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The Hopf algebra structure of $K_*(\Omega Sp(n))$

By

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Dedicated to Professor A. Komatu on his 70th birthday

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§ 0. Introduction

Let G be a compact connected Lie group and ΩG the space of loops on G. R. Bott introduced an idea "the generating variety" for ΩG and determined the bi-commutative Hopf algebra $H_*(\Omega G)$ for $G=SU(n)$, $Spin(n)$ and G_2 ([5]). Recently, F. Clarke determined the Hopf algebra structure of $K_*(\Omega G)$ for $G=SU(n)$, $Spin(n)$ and G_2 where $K_*(\ \ )$ is the $Z/2Z$-graded K-homology theory using the generating varieties ([8]). But the results for $G=Sp(n)$ is not known. In our recent paper [10], A. Kono and myself determined the Hopf algebra $H_*(\Omega Sp(n))$ and $h_*(\Omega Sp(n))\otimes Z[\frac{1}{2}]$ where $h_*(\ \ )$ is a complex oriented homology theory. However the method used there is not applicable for $h_*(\Omega Sp(n))$ with $h=K$ or MU.

The purpose of this paper is to determine $K_*(\Omega Sp(n))$ as a Hopf algebra over $\boldsymbol{Z}$.

By the result of R. Bott, ΩSU and BU are homotopy equivalent as an H-space, and the Hopf algebra $K_*(BU)=K_0(BU)\oplus K_1(BU)$ was determined by J. F. Adams [2]. In particular $K_1(BU)=0$.

As in proved in [10], we may consider $K_*(\Omega Sp)$ as a Hopf subalgebra of $K_*(\Omega SU)$ by $(\Omega c)_*$ where $c: Sp\to SU$ is the complexification map. Moreover $K_*(\Omega Sp(n))$ is a Hopf subalgebra of $K_*(\Omega Sp)$ (cf. Theorem 1.1).

Let R be a commutative ring with unit and $f(x)=\sum_{i\geq 0}f_i x^i$, $g(x)=\sum_{i\geq 0}g_i x^i$ $\in R[[x]]$. Define $(f\square g)(x)\in(R\otimes R)[[x]]$ to be $\sum_{i\geq 0}\left(\sum_{\substack{j+k=i\\ j,k\geq 0}}f_j\otimes g_k\right)x^i$. Then the main result of this paper is

Corollary 2.7. *$K_0(\Omega Sp(n))=\boldsymbol{Z}[r_1, r_3, \cdots, r_{2k-1}, \cdots, r_{2n-1}]$ as an algebra and the diagonal ϕ is given by*

$$\phi(r_{2k-1})=\left[\frac{(1\square\tilde{r}_n)(x)+(\tilde{r}_n\square 1)(x)+x\cdot(\tilde{r}_n\square\tilde{r}_n)(x)}{1\otimes 1+(\tilde{r}_n\square\tilde{r}_n)(x)}\right]_{2k-1},$$

where $\tilde{r}_n(x)=\sum_{i=1}^n r_{2i-1}x^{2i-1}$ and $[\sum a_i x^i]_j$ denotes the coefficient of x^j in $\sum a_i x^i$.

This paper is organized as follows:

In § 1, we construct an 'artificial' Hopf subalgebra Γ of $K_0(\Omega SU)$ and prove that it agrees with $\mathrm{Im}\,(\Omega c)_*$ in § 2. Thus we can determine $K_0(\Omega Sp)$ and we will reduce these results to the finite case $K_0(\Omega Sp(n))$.

Throughout the paper the binomial coefficient $\binom{n}{m}$ is equal to the coefficient of x^m in $(1+x)^n$ for $n\geq 0$ and is equal to zero for $n<0$.

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§ 1. Notations and the main theorem

First, recall some notations (see [10]).

Let $U(n)$, $Sp(n)$ be the n-th unitary and symplectic group, and U, Sp the infinite unitary and symplectic group, respectively.

Let q: $U(n)\rightarrow Sp(n)$ be a map induced by the natural inclusion $\boldsymbol{C}\subset\boldsymbol{H}$ where $\boldsymbol{C}$ (resp. $\boldsymbol{H}$) is the field of complex (resp. quaternion) numbers.

Let $a_{ij}\in\boldsymbol{H}$ and $a_{ij}=b_{ij}+jc_{ij}$ for b_{ij}, $c_{ij}\in\boldsymbol{C}$ and define a map c: $Sp(n)\rightarrow U(2n)$ by

$$c((a_{ij}))=\begin{pmatrix} N_{11} & \cdots & N_{1n} \\ & \cdots\cdots & \\ & \cdots\cdots & \\ N_{n1} & \cdots & N_{nn} \end{pmatrix},$$

where

$$N_{ij}=\begin{pmatrix} b_{ij} & -c_{ij} \\ \bar{c}_{ij} & \bar{b}_{ij} \end{pmatrix}.$$

Let BG be the classifying space of a topological group G and Bf: $BG\rightarrow BH$ the map induced by a continuous homorphism f: $G\rightarrow H$.

Let ΩX be the space of loops on a space X and Ωf: $\Omega X\rightarrow\Omega Y$ the map induced by a map f: $X\rightarrow Y$.

Let i_n: $Sp(n)\rightarrow Sp$ be the natural inclusion. Let

$$\lambda\colon \Omega SU\times\Omega SU\rightarrow\Omega SU$$

be the loop product and

$$\Delta\colon \Omega SU\rightarrow\Omega SU\times\Omega SU$$

the diagonal map. Also let J: $\Omega SU\rightarrow\Omega SU$ be the loop inverse of ΩSU, that is, $J(p(t))=p(-t)$ for every $p\in\Omega SU$. Define I: $U\rightarrow U$ by $I(A)=\bar{A}$. Then I induecs a map BI: $BU\rightarrow BU$.

Let g: $BU\rightarrow\Omega SU$ be the Bott map. For simplicity, we define ℓ: $\Omega SU\rightarrow\Omega SU$ to be $g\circ BI\circ g^{-1}$.

Put $\iota(x)=-x/(1+x)\in\boldsymbol{Z}[[x]]$.

Under these notations we can quote the results from our recent paper [10].

Theorem 1.1. (i) *$K_0(\Omega SU)=\boldsymbol{Z}[\beta_1, \beta_2, \cdots, \beta_n, \cdots]$ as an algebra and $K_1(\Omega SU)=0$. Moreover*

$$\bar{\phi}(\beta_i)=\sum_{\substack{j+k=i\\ j,k>0}} \beta_j \otimes \beta_k,$$

where $\bar{\phi}$ is the reduced diagonal map defined by Δ.

(ii) *The following diagram commutes:*

$$\begin{array}{ccccccc} \Omega SU & \xrightarrow{\quad \Omega q \quad} & & \Omega Sp & & \xrightarrow{\quad \Omega c \quad} & \Omega SU \\ \| & & & & & & \| \\ \Omega SU & \xrightarrow{\Delta} & \Omega SU\times \Omega SU & \xrightarrow{\mathrm{id}\times J\circ \ell} & \Omega SU\times\Omega SU & \xrightarrow{\lambda} & \Omega SU. \end{array}$$

Moreover, if we put $\beta(x)=\sum_{i\geq 0}\beta_i x^i$ ($\beta_0=1$) and extend J_*, ℓ_* or $\Omega(c\circ q)_*$ over $K_0(\Omega SU)[[x]]$ by the natural way, then

$$J_*\beta(x)=1/\beta(x)$$
$$\ell_*\beta(x)=\beta(\iota(x))$$

and

$$\Omega(c\circ q)_*\beta(x)=\beta(x)/\beta(\iota(x)).$$

(iii) *There are $z_{2k-1}\in K_0(\Omega Sp)$ such that*

$$K_0(\Omega Sp)=\boldsymbol{Z}[z_1, z_3, \cdots, z_{2k-1}, \cdots]$$

as an algebra and

$$\Omega c_* z_{2k-1}\equiv\beta_{2k-1}$$

modulo the subalgebra generated by $\beta_1, \beta_2, \cdots, \beta_{2k-2}$ in $K_0(\Omega SU)$. Thus Ωc_* is a split monomorphism.

(iv) *$(\Omega i_n)_*\colon K_0(\Omega Sp(n))\to K_0(\Omega Sp)$ is a split monomorphism and* $\mathrm{Im}\,(\Omega i_n)_*$ *is generated by $z_1, z_3, \cdots, z_{2n-1}$ as a subalgebra of $K_0(\Omega Sp)$.*

For the proofs of (i) and (ii), see § 1, § 2, and § 6 of [10]. (iii) and (iv) are obtained easily from § 6 of [10] and the naturality of the Atiyah-Hirzebruch spectral sequence.

Let R be a commutative ring with unit, and $R[[x]]$ the formal power series ring over R. If $f\colon R\to S$ is a ring homomorphism, we define $f\colon R[[x]]\to S[[x]]$ by

$$f(r)(x)=\sum_{i\geq 0} f(r_i)x^i \quad \text{where} \quad r(x)=\sum_{i\geq 0} r_i x^i\in R[[x]].$$

Put

$$b_{2n-1}=\sum_{i\geqq 0}\binom{n-1}{i}\beta_{2n-1-i}\in K_0(\Omega SU)$$

and

$$b_{2n}=\sum_{i\geqq 0}\binom{n-1}{i}\beta_{2n-i}\in K_0(\Omega SU)$$

for $n>0$.

We define also $b_{od}(x)$ and $b_{ev}(x)\in K_0(\Omega SU)[[x]]$ to be $\sum_{i>0} b_{2i-1}x^{2i-1}$ and $1+\sum_{i>0} b_{2i}x^{2i}$, respectively.

Then clearly $b_{ev}(x)$ is a unit in $K_0(\Omega SU)[[x]]$. Put $r(x)=\sum_{i>0} r_{2i-1}x^{2i-1} =b_{od}(x)/b_{ev}(x)$. Let Γ be a subalgebra of $K_0(\Omega SU)$ generated by r_{2k-1} $(k=1, 2, \cdots)$.

Now we can state our main result.

Theorem 1.2. $\operatorname{Im}(\Omega c)_*=\Gamma$ *so that* $K_0(\Omega Sp)\cong\Gamma$.

This result will be proved in § 2. The rest of this section is devoted to calculating $\phi(r_{2n-1})$. We must first calculate the diagonal formulas of b_{2n-1} and b_{2n}. We need some algebraic notations concerning formal power series.

Let R be a commutative ring with unit and $f(x)=\sum_{i\geqq 0} f_i x^i$, $g(x)=\sum_{i\geqq 0} g_i x^i\in R[[x]]$. Define $(f\square g)(x)\in(R\otimes R)[[x]]$ to be $\sum_{i\geqq 0}\left(\sum_{j+k=i,\ j,k\geqq 0} f_j\otimes g_k\right)x^i$.

If R is a commutative Hopf algebra, then the diagonal $\phi\colon R\to R\otimes R$ is a ring homomorphism. Thus we can obtain a ring homomorphism $\phi\colon R[[x]]\to(R\otimes R)[[x]]$.

Proposition 1.3.

(i) $(\phi b_{ev})(x)=(b_{ev}\square b_{ev})(x)+(b_{od}\square b_{od})(x)$

and

(ii) $(\phi b_{od})(x)=(b_{ev}\square b_{od})(x)+(b_{od}\square b_{ev})(x)+x\cdot(b_{od}\square b_{od})(x)$.

Proof. These statements are equivalent to

(i)′ $\bar{\phi}(b_{2n})=\sum_i b_i\otimes b_{2n-i}$

and

(ii)′ $\bar{\phi}(b_{2n-1})=\sum_i b_i\otimes b_{2n-1-i}+\sum_j b_{2j-1}\otimes b_{2(n-j)-1}$,

where $b_i=0$ for $i\leqq 0$ and the summations run through all non-zero terms. In the case (i) we have

$$\bar{\phi}(b_{2n})=\bar{\phi}\left(\sum_i\binom{n-1}{i}\beta_{2n-i}\right)=\sum_i\binom{n-1}{i}\left(\sum_j\beta_{2n-i-j}\otimes\beta_j\right)=\sum_{s,t}\binom{n-1}{2n-s-t}\beta_s\otimes\beta_t,$$

and

$$\sum_i b_i\otimes b_{2n-i}=\sum_i (b_{2i}\otimes b_{2n-2i}+b_{2i-1}\otimes b_{2n-2i+1})$$
$$=\sum_i \Big(\sum_j \binom{i-1}{j}\beta_{2i-j}\Big)\otimes\Big(\sum_k \binom{n-i-1}{k}\beta_{2(n-i)-k}\Big)$$
$$+\Big(\sum_j \binom{i-1}{j}\beta_{2i-1-j}\Big)\otimes\Big(\sum_k \binom{n-i}{k}\beta_{2n-2i+1-k}\Big)$$
$$=\sum_{s,t}\Big(\sum_{\substack{2i-j=s\\2(n-i)-k=t}}\binom{i-1}{j}\binom{n-i-1}{k}+\sum_{\substack{2i-j-1=s\\2(n-i)-j+1=t}}\binom{i-1}{j}\binom{n-i}{k}\Big)\beta_s\otimes\beta_t$$

So we must prove that, for $s, t\geqq 1$,

$$(1.4)\qquad \begin{aligned}\binom{n-1}{2n-s-t}&=\sum_{\substack{2i-j=s\\2(n-i)-k=t}}\binom{i-1}{j}\binom{n-i-1}{k}+\sum_{\substack{2i-j-1=s\\2(n-i)-k+1=t}}\binom{i-1}{j}\binom{n-i}{k}\\&=\sum_i\Big[\binom{i-1}{2i-s}\binom{n-i-1}{2(n-i)-t}+\binom{i-1}{2i-1-s}\binom{n-i}{2(n-i)+1-t}\Big].\end{aligned}$$

In the case (ii) we have only to prove the following equarion

$$\binom{n-1}{2n-1-s-t}=\sum_i\Big[\binom{i-1}{2i-s}\binom{n-i-1}{2(n-i)-1-t}+\binom{t-1}{2i-1-s}\binom{n-i-1}{2(n-i)-t}$$
$$+\binom{i-1}{2i-1-s}\binom{n-i-1}{2(n-i)-1-t}\Big]$$

obtained by the same manner as in the case (i). But the right hand of this equation equals to

$$\sum_i\Big[\binom{i-1}{2i-s}\binom{n-i-1}{2(n-i)-1-t}+\binom{i-1}{2i-1-s}\Big\{\binom{n-i-1}{2(n-i)-t}+\binom{n-i-1}{2(n-i)-1-t}\Big\}\Big]$$
$$=\sum_i\Big[\binom{i-1}{2i-s}\binom{n-i-1}{2(n-i)-1-t}+\binom{i-1}{2i-1-s}\binom{n-i}{2(n-i)-t}\Big]$$

(see Lemma A.1 in Appendix.). Thus the case (ii) also reduces to (1.4). It is easy but tedious to show (1.4). So we defer this to the appendix.

We need some technical lemmas to determine ϕr.

Lemma 1.5. *Let R be a commutative ring with unit. Let $d(x), e(x), f(x)$ and $g(x)\in R[[x]]$. Then $(d\Box e)(x)(f\Box g)(x)=(df\Box eg)(x)$ in $(R\otimes R)[[x]]$.*

Proof. Put $d(x)=\sum_{i\geqq 0}d_ix^i$, $e(x)=\sum_{i\geqq 0}e_ix^i$, $f(x)=\sum_{i\geqq 0}f_ix^i$ and $g(x)=\sum_{i\geqq 0}g_ix^i$. Then

$$(d\Box e)(x)(f\Box g)(x)$$
$$=\Big(\sum_k\Big(\sum_{\substack{i+j=k\\i,j\geqq 0}}d_i\otimes e_j\Big)x^k\Big)\Big(\sum_u\Big(\sum_{\substack{s+t=u\\s,t\geqq 0}}f_s\otimes g_t\Big)x^u\Big)$$

$$=\sum_m\left(\left(\sum_{\substack{i+j+s+t=m\\ i,j,s,t\geq 0}} d_i f_s\otimes e_j g_t\right)x^m\right)$$

$$=\sum_m\left(\left(\sum_{\substack{i+s=u\\ i,s\geq 0}} d_i f_s\right)\otimes\left(\sum_{\substack{j+t=m-u\\ j,t\geq 0}} e_j g_t\right)x^m\right)$$

$$=(df\,\square\, eg)(x).$$

Corollary 1.6. *Let $f(x)$ and $g(x)$ be multiplicative units in $R[[x]]$. Then $(f\square g)(x)$ is also a unit in $(R\otimes R)[[x]]$. In fact, $1/(f\square g)=(1/f)\square(1/g)$.*

Proof. $(f\square g)(1/f\square 1/g)=1\otimes 1$.

Theorem 1.7.

$$(\phi r)(x)=\frac{(r\square 1)(x)+(1\square r)(x)+x\cdot(r\square r)(x)}{1\otimes 1+(r\square r)(x)}.$$

Proof. $\phi r=\phi(b_{od}/b_{ev})=\phi b_{od}/\phi b_{ev}$. By (1.3),

$$(\phi r)(x)=\frac{(b_{ev}\square b_{od})(x)+(b_{od}\square b_{ev})(x)+x\cdot(b_{od}\square b_{od})(x)}{(b_{ev}\square b_{ev})(x)+(b_{od}\square b_{od})(x)}$$

Since b_{ev} is a unit, (1.6) asserts $b_{ev}\square b_{ev}$ is also a unit. Then

$$(\phi r)(x)=\frac{((b_{ev}\square b_{od})(x)+(b_{od}\square b_{ev})(x)+x\cdot(b_{od}\square b_{od})(x))(1/(b_{ev}\square b_{ev})(x))}{((b_{ev}\square b_{ev})(x)+(b_{od}\square b_{od})(x))(1/(b_{ev}\square b_{ev})(x))}$$

$$=\frac{(r\square 1)(x)+(1\square r)(x)+x\cdot(r\square r)(x)}{1\otimes 1+(r\square r)(x)}.$$

Corollary 1.8. *Let $\Gamma=\boldsymbol{Z}[r_1, r_3, \cdots, r_{2k-1}, \cdots]$. Then Γ is a Hopf subalgebra of $K_0(\Omega SU)$.*

Proof. If $f(x)\in R[[x]]$ is unit in $R[[x]]$, the coefficients of $(1/f)(x)$ is written by the polynomial of the coefficients of $f(x)$. Thus the coefficients of ϕr is in $\Gamma\otimes\Gamma$.

§ 2. $K_0(\Omega Sp)$ and $K_0(\Omega Sp(n))$

We must prove $\Gamma\subset\mathrm{Im}\,(\Omega c)_*$ to prove (1.2). We calculate first $\Omega(c\circ q)_* r(x)$. For this, we must calculate $J_* r(x)$ and $\ell_* r(x)$ as in (1.1).

Proposition 2.1. $J_* r(x)=-r(x)/(1+x\cdot r(x))$.

Proof. Since

$$\Omega SU\xrightarrow{\Delta}\Omega SU\times\Omega SU\xrightarrow{1\times J}\Omega SU\times\Omega SU\xrightarrow{\lambda}\Omega SU$$

is null-homotopic, we have easily the following equation:

$$\mu(1\otimes J_*)(\phi r)(x)=0$$

where $\mu\colon (K_0(\Omega SU)\otimes K_0(\Omega SU))[[x]]\to K_0(\Omega SU)[[x]]$ is the ring homomorphism induced from the product

$$\mu\colon K_0(\Omega SU)\otimes K_0(\Omega SU)\to K_0(\Omega SU).$$

Thus, by (1.7) and the fact that $\mu(f\square g)=f\cdot g$, we have

$$0=\frac{r(x)+J_*r(x)+x\cdot r(x)\cdot J_*r(x)}{1+r(x)\cdot J_*r(x)}.$$

So $J_*r(x)\cdot(1+x\cdot r(x))+r(x)=0$, and we have proved our proposition.

Proposition 2.2.
(i) $\ell_*b_{od}(x)=-b_{od}(x)$
and
(ii) $\ell_*b_{ev}(x)=b_{ev}(x)+x\cdot b_{od}(x)$.

Proof. Define $[f(x)]_n$ to be the coefficient of x^n in $f(x)\in R[[x]]$. Since

$$\begin{aligned}\beta(\iota(x))&=\sum_{i\geq 0}\beta_i\left(-\frac{x}{1+x}\right)^i\\ &=\sum_{i\geq 0}((-1)^i\beta_i x^i(1+x)^{-i})\\ &=\sum_{i\geq 0}\left\{(-1)^i\beta_i x^i\sum_{j\geq 0}\binom{i+j-1}{j}(-1)^j x^j\right\}\\ &=\sum_{n\geq 0}(-1)^n\left(\sum_{j\geq 0}\binom{n-1}{j}\beta_{n-j}\right)x^n,\end{aligned}$$

we see

$$\ell_*\beta_n=[\beta(\iota(x))]_n=(-1)^n\sum_{j\geq 0}\binom{n-1}{j}\beta_{n-j}.$$

Then

$$\begin{aligned}\ell_*b_{2n-\varepsilon}&=\ell_*\left(\sum_{i\geq 0}\binom{n-1}{i}\beta_{2n-\varepsilon-i}\right)\\ &=\sum_{i\geq 0}\left[\binom{n-1}{i}(-1)^{2n-\varepsilon-i}\sum_{j\geq 0}\binom{2n-\varepsilon-1-i}{j}\beta_{2n-\varepsilon-i-j}\right]\\ &=\sum_{s\geq 0}\left[\sum_{i+j=s}(-1)^{2n-\varepsilon-i}\binom{n-1}{i}\binom{2n-\varepsilon-1-i}{j}\right]\beta_{2n-\varepsilon-s}\end{aligned}$$

where $\varepsilon=0$ or 1. Since $\binom{n}{s}=\binom{n-1}{s}+\binom{n-1}{s-1}$ $(n,s>0)$, we have only to show that

$$(-1)^\varepsilon\binom{n-\varepsilon}{s}=\sum_{i+j=s}(-1)^{2n-\varepsilon-i}\binom{n-1}{i}\binom{2n-1-\varepsilon-i}{j}.$$

Since

$$\begin{aligned}
&\sum_{i+j=s}(-1)^{2n-\varepsilon-i}\binom{n-1}{i}\binom{2n-1-\varepsilon-i}{j}\\
&=(-1)^{\varepsilon}\sum_{i+j=s}(-1)^{i}\binom{n-1}{i}\binom{2n-1-\varepsilon-i}{j}\\
&=(-1)^{\varepsilon}\Big[\sum_{i\geqq 0}(-1)^{i}\binom{n-1}{i}(1+x)^{2n-1-\varepsilon-i}x^{i}\Big]_s\\
&=(-1)^{\varepsilon}\Big[(1+x)^{n-\varepsilon}\sum_{i\geqq 0}(-1)^{i}\binom{n-1}{i}(1+x)^{n-1-i}x^{i}\Big]_s\\
&=(-1)^{\varepsilon}[(1+x)^{n-\varepsilon}((1+x)-x)^{n-1}]_s\\
&=(-1)^{\varepsilon}[(1+x)^{n-\varepsilon}]_s=(-1)^{\varepsilon}\binom{n-\varepsilon}{s},
\end{aligned}$$

the proposition is proved.

Corollary 2.3. $\ell_* r(x)=-r(x)/(1+x\cdot r(x))$.

Proof. By the definition,

$$\begin{aligned}
\ell_* r(x)&=\ell_*(b_{od}(x)/b_{ev}(x))\\
&=\ell_* b_{od}(x)/\ell_* b_{ev}(x).
\end{aligned}$$

Applying (2.2), we obtain

$$\begin{aligned}
\ell_* r(x)&=-b_{od}(x)/(b_{ev}(x)+x\cdot b_{od}(x))\\
&=-\frac{b_{od}(x)}{b_{ev}(x)}\Big/\frac{b_{ev}(x)+x\cdot b_{od}(x)}{b_{ev}(x)}\\
&=-r(x)/(1+x\cdot r(x)).
\end{aligned}$$

Now we can easily calculate $\Omega(c\circ q)_* r(x)$.

Proposition 2.4.

$$\Omega(c\circ q)_* r(x)=\frac{2r(x)+x\cdot(r(x))^2}{1+(r(x))^2}.$$

Proof. Since $\ell_* r(x)=J_* r(x)$, we obtain

$$J_*\ell_* r(x)=J_*J_* r(x)=(J\circ J)_* r(x)=r(x),$$

So by (1.7),

$$\begin{aligned}
\Omega(c\circ q)_* r(x)&=\mu(1\otimes J_*\ell_*)(\phi r)(x)\\
&=\frac{r(x)+J_*\ell_* r(x)+x\cdot r(x)\cdot J_*\ell_* r(x)}{1+r(x)\cdot J_*\ell_* r(x)}
\end{aligned}$$

$$=\frac{2r(x)+x\cdot(r(x))^2}{1+(r(x))^2}.$$

As a corollary, we obtain the follwoing

Corollary 2.5. $\Gamma\subset\operatorname{Im}(\Omega c)_*$.

Proof. We prove this by induction. (iii) of (1.1) implies $(\Omega c)_* z_1=\beta_1$. By the definition, $r_1=\beta_1$. So $r_1\in\operatorname{Im}(\Omega c)_*$. Assume that $r_1, r_3, \cdots, r_{2k-1}\in\operatorname{Im}(\Omega c)_*$. Note that

$$\left[\frac{2r(x)+x\cdot(r(x))^2}{1+(r(x))^2}\right]_{2k+1}\equiv 2r_{2k+1}$$

modulo Γ_k where $\Gamma_k=Z[r_1, r_3, \cdots, r_{2k-1}]$. Since $\Gamma_k\subset\operatorname{Im}(\Omega c)_*$ by the assumption, we have $2r_{2k+1}\in\operatorname{Im}(\Omega c)_*$ by (2.4). But, by (iii) of (1.1), $\operatorname{Im}(\Omega c)_*$ is a split submodule of $K_0(\Omega SU)$. Thus $r_{2k+1}\in\operatorname{Im}(\Omega c)_*$.

We can now prove,

Theorem 1.2. $\operatorname{Im}(\Omega c)_*=\Gamma$.

Proof of (1.2). Since

$$r_{2k-1}\equiv\beta_{2k-1} \quad \bmod \quad Z[\beta_1, \beta_2, \cdots, \beta_{2k-2}],$$

(iii) of (1.1) and (2.5) assert the following equation:

$$r_{2k-1}\equiv(\Omega c)_*(z_{2k-1})$$

mod $Z[(\Omega c)_*(z_1), (\Omega c)_*(z_3), \cdots, (\Omega c)_*(z_{2k-3})]$. So

$$Z[r_1, r_3, \cdots, r_{2k-1}]=Z[(\Omega c)_*(z_1), (\Omega c)_*(z_3), \cdots, (\Omega c)_*(z_{2k-1})]$$

can be obtained by an easy induction. If we put $k=\infty$, we have

$$\begin{aligned}&Z[r_1, r_3, \cdots, r_{2n-1}, \cdots]\\&\quad=Z[(\Omega c)_*(z_1), (\Omega c)_*(z_3), \cdots, (\Omega c)_*(z_{2n-1}), \cdots]\\&\quad=\operatorname{Im}(\Omega c)_*.\end{aligned}$$

Corollary 2.6. $K_0(\Omega Sp)=Z[r_1, r_3, \cdots, r_{2k-1}, \cdots]$ *as an algebra and the diagonal is given by*

$$(\phi r)(x)=\frac{(1\,\square\, r)(x)+(r\,\square\, 1)(x)+x\cdot(r\,\square\, r)(x)}{1\otimes 1+(r\,\square\, r)(x)}.$$

From the proof of (2.5), we also obtain

$$Z[r_1, r_3, \cdots, r_{2n-1}]=(\Omega c)_*(Z[z_1, z_3, \cdots, z_{2n-1}]).$$

Then (iv) of (1.1) reduces (2.6) to the finite case:

Corollary 2.7. *$K_0(\Omega Sp(n))=Z[r_1, r_3, \cdots, r_{2k-1}, \cdots, r_{2n-1}]$ as an algebra and the diagonal is given by*

$$\phi(r_{2k-1})=\left[\frac{(1\square\tilde{r}_n)(x)+(\tilde{r}_n\square 1)(x)+x\cdot(\tilde{r}_n\square\tilde{r}_n)(x)}{1\otimes 1+(\tilde{r}_n\square\tilde{r}_n)(x)}\right]_{2k-1}$$

where $\tilde{r}_n(x)=\sum_{i=1}^{n} r_{2i-1}x^{2i-1}$ and the other notations are as in § 1.

Proof. The first half of (2.7) is clear. The second half follows from

$$[\tilde{r}_n(x)]_{2k-1}=[r(x)]_{2k-1}$$

and

$$\left[\frac{(1\square\tilde{r}_n)(x)+(\tilde{r}_n\square 1)(x)+x\cdot(\tilde{r}_n\square\tilde{r}_n)(x)}{1\otimes 1+(\tilde{r}_n\square\tilde{r}_n)(x)}\right]_{2k-1}$$
$$=\left[\frac{(1\square r)(x)+(r\square 1)(x)+x\cdot(r\square r)(x)}{1\otimes 1+(r\square r)(x)}\right]_{2k-1}$$

for $k\leqq n$.

Appendix.

Put

$$f(s,t)=\sum_i\left\{\binom{i-1}{2i-s}\binom{n-i-1}{2(n-i)-t}+\binom{i-1}{2i-1-s}\binom{n-i}{2(n-i)+1-t}\right\}$$

for $s, t\geqq 1$. Then the purpose of this appendix is to show

$$\binom{n-1}{2n-s-t}=f(s,t). \tag{1.4}$$

First recall that

Lemma A.1. *If $(a, b)\neq(0,0)$, then*

$$\binom{a}{b}=\binom{a-1}{b}+\binom{a-1}{b-1}.$$

Let $d(s,t)=f(s,t)-f(s-1,t+1)$ for $s\geqq 2$. Then we have

Lemma A.2. $d(s,t)=0$.

Proof.

$$d(s,t)=\sum_i\left\{\binom{i-1}{2i-s}\binom{n-i-1}{2(n-i)-t}+\binom{i-1}{2i-1-s}\binom{n-i}{2(n-i)+1-t}\right\}$$
$$-\sum_i\left\{\binom{i-1}{2i-s+1}\binom{n-i-1}{2(n-i)-t-1}+\binom{i-1}{2i-s}\binom{n-i}{2(n-i)-t}\right\}$$
$$=\sum_i\binom{i-1}{2i-s}\left\{\binom{n-i-1}{2(n-i)-t}-\binom{n-i}{2(n-i)-t}\right\}$$
$$+\sum_i\binom{i-1}{2i-1-s}\binom{n-i}{2(n-i)+1-t}-\sum_i\binom{i-1}{2i-s+1}\binom{n-i-1}{2(n-i)-t-1}.$$

Since $(n-i, 2(n-i)-t)\neq(0,0)$ we have

$$d(s,t)=-\sum_i\binom{i-1}{2i-s}\binom{i-1}{2(n-i)-1-t}+\sum_i\binom{i-1}{2i-1-s}\binom{n-i}{2(n-i)+1-t}$$
$$-\sum_i\binom{i-1}{2i-s+1}\binom{n-i-1}{2(n-i)-t-1}$$
$$=-\sum_i\left\{\binom{i-1}{2i-s}+\binom{i-1}{2i-s+1}\right\}\binom{n-i-1}{2(n-i)-t-1}$$
$$+\sum_i\binom{i-1}{2i-1-s}\binom{n-i}{2(n-i)+1-t}.$$

Also since $(i, 2i-s+1)\neq(0,0)$, we have

$$d(s,t)=-\sum_i\binom{i}{2i-s+1}\binom{n-i-1}{2(n-i)-t-1}+\sum_i\binom{i-1}{2i-1-s}\binom{n-i}{2(n-i)+1-t}$$
$$=-\sum_i\binom{i}{2i-s+1}\binom{n-i-1}{2(n-i)-t-1}$$
$$+\sum_i\binom{i-1}{2(i-1)-s+1}\binom{n-(i-1)-1}{2(n-(i-1))-t-1}$$
$$=0$$

Q.E.D.

Proof of (1.4). By Lemma A.2, we have

$$f(s,t)=f(1,t+s-1).$$

But

$$f(1,s+t-1)$$
$$=\sum_i\left\{\binom{i-1}{2i-1}\binom{n-i-1}{2(n-i)-s-t+1}+\binom{i-1}{2i-2}\binom{n-i}{2(n-i)+1-s-t+1}\right\}$$
$$=\binom{n-1}{2n-s-t},$$

since $\binom{i-1}{2i-1}=0$ for any i and $\binom{i-1}{2i-2}=0$ for $i\neq 1$. So the result follows.

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