On ideal transforms of noetherian rings, I

By

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Introduction

In [3], Matijevic has given a nice generalization of Krull-Akizuki Theorem :

Let A be a noetherian ring. We define the total transform $T(A)$ of A as follows: $T(A) = \{a \in Q(A) \text{ (=the total quotient ring of } A\text{); the conductor of } a \text{ to } A \text{ contains a power of a finite product of maximal ideals of } A\}$.

THEOREM: Let B be an A -algebra contained in $T(A)$. Then B/xB is a finite A -module for each non-zero-divisor x in A .

COROLLARY: If A is reduced, B is always noetherian.

PROOF consists of three steps:

Step 1. For any b in B , there exists a $k > 0$ such that $b \in Ax^{-k} + xB$.

Step 2. The descending chain of ideals $I_h = (x^h B \cap A, xA)$ stabilizes for some $h = n$.

Step 3. B is contained in $Ax^{-n} + xB$. This gives the assertion.

In this article, we give a general theory on ideal transforms of noetherian rings using **THEOREM** and **COROLLARY**. We also apply the technique of **PROOF** to some problems on the integral closures of noetherian domains.

In section 1, we begin by giving an elementary lemma on ideal transforms of noetherian rings. Then we treat the total transform of a local domain $(A, \mathfrak{m})$, which we call the global transform of A and write by $A(\mathfrak{m})$. Proposition (1.2) gives a sufficient condition of $A(\mathfrak{m})$ to be integral over A . Proposition (1.3) shows that, for a henselian local domain $(A, \mathfrak{m})$ of dimension greater than one the derived normal ring (=the integral closure of A in its field of quotients) $\bar{A} = \bigcap_{j=1}^k \bar{A}_{a_j}$, where $\mathfrak{m} = a_1 A + \cdots + a_k A$. This replaces [6, Proposition 5], so that we can constantly use henselizations instead of completions in the proof of [6, Proposition 6]. Lemma (1.4) is easy, but $A(\mathfrak{m})$ being noetherian is essential. Theorem (1.6) gives the relation between $A(\mathfrak{m})$ and the integral closure $\tilde{A}$ of A in $A(\mathfrak{m})$. Theorem (1.6) seems to be already known (cf. [3], [7], [8]), but we have not seen a complete proof. Corollary (1.7) gives a necessary (and sufficient) condition that $A(\mathfrak{m})$ is integral over A . Corollary (1.8) shows the explicit form of the ring which lies between $\tilde{A}$ and $A(\mathfrak{m})$.

In section 2, we continue to investigate the global transform of a local ring, but not necessarily an integral domain. We begin with Proposition (2.1) in order that we can apply COROLLARY and its related results to non-reduced cases (cf. Proposition (2.2), Corollary (2.4), Theorem (2.6) and Proposition (2.7)). Proposition (2.3) gives criteria of $A(\mathfrak{m})$ to be finite over A (or integral over A) in terms of associated primes of its completion $\hat{A}$. The proof of Proposition (2.3) is an easy exercise of commutative rings, but the union of Corollary (1.7) and Proposition (2.3) gives another proof of [5, (33.11)]:

Let $\bar{\mathfrak{p}}$ be a height one prime ideal of the derived normal ring of a noetherian domain A . Then $\mathfrak{p} = \bar{\mathfrak{p}} \cap A$ is an associated prime ideal of a principal ideal of A (i. e. $\text{prof } A_{\mathfrak{p}} = 1$).

In Theorem (2.6), we reduce the finiteness (or integrality) problem of a general ideal transform $A(I)$ to the same problem of global transforms of local rings $A_{\mathfrak{p}}$, where $\mathfrak{p}$ are prime ideals of A containing I . Consequently, we get an elementary proof of Ferrand-Raynaud [1, Proposition 1.1].

In section 3, we give criteria of the derived normal ring of a noetherian domain A to be a finite A -module. Propositions (3.1) and (3.2) are slight generalizations of [5, (35.3)] or [4, Lemma, p.239]. We hope the hypotheses given here are more acceptable than that of [5, (35.3)].

Finally, we emphasize that, even in Propositions (3.1) and (3.2), the technique of PROOF is essential.

Notations and terminology

In this paper, we mean by a ring a commutative ring with identity and by a local ring (resp. a semi-local ring) a noetherian ring with only one maximal ideal (resp. with finite number of maximal ideals). For a semi-local ring $(A, \mathfrak{m}_1, \dots, \mathfrak{m}_r)$ with radical $\mathfrak{m} = \mathfrak{m}_1 \cdots \mathfrak{m}_r$, $\hat{A}$ denotes the completion of A with respect to $\mathfrak{m}$. Let I be an ideal of a ring A , we define the ideal transform $A(I)$ of A with respect to I as follows:

$$A(I) = \{a \in Q(A) \mid (= \text{the total quotient ring of } A); \text{ there exists a natural number } n \text{ such that } I^n a \subset A\}.$$

We assume I contains a non-zero-divisor of A , whenever we define $A(I)$.

We call a semi-local domain A a well-branched ring if the number of maximal ideals of A is the same as that of $\bar{A}$ (=the derived normal ring of A). Let A, B be semi-local domains with common field of quotients. We call B a well-branched over-ring of A , if B is a finite A -module and a well-branched ring.

$\text{Ass}(A)$ denotes the set of associated primes of A and for a local ring $(A, \mathfrak{m})$ $\text{prof } A$ means the depth of A with respect to $\mathfrak{m}$.

1. Global transforms of local domains

Lemma (1.1) *Let I be an ideal of a noetherian ring A . Assume I contains a non-zero-divisor of A . Hence we can write $I = a_1 A + \dots + a_k A$ with non-zero-divisor a_j in A . Then*

$$(1.1.1) \quad A(I) = \bigcap_{j=1}^k A_{a_j}.$$

Therefore, if B is a flat A -algebra, we have:

$$(1.1.2) \quad B(IB) = A(I) \otimes_A B.$$

In particular, let $(A, \mathfrak{m}_1, \dots, \mathfrak{m}_r)$ be a semi-local ring with radical $\mathfrak{m} = \mathfrak{m}_1 \dots \mathfrak{m}_r$ (which contains a non-zero-divisor). For each maximal ideal $\mathfrak{m}_i$:

$$(1.1.3) \quad A(\mathfrak{m})_{\mathfrak{m}_i} = A_{\mathfrak{m}_i}(\mathfrak{m}_i).$$

For non-maximal prime ideal $\mathfrak{p}$:

$$(1.1.4) \quad A(\mathfrak{m})_{\mathfrak{p}} = A_{\mathfrak{p}}.$$

Hence there exists a unique prime ideal $\mathfrak{q}$ of $A(\mathfrak{m})$ such that:

$$(1.1.5) \quad \mathfrak{p} = \mathfrak{q} \cap A \quad \text{and} \quad A(\mathfrak{m})_{\mathfrak{q}} = A_{\mathfrak{p}}.$$

Proposition (1.2) (cf. [7, Remarque]) *Let $(A, \mathfrak{m})$ be a local domain. Suppose $\dim \bar{A}_{\bar{\mathfrak{m}}} > 1$ for any maximal ideal $\bar{\mathfrak{m}}$ of $\bar{A}$ (=the derived normal ring of A). Then*

$$(1.2.1) \quad A(\mathfrak{m}) \text{ is integral over } A.$$

Hence

$$(1.2.2) \quad (A(\mathfrak{m}), \mathfrak{n}_1, \dots, \mathfrak{n}_s) \text{ is a semi-local domain.}$$

$$(1.2.3) \quad \text{prof } A(\mathfrak{m})_{\mathfrak{n}_i} > 1 \text{ for each maximal ideal } \mathfrak{n}_i \text{ of } A(\mathfrak{m}).$$

Proof. Let $B = A(\mathfrak{m}) \cap \bar{A}$. Then $(B, \mathfrak{n}_1, \dots, \mathfrak{n}_s)$ is a semi-local domain with radical $\mathfrak{n} = \mathfrak{n}_1 \dots \mathfrak{n}_s$ by COROLLARY.

We claim: $B(\mathfrak{n}) = B$ and $A(\mathfrak{m}) = B$.

1) $\mathfrak{n} : \mathfrak{n} = B$. Since $\mathfrak{n} : \mathfrak{n} \subset A(\mathfrak{m}) : \mathfrak{m} = A(\mathfrak{m})$ and $\mathfrak{n} : \mathfrak{n} \subset \bar{B} = \bar{A}$, $B \subset \mathfrak{n} : \mathfrak{n} \subset A(\mathfrak{m}) \cap \bar{A} = B$.

2) $B(\mathfrak{n}) = B$. As $\dim P_{\mathfrak{n}_i} > 1$ for any i , $(B : \mathfrak{n}_i)_{\mathfrak{n}_i} = \mathfrak{n}_i$. Hence $B : \mathfrak{n} = \mathfrak{n} : \mathfrak{n} = B$.

3) $A(\mathfrak{m}) = B$. Since B is noetherian and $\text{rad}(\mathfrak{m}B) = \mathfrak{n}$, $A(\mathfrak{m}) = \bigcup_{\nu > 0} (A : \mathfrak{m}^\nu) \subset \bigcup_{\nu > 0} (B : \mathfrak{m}^\nu) = \bigcup_{\mu > 0} (B : \mathfrak{n}^\mu) = B(\mathfrak{n}) = B$.

Proposition (1.3) *Let $(A, \mathfrak{m})$ be a henselian local domain of dimension greater than one. Write $\mathfrak{m} = a_1 A + \dots + a_k A$. Then $\bar{A} = \bigcap_{j=1}^k \bar{A}_{a_j}$.*

Proof is easy, because $A(\mathfrak{m}) = \bigcap_{j=1}^k A_{a_j}$ is integral over A by (1.2.1).

Lemma (1.4) *Let $(A, \mathfrak{m}_1, \dots, \mathfrak{m}_r)$, $(B, \mathfrak{n}_1, \dots, \mathfrak{n}_s)$ be semi-local domains with common field of quotients. If B is a finite A -module, then $B(\mathfrak{n})$ is a finite $A(\mathfrak{m})$ -module.*

Proposition (1.5) *Let $(A, \mathfrak{m})$ be a local domain of dimension greater than one. Then*

$$(1.5.1) \quad (A(\mathfrak{m}), \mathfrak{n}_1, \dots, \mathfrak{n}_s) \text{ is a semi-local domain with } \dim A(\mathfrak{m}) = \dim A.$$

For each maximal ideal $\mathfrak{n}_i$ of $A(\mathfrak{m})$,

$$(1.5.2) \quad \mathfrak{n}_i \cap A = \mathfrak{m}, \text{ and}$$

$$(1.5.3) \quad \text{prof } A(\mathfrak{m})_{\mathfrak{n}_i} > 1.$$

Proof. Let $(B, \mathfrak{m}'_1, \dots, \mathfrak{m}'_r)$ be a well-branched over-ring of A with radical $\mathfrak{m}' = \mathfrak{m}'_1 \cdots \mathfrak{m}'_r$. Suppose $\dim B_{\mathfrak{m}'_i} > 1$ iff $i \leq r_0 (\leq r)$. Then $B(\mathfrak{m}') = B_S(\mathfrak{m}'_S)$, where $S = B - \bigcup_{i \leq r_0} \mathfrak{m}'_i$ by (1.1.3). Since $B_S(\mathfrak{m}'_S)$ is integral over B_S and $B(\mathfrak{m}')$ is a finite $A(\mathfrak{m})$ -module by (1.2.1) and Lemma (1.4), $(A(\mathfrak{m}), \mathfrak{n}_1, \dots, \mathfrak{n}_s)$ is a semi-local domain with $\mathfrak{n}_i \cap A = \mathfrak{m}$ and $\text{rad}(\mathfrak{m}A(\mathfrak{m})) = \mathfrak{n}_1 \cdots \mathfrak{n}_s$. Therefore we get the assertions.

Theorem (1.6) (cf. [7, Lemme 3], [8, Corollaires (I 4) et (I 10)]) *Let $(A, \mathfrak{m})$ be a local domain with field of quotients K . Let $P = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ be the set of height one maximal ideals of $\bar{A}$ (which may be empty).*

Put $\tilde{A} = A(\mathfrak{m}) \cap \bar{A}$ (=the integral closure of A in $A(\mathfrak{m})$). Then

$$(1.6.1) \quad (\tilde{A}, \mathfrak{m}_1, \dots, \mathfrak{m}_r) \text{ is a semi-local domain with radical } \mathfrak{m} = \mathfrak{m}_1 \cdots \mathfrak{m}_r.$$

$$(1.6.2) \quad \tilde{A} = A(\mathfrak{m}) \cap \bar{A}_{\mathfrak{p}_1} \cap \cdots \cap \bar{A}_{\mathfrak{p}_t}.$$

$$(1.6.3) \quad \bar{A} = \overline{A(\mathfrak{m})} \cap \bar{A}_{\mathfrak{p}_1} \cap \cdots \cap \bar{A}_{\mathfrak{p}_t}.$$

$$(1.6.4) \quad \tilde{A}(\mathfrak{m}) = A(\mathfrak{m}).$$

Divide the set $M = \{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$ of maximal ideals of $\tilde{A}$ into three classes:

$$M_0 = \{\mathfrak{m}_k \in M; \dim \tilde{A}_{\mathfrak{m}_k} = 0\}, \quad M_1 = \{\mathfrak{m}_j \in M; \dim \tilde{A}_{\mathfrak{m}_j} = 1\} \quad \text{and}$$

$$M_2 = \{\mathfrak{m}_i \in M; \dim \tilde{A}_{\mathfrak{m}_i} > 1\}. \quad \text{Then}$$

$$(1.6.5) \quad A(\mathfrak{m}) = \tilde{A}_S, \quad \text{where } S = \tilde{A} - \bigcup_{\mathfrak{m}_i \in M_2} \mathfrak{m}_i \quad (\text{if } M_2 \text{ is empty, } S = A - \{0\}).$$

P is not empty if and only if M_1 is not empty and:

$$(1.6.6) \quad \bar{A}_{\mathfrak{p}_1} \cap \cdots \cap \bar{A}_{\mathfrak{p}_t} = \tilde{A}_T, \quad \text{where } T = \tilde{A} - \bigcup_{\mathfrak{m}_j \in M_1} \mathfrak{m}_j.$$

Proof. 1) Let $(B, \mathfrak{n}_1, \dots, \mathfrak{n}_{s+t})$ be a well-branched over-ring of A such that $\dim B_{\mathfrak{n}_i} > 1$ iff $i \leq s$ (s may be zero).

Then

$$\begin{aligned} A &= A(\mathfrak{m}) \cap \bar{A} = A(\mathfrak{m}) \cap B(\mathfrak{n}) \cap \bar{B} = A(\mathfrak{m}) \cap \left(\bigcap_{i=1}^s B_{\mathfrak{n}_i}(\mathfrak{n}_i) \right) \cap \left(\bigcap_{j=1}^{s+t} B_{\mathfrak{n}_j} \right) \\ &= A(\mathfrak{m}) \cap B(\mathfrak{n}) \cap \left(\bigcap_{j=s+1}^{s+t} \bar{B}_{\mathfrak{n}_j} \right) = A(\mathfrak{m}) \cap \bar{A}_{\mathfrak{p}_1} \cap \cdots \cap \bar{A}_{\mathfrak{p}_t}. \end{aligned}$$

$\bar{A}_{\mathfrak{p}_j} = \bar{B}_{\mathfrak{n}_{s+j}}$ are discrete valuation rings (cf. [5, (33.2)]).

2) On the other hand,

$$\begin{aligned} \bar{A} = \bar{B} &= \bigcap_{i=1}^{s+t} \bar{B}_{\mathfrak{n}_i} = \left(\bigcap_{i=1}^s \bar{B}_{\mathfrak{n}_i} \right) \cap \left(\bigcap_{j=s+1}^{s+t} \bar{B}_{\mathfrak{n}_j} \right) \\ &= \overline{B(\mathfrak{n})} \cap \left(\bigcap_{j=1}^t \bar{A}_{\mathfrak{p}_j} \right) = \overline{A(\mathfrak{m})} \cap \bar{A}_{\mathfrak{p}_1} \cap \cdots \cap \bar{A}_{\mathfrak{p}_t} \\ &\quad (\text{cf. (1.2.1), Lemma (1.4)}). \end{aligned}$$

3) Since $A(\mathfrak{m})$ is noetherian, $A(\mathfrak{m}) \subset \tilde{A}(\mathfrak{m}) \subset A(\mathfrak{m})(\mathfrak{m}) \subset A(\mathfrak{m})(\mathfrak{m}) = A(\mathfrak{m})$.

4) Consider the pair of semi-local domains $\tilde{A}_S$ and $A(\mathfrak{m})$ with common field of quotients. Since $A(\mathfrak{m})_S = \tilde{A}(\mathfrak{m})_S = \tilde{A}(\mathfrak{m}) = A(\mathfrak{m})$, $\tilde{A}_S$ is integrally closed in $A(\mathfrak{m})$. As $\tilde{A}(\mathfrak{m})_{\mathfrak{m}_i} = \tilde{A}_{\mathfrak{m}_i}(\mathfrak{m}_i) = A(\mathfrak{m})_{\mathfrak{m}_i}$ for each $\mathfrak{m}_i$ in M_2 , for any prime ideal $\mathfrak{p}$ of $\tilde{A}_S$, there always exists a prime ideal $\mathfrak{q}$ of $A(\mathfrak{m})$ such that $\mathfrak{q} \cap \tilde{A}_S = \mathfrak{p}$ by (1.5.2) and (1.1.4). Hence $\tilde{A}_S = A(\mathfrak{m})$ by [5, (33.1)].

5) Since $\bar{A}_S = \tilde{A}_S = \overline{A(\mathfrak{m})} = \overline{B(\mathfrak{n})}$ has no maximal ideal of height one,

$$\begin{aligned} \tilde{A}_T &= A(\mathfrak{m})_T \cap (\bar{A}_{\mathfrak{p}_1})_T \cap \cdots \cap (\bar{A}_{\mathfrak{p}_t})_T = \tilde{A}(\mathfrak{m})_T \cap \bigcap_{j=1}^t \bar{A}_{\mathfrak{p}_j} \\ &= K \cap \bar{A}_{\mathfrak{p}_1} \cap \cdots \cap \bar{A}_{\mathfrak{p}_t} = \bar{A}_{\mathfrak{p}_1} \cap \cdots \cap \bar{A}_{\mathfrak{p}_t}. \end{aligned}$$

Corollary (1.7) *Let $(A, \mathfrak{m})$ be a local domain. Then $A(\mathfrak{m})$ is integral over A if and only if the derived normal ring $\bar{A}$ of A has no maximal ideal of height one (cf. (1.2.1)).*

Corollary (1.8) *With the notations as in Theorem (1.6). Let C be a ring such that $\tilde{A} \subset C \subset A(\mathfrak{m})$, then $C = A(\mathfrak{m}) \cap (\bigcap_{\mathfrak{p}_j \in Q} \bar{A}_{\mathfrak{p}_j})$, where Q is a subset of P .*

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