On local isometric immersions of the spaces of negative constant curvature into the euclidean spaces

By

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Introduction

In his paper [9], T. Otsuki obtained an estimate for the lower bound of the dimensions of the euclidean spaces in which a space of negative constant curvature can be locally isometrically immersed. He proved that any space of negative constant curvature of dimension n cannot be isometrically immersed into the euclidean space $\mathbf{R}^{2n-2}$.

The main purpose of this paper is to show the following

Theorem 5.2. *Any space of negative constant curvature of dimension n can be locally isometrically immersed into the euclidean space $\mathbf{R}^{2n-1}$.*

We wish to prove this theorem by a method based on the theory of non-linear partial differential equations established by M. Kuranishi [8] and H. Goldschmidt [4].

Let (M, g) be an n -dimensional Riemannian manifold. Let P be the differential equation of isometric immersions of (M, g) into the euclidean space $\mathbf{R}^m$ with $m \geq n$, which is a system of non-linear partial differential equations of order 1. Adjoining to this system P the equation of Gauss-Weingarten which is obtained by differentiating the equation of isometric immersions, we obtain the system $P^{(1)}$ of order 2, the first prolongation of P . Similarly adjoining to the system $P^{(1)}$ the equation which is obtained by differentiating the equation of Gauss-Weingarten, we obtain the system of order 3, the second prolongation of P . A formal solution of order 1 can be always be extended to a formal solution of order 2, while a formal solution of order 2 cannot be necessarily extended to a formal solution of order 3. There exists an obstruction to extending the formal solutions of order 2 to the formal solutions of order 3, which is called the equation of Gauss.

Recently J. Gasqui [3] gave a new proof of the famous theorem of Janet-Cartan, showing that the system Q which is obtained by adjoining the equation of Gauss to the system $P^{(1)}$ forms an involutive system under the assumption $m \geq \frac{1}{2}n(n+1)$. Another proof of the theorem of Janet-Cartan from a somewhat different viewpoint was delivered by N. Tanaka in his lecture at Kyoto University before

Gasqui's paper was published (cf. [6]). He proved that the symbol q of the system Q is isomorphic to the second prolongation $\mathfrak{h}^{(2)}$ of the symbol $\mathfrak{h}$ of the linear operator L which was first introduced by himself (see [10]), and proved that if $m \geq \frac{1}{2}n(n+1)$, then at each generic point of Q the symbol $\mathfrak{h}$ is involutive. Here we remark that the vanishing of the Spencer cohomology group of the symbol $\mathfrak{h}$ plays an important role in his proof of the theorem of Janet-Cartan.

In this paper we need to investigate the system Q under the assumption $m < \frac{1}{2}n(n+1)$. Unfortunately the symbol $\mathfrak{h}$ cannot be involutive in this case. Hence the method developed by N. Tanaka cannot be applied to our problem. However, by letting (M, g) be a space of negative constant curvature, we can prove the following

Theorem 5.1. *If $m = 2n - 1$, then there exists an open fibered submanifold $\pi_1^2: P_\#^{(1)} \rightarrow P$ of the vector bundle $\pi_1^2: P^{(1)} \rightarrow P$ such that the intersection $Q_\# = Q \cap P^{(1)}$ forms an involutive differential equation.*

Theorem 5.2 cited above now follows from Theorem 5.1.

Following the formulations given by N. Tanaka, we recall in § 1 the differential equations $P, P^{(1)}$, etc. In § 2, we define the formal Gaussian variety with respect to a curvature like tensor. § 3 and § 4 are devoted to the proof of Theorem 3.1 that describes the properties of the formal Gaussian variety with respect to a curvature like tensor of negative constant curvature. Finally in § 5 we prove Theorem 5.1.

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§ 1. The differential equations associated with isometric immersions

Let M be an n -dimensional differentiable manifold^(*) and let $T = T(M)$ (resp. $T^* = T^*(M)$) be the tangent (resp. cotangent) bundle of M . By $\otimes^k T^*$ (resp. $S^k T^*$), we mean the bundle of k -tensors (resp. symmetric k -tensors) on M . Let g be a Riemannian metric on M . We denote by ∇ (resp. R) the covariant differentiation (resp. the curvature tensor) associated with the Riemannian connection on M induced from g .

Let R^m be the m -dimensional euclidean space with $m \geq n$ and $\langle \cdot, \cdot \rangle$ be the standard inner product of R^m . We denote by $\bar{g}$ the canonical Riemannian metric of R^m induced from $\langle \cdot, \cdot \rangle$.

By definition an isometric immersion f of the Riemannian manifold (M, g) into the euclidean space R^m is an immersion of M into R^m which is a solution of the equation

$$f^* \bar{g} = g,$$

where $f^* g$ stands for the Riemannian metric on M induced from g by f .

Let f be an isometric immersion of (M, g) into R^m . Then at each $p \in M$, we have the following equalities (cf. Proposition 2 of Appendix in [6]):

(*) Throughout this paper we shall assume the differentiability of class C^∞ .

$$(1.1) \quad \langle \nabla_x f, \nabla_y f \rangle = g(x, y),$$

$$(1.2) \quad \langle \nabla_z \nabla_x f, \nabla_y f \rangle = 0,$$

$$(1.3) \quad \langle \nabla_u \nabla_z \nabla_x f, \nabla_y f \rangle + \langle \nabla_z \nabla_x f, \nabla_u \nabla_y f \rangle = 0,$$

$$(1.4) \quad \langle \nabla_z \nabla_x f, \nabla_u \nabla_y f \rangle - \langle \nabla_u \nabla_x f, \nabla_z \nabla_y f \rangle = -g(R(z, u)x, y),$$

$$(1.5) \quad \begin{aligned} & \langle \nabla_v \nabla_z \nabla_x f, \nabla_u \nabla_y f \rangle + \langle \nabla_z \nabla_x f, \nabla_v \nabla_u \nabla_y f \rangle - \langle \nabla_v \nabla_u \nabla_x f, \nabla_z \nabla_y f \rangle \\ & - \langle \nabla_u \nabla_x f, \nabla_v \nabla_z \nabla_y f \rangle = -g(\nabla_v R(z, u)x, y) \quad \text{for } x, y, z, u, v \in T_p. \end{aligned}$$

Classically the equation (1.2) (resp. (1.4)) is called the equation of *Gauss-Weingarten* (resp. the equation of *Gauss*).

Let $J^k(M, m)$ be the vector bundle of all k -jets of local differentiable maps of M into $\mathbf{R}^m$. By π_{k-1}^k we mean the canonical projection of $J^k(M, m)$ onto $J^{k-1}(M, m)$ and by π_{-1}^k the source map of $J^k(M, m)$ onto M . As usual the set of formal solutions of order k are represented by a subvariety of $J^k(M, m)$. We denote by P the subvariety of $J^1(M, m)$ composed of all 1-jets satisfying the equation (1.1) and by $P^{(1)}$ the subvariety of $J^2(M, m)$ composed of all 2-jets satisfying the system of equations (1.1) and (1.2). We also denote by Q the subvariety of $J^2(M, m)$ composed of all 2-jets satisfying the system of equations (1.1), (1.2) and (1.4) and by $Q^{(1)}$ the subvariety of $J^3(M, m)$ composed of all 3-jets satisfying the system of equations (1.1)~(1.5). Note that $P^{(1)}$ (resp. $Q^{(1)}$) is the first prolongation of P (resp. Q) in the usual sense.

We now give the explicit expressions of the varieties P , $P^{(1)}$, Q and $Q^{(1)}$.

Let $\otimes^k T^* \otimes \mathbf{R}^m$ be the vector bundle of all $\mathbf{R}^m$ -valued k -tensors on M . Let us set $T^k(M, m) = \sum_{i=0}^k \otimes^i T^* \otimes \mathbf{R}^m$. We shall represent every element $\omega \in T^k(M, m)$ by $\omega = (p; \omega_0, \omega_1, \dots, \omega_k)$, where p is the origin of ω and $\omega_i \in \otimes^i T_p^* \otimes \mathbf{R}^m$ ($i=0, 1, \dots, k$). As in Appendix in [6], we shall consider the vector bundle $J^k(M, m)$ as a subbundle of the vector bundle $T^k(M, m)$. We have $J^0(M, m) = T^0(M, m)$ and $J^1(M, m) = T^1(M, m)$. The bundles $J^2(M, m)$ and $J^3(M, m)$ can be characterized as follows: $J^2(M, m)$ consists of all $(p; \omega_0, \omega_1, \omega_2) \in T^2(M, m)$ such that

$$\omega_2(x, y) = \omega_2(y, x) \quad \text{for any } x, y \in T_p.$$

$J^3(M, m)$ consists of all $(p; \omega_0, \omega_1, \omega_2, \omega_3) \in T^3(M, m)$ such that

$$\begin{aligned} & (p; \omega_0, \omega_1, \omega_2) \in J^2(M, m); \\ & \omega_3(x, y, z) = \omega_3(y, x, z) - \omega_1(R(x, y)z), \\ & \omega_3(x, y, z) = \omega_3(x, z, y) \quad \text{for any } x, y, z \in T_p. \end{aligned}$$

These being prepared, we give the explicit expressions of the varieties P , $P^{(1)}$, Q and $Q^{(1)}$.

The variety P is composed of all $(p; \omega_0, \omega_1) \in J^1(M, m)$ satisfying

$$(1.6) \quad \langle \omega_1(x), \omega_1(y) \rangle = g(x, y) \quad \text{for any } x, y \in T_p.$$

It is easily observed that P is a fibered submanifold of $\pi_{-1}^1: J^1(M, m) \rightarrow M$. Let $\alpha = (p; \omega_0, \omega_1) \in P$. We mean by N_α the orthogonal complement of the subspace $\omega_1(T_p)$ in $\mathbb{R}^m$. Then the union $N = \bigcup_{\alpha \in P} N_\alpha$ forms a vector bundle over P .

The variety $P^{(1)}$ is composed of all $(P; \omega_0, \omega_1, \omega_2) \in J^2(M, m)$ satisfying $(p; \omega_0, \omega_1) \in P$ and

$$(1.7) \quad \langle \omega_2(z, x), \omega_1(y) \rangle = 0 \quad \text{for any } x, y, z \in T_p.$$

We can easily see that $P^{(1)}$ forms a vector bundle over P which is isomorphic to the vector bundle $S^2 T^* \otimes_P N$.

Analogously the variety Q consists of all $(p; \omega_0, \omega_1, \omega_2) \in P^{(1)}$ satisfying

$$(1.8) \quad \langle \omega_2(z, x), \omega_2(u, y) \rangle - \langle \omega_2(u, x), \omega_2(z, y) \rangle = -g(R(z, u)x, y) \\ \text{for any } x, y, z, u \in T_p,$$

and the variety $Q^{(1)}$ is composed of all $(p; \omega_0, \omega_1, \omega_2, \omega_3) \in J^3(M, m)$ satisfying $(p; \omega_0, \omega_1, \omega_2) \in Q$ and

$$(1.9) \quad \langle \omega_3(u, z, x), \omega_1(y) \rangle + \langle \omega_2(z, x), \omega_2(u, y) \rangle = 0,$$

$$(1.10) \quad \langle \omega_3(v, z, x), \omega_2(u, y) \rangle + \langle \omega_2(z, x), \omega_3(v, u, y) \rangle - \langle \omega_3(v, u, x), \omega_2(z, y) \rangle \\ - \langle \omega_2(u, x), \omega_3(v, z, y) \rangle = -g(\nabla_v R(z, u)x, y) \quad \text{for any } x, y, z, u, v \in T_p.$$

In the subsequent sections we shall mainly concerned with the variety Q . In connection with the variety Q we make some definitions.

Let $\beta = (p; \omega_0, \omega_1, \omega_2) \in Q$. By definition the symbol q_β of the variety Q at β is the subspace of $S^2 T_p^* \otimes N_\alpha$ consisting of all $\xi \in S^2 T_p^* \otimes N_\alpha$ such that

$$(1.11) \quad \langle \xi(z, x), \omega_2(u, y) \rangle + \langle \omega_2(z, x), \xi(u, y) \rangle - \langle \xi(u, x), \omega_2(z, y) \rangle \\ - \langle \omega_2(u, x), \xi(z, y) \rangle = 0 \quad \text{for any } x, y, z, u \in T_p,$$

where we set $\alpha = \pi_1^2(\beta)$. We also denote by $q_\beta^{(1)}$ the *first prolongation* of the symbol q_β , i.e., $q_\beta^{(1)} = T^* \otimes q_\beta \cap S^3 T_p^* \otimes N_\alpha$.

§ 2. Formal Gaussian varieties

Let T be a finite dimensional real vector space and T^* be the dual vector space of T . By $\otimes^k T^*$ (resp. $S^k T^*$) we mean the vector space of covariant k -tensors (resp. symmetric k -tensors) of T .

By definition an element $C \in \otimes^4 T^*$ is called *curvature like* if it satisfies the following:

$$C(x, y, z, w) = -C(y, x, z, w) = -C(x, y, w, z), \\ C(x, y, z, w) + C(y, z, x, w) + C(z, x, y, w) = 0 \quad \text{for } x, y, z, w \in T.$$

We denote by $K(T)$ the vector space of all curvature like tensors.

Let N be another finite dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle$. For each $\alpha \in S^2 T^* \otimes N$, we denote by $\Omega(\alpha)$ the element in $K(T)$ defined by

$$\Omega(\alpha)(x, y, z, w) = \langle \alpha(x, z), \alpha(y, w) \rangle - \langle \alpha(x, w), \alpha(y, z) \rangle \quad \text{for } x, y, z, w \in T.$$

Let C be any element in $K(T)$. By $\mathcal{G}(C)$ we mean the inverse image of C by the map $S^2 T^* \otimes N \ni \alpha \rightarrow \Omega(\alpha) \in K(T)$. $\mathcal{G}(C)$ is called the *formal Gaussian variety* with respect to C .

Let $\alpha \in \mathcal{G}(C)$. Define a linear map $\Omega_{*\alpha}: S^2 T^* \otimes N \rightarrow K(T)$ by setting

$$\begin{aligned} \Omega_{*\alpha}(\beta)(x, y, z, w) &= \langle \beta(x, z), \alpha(y, w) \rangle + \langle \alpha(x, z), \beta(y, w) \rangle \\ &\quad - \langle \beta(x, w), \alpha(y, z) \rangle - \langle \alpha(x, w), \beta(y, z) \rangle \\ &\quad \text{for } \beta \in S^2 T^* \otimes N, x, y, z, w \in T. \end{aligned}$$

We denote by $\mathfrak{g}_\alpha$ the kernel of the map $\Omega_{*\alpha}$, which may be identified with the tangent space to the variety $\mathcal{G}(\Omega(\alpha))$ at α . We also denote by $\mathfrak{g}_\alpha^{(1)}$ the first prolongation of the subspace $\mathfrak{g}_\alpha$ of $S^2 T^* \otimes N$, i.e., $\mathfrak{g}_\alpha^{(1)} = T^* \otimes \mathfrak{g}_\alpha \cap S^3 T^* \otimes N$.

Let $\langle \cdot, \cdot \rangle$ be an inner product of T . For each $k \in \mathbb{R}$ let us define $C_k \in K(T)$ by

$$C_k(x, y, z, w) = k(\langle x, z \rangle \langle y, w \rangle - \langle x, w \rangle \langle y, z \rangle) \quad \text{for } x, y, z, w \in T.$$

We say that C_k is the curvature like tensor of constant curvature with sectional curvature k .

Let $O(T)$ (resp. $O(N)$) be the orthogonal group of T (resp. N) with respect to the given inner product $\langle \cdot, \cdot \rangle$ of T (resp. N) and $\mathfrak{o}(T)$ (resp. $\mathfrak{o}(N)$) be the Lie algebra of $O(T)$ (resp. $O(N)$).

Let $a \in O(N)$, $t \in O(T)$, $\chi \in S^1 T^* \otimes N$ and $C \in K(T)$. Let us define $a\chi^t \in S^1 T^* \otimes N$ and $C^t \in K(T)$ by setting

$$\begin{aligned} a\chi^t(x_1, \dots, x_t) &= a(\chi(tx_1, \dots, tx_t)) \\ C^t(x_1, x_2, x_3, x_4) &= C(tx_1, tx_2, tx_3, tx_4) \quad \text{for } x_1, \dots, x_4, \dots, x_t \in T. \end{aligned}$$

It is straightforward to see that if $\alpha \in \mathcal{G}(C)$, then $a\alpha^t \in \mathcal{G}(C^t)$. Similarly if $\beta \in \mathfrak{g}_\alpha$ (resp. $\gamma \in \mathfrak{g}_\alpha^{(1)}$), then $a\beta^t \in \mathfrak{g}_{a\alpha^t}$ (resp. $a\gamma^t \in \mathfrak{g}_{a\alpha^t}^{(1)}$).

Let us consider the case $C = C_k$. Since $C_k^t = C_k$ holds for any $t \in O(T)$, we have

Proposition 2.1. *The formal Gaussian variety $\mathcal{G}(C_k)$ is invariant under the action of the product group $O(N) \times O(T)$ on $S^2 T^* \otimes N$ defined by*

$$O(N) \times O(T) \times S^2 T^* \otimes N \ni ((a, t), \alpha) \rightarrow a\alpha^t \in S^2 T^* \otimes N.$$

Proposition 2.1 is useful in the consideration of the formal Gaussian variety $\mathcal{G}(C_k)$.

§ 3. The formal Gaussian variety $\mathcal{G}(C_k)$ with $k < 0$

In this and the next sections we shall investigate the formal Gaussian variety

$\mathcal{G}(C_k)$ with $k < 0$. Our main aim is to show

Theorem 3.1. *Assume that $\dim N = \dim T - 1 = n$ and $k < 0$. Then there exists an open set O in $S^2 T^* \otimes N$ such that:*

- (1) $\mathcal{G}(C_k) \cap O$ is an $n(n+1)$ -dimensional submanifold of $S^2 T^* \otimes N$.
- (2) For each $\alpha \in \mathcal{G}(C_k) \cap O$, there exists a vector $e \in T$ such that

$$e \lrcorner \beta \neq 0 \quad \text{for any } \beta \in \mathfrak{g}_\alpha, \beta \neq 0.$$

- (3) $\dim \mathfrak{g}_\alpha^{(1)} = n(n+1)$ for any $\alpha \in \mathcal{G}(C_k) \cap O$.

Remark. In order to prove the theorem we have only to prove it in the case where $k = -1$. In fact consider the linear endomorphism of $S^2 T^* \otimes N$ given by $S^2 T^* \otimes N \ni \alpha \rightarrow \sqrt{-k} \alpha \in S^2 T^* \otimes N$. It is clear that this endomorphism maps $\mathcal{G}(C_{-1})$ onto $\mathcal{G}(C_k)$. Moreover we have $\mathfrak{g}_\alpha = \mathfrak{g}_{\sqrt{-k}\alpha}$ and hence $\mathfrak{g}_\alpha^{(1)} = \mathfrak{g}_{\sqrt{-k}\alpha}^{(1)}$.

In the following we shall simply write $\mathcal{G}$ instead of $\mathcal{G}(C_{-1})$.

Let $\{e_a\}_{0 \leq a \leq n}$ (resp. $\{v_i\}_{1 \leq i \leq n}$) be an orthonormal basis of T (resp. N). Making use of these basis, we shall express N -valued covariant tensors of T in terms of their coefficients. Let $\chi \in \otimes^l T^* \otimes N$. Define an element $X = (X_{a_1, \dots, a_l}^k)_{\substack{0 \leq a_1, \dots, a_l \leq n \\ 1 \leq k \leq n}} \in \mathbf{R}^{n(n+1)^l}$ by

$$X_{a_1, \dots, a_l}^k = \langle \chi(e_{a_1}, \dots, e_{a_l}), v_k \rangle \quad \text{for } 0 \leq a_1, \dots, a_l \leq n, 1 \leq k \leq n.$$

Let $\alpha \in \mathcal{G}$ and let $A = (A_{ab}^k)_{\substack{0 \leq a, b \leq n \\ 1 \leq k \leq n}} \in \mathbf{R}^{n(n+1)^2}$ be the coefficients of α . Then we have

$$(3.1) \quad A_{ab}^k = A_{ba}^k,$$

$$(3.2) \quad \sum_{p=1}^n (A_{ac}^p A_{bd}^p - A_{ad}^p A_{bc}^p) = -(\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc})$$

for $0 \leq a, b, c, d \leq n, 1 \leq k \leq n$,

where δ means the Kronecker's delta. Conversely, it is clear that any $\alpha \in \otimes^2 T^* \otimes N$ whose coefficients $A = (A_{ab}^k)_{\substack{0 \leq a, b \leq n \\ 1 \leq k \leq n}}$ satisfy (3.1) and (3.2) is contained in $\mathcal{G}$.

Let $\alpha \in \mathcal{G}$. Assume that α satisfies the following

$$(\#) \quad \alpha(e_0, e_0) = 0, \quad \alpha(e_0, e_i) = v_i \quad \text{for } 1 \leq i \leq n.$$

Then we have $A_{00}^k = 0$, $A_{0i}^k = \delta_{ik}$ for $1 \leq i, k \leq n$. Hence by (3.1) and (3.2) we obtain

$$(3.3) \quad A_{ij}^k = A_{ji}^k = A_{ik}^j,$$

$$(3.4) \quad \sum_{p=1}^n (A_{ik}^p A_{jl}^p - A_{il}^p A_{jk}^p) = -(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk})$$

for $1 \leq i, j, k, l \leq n$.

Let us denote by $\mathcal{G}$ the set of all α satisfying $(\#)$ and by $\mathcal{G}_n$ the subvariety of $\mathbf{R}^{n^3}$ of all $A = (A_{ij}^k)_{1 \leq i, j, k \leq n} \in \mathbf{R}^{n^3}$ satisfying the system of equations (3.3) and (3.4). Clearly we can identify $\mathcal{G}$ and $\mathcal{G}_n$ in a natural way.

Let $A = (A_{ij}^k)_{1 \leq i, j, k \leq n} \in \bar{\mathcal{G}}_n$. By $(\bar{\mathfrak{g}}_n)_A$, we mean the vector space of all $B = (B_{ij}^k)_{1 \leq i, j, k \leq n} \in \mathbb{R}^{n^3}$ such that

$$(3.5) \quad B_{ij}^k = B_{ji}^k = B_{ik}^j,$$

$$(3.6) \quad \sum_{p=1}^n (B_{ik}^p A_{jl}^p + A_{ik}^p B_{jl}^p - B_{il}^p A_{jk}^p - A_{il}^p B_{jk}^p) = 0$$

for $1 \leq i, j, k, l \leq n$.

Naturally $(\bar{\mathfrak{g}}_n)_A$ may be identified with the tangent space to the subvariety $\bar{\mathcal{G}}_n$ at A . We also denote by $(\bar{\mathfrak{g}}_n^{(1)})_A$ the vector space of all $C = ({}^l C_{ij}^k)_{1 \leq i, j, k, l \leq n} \in \mathbb{R}^{n^4}$ such that

$$(3.7) \quad {}^l C_{ij}^k = {}^l C_{lj}^k$$

$$(3.8) \quad {}^l C_{ij}^k = {}^l C_{ji}^k = {}^l C_{ik}^j,$$

$$(3.9) \quad \sum_{p=1}^n ({}^q C_{ik}^p A_{jl}^p + A_{ik}^p {}^q C_{jl}^p - {}^q C_{il}^p A_{jk}^p - A_{il}^p {}^q C_{jk}^p) = 0$$

for $1 \leq i, j, k, l \leq n$.

We prove

Proposition 3.2. *There exists an open set U of $\mathbb{R}^{n^3}$ such that:*

- (1) $\bar{\mathcal{G}}_n \cap U$ is a $\frac{1}{2}n(n+1)$ -dimensional submanifold of $\mathbb{R}^{n^3}$.
- (2) For each $A \in \bar{\mathcal{G}}_n \cap U$, there exists a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ having the following property:

If $B = (B_{ij}^k)_{1 \leq i, j, k \leq n} \in (\bar{\mathfrak{g}}_n)_A$ and $B \neq 0$, then

$$\sum_{p=1}^n x_p B_{ij}^p \neq 0 \quad \text{for some } (i, j) \ (1 \leq i, j \leq n).$$

- (3) $\dim (\bar{\mathfrak{g}}_n^{(1)})_A = \frac{1}{2}n(n+1)$ for any $A \in \bar{\mathcal{G}}_n \cap U$.

Before proceeding to the proof of Proposition 3.2, we first note the following. Let $a = (a_i^j)_{1 \leq i, j \leq n}$ be an orthogonal matrix, i.e., $\sum_{k=1}^n a_k^i a_k^j = \delta_{ij}$ and let $X = (X_{ij}^k)_{1 \leq i, j, k \leq n} \in \mathbb{R}^{n^3}$ and $Y = ({}^l Y_{ij}^k)_{1 \leq i, j, k, l \leq n} \in \mathbb{R}^{n^4}$. Define $X^a = ((X^a)_{ij}^k)_{1 \leq i, j, k \leq n} \in \mathbb{R}^{n^3}$ and $Y^a = ({}^l (Y^a)_{ij}^k)_{1 \leq i, j, k, l \leq n} \in \mathbb{R}^{n^4}$ by setting

$$(X^a)_{ij}^k = \sum_{p, q, r=1}^n a_i^p a_j^q a_k^r X_{pq}^r,$$

$${}^l (Y^a)_{ij}^k = \sum_{p, q, r, s=1}^n a_i^p a_j^q a_k^r {}^s Y_{pq}^s \quad \text{for } 1 \leq i, j, k, l \leq n.$$

By simple calculations we know that if $A \in \bar{\mathcal{G}}_n$ then $A^a \in \bar{\mathcal{G}}_n$ and that if $B \in (\bar{\mathfrak{g}}_n)_A$ (resp. $C \in (\bar{\mathfrak{g}}_n^{(1)})_A$), then $B^a \in (\bar{\mathfrak{g}}_n)_{A^a}$ (resp. $C^a \in (\bar{\mathfrak{g}}_n^{(1)})_{A^a}$).

From now on let us assume that $n \geq 2$ and $\bar{\mathcal{G}}_{n-1} \neq \emptyset$. Let $\lambda \in \mathbb{R}_* = \mathbb{R} - \{0\}$ and let $\hat{A} = (\hat{A}_{ij}^k) \in \bar{\mathcal{G}}_{n-1}$. Define an element $\varphi(\lambda, \hat{A}) = (\varphi(\lambda, \hat{A})_{ij}^k)_{1 \leq i, j, k \leq n} \in \mathbb{R}^{n^3}$ by

$$\varphi(\lambda, \hat{A})_{ij}^k = \sqrt{1 + \lambda^2} \hat{A}_{ij}^k,$$

$$\begin{aligned}
 (*) \quad & \varphi(\lambda, \hat{A})_{ij}^n = \varphi(\lambda, \hat{A})_{in}^j = \varphi(\lambda, \hat{A})_{ni}^j = \lambda \delta_{ij}, \\
 & \varphi(\lambda, \hat{A})_{nn}^i = \varphi(\lambda, \hat{A})_{ni}^n = \varphi(\lambda, \hat{A})_{in}^n = 0, \quad \varphi(\lambda, \hat{A})_{nn}^n = \lambda - 1/\lambda \\
 & \text{for } 1 \leq i, j, k \leq n-1.
 \end{aligned}$$

It is easily observed that $\varphi(\lambda, \hat{A}) \in \bar{\mathcal{G}}_n$. We now define a map $\Phi: \mathbf{R}_* \times \bar{\mathcal{G}}_{n-1} \times \mathbf{R}^{n-1} \rightarrow \mathbf{R}^{n^3}$ by

$$\Phi(\lambda, \hat{A}, \xi) = \varphi(\lambda, \hat{A})^{\exp \xi} \quad \text{for } (\lambda, \hat{A}, \xi) \in \mathbf{R}_* \times \bar{\mathcal{G}}_{n-1} \times \mathbf{R}^{n-1},$$

where, for $\xi = (\xi_1, \dots, \xi_{n-1}) \in \mathbf{R}^{n-1}$, we mean by ξ the skew symmetric matrix of the form:

$$\xi = \begin{pmatrix} 0 & -\xi_1 & & \\ & \ddots & & \\ & & 0 & \\ \xi_1 & \dots & \xi_{n-1} & 0 \end{pmatrix}.$$

Since $\varphi(\lambda, \hat{A}) \in \bar{\mathcal{G}}_n$, we have $\Phi(\lambda, \hat{A}, \xi) \in \bar{\mathcal{G}}_n$. Let $(\lambda, \hat{A}) \in \mathbf{R}_* \times \bar{\mathcal{G}}_{n-1}$. We denote by $\Phi_* = \Phi_{*(\lambda, \hat{A}, 0)}$ the differential of the map Φ at $(\lambda, \hat{A}, 0)$, where 0 means the zero vector $(0, \dots, 0)$ in $\mathbf{R}^{n-1}$. In a natural way, the tangent space the variety $\mathbf{R}_* \times \bar{\mathcal{G}}_{n-1} \times \mathbf{R}^{n-1}$ at $(\lambda, \hat{A}, 0)$ may be identified with the direct sum $\mathbf{R} \oplus (\bar{\mathfrak{g}}_{n-1})_{\hat{A}} \oplus \mathbf{R}^{n-1}$. Hence Φ_* induces a linear map of $\mathbf{R} \oplus (\bar{\mathfrak{g}}_{n-1})_{\hat{A}} \oplus \mathbf{R}^{n-1}$ into $\mathbf{R}^{n^3}$. We also denote it by Φ_* .

Let $\mu \in \mathbf{R}$, $\hat{B} = (\hat{B}_{ij}^k)_{1 \leq i, j, k \leq n-1} \in (\bar{\mathfrak{g}}_{n-1})_{\hat{A}}$ and $\xi = (\xi_1, \dots, \xi_{n-1}) \in \mathbf{R}^{n-1}$. We note that the vectors $\Phi_*(\mu)$, $\Phi_*(\hat{B})$ and $\Phi_*(\xi)$ in $\mathbf{R}^{n^3}$ are necessarily contained in $(\bar{\mathfrak{g}}_n)_{\varphi(\lambda, \hat{A})}$. By using the coefficients they are explicitly represented as follows:

$$\begin{aligned}
 (**) \quad & \Phi_*(\mu)_{ij}^k = (\lambda/\sqrt{1+\lambda^2})\mu \hat{A}_{ij}^k, \quad \Phi_*(\mu)_{ij}^n = \mu \delta_{ij}, \\
 & \Phi_*(\mu)_{in}^n = 0, \quad \Phi_*(\mu)_{nn}^n = (1/\lambda^2)(1+\lambda^2)\mu; \\
 & \Phi_*(\hat{B})_{ij}^k = \sqrt{1+\lambda^2} \hat{B}_{ij}^k, \\
 & \Phi_*(\hat{B})_{ij}^n = \Phi_*(\hat{B})_{in}^n = \Phi_*(\hat{B})_{nn}^n = 0; \\
 & \Phi_*(\xi)_{ij}^k = \lambda(\xi_i \delta_{jk} + \xi_j \delta_{ki} + \xi_k \delta_{ij}), \\
 & \Phi_*(\xi)_{ij}^n = -\sqrt{1+\lambda^2} (\sum_{p=1}^{n-1} \xi_p \hat{A}_{ij}^p), \\
 & \Phi_*(\xi)_{in}^n = -(1/\lambda)(1+\lambda^2)\xi_i, \quad \Phi_*(\xi)_{nn}^n = 0 \quad \text{for } 1 \leq i, j, k \leq n-1.
 \end{aligned}$$

Lemma 3.3. *For each $B \in (\bar{\mathfrak{g}}_n)_{\varphi(\lambda, \hat{A})}$, there exists a unique $(\mu, \hat{B}, \xi) \in \mathbf{R} \oplus (\bar{\mathfrak{g}}_{n-1})_{\hat{A}} \oplus \mathbf{R}^{n-1}$ such that*

$$B = \Phi_*(\mu) + \Phi_*(\hat{B}) + \Phi_*(\xi).$$

Proof. We first suppose that $\Phi_*(\mu) + \Phi_*(\hat{B}) + \Phi_*(\xi) = 0$ for some $(\mu, \hat{B}, \xi) \in \mathbf{R} \oplus (\bar{\mathfrak{g}}_{n-1})_{\hat{A}} \oplus \mathbf{R}^{n-1}$. Then by (**) we can easily obtain $\mu = \hat{B} = \xi = 0$. This implies the uniqueness. Next we show the decomposition. Let $B = (B_{ij}^k)_{1 \leq i, j, k \leq n} \in (\bar{\mathfrak{g}}_n)_{\varphi(\lambda, \hat{A})}$. Take $\mu \in \mathbf{R}$ and $\xi = (\xi_1, \dots, \xi_{n-1}) \in \mathbf{R}^{n-1}$ so that $B_{nn}^n = (1/\lambda^2)(1+\lambda^2)\mu$, $B_{in}^n = -(1/\lambda)(1+\lambda^2)\xi_i$ for $1 \leq i \leq n$. We set $\bar{B} = B - (\Phi_*(\mu) + \Phi_*(\xi))$. Then $\bar{B} \in (\bar{\mathfrak{g}}_n)_{\varphi(\lambda, \hat{A})}$, $\bar{B}_{nn}^n = 0$ and $\bar{B}_{in}^n$

$=0$ for $1 \leq i \leq n$. Substituting $B = \bar{B}$ into (3.6), we have $\bar{B}_{ij}^n = 0$ for $1 \leq i, j \leq n-1$ and

$$\sum_{p=1}^n (\bar{B}_{ik}^p \hat{A}_{jl}^p + \hat{A}_{ik}^p \bar{B}_{jl}^p - \bar{B}_{il}^p \hat{A}_{jk}^p - \hat{A}_{il}^p \bar{B}_{jk}^p) = 0$$

for $1 \leq i, j, k, l \leq n-1$.

Thus if we set $\hat{B}_{ij}^k = (1/\sqrt{1+\lambda^2})\bar{B}_{ij}^k$ for $1 \leq i, j, k \leq n-1$, then $\hat{B} = (\hat{B}_{ij}^k)_{1 \leq i, j, k \leq n-1} \in (\bar{\mathfrak{g}}_{n-1})_{\hat{A}}$ and $\bar{B} = \Phi_*(\hat{B})$. Hence we have $B = \Phi_*(\mu) + \Phi_*(\hat{B}) + \Phi_*(\xi)$. Q.E.D.

Proof of Proposition 3.2. We proceed by induction on n . It is easy to see that the proposition holds for $n=1$. We now assume that the proposition holds $n-1$ with $n \geq 2$. Then there exists an open set $\hat{U}$ in $\mathbf{R}^{(n-1)^3}$ such that:

(1') $\mathcal{G}_{n-1} \cap \hat{U}$ is a $\frac{1}{2}n(n-1)$ -dimensional submanifold of $\mathbf{R}^{(n-1)^3}$.

(2') For each $\hat{A} \in \mathcal{G}_{n-1} \cap \hat{U}$, there exists a vector $x = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$ having the following property:

If $\hat{B} = (\hat{B}_{ij}^k)_{1 \leq i, j, k \leq n-1} \in (\bar{\mathfrak{g}}_{n-1})_{\hat{A}}$ and $B \neq 0$, then

$$\sum_{p=1}^{n-1} x_p B_{ij}^k \neq 0 \quad \text{for some } (i, j) \text{ } (1 \leq i, j \leq n-1).$$

(3') $\dim(\bar{\mathfrak{g}}_{n-1})_{\hat{A}} = \frac{1}{2}n(n-1)$ for any $\hat{A} \in \mathcal{G}_{n-1} \cap \hat{U}$.

Let us take any $\lambda_0 \in \mathbf{R}_*$, $\hat{A}_0 \in \mathcal{G}_{n-1} \cap \hat{U}$ and a sufficiently small open set $\tilde{U}$ in $\mathbf{R}_* \times (\mathcal{G}_{n-1} \cap \hat{U}) \times \mathbf{R}^{n-1}$ containing $(\lambda_0, \hat{A}_0, 0)$. In view of Lemma 3.3, we know that the differential Φ_* of at $(\lambda_0, \hat{A}_0, 0)$ is injective. Hence we may assume that the restriction $\Phi|_{\tilde{U}}: \tilde{U} \rightarrow \mathbf{R}^{n^3}$ of the map Φ to $\tilde{U}$ is an imbedding. Because of the assumption (1'), the image $\Phi(\tilde{U})$ of $\tilde{U}$ forms a $\frac{1}{2}n(n-1)$ -dimensional submanifold of $\mathbf{R}^{n^3}$. Moreover we know $(\bar{\mathfrak{g}}_n)_{\varphi(\lambda_0, \hat{A}_0)} = T_{\varphi(\lambda_0, \hat{A}_0)}\Phi(\tilde{U})$. We now notice the following

Lemma 3.4. Let $f_1, \dots, f_r$ be differentiable functions on a manifold M and let $\mathcal{V}$ be the subvariety of M defined by the system of equations $f_1 = \dots = f_r = 0$. Assume that there is a submanifold S of M and a point $p \in S$ such that

(i) $S \subset \mathcal{V}$;

(ii) $T_p(S) = \{x \in T_p(M) \mid df_1(x) = \dots = df_r(x) = 0\}$.

Then there exists a neighborhood U of p in M such that $\mathcal{V} \cap U = S \cap U$.

The proof of the lemma is left to the readers.

By virtue of Lemma 3.4, we know that there exists an open neighborhood U of $\varphi(\lambda_0, \hat{A}_0)$ in $\mathbf{R}^{n^3}$ such that $\mathcal{G}_n \cap U = \Phi(\tilde{U}) \cap U$. This implies (1) of the proposition.

Next we show (2). Let $A \in \mathcal{G}_n \cap U$. Since $A \in \Phi(\tilde{U})$, we may write $A = \varphi(\lambda, \hat{A})^{\exp \xi}$ by using a suitable $(\lambda, \hat{A}, \xi) \in \tilde{U}$. Hence, in order to show (2), it suffices to deal with the case $A = \varphi(\lambda, \hat{A})$. Let $\hat{x} = (\hat{x}_1, \dots, \hat{x}_{n-1}) \in \mathbf{R}^{n-1}$ be the vector stated in (2'). We may assume that $|\hat{x}_i| \ll 1$ for $1 \leq i \leq n-1$. We define $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ by $x_i = \hat{x}_i$ for $1 \leq i \leq n-1$ and $x_n = 1$. Let us show the vector x defined above has the property stated in (2). Let $B = (B_{pq}^r)_{1 \leq p, q, r \leq n} \in (\bar{\mathfrak{g}}_n)_{\varphi(\lambda, \hat{A})}$ and suppose $\sum_{r=1}^n x_r B_{pq}^r = 0$ for any (p, q) ($1 \leq p, q \leq n$). By using the decomposition $B = \Phi_*(\mu) + \Phi_*(\hat{B}) + \Phi_*(\xi)$ in

Lemma 3.3 and the formulas (**) for $\Phi_*(\mu)$, $\Phi_*(\hat{B})$, $\Phi_*(\xi)$, we have

$$(3.10) \quad \sum_{k=1}^{n-1} \{ \lambda(\hat{x}_j \delta_{ik} + \hat{x}_i \delta_{jk} + \hat{x}_k \delta_{ij}) - \sqrt{1+\lambda^2} \hat{A}_{ij}^k \} \xi_k \\ + \left(\delta_{ij} + \frac{\lambda}{\sqrt{1+\lambda^2}} \sum_{k=1}^{n-1} \hat{x}_k \hat{A}_{ij}^k \right) \mu + \sqrt{1+\lambda^2} \left(\sum_{k=1}^{n-1} \hat{x}_k \hat{B}_{ij}^k \right) = 0 \\ \text{for } 1 \leq i, j \leq n-1;$$

$$(3.11) \quad -\frac{1}{\lambda}(1+\lambda^2) \sum_{j=1}^{n-1} \left(\delta_{ij} + \frac{\lambda}{\sqrt{1+\lambda^2}} \sum_{j=1}^{n-1} \hat{x}_k \hat{A}_{ij}^k \right) \xi_j + \hat{x}_i \mu = 0 \\ \text{for } 1 \leq i \leq n-1;$$

$$(3.12) \quad \mu - \lambda \sum_{k=1}^{n-1} \hat{x}_k \xi_k = 0.$$

Since $|\hat{x}_i| \ll 1$, it follows from (3.11) and (3.12) that $\mu = \xi_1 = \dots = \xi_{n-1} = 0$. Hence by (3.10) we have $\sum_{k=1}^{n-1} \hat{x}_k \hat{B}_{ij}^k = 0$ for any $1 \leq i, j \leq n-1$. Therefore by the assumption (2'), we obtain $\hat{B} = 0$. This completes the proof of (2).

Finally we show (3) of the proposition. As in the proof of (2), we may assume $A = \varphi(\lambda, \hat{A})$. Let $C = ({}^s C_{pq}^r)_{1 \leq p, q, r, s \leq n} \in (\bar{\mathfrak{g}}_n^{(1)})_{\varphi(\lambda, \hat{A})}$. For each s ($1 \leq s \leq n$), we denote by ${}^s C$ the element of R^{n^3} given by ${}^s C = ({}^s C_{pq}^r)_{1 \leq p, q, r \leq n}$. Clearly we have ${}^s C \in (\bar{\mathfrak{g}}_n)_{\varphi(\lambda, \hat{A})}$ for $1 \leq s \leq n$. By Lemma 3.3, there exist ${}^s \mu \in R$, ${}^s \hat{B} \in (\bar{\mathfrak{g}}_{n-1})_{\hat{A}}$ and ${}^s \xi \in R^{n-1}$ such that

$${}^s C = \Phi_*({}^s \mu) + \Phi_*({}^s \hat{B}) + \Phi_*({}^s \xi).$$

Since ${}^s C_{pq}^r = {}^r C_{sq}^r$, we have

$$(3.13) \quad {}^i \mu = -\lambda {}^n \xi_i \quad \text{for } 1 \leq i \leq n-1;$$

$$(3.14) \quad {}^j \xi_i = \frac{\lambda}{\sqrt{1+\lambda^2}} \left(\sum_{l=1}^{n-1} {}^n \xi_l \hat{A}_{ij}^l - \frac{1}{\sqrt{1+\lambda^2}} {}^n \mu \delta_{ij} \right) \quad \text{for } 1 \leq i, j \leq n-1;$$

$$(3.15) \quad {}^n \hat{B}_{ij}^k = -\frac{\lambda}{\sqrt{1+\lambda^2}} (2 {}^n \xi_i \delta_{jk} + {}^n \xi_j \delta_{ik} + {}^n \xi_k \delta_{ij}) - \frac{\lambda}{\sqrt{1+\lambda^2}} \left(\sum_{p, q=1}^{n-1} {}^n \xi_q \hat{A}_{iq}^p \hat{A}_{jk}^p \right) \\ \text{for } 1 \leq i, j, k \leq n-1;$$

$$(3.16) \quad {}^i \hat{B}_{ij}^k - {}^i \hat{B}_{lj}^k = \frac{\lambda}{\sqrt{1+\lambda^2}} ({}^i \xi_j \delta_{lk} + {}^i \xi_k \delta_{lj} - {}^l \xi_j \delta_{lk} - {}^l \xi_k \delta_{lj}) \\ + \frac{\lambda}{1+\lambda^2} ({}^i \mu \hat{A}_{ij}^k - {}^l \mu \hat{A}_{lj}^k) \quad \text{for } 1 \leq i, j, k, l \leq n-1.$$

Here we set

$$(3.17) \quad {}^i_0 \hat{B}_{ij}^k = -\frac{\lambda}{\sqrt{1+\lambda^2}} (2 {}^i \xi_i \delta_{jk} + {}^i \xi_j \delta_{ik} + {}^i \xi_k \delta_{ij}) \\ - \frac{\lambda}{\sqrt{1+\lambda^2}} \left(\sum_{p, q=1}^{n-1} {}^i \xi_p \hat{A}_{ip}^q \hat{A}_{jk}^q \right) \quad \text{for } 1 \leq i, j, k, l \leq n-1.$$

Then by (3.13) and (3.14) we have ${}^l_0\hat{B} = ({}^l_0\hat{B}_{ij}{}^k)_{1 \leq i, j, k \leq n-1} \in (\bar{\mathfrak{g}}_{n-1})_{\hat{A}}$ for $1 \leq l \leq n-1$ and ${}^l\hat{B}_{ij}{}^k - {}^l_0\hat{B}_{ij}{}^k = {}^l\hat{B}_{ij}{}^k - {}^l_0\hat{B}_{ij}{}^k$ for $1 \leq i, j, k, l \leq n-1$. Hence if we set

$$(3.18) \quad {}^l\hat{C}_{ij}{}^k = {}^l\hat{B}_{ij}{}^k - {}^l_0\hat{B}_{ij}{}^k \quad \text{for } 1 \leq i, j, k, l \leq n-1,$$

then $\hat{C} = ({}^l\hat{C}_{ij}{}^k)_{1 \leq i, j, k, l \leq n-1} \in (\bar{\mathfrak{g}}_{n-1}^{(1)})_{\hat{A}}$. By these arguments we know that any $C \in (\bar{\mathfrak{g}}_n^{(1)})_{\varphi(\lambda, \hat{A})}$ can be completely determined by ${}^n\eta \in \mathbf{R}$, $\hat{C} \in (\bar{\mathfrak{g}}_{n-1}^{(1)})_{\hat{A}}$ and ${}^n\xi = ({}^n\xi_1, \dots, {}^n\xi_{n-1}) \in \mathbf{R}^{n-1}$. Conversely it is clear that these variables are independent. Hence we have

$$\dim (\bar{\mathfrak{g}}_n^{(1)})_{\varphi(\lambda, \hat{A})} = 1 + \frac{1}{2}n(n-1) + n-1 = \frac{1}{2}n(n+1).$$

This completes the proof of the proposition.

Q.E.D.

§ 4. The proof of Theorem 3.1

As in the previous section, we shall fix an orthonormal basis $\{e_a\}_{0 \leq a \leq n}$ (resp. $\{v_i\}_{1 \leq i \leq n}$) of T (resp. N). As usual every $\sigma \in \mathfrak{o}(T)$ (resp. $\rho \in \mathfrak{o}(N)$) can be represented by a skew symmetric matrix $(\sigma_a{}^b)_{0 \leq a, b \leq n}$ (resp. $(\rho_i{}^j)_{1 \leq i, j \leq n}$) with respect to $\{e_a\}$ (resp. $\{v_i\}$).

Let $A \in \bar{\mathcal{G}}_n$. By α_A , we mean the element of $\hat{\mathcal{G}}$ corresponding to A . Let us define a map $\psi: \mathfrak{o}(N) \times \bar{\mathcal{G}}_n \times \mathbf{R}^n \rightarrow S^2T^* \otimes N$ by

$$\psi(\rho, A, \tau) = \exp \rho \alpha_A^{\exp \tau} \quad \text{for } (\rho, A, \tau) \in \mathfrak{o}(N) \times \bar{\mathcal{G}}_n \times \mathbf{R}^n,$$

where for each $\tau = (\tau_1, \dots, \tau_n)$ we denote by $\hat{\tau}$ the element of $\mathfrak{o}(T)$ such that

$$\hat{\tau} = \begin{pmatrix} 0, & \tau_1, & \dots, & \tau_n \\ -\tau_1 & & & \\ \vdots & & 0 & \\ -\tau_n & & & \end{pmatrix}.$$

Note that since $\alpha_A \in \hat{\mathcal{G}}$, we have $\psi(\rho, A, \tau) \in \mathcal{G}$. Let us denote by $\psi_* = \psi_{*(0, A, 0)}$ the differential of ψ at $(0, A, 0) \in \mathfrak{o}(N) \times \bar{\mathcal{G}}_n \times \mathbf{R}^n$, where 0 means the zero matrix of degree n or the vector $(0, \dots, 0) \in \mathbf{R}^n$. Naturally the tangent space to the variety $\mathfrak{o}(N) \times \bar{\mathcal{G}}_n \times \mathbf{R}^n$ at $(0, A, 0)$ may be identified with the direct sum $\mathfrak{o}(N) \oplus (\bar{\mathfrak{g}}_n)_A \oplus \mathbf{R}^n$. Hence ψ_* induces a linear map of $\mathfrak{o}(N) \oplus (\bar{\mathfrak{g}}_n)_A \oplus \mathbf{R}^n$ into $S^2T^* \otimes N$. We also denote it by ψ_* .

Let $\rho = (\rho_i{}^j)_{1 \leq i, j \leq n} \in \mathfrak{o}(N)$, $B = (B_{ij}{}^k)_{1 \leq i, j, k \leq n} \in (\bar{\mathfrak{g}}_n)_A$, and $\tau = (\tau_1, \dots, \tau_n) \in \mathbf{R}^n$. It is noted that the vectors $\psi_*(\rho)$, $\psi_*(B)$ and $\psi_*(\tau)$ in $S^2T^* \otimes N$ are necessarily contained in $\mathfrak{g}_{\alpha_A}$. In terms of coefficients they are represented as follows:

$$\begin{aligned} \psi_*(\rho)_{00}{}^k &= 0, \quad \psi_*(\rho)_{0i}{}^k = \delta_{ik}, \quad \psi_*(\rho)_{ij}{}^k = \sum_{l=1}^n \rho_i{}^l A_{lj}{}^k; \\ (***) \quad \psi_*(B)_{00}{}^k &= \psi_*(B)_{0i}{}^k = 0, \quad \psi_*(B)_{ij}{}^k = B_{ij}{}^k; \\ \psi_*(\tau)_{00}{}^k &= -2\tau_k, \quad \psi_*(\tau)_{0i}{}^k = -\sum_{l=1}^n \tau_l A_{li}{}^k, \quad \psi_*(\tau)_{ij}{}^k = \tau_i \delta_{jk} + \tau_j \delta_{ik} \\ &\quad \text{for } 1 \leq i, j, k \leq n. \end{aligned}$$

Lemma 4.1. *For each $\beta \in \mathfrak{g}_{\alpha_A}$, there exists a unique $(\rho, B, \tau) \in \mathfrak{o}(N) \oplus (\bar{\mathfrak{g}}_n)_A \oplus \mathbb{R}^n$ such that*

$$\beta = \psi_*(\rho) + \psi_*(B) + \psi_*(\tau).$$

Proof. We first assume that $\psi_*(\rho) + \psi_*(B) + \psi_*(\tau) = 0$ for some $(\rho, B, \tau) \in \mathfrak{o}(N) \oplus (\bar{\mathfrak{g}}_n)_A \oplus \mathbb{R}^n$. Then by (**), we immediately have $\rho = B = \tau = 0$. This proves the uniqueness. We next show the decomposition. Let $\beta \in \mathfrak{g}_{\alpha_A}$ and let $\bar{B} = (\bar{B}_{ab}^k)_{\substack{0 \leq a, b \leq n \\ 1 \leq k \leq n}} \in \mathbb{R}^{n(n+1)^2}$ be the coefficients of β . Take $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n$ so that $\tau_k = -\frac{1}{2}\bar{B}_{00}^k$ for $1 \leq k \leq n$. We set $\bar{\beta} = \beta - \psi_*(\tau)$. Since $\bar{\beta} \in \mathfrak{g}_{\alpha_A}$, we have $\Omega_{*\alpha_A}(\bar{\beta}) = 0$, i.e.,

$$(4.1) \quad \begin{aligned} &\langle \bar{\beta}(x, z), \alpha_A(y, w) \rangle + \langle \alpha_A(x, z), \bar{\beta}(y, w) \rangle - \langle \bar{\beta}(x, w), \alpha_A(y, z) \rangle \\ &\quad - \langle \alpha_A(x, w), \bar{\beta}(y, z) \rangle = 0 \quad \text{for } x, y, z, w \in T. \end{aligned}$$

Let $(\bar{B}_{ab}^k)_{\substack{0 \leq a, b \leq n \\ 1 \leq k \leq n}}$ be the coefficients of $\bar{\beta}$. By the choice of τ , we have $\bar{B}_{00}^k = 0$ for $1 \leq i \leq n$. Thus if we put $x = e_0, y = e_i, z = e_0, w = e_j$ ($1 \leq i, j \leq n$) into (4.1), then we obtain $\bar{B}_{0i}^j + \bar{B}_{0j}^i = 0$. Let us set $\rho = (\bar{B}_{0i}^j)_{1 \leq i, j \leq n}$ and set $\dot{\beta} = \bar{\beta} - \psi_*(\rho) = \beta - (\psi_*(\tau) + \psi_*(\rho))$. Since $\rho \in \mathfrak{o}(N)$, we have $\dot{\beta} \in \mathfrak{g}_{\alpha_A}$. Hence

$$(4.2) \quad \begin{aligned} &\langle \dot{\beta}(x, z), \alpha_A(y, w) \rangle + \langle \alpha_A(x, z), \dot{\beta}(y, w) \rangle - \langle \dot{\beta}(x, w), \alpha_A(y, z) \rangle \\ &\quad - \langle \alpha_A(x, w), \dot{\beta}(y, z) \rangle = 0 \quad \text{for } x, y, z, w \in T. \end{aligned}$$

Let $\dot{B} = (\dot{B}_{ab}^k)_{\substack{0 \leq a, b \leq n \\ 1 \leq k \leq n}}$ be the coefficients of $\dot{\beta}$. Then we have $\dot{B}_{00}^k = \dot{B}_{0i}^k = 0$ for $1 \leq i, k \leq n$. By this relation, the equation (4.2) may be reduced to the system of equations:

$$\begin{aligned} \dot{B}_{ij}^k &= \dot{B}_{ik}^j; \quad \sum_{p=1}^n (\dot{B}_{ik}^p A_{jl}^p + A_{ik}^p \dot{B}_{jl}^p - \dot{B}_{il}^p A_{jk}^p - A_{il}^p \dot{B}_{jk}^p) = 0 \\ &\quad \text{for } 1 \leq i, j, k, l \leq n. \end{aligned}$$

Thus if we set $B = (\dot{B}_{ij}^k)_{1 \leq i, j, k \leq n}$, we have $B \in (\bar{\mathfrak{g}}_n)_A$ and $\dot{\beta} = \psi_*(B)$. Hence we have $\beta = \psi_*(\rho) + \psi_*(B) + \psi_*(\tau)$. Q.E.D.

These being prepared, we start the proof of Theorem 3.1. Let U be the open set in $\mathbb{R}^{n^3}$ stated in Proposition 3.2. Let $A_0 \in \bar{\mathcal{G}}_n \cap U$. By Lemma 4.1, we know that the differential ψ_* of ψ at $(0, A_0, 0)$ is injective. Taking a sufficiently small open neighborhood $\tilde{O}$ of $(0, A_0, 0)$ in $\mathfrak{o}(N) \times (\bar{\mathcal{G}}_n \cap U) \times \mathbb{R}^n$, we may assume that the restriction of the map ψ to $\tilde{O}$ is an imbedding. Therefore the image $\psi(\tilde{O})$ of $\tilde{O}$ forms an $n(n-1)$ -dimensional submanifold of $S^2 T^* \otimes N$. Note that $\bar{\mathcal{G}}_n \cap U$ is a $\frac{1}{2}n(n-1)$ dimensional submanifold of $\mathbb{R}^{n^3}$. Moreover we know $\mathfrak{g}_{\alpha_{A_0}} = T_{\alpha_{A_0}} \psi(\tilde{O})$. Hence by Lemma 3.4, there exists an open neighborhood O of α_{A_0} in $S^2 T^* \otimes N$ such that $\mathcal{G} \cap O = \psi(\tilde{O}) \cap O$. This shows (1) of the theorem.

Next we show (2) of the theorem. Let $\alpha \in \mathcal{G} \cap O$. Since $\alpha \in \psi(\tilde{O})$, we may write $\alpha = \psi(\rho, A, \tau)$ by using suitable $(\rho, A, \tau) \in \mathfrak{o}(N) \times (\bar{\mathcal{G}}_n \cap U) \times \mathbb{R}^n$. Hence in order to show (2), it suffices to deal with the case where $\alpha = \psi(0, A, 0) = \alpha_A$. Since $A \in \bar{\mathcal{G}}_n \cap U$ there exists a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ satisfying (2) of Proposition 3.2. Here we may assume that $|x_i| \ll 1$ for $1 \leq i \leq n$. We set $e = e_0 + \sum_{i=1}^n x_i e_i$. Then

we can see that the vector e satisfies the condition in (2). In fact, let us suppose $e \lrcorner \beta = 0$ for some $\beta \in \mathfrak{g}_{\alpha_A}$. Then by using the decomposition $\beta = \psi_*(\rho) + \psi_*(B) + \psi_*(\tau)$ in Lemma 4.1. and the formulas (**), we have

$$(4.3) \quad -2\tau_k + \sum_{i=1}^n x_i \left(\rho_i^k - \sum_{j=1}^n \tau_j A_{ji}^k \right) = 0 \quad \text{for } 1 \leq k \leq n;$$

$$(4.4) \quad \begin{aligned} & \rho_i^k - \sum_{j=1}^n \tau_j A_{ji}^k + \sum_{j=1}^n x_j \left(\tau_j \delta_{ik} + \tau_i \delta_{jk} + \sum_{l=1}^n \rho_l^k A_{jl}^l \right) \\ & + \sum_{j=1}^n x_j B_{ji}^k = 0 \quad \text{for } 1 \leq i, k \leq n. \end{aligned}$$

The equation (4.3) and the skew symmetric part of (4.4) with respect to the pair (i, k) form a system of homogeneous linear equations with variables τ_i, ρ_j^k ($1 \leq i, j, k \leq n$). Since $|x_i| \ll 1$ it follows that $\tau_i = \rho_j^k = 0$ for $1 \leq i, j, k \leq n$. Hence we have $\sum_{j=1}^n x_j B_{ji}^k = 0$ for any $1 \leq i, k \leq n$. Therefore we obtain $B_{ji}^k = 0$ for any $1 \leq i, j, k \leq n$. This shows $\beta = 0$.

Finally we show (3) of the theorem. As in the proof of (2), we may assume that $\alpha = \alpha_A$. Let $\gamma \in \mathfrak{g}_{\alpha_A}^{(1)}$. Since $e_s \lrcorner \gamma \in \mathfrak{g}_{\alpha_A}$ for $0 \leq s \leq n$, there are ${}^s\rho = ({}^s\rho_i^j)_{1 \leq i, j \leq n} \in \mathfrak{o}(N)$, ${}^sB = ({}^sB_{ij}^k)_{1 \leq i, j, k \leq n} \in (\mathfrak{g}_n)_A$ and ${}^s\tau = ({}^s\tau_1, \dots, {}^s\tau_n) \in \mathbb{R}^n$ such that $e_s \lrcorner \gamma = \Psi_*({}^s\rho) + \Psi_*({}^sB) + \Psi_*({}^s\tau)$. Then, by the relation $e_t \lrcorner e_s \lrcorner \gamma = e_t \lrcorner e_s \lrcorner \gamma$ for $1 \leq s, t \leq n$, we have

$$(4.5) \quad {}^t\tau_k = \frac{1}{2} {}^0\rho_k^i + \frac{1}{2} \sum_{j=1}^n {}^0\tau_j A_{jk}^i \quad \text{for } 1 \leq i, k \leq n;$$

$$(4.6) \quad \begin{aligned} & {}^i\rho_j^k - \sum_{l=1}^n {}^t\tau_l A_{lj}^k \\ & = {}^0\tau_i \delta_{jk} + {}^0\tau_j \delta_{ik} + \sum_{l=1}^n {}^0\rho_l^k A_{il}^l + {}^0B_{ij}^k \quad \text{for } 1 \leq i, j, k \leq n; \end{aligned}$$

$$(4.7) \quad \begin{aligned} & {}^lB_{ij}^k - {}^tB_{lj}^k = \frac{1}{2} ({}^0\rho_l^i \delta_{jk} + {}^0\rho_i^j \delta_{lk} + {}^0\rho_l^k \delta_{ji}) - \frac{1}{2} ({}^0\rho_i^l \delta_{jk} + {}^0\rho_l^j \delta_{ik} + {}^0\rho_i^k \delta_{jl}) \\ & + \frac{1}{2} \sum_{p, q=1}^n {}^0\rho_p^q (A_{ik}^p A_{lj}^q - A_{lk}^p A_{ij}^q) \quad \text{for } 1 \leq i, j, k, l \leq n. \end{aligned}$$

From (4.6) we obtain

$$(4.8) \quad {}^i\rho_j^k = \frac{1}{2} ({}^0\tau_j \delta_{ik} - {}^0\tau_k \delta_{ij}) + \frac{1}{2} \sum_{l=1}^n ({}^0\rho_l^k A_{ij}^l - {}^0\rho_l^j A_{ik}^l);$$

$$(4.9) \quad \begin{aligned} & {}^0B_{ij}^k = -\frac{1}{2} ({}^0\tau_j \delta_{ik} + {}^0\tau_k \delta_{ij} + 2{}^0\tau_i \delta_{jk}) - \frac{1}{2} \sum_{p, q=1}^n {}^0\tau_q A_{qi}^p A_{jk}^p \\ & - \frac{1}{2} \sum_{p=1}^n ({}^0\rho_p^k A_{ij}^p + {}^0\rho_p^j A_{ik}^p + {}^0\rho_p^i A_{jk}^p) \quad \text{for } 1 \leq i, j, k \leq n. \end{aligned}$$

We set

$$(4.10) \quad \begin{aligned} & {}^lB_{ij}^k = -\frac{1}{2} ({}^l\tau_j \delta_{ik} + {}^l\tau_k \delta_{ij} + 2{}^l\tau_i \delta_{jk}) - \frac{1}{2} \sum_{p, q=1}^n {}^l\tau_q A_{qi}^p A_{jk}^p \\ & - \frac{1}{2} \sum_{p=1}^n ({}^l\rho_p^k A_{ij}^p + {}^l\rho_p^j A_{ik}^p + {}^l\rho_p^i A_{jk}^p) \quad \text{for } 1 \leq i, j, k, l \leq n. \end{aligned}$$

Then we have ${}^l_0B = ({}^l_0B_{ij}{}^k)_{1 \leq i, j, k \leq n} \in (\bar{g}_n)_A$ for $1 \leq l \leq n$. Moreover by (4.5), (4.7) and (4.8) we obtain ${}^lB_{ij}{}^k - {}^l_0B_{ij}{}^k = {}^lB_{ij}{}^k - {}^l_0B_{ij}{}^k$ for $1 \leq i, j, k, l \leq n$. Let us set

$$(4.11) \quad {}^lC_{ij}{}^k = {}^lB_{ij}{}^k - {}^l_0B_{ij}{}^k \quad \text{for } 1 \leq i, j, k, l \leq n.$$

Then we have $C = ({}^lC_{ij}{}^k)_{1 \leq i, j, k, l \leq n} \in (\bar{g}_n^{(1)})_A$. Therefore any $\gamma \in g_{\alpha A}^{(1)}$ can be completely determined by ${}^0\rho = ({}^0\rho_i{}^j)_{1 \leq i, j \leq n} \in o(N)$, $C = ({}^lC_{ij}{}^k)_{1 \leq i, j, k, l \leq n} \in (\bar{g}_n^{(1)})_A$ and ${}^0\tau = ({}^0\tau_1, \dots, {}^0\tau_n) \in \mathbb{R}^n$. Conversely it is clear that these variables are independent. Hence we have $\dim g_{\alpha A}^{(1)} = \frac{1}{2}n(n-1) + \frac{1}{2}n(n+1) + n = n(n+1)$. Thus we have completed the proof of Theorem 3.1.

§ 5. Isometric immersions of the spaces of negative constant curvature

Let (M, g) be the space of constant curvature of dimension n with sectional curvature $k < 0$. Then at each $p \in M$, we have

$$-g(R(x, y)z, w) = k\{g(x, z)g(y, w) - g(x, w)g(y, z)\} \\ \text{for } x, y, z, w \in T_p.$$

Moreover we have $\nabla R \equiv 0$ on M .

We now show the following

Theorem 5.1. *If $m = 2n - 1$, then there exists an open fibered submanifold $\pi_1^2: P_{\#}^{(1)} \rightarrow P$ of the vector bundle $\pi_1^2: P^{(1)} \rightarrow P$ such that the intersection $Q_{\#} = Q \cap P_{\#}^{(1)}$ forms an involutive differential equation. (**)*

Proof. Let $\alpha \in P$ and let V be a sufficiently small neighborhood of α in P . From the local triviality of the vector bundle $\pi: N \rightarrow P$, we may assume that $\pi^{-1}(V)$ is isomorphic to the vector bundle $V \times N$, where $N = N_{\alpha}$. Furthermore we may assume that the isomorphism gives an isometric isomorphism between each fiber of $\pi^{-1}(V)$ and N . (Note that since each fiber of the vector bundle $\pi: N \rightarrow P$ is a subspace of $\mathbb{R}^{2n-1}$, it is endowed with an inner product.) Similarly the restriction $T_{|\pi^{-1}_1(V)}$ of the tangent bundle $T = T(M)$ to $\pi^{-1}_1(V)$ may be assumed to be isomorphic to the bundle $\pi^{-1}_1(V) \times T$, where we set $T = T_p$. We may assume that the isomorphism gives an isometric isomorphism between each fiber of $T_{|\pi^{-1}_1(V)}$ and T with respect to the given Riemannian metric g . Under these observations the restriction $P_{|V}^{(1)}$ of the vector bundle $\pi_1^2: P^{(1)} \rightarrow P$ to V may be considered to be isomorphic to the bundle $V \times S^2T^* \otimes N$. By this isomorphism the set $Q \cap (\pi_1^2)^{-1}(V)$ is mapped onto $V \times \mathcal{G}(G_k)$ (see the equation (1.8)). Hence there exists an open set O in $S^2T^* \otimes N$ having the property stated in Theorem 3.1. By O_{α} we denote the open set in $P^{(1)}$ that corresponds to the open set $V \times O$ in $V \times S^2T^* \otimes N$. For any $\alpha \in P$, we take such an open set O_{α} in $P^{(1)}$ and set $P_{\#}^{(1)} = \bigcup_{\alpha \in P} O_{\alpha}$. Then it is clear that $\pi_1^2: P_{\#}^{(1)} \rightarrow P$ forms an open fibered submanifold of the vector bundle $\pi_1^2: P^{(1)} \rightarrow P$. We set $Q_{\#} = Q \cap P_{\#}^{(1)}$ and

(**) For the definition of involutive differential equations, see [8] or [4].

$Q_{\#}^{(1)} = Q^{(1)} \cap (\pi_2^3)^{-1}(P_{\#}^{(1)})$. Then we can easily verify that $Q_{\#}$ forms a fibered submanifold of the vector bundle $\pi_1^2: P^{(1)} \rightarrow P$. Hence $Q_{\#}$ is a differential equation. In order to show the involutiveness of $Q_{\#}$, we must show the following:

- (a) The map $\pi_2^3: Q_{\#}^{(1)} \rightarrow Q_{\#}$ is surjective.
- (b) The union $q^{(1)} = \bigcup_{\beta \in Q_{\#}} q_{\beta}^{(1)}$ is a vector bundle over $Q_{\#}$.
- (c) For each $\beta \in Q_{\#}$, the symbol q_{β} of $Q_{\#}$ at β is involutive.

Proof of (a). Let $\beta = (p; \omega_0, \omega_1, \omega_2) \in Q_{\#}$. Let us define $\bar{\omega}_3 \in \otimes^3 T_p^* \otimes \mathbb{R}^{2n-1}$ by setting

$$\begin{aligned} \langle \bar{\omega}_3(w, z, x), \omega_1(y) \rangle &= -\langle \omega_2(z, x), \omega_2(w, y) \rangle, \\ \langle \bar{\omega}_3(w, z, x), n \rangle &= 0 \quad \text{for } x, y, z, w \in T_p, n \in N_{\alpha}(\alpha = \pi_1^2(\beta)). \end{aligned}$$

Then we have $\gamma = (p; \omega_0, \omega_1, \omega_2, \bar{\omega}_3) \in J^3(M, m)$ and $\pi_2^3(\gamma) = \beta$. Moreover since $\nabla R \equiv 0$ we have $\gamma \in Q_{\#}^{(1)}$ (see the equations (1.9) and (1.10)). Hence the map $\pi_2^3: Q_{\#}^{(1)} \rightarrow Q_{\#}$ is surjective.

Proof of (b) and (c). Let $\beta = (p; \omega_0, \omega_1, \omega_2) \in Q_{\#}$. We set $\alpha = \pi_1^2(\beta)$. We may assume $\beta \in O_{\alpha}$. Then by the definitions of the vector spaces q_{β} and $q_{\beta}^{(1)}$, we have $q_{\beta} = g_{\omega_2}$ and $q_{\beta}^{(1)} = g_{\omega_2}^{(1)}$ (Note that we are assuming $T = T_p$ and $N = N_{\alpha}$.) Since $\dim N = \dim T - 1 = n - 1$, we have $\dim q_{\omega_2}^{(1)} = \dim g_{\omega_2}^{(1)} = n(n-1)$. This indicates that the union $q^{(1)} = \bigcup_{\beta \in Q_{\#}} q_{\beta}^{(1)}$ is a vector bundle over $Q_{\#}$. We next show (c). Since $\beta \in O_{\alpha}$, there exists a vector $e \in T = T_p$ such that $e \lrcorner \xi \neq 0$ for any $\xi \in q_{\beta}$; $\xi \neq 0$. Then we can easily see that any basis $\{e_1, \dots, e_n\}$ of $T = T_p$ such that $e_1 = e$ is regular for the symbol q_{β} . Hence q_{β} is involutive. Thus we have completed the proof of Theorem 5.1. Q.E.D.

Note that any space of constant curvature is a real analytic Riemannian manifold. Then the varieties $P, P^{(1)}, Q^{(1)}$ and Q are also considered to be real analytic. We can easily see that Theorem 5.1 still holds if we consider everything in the real analytic category. Then $Q_{\#}$ forms a real analytic differential equation. From the existence theorem of local solutions of real analytic involutive differential equations (cf. [8], [4]) follows

Theorem 5.2. *Any space of negative constant curvature of dimension n can be locally isometrically immersed into the euclidean space $\mathbb{R}^{2n-1}$.*

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Added in proof: After submitting this paper, the author knew the following classical work:

E. Cartan, *Sur les variétés de courbure constante d'un espace euclidien ou non-euclidean*, *Oeuvres complètes*, Partie III, vol. 1, Gauthier-Villars, Paris, 1955.