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On homotopy equivalences of $S^2\times RP^2$ to itself

By

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Dedicated to Professor A. Komatu on his 70th birthday

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In this note the general exact sequence method to calculate the based homotopy set $[X, Y]_0$ is presented, including the case in which Y is not simple (§0). As an application, we shall determine the based homotopy set $[S^2\times RP^2, S^2\times RP^2]_0$ and the group $\mathcal{E}(S^2\times RP^2)$ of self homotopy equivalences, where S^2 and RP^2 denote the 2-sphere and the real projective plane respectively (Theorem 1, Lemmas 1.1 and 1.4).

Which homotopy classes are representable by diffeomorphisms (Corollary 2.1)? This question leads us to study the homotopy smoothings from the surgery theoretical point of view. We shall show that any homotopy smoothing of $S^2\times RP^2$ is s-cobordant to a homotopy equivalence of $S^2\times RP^2$ to itself (Corollary 2.2). Similarly, any smooth s-cobordism of $S^1\times RP^2$ to itself is shown to be s-cobordant to the product cobordism $S^1\times RP^2\times I$ relative to the boundary (Proposition 3). We refer [11] for the topological s-cobordisms.

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§0. Generalities on the based homotopy set $[X, Y]_0$

Let X and Y be connected CW complexes with based point. We are concerned with the space $Map_0(X, Y)$ of based point preserving maps of X into Y equipped with compact-open topology. In order to study the set of based homotopy classes $\pi_0(Map_0(X, Y))$, or in a more familiar notation $[X, Y]_0$, we can use two types of filtrations.

The first one comes from the structure of a CW complex with unique 0-cell having the same homotopy type as X. Let X^n denote the n-skeleton of this CW complex. Then, we have the Puppe cofibering sequence,

$$S^{n-1}\vee\cdots\vee S^{n-1}\xrightarrow{f} X^{n-1}\xrightarrow{j} Cf=X^n\longrightarrow S^n\vee\cdots\vee S^n\xrightarrow{S(f)} SX^{n-1}\longrightarrow\cdots$$

and the following induced exact sequence in the category of pointed sets.

$$(0.1)\quad \cdots \longrightarrow [SX^{n-1}, Y]_0 \xrightarrow{S(f)^*} [S^n \vee \cdots \vee S^n, Y]_0$$
$$\longrightarrow [X^n, Y]_0 \xrightarrow{j^*} [X^{n-1}, Y]_0 \xrightarrow{f^*} [S^{n-1} \vee \cdots \vee S^{n-1}, Y]_0$$

Here, the group $[S^n \vee \cdots \vee S^n, Y]_0$ operates naturally on $[X^n, Y]_0$ and its orbit space corresponds bijectively to the preimage $(j^*)^{-1}(0)$.

To describe the preimage $(j^*)^{-1}([u])$ for any based point preserving map $u: X^{n-1} \to Y$ which extends over X^n, i.e.: $u \circ f \simeq 0$, we observe the following diagram,

$$\begin{array}{ccc}
Map_0(S^n \vee \cdots \vee S^n, Y) & = & Map_0(S^n \vee \cdots \vee S^n, Y) \\
\downarrow & & \downarrow \\
\{\bar{v} \in Map_0(X^n, Y); \bar{v} \circ j \simeq u\} & \longrightarrow & Map_0(D^n \vee \cdots \vee D^n, Y) \\
\downarrow j^* & & \downarrow j_0^* \\
\{v \in Map_0(X^{n-1}, Y); v \simeq u\} & \xrightarrow{f^*} & Map_0(S^{n-1} \vee \cdots \vee S^{n-1}, Y)
\end{array}$$

in which j_0^* is a fiber space in the sense of Serre and j^* is the induced fiber space by f^*. Since the right vertical sequence is a universal loop fibering up to homotopy type,

$$(0.2)\quad \{\bar{v} \in Map_0(X^n, Y); \bar{v} \circ j \simeq u\} \xrightarrow{j^*} \{v \in Map_0(X^{n-1}, Y); v \simeq u\}$$
$$\xrightarrow{f^*} Map_0(S^{n-1} \vee \cdots \vee S^{n-1}, Y)$$

is a homotopy fibering. From the long exact sequence of the homotopy groups, we get the following bijection compatible with the natural group operation (Cf. Barcus-Barratt [1]).

$$(0.3)\qquad [S^n \vee \cdots \vee S^n, Y]_0 / \mathrm{Im}\, f_u \longrightarrow (j^*)^{-1}([u])$$

Here, the homomorphism $f_u: \pi_1(Map_0(X^{n-1}, Y), u) \to [S^n \vee \cdots \vee S^n, Y]_0$ is defined to be the composite of f^* with the isomorphisms,

$$\pi_1(Map_0(S^{n-1} \vee \cdots \vee S^{n-1}, Y), u \circ f)$$
$$\cong \pi_1(Map_0(S^{n-1} \vee \cdots \vee S^{n-1}, Y), 0) \cong [S^n \vee \cdots \vee S^n, Y]_0.$$

The dual of the CW complex filtration is given by the Postnikov system of Y, a system $\{Y_n, f_n, p_n\}$ with the following property:

(i) The map $f_n: Y \to Y_n$ induces an isomorphism on π_r if $r \leqq n$ and $\pi_r(Y_n) = 0$ if $r > n$,

(ii) $p_n: Y_n \to Y_{n-1}$ is a fiber space in the sense of Serre with an Eilenberg-MacLane space $K(\pi_n(Y), n)$ as fiber, and

(iii) $p_n \circ f_n$ is homotopic to f_{n-1}.

We may assume that each Y_n has the homotopy type of a CW complex with based point and each of f_n and p_n preserves the based points.

If Y is *n-simple* i.e.: the operation of $\pi_1(Y, *)$ on $\pi_n(Y, *)$ is trivial, the fibering in the condition (ii) is induced up to homotopy type from the universal loop fibering, $K(\pi_n(Y), n) \to * \to K(\pi_n(Y), n+1)$, by the k-invariant $k: Y_{n-1} \to K(\pi_n(Y), n+1)$. Thus, we have a fibering sequence,

$$\cdots \longrightarrow Y_{n-1} \longrightarrow K(\pi_n(Y), n) \longrightarrow Y_n \xrightarrow{p_n} Y_{n-1} \xrightarrow{k} K(\pi_n(Y), n+1).$$

Taking the based homotopy set of the based point preserving maps of X, we get the following exact sequence in the category of pointed sets.

$$(0.4) \quad \cdots \longrightarrow [X, \Omega Y_{n-1}]_0 \xrightarrow{\Omega(k)_*} [X, K(\pi_n(Y), n)]_0$$
$$\longrightarrow [X, Y_n]_0 \xrightarrow{p_*} [X, Y_{n-1}]_0 \xrightarrow{k_*} [X, K(\pi_n(Y), n+1)]_0$$

Moreover, $[X, K(\pi_n(Y), n)]_0$ is isomorphic to the cohomology group $H^n(X; \pi_n(Y))$ which operates naturally on $[X, Y_n]_0$ and the orbit space corresponds bijectively to $(p_*)^{-1}(0)$.

Let $n \geqq 2$ and π be an abelian group. Before handling the general case, we study the fiberings with $K(\pi, n)$ as fiber. In the semi-simplicial setting, P. May proved in his book [3, p. 100] that any fibering with the Eilenberg-MacLane complex $K(\pi, n)$ as fiber has a semi-simplicial structural group $A(K(\pi, n))$ which is identified with $\mathrm{Aut}(\pi) \times K(\pi, n)$ whose semi-simplicial group operation is defined on q-simplexes $\mathrm{Aut}(\pi)_q \times K(\pi, n)_q$ by $(f, x)\cdot(g, y) = (fg, f(y)+x)$. This was essentially proved by R. Thom [7] already in 1956. The fiberings are classified by the homotopy class of a map $k: X \to BA(K(\pi, n))$ or an element of the cohomology group $H^{n+1}(X; \pi_{loc})$ in some local coefficient system. Actually, the obstruction map,

$$(0.5) \qquad [X, BA(K(\pi, n))]_0 \xrightarrow{\cong} \bigcup H^{n+1}(X; \pi_{loc}) \quad \text{(disjoint union)},$$

is a bijection for any connected CW complex with based point. This fact does not seem well-known, but is easily proved because $\pi_1(BA(K(\pi, n)), *) = \mathrm{Aut}(\pi)$ and the induced homomorphism $\pi_1(X, *) \to \mathrm{Aut}(\pi)$ determines a local coefficient system in which the argument of the classical obstruction theory remains powerful for the based point preserving homotopies (Cf. P. Olum [4]).

In the general case, a based point preserving map $u: X \to Y_{n-1}$ is assumed to have a lifting $\bar{u}: X \to Y_n$ such that $p_n \circ \bar{u} = u$, which is equivalent to say $k_* u = 0$ in $H^{n+1}(X, \pi_n(Y)_u)$ where $\pi_n(Y)_u$ is the local coefficient system determined by the homomorphism $u_*: \pi_1(X, *) \to \pi_1(Y_{n-1}, *) = \pi_1(Y, *)$. Then, we shall prove that

$$(0.6) \quad \{\bar{v} \in Map_0(X, Y_n);\, p_n \circ \bar{v} \simeq u\} \xrightarrow{(p_n)_*} \{v \in Map_0(X, Y_{n-1});\, v \simeq u\}$$
$$\xrightarrow{k_*} \{\underline{v} \in Map_0(X, BA(K(\pi_n(Y), n)));\, \underline{v} \simeq k \circ u\}$$

is a homotopy fibering. In fact, $(p_n)_*$ is a fiber space in the sense of Serre and its fiber is identified with the space $\Gamma(X, Q_u)$ of sections of the induced fiber space (=pull-back) Q_u over X by u from p_n, or by $k\circ u$ from the universal fiber space E with fiber $K(\pi_n(Y), n)$.

$$\begin{array}{ccccc} Q_u & \longrightarrow & Y_n & \longrightarrow & E \\ \downarrow & u & \downarrow p_n & k & \downarrow \pi \\ X & \longrightarrow & Y_{n-1} & \longrightarrow & BA(K(\pi_n(Y), n)) \end{array}$$

Hence, $(p_n)_*$ in (0.6) is an induced fiber space by k from the following fiber space.

$$\Gamma(X, Q_u) \longrightarrow \{v\in Map_0(X, E)\,;\, \pi\circ v\simeq k\circ u\}$$
$$\longrightarrow \{\underline{v}\in Map(X, BA(K(\pi_n(Y), n)))\,;\, \underline{v}\simeq k\circ u\}$$

Now, we observe that the total space of this fiber space is contractible, because it is a connected component of $Map_0(X, E)$ and $\pi_i(Map_0(X, E))\cong\tilde{H}^{1-i}(X, \mathrm{Aut}(\pi))$. The last isomorphism comes from the fact that E is an Eilenberg-MacLane space $K(\mathrm{Aut}(\pi), 1)$. Therefore, the above fiber space is a universal loop fibering up to homotopy type and hence (0.6) is a homotopy fibering.

For applications, we should remark the existence of the isomorphism,

(0.7) $\quad \pi_i(\{\underline{v}\in Map_0(X, BA(K(\pi_n(Y), n))\,;\, \underline{v}\simeq k\circ u\})\cong\tilde{H}^{n+1-i}(X\,;\, \pi_n(Y)_u)$,

defined by the difference cocycles. Then, we get a bijection,

(0.8) $$H^n(X\,;\,\pi_n(Y)_u)/\mathrm{Im}\, k_u \xrightarrow{\cong} (p_{n*})^{-1}([u]),$$

where $k_u : \pi_1(\{v\in Map_0(X, Y_{n-1})\,;\, v\simeq u\}, u)\to H^n(X, \pi_n(Y)_u)$ is the composite of k_* with the isomorphism of (0.7). Here, $P_{n*} : [X, Y_n]_0\to[X, Y_{n-1}]_0$.

§1. Self homotopy equivalence group of $S^2\times RP^2$

We shall determine the group $\mathcal{E}(S^2\times RP^2)$ formed by the free homotopy classes of homotopy equivalences of $S^2\times RP^2$ to itself with the operation induced by the composition of maps. Let $\alpha : S^2\times RP^2\to S^2\times RP^2$ be the map defined by the antipodal of S^2 cross the identity of RP^2. Knowing that $[RP^2, SO(3)]_0\cong \boldsymbol{Z}_2$, we define β to be the unique non-trivial $SO(3)$-bundle automorphism of the product bundle $S^2\times RP^2$ over RP^2. Choose an embedding $D^4\subset M^4=S^2\times RP^2$. If D^4 is shrunk to a point, the result is homeomorphic to M^4. Shrink instead ∂D^4 to a point to give a map, $c : M^4\to M^4\vee S^4$. Now let $\eta^2 : S^4\to S^2$ be an essential map and $s : S^2\to S^2\times RP^2$ and $t : S^2\to S^2\times RP^2$ be the composites of

$$S^2=S^2\times *\hookrightarrow S^2\times RP^2, \quad S^2\xrightarrow{p} RP^2=*\times RP^2\hookrightarrow S^2\times RP^2$$

respectively, where $p : S^2\to RP^2$ is the natural covering map. We define σ and τ to be the composites of

$$S^2\times RP^2\xrightarrow{c} S^2\times RP^2\vee S^4\xrightarrow{id\vee\eta^2} S^2\times RP^2\vee S^2\xrightarrow{(id,\, x)} S^2\times RP^2$$

where we take $x=s$ to define σ and $x=t$ to define τ.

Theorem 1. *$\mathcal{E}(S^2\times RP^2)$ is generated by the four generators induced by α, β, σ and τ. The generators are of order 2 and mutually commutative. In particular, $\mathcal{E}(S^2\times RP^2)$ is isomorphic to $\boldsymbol{Z}_2\times\boldsymbol{Z}_2\times\boldsymbol{Z}_2\times\boldsymbol{Z}_2$.*

Since $\mathcal{E}(S^2\times RP^2)$ is a quotient group of the unit group $\mathcal{E}_0(S^2\times RP^2)$ of the based homotopy set $[S^2\times RP^2, S^2\times RP^2]_0$ for the composition, we study first $[S^2\times RP^2, S^2\times RP^2]_0$.

Lemma 1.1. *There is a bijection,*

$$\lambda_1 : \pi_4(S^2)\times[S^2, S^2]_0\times[RP^2, S^2]_0 \longrightarrow [S^2\times RP^2, S^2]_0$$

Proof. Let $X=S^2\times RP^2$ and $Y=S^2$. Then, $Y_2=K(\boldsymbol{Z}, 2)$ and the first k-invariant of S^2, $k_3 : K(\boldsymbol{Z}, 2)\rightarrow K(\boldsymbol{Z}, 4)$ induces the squaring $(k_3)_*=Sq^2 : H^2(X, \boldsymbol{Z}) \rightarrow H^4(X;\boldsymbol{Z})$, i.e.: $(k_3)_*(x)=x^2$, and in particular, $(k_3)_*(H^2(X;\boldsymbol{Z}))=0$. So, in the following exact sequence,

$$H^3(X;\boldsymbol{Z})\longrightarrow [X, Y_3]_0 \xrightarrow{p_*} [X, K(\boldsymbol{Z}, 2)]_0 \xrightarrow{k_*} H^4(X;\boldsymbol{Z}),$$

we see that p_* is a bijection, because $H^3(X;\boldsymbol{Z})=0$. On the other hand, $X^2=S^2\vee RP^2$ and by cohomological calculation, $j^* : [X, K(\boldsymbol{Z}, 2)]_0\rightarrow [S^2\vee RP^2, K(\boldsymbol{Z}, 2)]_0$ is an isomorphism. By a trivial reason, $p_* : [S^2\vee RP^2, Y]_0\rightarrow [S^2\vee RP^2, K(\boldsymbol{Z}, 2)]_0$ is an isomorphism. Hence, we get a natural bijection,

$$[S^2\vee RP^2, Y]_0(\cong[S^2, Y]_0\times[RP^2, Y]_0)\longrightarrow [X, Y_3]_0.$$

Now, we compare the operations of $\pi_4(Y)$ and $H^4(X;\boldsymbol{Z}_2)$ in the following commutative diagram in which each sequence is a part of long exact sequence.

$$\begin{array}{ccccccc}
\pi_4(K(\boldsymbol{Z}_2, 4)) & = & \pi_4(Y) & \longrightarrow & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
H^4(X;\boldsymbol{Z}_2) & \longrightarrow & [X, Y]_0 & \longrightarrow & [X, Y_3]_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & [X^3, Y]_0 & \longrightarrow & [X^3, Y_3]_0 & &
\end{array}$$

The operation of $H^4(X;\boldsymbol{Z}_2)$ on $[X, Y]_0$ is identified with the operation of $\pi_4(Y)$ on $[X, Y]_0$ defined by $g\cdot x=(x, g)\circ c$ for $g\in\pi_4(Y)$ and $x\in[X, Y]_0$.

Choosing some lifting $[X, Y_3]_0\rightarrow[X, Y]_0$, we can define a map, $\lambda_1 : \pi_4(Y)\times [S^2, Y]_0\times[RP^2, Y]_0\rightarrow[X, Y]_0$. To see that λ_1 is a bijection, it suffices to prove that the image of k_u, defined in (0.8), reduces to the zero element of $H^4(X;\pi_4(Y)_u)$ for each map $u : X\rightarrow Y_3$. Observe the following commutative diagram in which the upper horizontal sequence is exact; abbreviating

$$\pi_1'=\pi_1(Map_0(X, K(\boldsymbol{Z}, 3)), 0),\quad \pi_1=\pi_1(\{v\in Map_0(X, Y_3); v\simeq u\}, u)$$

and

$$\pi_1''=\pi_1(\{\underline{v}\in Map_0(X, K(\boldsymbol{Z}, 2)); \underline{v}\simeq p\circ u\}, p\circ u),$$

$$\begin{array}{ccccc} \pi_1' & \xrightarrow{(i_3)_*} & \pi_1 & \xrightarrow{p_*} & \pi_1'' \\ \downarrow & & \downarrow k_u & & \\ H^2(X;\boldsymbol{Z}) & \xrightarrow{Sq^2} & H^4(X;\boldsymbol{Z}_2) & & \end{array}$$

where $Sq^2: H^2(X;\boldsymbol{Z}) \to H^4(X;\boldsymbol{Z}_2)$ is induced by the composition $k_4 \circ i_3: K(\boldsymbol{Z}, 3) \to Y_3 \to K(\boldsymbol{Z}_2, 5)$. (We need only the fact that, if $k_4 \circ i_3$ is essential, then it induces Sq^2.) Then, since $\pi_1'' \cong H^1(X;\boldsymbol{Z})=0$ and $Sq^2=0$, we get $k_u=0$.

q. e. d.

For the even-dimensional real projective space RP^{2n}, we define $u_k(0)$ by the composite of

$$RP^{2n} \xrightarrow{c} RP^{2n} \vee S^{2n} \xrightarrow{0 \vee k} RP^{2n} \vee S^{2n} \xrightarrow{(id,\ p)} RP^{2n}$$

where $0: RP^{2n} \to RP^{2n}$ is the trivial map, $k: S^{2n} \to S^{2n}$ is the degree k map and $p: S^{2n} \to RP^{2n}$ is the natural covering map. In the same way we define $u_k(id)$ by replacing the trivial map with the identity map of RP^{2n}.

Lemma 1.2. *$[RP^{2n}, RP^{2n}]_0$ consists of the following mutually non-homotopic maps: 1) the trivial map, 2) $u_1(0)$, 3) the identity map$=u_0(id)$ and 4) $u_k(id)$ for $k \in \boldsymbol{Z}-\{0\}$. Moreover, $u_k(id)$ and $u_{-1-k}(id)$ are homotopic with the based point moving through the non-trivial element of $\pi_1(RP^{2n})$.*

Proof. The trivial map and the identity map are liftings of two elements 0 and id' respectively of $[RP^{2n}, K(\boldsymbol{Z}_2, 1)]_0$. Since

$$\begin{aligned}\pi_1(Map_0(RP^{2n}, K(\boldsymbol{Z}_2, 1)), id') &\cong \pi_1(Map_0(RP^{2n}, K(\boldsymbol{Z}_2, 1)), 0) \\ &\cong \tilde{H}^0(RP^{2n};\boldsymbol{Z}_2)=0,\end{aligned}$$

the operation of $H^{2n}(RP^{2n};\boldsymbol{Z}_u)$ is effective for $u=id'$ or $u=0$ by (0.8). Remark that $H^{2n}(RP^{2n}, \boldsymbol{Z}_u)$ is isomorphic to $\boldsymbol{Z}$ if $u=id'$ and to $\boldsymbol{Z}_2$ if $u=0$. So, as in the proof of Lemma 1.1 the comparison with the operation of $\pi_{2n}(RP^{2n})$ gives the result on $[RP^{2n}, RP^{2n}]_0$. For the free homotopy classes, we refer the paper of P. Olum [5; Th. IIb, p. 464] q. e. d.

Lemma 1.3. *Unless $n=2$, RP^2 is n-simple.*

Proof. Since the antipodal map on S^3 is homotopic to the identity, RP^3 is a simple space. But by using the fibering $S^1 \to RP^3 \to RP^2$, we see that the operation of $\pi_1(RP^2)$ on $\pi_n(RP^2)$ is induced from that of $\pi_1(RP^3)$ on $\pi_n(RP^3)$ unless $n=2$. q. e. d.

We define the subset T of $[S^2, RP^2]_0 \times [RP^2, RP^2]_0$ by

$$T=T_1 \cup \{0 \times u_k(id);\ k \in \boldsymbol{Z}\}$$

and

$$T_1=\{n \times 0;\ n \in \boldsymbol{Z}\} \cup \{n \times u_1(0);\ n \in \boldsymbol{Z}\}$$

where $n: S^2 \to RP^2$ is the composite of a degree n map: $S^2 \to S^2$ with the natural covering map: $S^2 \to RP^2$.

Lemma 1.4. *There is a bijection,*

$$\lambda_2 : \pi_4(RP^2)\times T \longrightarrow [S^2\times RP^2, RP^2]_0.$$

Proof. Let $X=S^2\times RP^2$ and $Y'=RP^2$. We consider the exact sequence induced by the cofibering $S^2 \xrightarrow{f} S^2 \vee RP^2 \to X^3$,

$$(0=)\pi_3(Y_2') \longrightarrow [X^3, Y_2']_0 \xrightarrow{j^*} [S^2 \vee RP^2, Y_2']_0 \xrightarrow{f^*} \pi_2(Y_2').$$

The fact that f^* is the zero map is easily seen, because $c_1 \circ f: S^2 \to S^2$ and $c_2 \circ f: S^2 \to RP^2$ are zero maps where c_i is the collapsing of S^2 or RP^2 in $S^2 \vee RP^2$ for $i=1$ or 2 respectively. Hence, j^* in the above sequence is a bijection. On the other hand, by a trivial reason, the natural maps $[X, Y_2']_0 \to [X^3, Y_2']_0$ and $[S^2 \vee RP^2, Y']_0 \to [S^2 \vee RP^2, Y_2']_0$ are bijections. Therefore, we have a natural bijection,

$$[S^2, Y']_0\times[RP^2, Y']_0(\cong[S^2 \vee RP^2, Y']_0) \longrightarrow [X, Y_2']_0.$$

Now, we consider the following commutative diagram in which the horizontal sequence is exact.

$$\begin{array}{ccccccc} (0=)H^3(X;\boldsymbol{Z}) & \longrightarrow & [X, Y_3']_0 & \longrightarrow & [X, Y_2']_0 & \xrightarrow{k_*} & H^4(X;\boldsymbol{Z}) \\ & & & & \uparrow i_* & \nearrow Sq^2=0 & \\ & & & & H^2(X;\boldsymbol{Z}) & & \end{array}$$

Any element of the image of i_*, which is exactly the subset T_1 of $[X, Y_2']_0 \cong [S^2 \vee RP^2, Y']_0$, lifts to an element of $[X, Y_3']_0$. The other elements of T, i.e.: $0\times u_k(id)$, are extended over X if they are considered to be maps of $S^2 \vee RP^2$ into $Y'=RP^2$, and hence lift to some elements of $[X, Y_3']$.

We remark that the $p_*: [X, Y']_0 \to [X, Y_3']_0$ is a surjection because $H^5(X;\boldsymbol{Z}_2)=0$. So, if $u': X \to Y_2'$ can be lifted over Y_3', it can be lifted over Y'. This means that the corresponding map $u: S^2 \vee RP^2 \to RP^2$ can be extended over X. The covering map $\bar{u}: S^2 \vee S^2 \to S^2$ in the following diagram may be extended over $S^2\times S^2$.

$$\begin{array}{ccccc} S^2\times S^2 & \supset & S^2 \vee S^2 & \xrightarrow{\bar{u}} & S^2 \\ \downarrow id\times p & & \downarrow id \vee p & & \downarrow p \\ S^2\times RP^2 & \supset & S^2 \vee RP^2 & \xrightarrow{u} & RP^2 \end{array}$$

Here $p: S^2 \to RP^2$ is the standard covering map. Any element of $[S^2 \vee RP^2, Y']_0$ outside T has a form $u=(n, u_k(id)): S^2 \vee RP^2 \to RP^2$ with $n \neq 0$ and induces the map $\bar{u}=(n, 2k+1): S^2 \vee S^2 \to S^2$. So, it cannot be extended over $S^2\times S^2$ because the Whitehead product $[n\iota, (2k+1)\iota]$ does not vanish. This implies that $k_*(u') \neq 0$ unless $u' \in T$.

Therefore, we can define a natural bijection,

$$\lambda_2' : T \longrightarrow [X, Y_3']_0$$

by the inverse of the bijection of $[X, Y_3']_0$ onto the kernel of k_*. Since RP^2 is 4-simple, the definition of λ_2 is the same as that of λ_1 in the proof of Lemma 1.1. By using the homotopy fibering (0.6) as in the proof of Lemma 1.1 we see that the operation of $H^4(X; \boldsymbol{Z}_2)$ is effective on the element u of $[X^3, Y']_0$ which satisfies $f^*u=0$, if the induced map $u_* : \pi_1(X^3) \to \pi_1(Y')$ is trivial. Otherwise, that is, if $u_* : \pi_1(X^3) \to \pi_1(Y')$ is a non-trivial map, we observe the effectivity of the operation of $\pi_4(Y')$ from the homotopy fibering (0.2) with the help of a geometric consideration on the attaching maps of cells. (If $u=\lambda_2'(0\times id)$, Lemma 2.4 reprove the effectivity of the operation of $\pi_4(Y')$. The argument in the proof of Lemma 2.4 is also applicable to the other cases, i.e.: $u=\lambda_2'(0\times u_k(id))$.) Hence, we see that λ_2 is a bijection. q.e.d.

Proof of Theorem 1. Let $f: S^2\times RP^2 \to S^2\times RP^2$ be a based point preserving homotopy equivalence of $S^2\times RP^2$ to itself. Since f induces an isomorphism on $\pi_1(S^2\times RP^2)$, the composition $p_2\circ f$ of f with the projection p_2 on RP^2 is contained in $\lambda_2(\pi_4(RP^2)\times \{0\} \times u_k(id))$. We consider the covering $\bar{f}: S^2\times S^2 \to S^2\times S^2$ of f and know that $\bar{f}$ transforms the basis x and y of the 2-dimensional homology group into $ax+by$ and $cx+(2k+1)y$. Then, $x^2=y^2=(ax+by)^2=(cx+(2k+1)y)^2=0$, $xy=yx=1$ and $(ax+by)(cx+(2k+1)y)=\pm 1$ implies that $a=\pm 1$, $b=c=0$ and $k=0$ or $k=-1$. Hence,

$$[f] \in \lambda_1(\pi_4(S^2)\times \{\pm 1\} \times [RP^2, S^2]_0)\times \lambda_2(\pi_4(RP^2)\times \{0\} \times \{id, u_{-1}(id)\}).$$

Let α_0 be a based point preserving homotopy equivalence free homotopic to α and $\gamma=(id|S^2)\times(u_{-1}(id)|RP^2)$. The operations of $\pi_4(S^2)$ and $\pi_4(RP^2)$ correspond to the composition with σ and τ respectively. The homotopy commutativity of σ or τ with each of α_0, β and γ is easily checked in a geometric way. Each pair consisting of two among α_0, β and γ is also homotopy commutative because the invariants do not depend on the order of composition in each case. We remark here the fact that the invariant of β is the non-trivial element of $[RP^2, S^2]_0$; in fact, the composite of

$$RP^2 = *\times RP^2 \hookrightarrow S^2\times RP^2 \xrightarrow{\beta} S^2\times RP^2 \xrightarrow{p_1} S^2$$

is an essential map, because the operation of $SO(3)$ on S^2 induces an isomorphism: $[RP^2, SO(3)]_0 \to [RP^2, S^2]_0$. Since all the possible 32 combinations of invariants are representable by the compositions of σ, α_0, β, τ and γ, we see that the unit group $\mathcal{E}_0(S^2\times RP^2)$ of $[S^2\times RP^2, S^2\times RP^2]_0$ for the composition is the commutative group generated by α_0, β, γ, σ and τ.

The self homotopy equivalence group $\mathcal{E}(S^2\times RP^2)$ is a quotient group of $\mathcal{E}_0(S^2\times RP^2)$. The $\mathcal{E}(S^2\times RP^2)$ modulo the operation of $\pi_4(S^2\times RP^2)$ consists of id, α, β and $\alpha\circ\beta$ because the factor α is distinguished by the induced automorphism on $H_2(S^2\times RP^2; \boldsymbol{Z})$ and the factor β is distinguished by the map

$RP^2 \to S^2$ obtained by composing the inclusion of RP^2 from the left and the projection on S^2 from the right. We observe moreover that the operation of $\pi_4(S^2 \times RP^2)$ is still effective. In fact, if $\rho \cdot \delta$ is homotopic to δ with the based point moving through the non-trivial element of $\pi_1(S^2 \times RP^2)$ for $\rho \in \pi_4(S^2 \times RP^2)$ and $\delta \in \mathcal{E}_0(S^2 \times RP^2)$, then $\rho \cdot \delta$ is based homotopic to $\delta \circ \gamma$. But this is impossible because $\rho \cdot \delta$ and $\delta \circ \gamma$ induce the different elements in $[X^3, S^2 \times RP^2]_0$.

q. e. d.

§2. Homotopy smoothings of $S^2 \times RP^2$

A homotopy smoothing of a Poincaré complex M is by definition a simple homotopy equivalence $f : Q \to M$ such that Q is a closed smooth manifold. The normal invariant $\eta(f)$ is defined by the class of the following induced normal map with the natural trivialization of $\nu(Q) \oplus \tau(Q)$,

$$\begin{array}{ccccc} \nu(Q) & \xrightarrow{\bar{f}} & (f^{-1})^* \nu(Q) & = & \nu \\ \downarrow & & \downarrow & & \downarrow \\ Q & \xrightarrow{f} & M & = & M \end{array}$$

where $\nu(Q)$ is the stable normal bundle and $\tau(Q)$ is the tangent bundle. Two normal maps (suffixed by 1 and 2) are of the same class if there exists a bundle equivalence $\phi : \nu_1 \to \nu_2$ such that $(f_1, \phi\nu_1)$ and (f_2, ν_2) are normally cobordant. If M is a closed smooth manifold, we consider that $\eta(f)$ is an element of $[M, G/O]$ by the Sullivan's argument [13]. The smooth normal invariant $\eta(f)$ induces the topological normal invariant $\eta_{TOP}(f)$ by the natural map $G/O \to G/TOP$. The normal invariant $\eta(f)$ depends only on the homotopy class of f. In the case that $M = S^2 \times RP^2$, there is no difference between homotopy equivalences and simple homotopy equivalences, because $Wh(Z_2) = 0$.

Proposition 2. *Let* $M = S^2 \times RP^2$.

(i) $\eta(\alpha^a \circ \beta^b \circ \sigma^s \circ \tau^t) = 0$ *if and only if* $s \equiv t \equiv 0 \pmod 2$.

Moreover, $\eta(\sigma)$, $\eta(\tau)$ *and* $\eta(\sigma \circ \tau)$ *are all distinct.*

(ii) $\eta_{TOP}(\alpha^a \circ \beta^b \circ \sigma^s \circ \tau^t) = 0$ *if and only if* $s \equiv 0 \pmod 2$.

Corollary 2.1. *Only the four homotopy classes id,* α, β *and* $\alpha \circ \beta$ *are representable by diffeomorphisms.*

Remark. We do not know whether τ is representable by a homeomorphism or not.

Corollary 2.2. *For any homotopy smoothing* $f : Q^4 \to S^2 \times RP^2$, *there exists a smooth s-cobordism between* Q^4 *and* $S^2 \times RP^2$.

Proof of Corollaries assuming Proposition 2. Corollary 2.1 is immediate from Theorem 1 and Proposition 2, because the four classes are representable by diffeomorphisms and any class represented by a diffeomorphism must have the trivial normal invariant.

For Corollary 2.2, we observe the Sullivan-Wall sequence associated to the surgery theory,

$$(0=)L_5(Z_2^-) \longrightarrow \mathcal{S}(S^2\times RP^2) \xrightarrow{\eta} [S^2\times RP^2, G/O] \xrightarrow{\theta} L_4(Z_2^-)(\cong Z_2)$$

where $\mathcal{S}(S^2\times RP^2)$ is the degree 1 homotopy smoothing classes modulo s-cobordism (Cf. [15; §16]). The sequence may not be exact at $[S^2\times RP^2, G/O]$, but $L_5(Z_2^-)=0$ implies that $\eta: \mathcal{S}(S^2\times RP^2)\to[S^2\times RP^2, G/O]$ is a one-to-one map into the kernel of θ. The Wall group $L_4(Z_2^-)$ is isomorphic to Z_2 and the obstruction map θ is a surjection because θ is given by the Kervaire-Arf invariant $c(g)=k_2(g)w_2(S^2\times RP^2)+k_2(g)^2$ for $g: S^2\times RP^2\to G/O$, where k_2 is a characteristic class defined by the generator of $H^2(G/O; Z_2)$ [15; Th. 13B. 5]. We remember that for the 5-th stages of the Postnikov decompositions of G/TOP and G/O,

$$(G/TOP)_5\simeq K(Z_2, 2)\times K(Z, 4),$$

and

$$K(Z, 4) \longrightarrow (G/O)_5 \longrightarrow K(Z_2, 2)$$

is a fibering with $\delta Sq^2\in H^5(K(Z_2, 2); Z)$ as k-invariant [10] [13]. So, we have another sequence of groups which is exact,

$$[S(S^2\times RP^2), G/TOP] \longrightarrow H^3(S^2\times RP^2; Z_2) \longrightarrow [S^2\times RP^2, G/O]$$
$$\longrightarrow [S^2\times RP^2, G/TOP] \longrightarrow H^4(S^2\times RP^2; Z_2),$$

where $[S^2\times RP^2, G/TOP]$ can be identified with $H^2(S^2\times RP^2; Z_2)\oplus H^4(S^2\times RP^2; Z)$ and the last homomorphism is a surjection because (x, y) is mapped to $x^2+y \bmod 2$. In the same way the first homomorphism is a null-map and hence the second one is injective. So, $[S^2\times RP^2, G/O]$ has 8 elements and $\ker\theta$ consists of 4 elements, which must be 0, $\eta(\sigma)$, $\eta(\tau)$ and $\eta(\sigma\circ\tau)$. Therefore, the Sullivan-Wall sequence above is actually exact and the natural map,

$$\{f\in\mathcal{E}(S^2\times RP^2); \deg f=1\} \longrightarrow \mathcal{S}(S^2\times RP^2)$$

is a surjection. Since α is represented by a degree -1 diffeomorphism h of $S^2\times RP^2$ to itself, any degree -1 homotopy smoothing is representable by the composite of some degree 1 homotopy smoothing with h. This suffices to prove Corollary 2.2. q. e. d.

The proof of Proposition 2 is carried out by the following four lemmas.

Lemma 2.1. *Let $f_i: S^2\times RP^2\to S^2\times RP^2$ be homotopy equivalence for $i=0, 1$.*

(i) *If $\eta(f_0)=0$, then $\eta(f_0\circ f_1)=\eta(f_1)$.*

(ii) *If $\eta_{TOP}(f_0)=0$, then $\eta_{TOP}(f_0\circ f_1)=\eta_{TOP}(f_1)$.*

Proof. If $\eta(f_0)=0$, there exists a normal cobordism connecting f_0 and *id*. Hence, composing with f_1, we get a normal cobordism between $f_0 \circ f_1$ and f_1. In particular, $\eta(f_0 \circ f_1)=\eta(f_1)$. The same proof applies for η_{TOP} as well.

q. e. d.

Lemma 2. 2. $\eta(\alpha)=\eta(\beta)=0$.

Proof. Since α and β are represented by diffeomorphisms, $\eta(\alpha)=\eta(\beta)=0$ by the definition of η. q. e. d.

Lemma 2. 3. $\eta_{TOP}(\sigma)\neq 0$.

Proof. We remark that the Sullivan's theory of characteristic variety remains valid in this case [13], [13′]. Actually, σ is detected by the characteristic variety $*\times RP^2$ of $S^2\times RP^2$. In fact, $\sigma^{-1}(*\times RP^2)$ may be assumed to be $W\cup *\times RP^2$ with W framed, where W is the preimage of one point under the generator $\eta^2 : S^4 \rightarrow S^2$ of $\pi_4(S^2)\cong \boldsymbol{Z}_2$. The splitting invariant is the Arf invariant of the framed W, which is equal to one. q. e. d.

Lemma 2. 4. $\eta_{TOP}(\tau)=0$ *and* $\eta(\tau)\neq 0$.

Proof. To see that $\eta_{TOP}(\tau)=0$, we have only to calculate the splitting invariants along the characteristic varieties $S^2\times *$ and $*\times RP^2$ of $S^2\times RP^2$, because $\eta_{TOP}(\tau)$ belongs to the image of the natural map $[S^2\times RP^2, G/O]\rightarrow [S^2\times RP^2, G/TOP]$ which corresponds to the subgroup $H^2(S^2\times RP^2; \boldsymbol{Z}_2)$ of $H^2(S^2\times RP^2; \boldsymbol{Z}_2)\oplus H^4(S^2\times RP^2; \boldsymbol{Z})$. (If one does not like this reasoning for $\eta_{TOP}(\tau)=0$, one can take another argument which will be explained in Remark succeeding the proof of this Lemma.)

On the other hand, we shall use the S-theory to verify that $\eta(\tau)\neq 0$. At first, we observe that the Thom space $T\nu(S^2\times RP^2)$ of the stable normal bundle of $S^2\times RP^2$ has the homotopy type of

$$S^n(S^2\times RP^3/*\times RP^3)\simeq S^{n+2}RP^3\vee S^nRP^3\simeq S^{n+2}RP^2\vee S^{n+5}\vee S^nRP^2\vee S^{n+3}.$$

(The suspension SRP^3 is the mapping cone of the suspension of the natural 2-covering map $S^2\rightarrow RP^2$ and has the homotopy type of $SRP^2\vee S^3$ because the Steenrod operation Sq^2 is trivial.) The possible normal invariants are the degree ± 1 maps of S^{n+5} into $T\nu(S^2\times RP^2)$. We have the S-duality given by a map $u : T\nu(S^2\times RP^2)\wedge S^m(S^2\times RP^2_+)\rightarrow S^{m+n+5}$. By this map $[S^{n+5}, S^n(S^2\times RP^3/*\times RP^3)]_0$ corresponds bijectively to the cohomotopy set $[S^{m+n+5}(S^2\times RP^2_+), S^{m+n+5}]_0$. Moreover, since $m+n$ is sufficiently large, the subset consisting of degree 1 maps corresponds to the classes $[S^2\times RP^2, G]$ of the stable fiber homotopy trivializations (Cf. [16; Th. 3. 5]). Because the restriction of the S-duality map $u : (S^{n+2}RP^3\vee S^nRP^3)\wedge S^m(S^2\times RP^2_+)\rightarrow S^{m+n+5}$ on $S^{n+2}RP^3\wedge S^m(RP^2_+)$ gives the S-duality map for RP^2_+, the subset $[RP^2, G]$ corresponds to the maps,

$$S^{n+5}\xrightarrow{g\vee id} S^{n+2}RP^2\vee S^{n+5}\subset T\nu(S^2\times RP^2),$$

with $g\in \pi_{n+5}(S^{n+2}RP^2)(\cong \boldsymbol{Z}_4)$. (The calculation of the order of $\pi_{n+i}(S^{n+2}RP^2)$ is easily reduced to that of stable homotopy groups of spheres by the exact sequence of homotopy groups associated to the cofibering $S^{n+3} \xrightarrow{2} S^{n+3} \to S^{n+2}RP^2$ in the suspension category [12].) The group $[S^2 \times RP^2, G]$ has 16 elements and the other generators correspond to the following two maps,

$$\omega : S^{n+5} \xrightarrow{\eta^2 \vee id} S^{n+3} \vee S^{n+5} \subset T\nu(S^2 \times RP^2)$$

and

$$\tau_0 : S^{n+5} \xrightarrow{t_0 \vee id} S^n RP^2 \vee S^{n+5} \subset T\nu(S^2 \times RP^2)$$

where η^2 is the generator of $\pi_{n+5}(S^{n+3}) \cong \boldsymbol{Z}_2$ and t_0 is the generator of $\pi_{n+5}(S^n RP^2) \cong \boldsymbol{Z}_2$.

The homotopy equivalence $\tau : S^2 \times RP^2 \to S^2 \times RP^2$ can be coverd by a vector bundle homomorphism as follows; the maps in the upper line of the following diagram restrict linear on each fiber of the induced normal vector bundles and their composite covers the composite of the maps in the lower line.

$$\begin{array}{ccccccc}
S^2 \times RP^3 & \xrightarrow{c} & S^2 \times RP^3 \vee S^5 & \xrightarrow{id \vee S\eta^2} & S^2 \times RP^3 \vee S^3 & \xrightarrow{(id,\, p)} & S^2 \times RP^3 \\
\cup & & \cup & & \cup & & \cup \\
S^2 \times RP^2 & \xrightarrow{c} & S^2 \times RP^2 \vee S^4 & \xrightarrow{id \vee \eta^2} & S^2 \times RP^2 \vee S^2 & \xrightarrow{(id,\, p)} & S^2 \times RP^2
\end{array}$$

Here, p denotes either the natural covering $S^3 \to RP^3$ or $S^2 \to RP^2$. Therefore, the induced map of $T\nu(S^2 \times RP^2)$ to itself determines the class of the map

$$\tau_1 : S^{n+5} \xrightarrow{S^n h \vee id} S^n RP^3 \vee S^{n+5} \subset T\nu(S^2 \times RP^2)$$

to which $\eta(\tau)$ belongs modulo the image of $[S^2 \times RP^2, 0]$, where

$$h = p \cdot S\eta^2 : S^5 \xrightarrow{S\eta^2} S^3 \xrightarrow{p} RP^3.$$

It is easy to see that the composite of

$$S^6 \xrightarrow{Sh} SRP^3 \xrightarrow{\simeq} SRP^2 \vee S^4 \xrightarrow{c_1} S^4$$

is homotopic to the trivial map, where c_1 is the collapsing of SRP^2. We observe the homotopy commutativity of the following diagram from the fact that the mapping cone of Sp is SRP^4 and $Sq^2 : H^3(SRP^4; \boldsymbol{Z}_2) \to H^5(SRP^4; \boldsymbol{Z}_2)$ is non-trivial:

$$\begin{array}{ccccccc}
S^4 & \xrightarrow{Sp} & SRP^3 & \xrightarrow{\simeq} & SRP^2 \vee S^4 & \xrightarrow{c_2} & SRP^2 \\
\| & & & & & & \downarrow Sd \\
S^4 & & & \xrightarrow{\quad \eta \quad} & & & S^3
\end{array}$$

Here Sp is the suspension of the natural covering map, c_2 is the collapsing of S^4, Sd is the suspension of the collapsing of the subset RP^1 of RP^2 and η is the suspension of the Hopf map. Let $t_1 = S^{n-1}c_2 \circ S^n h : S^{n+5} \to S^n RP^2$. Then, $Sd \circ c_2 \circ Sp \simeq \eta$ implies $S^n d \circ t_1 \simeq \eta^3$, which means that t_1 is the generator of $\pi_{n+5}(S^n RP^2)$. In fact, in the following long exact sequence associated to the cofibering $S^{n+1} \xrightarrow{2} S^{n+1} \to S^n RP^2$ in the suspension category, ∂ is induced by $S^{n-1}d$ and η^3 is the 12 times of the generator of $\pi_{n+5}(S^{n+2}) \cong \boldsymbol{Z}_{24}$ (Cf. [14]).

$$(0=)\,\pi_{n+5}(S^{n+1}) \longrightarrow \pi_{n+5}(S^n RP^2) \xrightarrow{\partial} \pi_{n+5}(S^{n+2}) \xrightarrow{\times 2} \pi_{n+5}(S^{n+2})$$

Hence, $t_1 = t_0 \in \pi_{n+5}(S^n RP^2)$ and in particular, $\tau_1 \simeq \tau_0$.

We consider now the following commutative diagram whose horizontal sequences are exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \boldsymbol{Z}_2 & \longrightarrow & \boldsymbol{Z}_4 & \longrightarrow & \boldsymbol{Z}_2 & \longrightarrow & 0 \\
 & & \uparrow \cong & & \uparrow \cong & & \uparrow \cong & & \\
 & & [RP^2, O] & \longrightarrow & [RP^2, G] & \longrightarrow & [RP^2, G/O] & & \\
 & & \downarrow \cong & & \cap & & \cap & & \\
 & & [S^2 \times RP^2, O] & \longrightarrow & [S^2 \times RP^2, G] & \xrightarrow{\mu} & [S^2 \times RP^2, G/O] & & \\
 & & & & \uparrow \cong & & & & \\
 & & & & \{f \in \pi_{n+5}(T\nu(S^2 \times RP^2))\,;\, \deg f = 1\} & & & &
\end{array}$$

Then, μ in the diagram above induces an injection,

$$\begin{aligned} &\mathrm{Coker}([RP^2, G] \longrightarrow [S^2 \times RP^2, G]) \\ &\qquad \longrightarrow \mathrm{Coker}([RP^2, G/O] \longrightarrow [S^2 \times RP^2, G/O]), \end{aligned}$$

which turns out to be a bijection because $[S^2 \times RP^2, G]$ consists of 16 elements and $[S^2 \times RP^2, G/O]$ of 8 elements. In particular, $\mu(\tau_0) \neq 0$. But $\mu(\tau_0)$ is identified with $\eta(\tau)$ because the normal map induced by τ is shown to represent $\mu(\tau_0)$. q. e. d.

Remark. We can verify that $\theta(\mu(\omega)) \neq 0$, if we use the formula for the Kervaire-Arf invariant and the projection $S^2 \times RP^2 \to S^2$. Hence, the image of $\ker \theta$ by the natural map $[S^2 \times RP^2, G/O] \to [S^2 \times RP^2, G/TOP]$ consists of two distinct elements $\eta_{TOP}(\sigma)$ and 0. Then, since $\eta_{TOP}(\tau)$ is not detected by $* \times RP^2$, $\eta_{TOP}(\tau)$ must be equal to 0.

The proof of Proposition 2 is immediate from Lemmas 2.1-2.4. In fact, $\eta(\alpha^a \circ \beta^b \circ \sigma^s \circ \tau^t) = \eta(\sigma^s \circ \tau^t)$ and $\eta_{TOP}(\alpha^a \circ \beta^b \circ \sigma^s \circ \tau^t) = \eta_{TOP}(\sigma^s)$. And $\eta_{TOP}(\sigma \circ \tau) \neq 0$ implies $\eta(\sigma \circ \tau) \neq 0$. Also, $\eta(\sigma) \neq \eta(\tau)$ and $\eta(\sigma \circ \tau) \neq \eta(\tau)$ because $\eta_{TOP}(\sigma \circ \tau) = \eta_{TOP}(\sigma) \neq \eta_{TOP}(\tau)$. Since the operation of $H^3(S^2 \times RP^2; \boldsymbol{Z}_2)$ on $[S^2 \times RP^2, G/O]$ is effective and moreover identified with the composition with τ, $\eta(\sigma \circ \tau) \neq \eta(\sigma)$.

§ 3. Smooth s-cobordism of $S^1\times RP^2$ to itself

We shall study the homotopy smoothings of $S^1\times RP^2\times I$ relative to the boundary. As a result we shall get the following proposition.

Proposition 3. *Any smooth s-cobordism of $S^1\times RP^2$ to itself is smoothly s-cobordant to the product cobordism $S^1\times RP^2\times I$ relative to the boundary.*

Let τ' and σ'' be the following composite of

$$\tau' : S^1\times RP^2\times I \xrightarrow{c} S^1\times RP^2\times I \vee S^4 \xrightarrow{(id,\, t)} S^1\times RP^2\times I$$

and

$$\sigma'' : RP^2\times I \xrightarrow{c} RP^2\times I \vee S^3 \xrightarrow{(id,\, s')} RP^2\times I$$

where t is the generator of $\pi_4(RP^2)\cong \boldsymbol{Z}_2$ and s' is a generator of $\pi_3(RP^2)\cong \boldsymbol{Z}$. We define a map $\sigma' : S^1\times RP^2\times I \to S^1\times RP^2\times I$ by $\sigma'=(id|S^1)\times\sigma''$. Since σ' and τ' induce the identity on the homotopy groups, they are homotopy equivalences. Moreover, since σ' and τ' induce the identity on the homology group of the universal covering, they are simple homotopy equivalences which restrict to the identity on the boundary. In other words, they are homotopy smoothings of $S^1\times RP^2\times I$ relative to the boundary.

The following Sullivan-Wall sequence is fundamental in our argument.

$$[S^1\times RP^2\times I\times I/\partial, G/O] \xrightarrow{\theta} L_5(\boldsymbol{Z}\oplus \boldsymbol{Z}_2^-) \longrightarrow \mathcal{S}(S^1\times RP^2\times I, \partial)$$
$$\xrightarrow{\eta} [S^1\times RP^2\times I/\partial, G/O] \xrightarrow{\theta} L_4(\boldsymbol{Z}\oplus \boldsymbol{Z}_2^-)$$

We consider at first the collapsing j of the complement of the embedding $S^1\times RP^2\times I \subset S^2\times RP^2$ and the following induced commutative diagram.

$$\begin{array}{ccc} \mathcal{S}(S^1\times RP^2\times I, \partial) & \longrightarrow & [S^1\times RP^2\times I/\partial, G/O] \\ \downarrow j^* & & \downarrow j^* \\ \mathcal{S}(S^2\times RP^2) & \longrightarrow & [S^2\times RP^2, G/O] \end{array}$$

Then, since $j^*(\text{the class of } \tau')=\text{the class of } \tau$, $j^*(\eta(\tau'))=\eta(\tau)$. Hence, $\eta(\tau)\neq 0$ implies $\eta(\tau')\neq 0$. To see that $\eta(\sigma')\neq 0$, we use the same method as in the proof of the fact that $\eta(\tau)\neq 0$. We consider the homotopy equivalence $\sigma'' \cup (id|RP^2\times I) : RP^2\times S^1 \to RP^2\times S^1$ which is covered by the normal vector bundle homomorphism in the following diagram.

$$\begin{array}{ccccccc} RP^3\times S^1 & \xrightarrow{c} & RP^3\times S^1 \vee S^4 & \xrightarrow{id\vee S\eta} & RP^3\times S^1 \vee S^3 & \xrightarrow{(id,\, p)} & RP^3\times S^1 \\ \cup & & \cup & & \cup & & \cup \\ RP^2\times S^1 & \xrightarrow{c} & RP^2\times S^1 \vee S^3 & \xrightarrow{id\vee \eta} & RP^2\times S^1 \vee S^2 & \xrightarrow{(id,\, p)} & RP^2\times S^1 \end{array}$$

The normal invariant $\eta(\sigma'' \cup (id|RP^2\times I))$ comes from the element corresponding to the map,

$$\sigma_1 : S^{n+5} \xrightarrow{s'' \vee id} S^{n+1}RP^2 \vee S^{n+5} \subset T\nu(RP^2\times S^1)$$

where n is sufficiently large and $s''=c_2\circ S^{n+1}(p\circ S\eta)$, c_2 denoting the collapsing of the subset S^{n+4} of $S^{n+1}RP^2 \vee S^{n+4} \simeq S^{n+1}RP^3$. Whence, s'' is the element of $\pi_{n+5}(S^{n+1}RP^2)$ such that $S^{n+2}d\circ s'' \simeq \eta^2$ in $\pi_{n+5}(S^{n+3})$, where d denotes the collapsing of the subset RP^1 of RP^2. We observe the following commutative diagram,

$$\begin{array}{ccccc} s''\in \pi_{n+5}(S^{n+1}RP^2) & \xrightarrow{\cong} & [RP^2\times I/\partial, G] & \longrightarrow & [RP^2\times I/\partial, G/O] \\ \downarrow \quad \downarrow & & \downarrow & & \downarrow \\ \eta^2\in \pi_{n+5}(S^{n+3}) & \xrightarrow{\cong} & [S^2, G] & \longrightarrow & [S^2, G/O] \end{array}$$

where the first horizontal isomorphisms are the S-dualities associated to the spaces S^2 and SRP^2 and the second horizontal maps are induced by the natural map $G\to G/O$. The commutativity of the first square follows because the inclusion $S^2=SRP^1\to SRP^2$ is the S-dual of the collapsing of the subset SRP^1 of SRP^2. We see easily that $[RP^2\times I/\partial, G/O]\cong \boldsymbol{Z}_2$ and its generator is the image of s''. In particular, $\eta(\sigma'' \cup (id|RP^2\times I))=(j_1)^*\eta(\sigma'')$ for $j_1 : RP^2\times S^1\to RP^2\times I/\partial$, and $\eta(\sigma'')$ is the generator of $[RP^2\times I/\partial, G/O]$. So, $\eta(\sigma')\neq 0$ because the map $[RP^2\times I/\partial, G/O]\to[S^1\times RP^2\times I/\partial, G/O]$ induced by crossing with S^1 is injective.

The natural map $[RP^2\times I/\partial, G/O]\to[RP^2\times I/\partial, G/TOP]$ is a bijection because the both sides of the map are naturally isomorphic to $H^2(SRP^2; \boldsymbol{Z}_2)$. Hence, $\eta_{TOP}(\sigma')\neq 0$ as well. On the other hand, it is not difficult to see that $\eta_{TOP}(\tau')=0$, if we use the fact that $j^*\eta_{TOP}(\tau')=\eta_{TOP}(\tau)=0$. Therefore, as in the proof of Proposition 2, the four elements 0, $\eta(\sigma')$, $\eta(\tau')$ and $\eta(\sigma'\circ\tau')$ are all distinct.

The Wall group $L_4(\boldsymbol{Z}\oplus \boldsymbol{Z}_2^-)$ is isomorphic to $\boldsymbol{Z}_2$ and the surgery obstruction map $\theta : [S^1\times RP^2\times I/\partial, G/O]\to L_4(\boldsymbol{Z}\oplus \boldsymbol{Z}_2^-)$ is given by the Kervaire-Arf invariant $c(S^1\times RP^2\times I, \partial ; g)=k_2(g)w_2(S^1\times RP^2\times I)+k_2(g)^2$ where $g : S^1\times RP^2\times I/\partial \to G/O$ and k_2 is a characteristic class defined by the generator of $H^2(G/O ; \boldsymbol{Z}_2)$. By a simple calculation, θ is a surjection and $[S^1\times RP^2\times I/\partial, G/O]$ consists of 8 elements. Hence $\ker\theta$ consists of 4 elements which must coincide with 0, $\eta(\sigma')$, $\eta(\tau')$ and $\eta(\sigma'\circ\tau')$. Similarly, the surgery obstruction map for $[RP^2\times I \times I/\partial, G/O]$ is a surjection onto $L_4(\boldsymbol{Z}_2^-)\cong \boldsymbol{Z}_2$. By the following commutative diagram

$$\begin{array}{ccc} [RP^2\times I\times I/\partial, G/O] & \longrightarrow & L_4(\boldsymbol{Z}_2^-) \\ \downarrow & & \downarrow \cong \\ [S^1\times RP^2\times I\times I/\partial, G/O] & \longrightarrow & L_5(\boldsymbol{Z}\oplus \boldsymbol{Z}_2^-) \end{array}$$

where the vertical maps are induced by taking direct product with S^1 and the induced map $L_4(\boldsymbol{Z}_2^-)\rightarrow L_5(\boldsymbol{Z}\oplus\boldsymbol{Z}_2^-)$ is an isomorphism, the surgery obstruction map for $[S^1\times RP^2\times I\times I/\partial, G/O]$ is a surjection onto $L_5(\boldsymbol{Z}\oplus\boldsymbol{Z}_2^-)$ as well. Since the Sullivan-Wall sequence is exact at $L_5(\boldsymbol{Z}\oplus\boldsymbol{Z}_2^-)$, the operation of $L_5(\boldsymbol{Z}\oplus\boldsymbol{Z}_2^-)$ on $\mathcal{S}(S^1\times RP^2\times I, \partial)$ is trivial. Therefore, we have proved the following proposition.

Proposition 3.1. *$\mathcal{S}(S^1\times RP^2\times I, \partial)$ consists of the four distinct classes represented by the simple homotopy equivalences id, σ', τ' and $\sigma'\circ\tau'$ of $S^1\times RP^2\times I$ to itself which restrict to the identity on the boundary.*

Proof of Proposition 3. If Q is a smooth s-cobordism of $S^1\times RP^2$ to itself, we have some simple homotopy equivalence

$$f: Q\longrightarrow S^1\times RP^2\times I$$

which restricts to a diffeomorphism on the boundary. Then, by Proposition 3.1 the homotopy smoothing f is of the same class in $\mathcal{S}(S^1\times RP^2\times I, \partial)$ as a simple homotopy equivalence of $S^1\times RP^2\times I$ to itself which restricts to the identity on the boundary. This implies that there exists a smooth s-cobordism between Q and $S^1\times RP^2\times I$ which restricts to $(S^1\times RP^2\times\partial I)\times I$ on the boundary, where the boundary of Q is identified with $S^1\times RP^2\times\partial I$ by $f|\partial Q$. q. e. d.

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References for §0 and §1

[1] W. D. Barcus and M. G. Barratt: On the homotopy classification of the extensions of a fixed map, Trans. Amer. Math. Soc. **88** (1958), 57-74.
[2] I. James and E. Thomas: On the enumeration of cross-sections, Topology **5** (1966), 95-114.
[3] P. May: Simplicial objects in algebraic topology, D. Van Nostrand, Princeton, 1967.
[4] P. Olum: Obstructions to extensions and homotopies, Ann. of Math. **52** (1950), 1-50.
[5] ———: Mappings of manifolds and the notion of degree, Ann. of Math. **58** (1953), 458-480.
[6] J. W. Rutter: A homotopy classification of maps into an induced fibre space, Topology **6** (1967), 379-403.
[7] R. Thom: L'homologie des espaces fonctionnels, Colloque de topologie algébrique tenu à Louvain 1956, Georges Thone, Liège (1957), 29-39.

References for §2 and §3

[8] M. F. Atiyah: Thom complexes, Proc. London Math. Scc. (3), **11** (1961), 291-310.
[9] S. E. Cappell and J. L. Shaneson: Some new four-manifolds, Ann. of Math. **104** (1976), 61-72.
[10] R. Kirby and L. C. Siebenmann: Foundational essays on topological manifolds, smoothings, and triangulations, Princeton Univ. Press, Princeton, 1977.

[11] T. Matumoto and L.C. Siebenmann: The topological s-cobordism theorem fails in dimension 4 or 5, Math. Proc. Camb. Phil. Soc. **84** (1978), 85-87.

[12] E.H. Spanier and J.H.C. Whitehead: The theory of carriers and S-theory, Algebraic geometry and topology (a Symposium in honor of S. Lefshetz), Princeton Univ. Press, Princeton (1957), 330-360.

[13] D. Sullivan: Triangulating and smoothing homotopy equivalences and homeomorphisms, Geometric topology seminar notes, Princeton Univ., 1967.

[13'] ———: Triangulating homotopy equivalences, Ph. D. thesis, Princeton Univ., 1965.

[14] H. Toda Composition methods in homotopy groups of spheres, Princeton Univ. Press, Princeton, 1962.

[15] C.T.C. Wall: Surgery on compact manifolds. Academic Press, London, 1971.

[16] ———: Poincaré complex I, Ann. of Math. **86** (1967), 213-245.