sqrt{C_0}$. Thus by Theorem 5.2 the initial value problem (2.1)-(2.3) for $t \geqq T_1$ with the initial data $(\rho, u, \theta)(T_1)$ has again a unique solution $(\rho, u, \theta) \in \mathcal{E}(T_1, 2T_1; H^3)$ satisfying the estimate

(7.6) $$\sup_{T_1 \leq t \leq 2T_1} \|(\rho, u, \theta)(t)\|_3 \leqq 2\|(\rho, u, \theta)(T_1)\|_3 \leqq \varepsilon_0$$

by (7.5).

Then by (7.4) (7.6) and by Theorem 6.4 we have

$$E_2 \equiv \sup_{0 \leq t \leq 2T_1} \|(\rho, u, \theta)(t)\|_3 \leqq \sqrt{C_0}\, E_0 \leqq \varepsilon_0/2$$

provided $E_0 < \varepsilon_0/2\sqrt{C_0}$. Thus we can continue the same process for $0 \leqq t \leqq nT_1$, $n = 3, 4, 5, \cdots$ and finally get a global solution $(\rho, u, \theta)(t) \in \mathcal{E}(0, \infty; H^3)$ satisfying the estimate for any $t \geqq 0$

$$\|(\rho, u, \theta)(t)\|_3^2 + \nu_0 \int_0^t \|D(\rho, u, \theta)(\tau)\|_2^2 + \|D(u, \theta)(\tau)\|_3^2 d\tau \leqq C_0 E_0^2 \leqq \varepsilon_0/2 .$$

Especially we have

$$(7.7)\qquad \begin{cases} D\rho(t)\in L_2(0,\infty;H^2) \\ D(u,\theta)(t)\in L_2(0,\infty;H^3). \end{cases}$$

Thus using the equation (2.1)-(2.3)

$$\frac{\partial}{\partial t}(\rho,u,\theta)(t)\in L_2(0,\infty;H^2).$$

Therefore the solution decays to zero:

$$\|D(\rho,u,\theta)(t)\|_1 \longrightarrow 0 \quad \text{as} \quad t\longrightarrow\infty.$$

With the aide of (7.2) and Nirenberg's inequality this implies the decay of maximum norm of the solution.

Now we have to show that the solution is a classical one for $t>0$. First by Lemma 2.2 we know

$$(7.8)\qquad \begin{cases} \rho(t)\in C^0(0,T;H^3)\cap C^1(0,T;H^2) \\ \qquad \subset C^0(0,T;\mathscr{B}^{1+\sigma})\cap C^1(0,T;\mathscr{B}^\sigma) \\ (u,\theta)(t)\in C^0(0,T;H^3)\cap C^1(0,T;H^1) \\ \qquad \subset C^0(0,T;\mathscr{B}^{1+\sigma}) \end{cases}$$

for any $\sigma\in(0,1/2)$. Thus $\rho(t)$ is a classical solution of (2.1) for $t\geqq0$. As for $(u,\theta)(t)$ we consider (2.2) (2.3) as a linear equation for $\tilde{u}=(\tilde{u}^1,\tilde{u}^2,\tilde{u}^3,\tilde{\theta})$:

$$(7.9)\qquad \begin{cases} L^i_\rho(\tilde{u})=g^i(\rho,u,\theta), \qquad i=1,2,3, \\ L^4_\rho(\tilde{u})=h(\rho,u,\theta)\equiv g^4, \\ \tilde{u}(0)=(u,\theta)(0)\in H^3. \end{cases}$$

The system (7.9) for $\tilde{u}$ is uniformly parabolic in the sense of Petrowski in $t\geqq0$, $x\in\boldsymbol{R}^3$. Since $(1+\rho)^{-1}\in C^0(0,\infty;\mathscr{B}^{1+\sigma})$, $g^i\in C^0(0,\infty;\mathscr{B}^\sigma)$, $i=1,2,3,4$, and $\tilde{u}(0)\in\mathscr{B}^{1+\sigma}$ for any $\sigma\in(0,1/2)$, it follows the argument in Chapter 9 [1] that there exists a fundamental solution $\Gamma(t,x;\tau,\xi)$ for (7.9) such that the solution of (7.9) is represented by

$$(7.10)\qquad \begin{aligned} \tilde{u}(t,x)=&\int\Gamma(t,x;0,\xi)\tilde{u}(0,\xi)d\xi \\ &+\int_0^t\!\!\int\Gamma(t,x;\tau,\xi)g(\tau,\xi)d\xi d\tau, \end{aligned}$$

which has the property:

$$\begin{cases} \tilde{u} \text{ is continuous in } [0,\infty)\times\boldsymbol{R}^3, \\ \tilde{u}_{x_j},\ \tilde{u}_{x_jx_k},\ \tilde{u}_t \text{ are continuous in } (0,\infty)\times\boldsymbol{R}^3 \end{cases}$$

and $\tilde{u}$ is a classical solution of (7.9) for $t>0$. On the other hand we obtained a solution $\bar{u}(t)=(u,\theta)(t)\in C^0(0,\infty;H^3)\cap C^1(0,\infty;H^1)$ of (7.9) in the former part of Theorem 7.1. Thus it is sufficient for our purpose to see that $\tilde{u}(t)=\bar{u}(t)$ in $t\geqq0$. To prove the coincidence we use the energy inequality for the weak solution $v(t)$ of the system (7.9) in the sense that $v(t)\in L^0_\infty(0, T; H^1)$ and $v_t(t)\in L^0_\infty(0, T; H^{-1})$ for $T>0$, where H^{-1} is the dual space of H^1:

$$\|v(t)\|_2\leqq e^{Ct}(\|v(0)\|_2+\int_0^t\|g(\tau)\|^2d\tau) \quad \text{for any} \quad t\in[0, T]. \tag{7.11}$$

This inequality is given by the argument for (3.18) using the mollifier in the same way as the proof for Proposition 3.3. Hence the proof of Theorem finishes, if we show the solution $\tilde{u}(t)$ given by (7.10) satisfies $\tilde{u}(t)\in L^0_\infty(0, T; H^1)$ and $\tilde{u}_t(t)\in L^0_\infty(0, T; H^{-1})$ for any $T\geqq 0$ by the following lemma.

Lemma 7.2.

$$\begin{cases} |D_x^m\Gamma(t, x; \tau, \xi)|\leqq C(t-\tau)^{-(3+|m|)/2}\exp\left(-\dfrac{\lambda|x-\xi|^2}{t-\tau}\right) \\ |D_\xi^l\Gamma(t, x+\xi; \tau, \xi)|\leqq C(t-\tau)^{-3/2}\exp\left(-\dfrac{\lambda|x|^2}{t-\tau}\right) \end{cases} \tag{7.12}$$

for any $m=0, 1, 2$, $l=0, 1$ and some constant $\lambda>0$.

Proof. See Theorem 7, Chapter 9 [1] for example.
In fact by (7.10)

$$\begin{aligned} \tilde{u}_{x_i}(t, x)&=\frac{\partial}{\partial x_i}\int\Gamma(t, x; 0, x+y)\tilde{u}(0, x+y)dy \\ &\quad+\int_0^t\int\frac{\partial}{\partial x_i}\Gamma(t, x; \tau, \xi)g(\tau, \xi)d\xi d\tau \\ &=\int\Gamma(t, x; 0, \xi)\frac{\partial\tilde{u}(0, \xi)}{\partial\xi_i}d\xi \\ &\quad+\int\frac{\partial}{\partial\xi_i}\Gamma(t, \xi-y; \tau, \xi)\Big|_{\xi=x+y}\tilde{u}(0, x+y)dy \\ &\quad+\int_0^t\int\frac{\partial}{\partial x_i}\Gamma(t, x; \tau, \xi)g(\tau, \xi)d\xi d\tau \\ &\equiv I_1+I_2+I_3. \end{aligned}$$

$$\begin{aligned} \|I_1\|^2&\leqq\int\left(\int|\Gamma(t, x; 0, \xi)|d\xi\int|\Gamma(t, x; 0, \xi)||\tilde{u}(0, \xi)_{\xi_i}|^2d\xi\right)dx \\ &\leqq\sup_x\int|\Gamma(t, x; 0, \xi)|d\xi\cdot\sup_\xi\int|\Gamma(t, x; 0, \xi)|dx\|D\tilde{u}(0)\|^2 \\ &\leqq C\|\tilde{u}(0)\|_1^2, \end{aligned}$$

$$\begin{aligned} \|I_2\|^2\leqq C\int\Big(&\int t^{-(3/2)}\exp\left(-\lambda\frac{|y|^2}{t}\right)dy \\ &\cdot\int t^{-(3/2)}\exp\left(-\lambda\frac{|y|^2}{t}\right)|\tilde{u}(0, x+y)|^2dy\Big)dx \end{aligned}$$

$$\leqq C\|\tilde{u}(0)\|^2,$$

$$\|I_3\|^2 \leqq \int_0^t \left(\int \left|\frac{\partial}{\partial x_i}\Gamma(t, x; \tau, \xi)g(\tau, \xi)d\xi\right|^2 dx\right)^{1/2} d\tau$$

$$\leqq \int_0^t \sup_x \left(\int |D_x\Gamma| d\xi\right)^{1/2} \sup_\xi \left(\int |D_x\Gamma| dx\right)^{1/2} \cdot \left(\int |g(\tau, \xi)|^2 d\xi\right)^{1/2} d\tau$$

$$\leqq C \sup_{0\leq\tau\leq t} \|g(\tau)\| \int_0^t (t-\tau)^{-(1/2)} d\tau$$

$$\leqq C \sup_{0\leq\tau\leq t} \|g(\tau)\| t^{1/2}.$$

Therefore we have

$$\sup_{0\leq t\leq T} \|D\tilde{u}(t)\| \leqq C(\|\tilde{u}(0)\|_1 + T^{1/2} \sup_{0\leq t\leq T} \|g(t)\|).$$

This completes the proof of Theorem 7.1.

DEPARTMENT OF APPLIED MATHEMATICS
AND PHYSICS
KYOTO UNIVERSITY

References

[1] Friedman, A., Partial Differential Equation of Parabolic Type, Prentice-Hall, Inc., (1964).

[2] Itaya, N., On the Cauchy problem for the system of fundamental equations describing the movement of compressible viscous fluid, Kōdai Math. Sem. Rep., **23** (1971) 60-120.

[3] Itaya, N., On the initial value problem of the motion of compressible viscous fluid, especially on the problem of uniqueness, J. Math., Kyoto Univ., **16** (1976) 413-427.

[4] Itaya, N., A survey on the generalized Burgers' equation with a pressure model term, J. Math. Kyoto Univ., **16** (1976) 1-18.

[5] Itaya, N., A survey on two model equations for compressible viscous fluid J. Math. Kyoto Univ. **19** (1979) 293-300.

[6] Kanel', Ya. I., On a model system of equations for one-dimensional gas motion, Diff. Eq. (in Russian) **4** (1968) 721-734.

[7] Kato, T., Linear evolution equations of hyperbolic type, J. Fac. Sci Univ. Tokyo, **17** (1970) 241-258.

[8] Kazhikhov, A. V., Sur la solubilité globale des problème monodimensionnelle aux valeurs initiales-limitées pour les équations du gaz visqueux et calorifère, C. R. Acad.Sci., Paris, **284** (1977) Ser. A, 317-320.

[9] Kazhikhov, A. V. and Shelukhin, V. V., Unique global solution in time of initial-boundary value problems for one-dimensional equations of a viscous gas, Prikl. Mat. Mech., **41** (1977) 282-291.

[10] Landau, L. D. and Lifshitz, E. M., Fluid Mechanics, Pergamon Press, (1959).

[11] Lamb, H., Hydrodynamics, 6th ed., Cambridge Univ. Press, (1932).

[12] Matsumura, A., Global existence and asymptotics of the solutions of the second-order quasilinear hyperbolic equations with the first-order dissipation, Publ. RIMS, Kyoto Univ., **13** (1977) 349-379.

[13] Mizohata, S., Theory of Partial Differential Equations, Cambridge Univ. Press, (1973).
[14] Nash, J., Le problème de Cauchy pour les équations différentielles d'un fluide général, Bull. Soc. Math. France, **90** (1962) 487-497.
[15] Tani, A., On the first initial-boundary problem of compressible viscous fluid motion, Publ. RIMS, Kyoto Univ., **13** (1977) 193-253.