On a decomposition of Thom-Boardman singularities of order two

By

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Dedicated to Professor A. Komatu on his 70th birthday

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§0. Introduction

In this paper we consider the space of differentiable maps having no Thom-Boardman singularities of certain type and a decomposition of Thom-Boardman singularities.

Let Ω be an open subbundle of a jet bundle $J^r(N, P)$ over N . A differentiable map $f: N \rightarrow P$ is called Ω -regular, if the image of $j^r f: N \rightarrow J^r(N, P)$ is contained in Ω . Let $C_\Omega^r(N, P)$ be the space of Ω -regular differentiable maps with C^r -topology and $\Gamma_N(\Omega)$ the space of differentiable cross-sections of the subbundle Ω over N with compact-open topology. If $j^r: C_\Omega^r(N, P) \rightarrow \Gamma_N(\Omega)$ is a weak homotopy equivalence, then Ω is called *integrable*.

Let I be an arbitrary series of r non-negative integers. We shall denote by Σ^I the Thom-Boardman singularities of the jet bundle $J^r(N, P)$ with symbol I . Let Ω^I be the union of all Thom-Boardman singularities Σ^K with symbol K which is smaller than or equal to I under the lexicographic order.

We denote $n = \dim N$ and $p = \dim P$. In [4], A. du Plessis has shown the following theorem.

Theorem ([4]). *Let $I = (i_1, \dots, i_r)$ and let d^I be the number of j 's such that $i_j - i_{j+1} > 1$. Then the inequality $i_r > n - p - d^I$ implies that Ω^I is integrable.*

The purpose of the present paper is to show how the subbundle Ω^I fails to be integrable and how the map $j^r: C_{\Omega^I}^r(N, P) \rightarrow \Gamma_N(\Omega^I)$ is close to a weak homotopy equivalence when $i_r \leq n - p - d^I$.

Theorem. *Let $r=2$ and $I=(i, j)$. If $p - n + i = 2$, then there exists the following decomposition $\{\Sigma^I(t, c): t \text{ and } c \text{ are non-negative integers}\}$ of the Thom-Boardman singularities Σ^I of $J^2(N, P)$:*

- i) $\Sigma^I(t, c)$ is a regular submanifold of $J^2(N, P)$,
- ii) $\Sigma^I(t, c)$ is non-empty if and only if
 - a) Σ^I is non-empty
 - b) either $c = t = i - j$

or $c < t < i - j$, $t \leq 2c$ and $i - j - c \leq 2(t - c)$,

iii) the codimension of $\Sigma^I(t, c)$ in $J^2(N, P)$ is

$$(i - j - c)(i - j + 2c - 2t + 1) - (t - c)(2c - t) + 2i + j(i + 1),$$

iv) Σ^I is the union of all $\Sigma^I(t, c)$,

v) $\Omega^I - \bigcup_{\substack{h \leq j \\ j - i + t < 0 \\ c \leq t}} \Sigma^{i, h}(t, c)$ is integrable.

Corollary. Let $I = (i, 0)$ and $p - n + i \geq 2$. Then the induced homomorphism $(j^2)_*: \pi_k(C_{\Omega^I}^\infty(N, P)) \rightarrow \pi_k(\Gamma_N(\Omega^I))$ is isomorphic for $k < 3i - n$ and surjective for $k = 3i - n$.

For general r we shall obtain Theorem 4.1 which is a direct consequence of Transversality lemma due to A. du Plessis [5] and his Theorem ([4]). It does not follow from Theorem 4.1 that $(j^2)_*$ is surjective for $k = 3i - n$ in the above corollary.

Let $\widetilde{\Omega^I}$ be the union of all Thom-Boardman singularities of $J^r(N \times \mathbf{R}, P)$. Let $i: J^r(N \times \mathbf{R}, P) \rightarrow J^r(N, P)$ be the canonical map induced from the inclusion of N into $N \times 0$. Then we know that $i(\widetilde{\Omega^I})$ is integrable and contained in Ω^I .

The decomposition of the singularities Σ^I into the submanifolds $\Sigma^K(t, c)$ is constructed in §1 and §2. We show in §3 that $i(\widetilde{\Omega^I})$ is represented as the disjoint union of certain submanifolds $\Sigma^K(t, c)$. By the transversality argument we obtain the above corollary.

Throughout the paper all manifolds and differentiable maps should be of class C^∞ and all manifolds should satisfy the second axiom of countability.

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§1. Preliminaries and lemmas

We begin by recalling the definition of Thom-Boardman singularities of $J^2(n, p)$ (see, for example, [7]). Let $\mathbf{R}$ be the field of real numbers. Let $\mathbf{R}[x_1, \dots, x_n]$ be the polynomial ring over $\mathbf{R}$ with n variables $x_1, \dots, x_n$ and $\mathfrak{M}$, its unique maximal ideal $(x_1, \dots, x_n)$.

Let $\mathfrak{I}$ be an ideal of $\mathfrak{M}/\mathfrak{M}^3$. The *Jacobian extension* of $\mathfrak{I}$ is defined as follows. Let $f_1, \dots, f_a$ be a set of generators of $\mathfrak{I}$. Let $\mathfrak{I}'$ denote the ideal generated by all $s \times s$ minor determinants of the Jacobian matrix $(\partial f_i / \partial x_j)$. Then we define $\Delta^s \mathfrak{I} = \mathfrak{I} + \mathfrak{I}'$. We call a set $y_1, \dots, y_n$ of $\mathfrak{M}/\mathfrak{M}^3$ a *system of coordinates*, if $y_1, \dots, y_n$ span $\mathfrak{M}/\mathfrak{M}^3$. If $y_1, \dots, y_n$ is a system of coordinates for $\mathfrak{M}/\mathfrak{M}^3$, then $\Delta^s \mathfrak{I}$ is generated by $f_1, \dots, f_a$ and all $s \times s$ minor determinants of Jacobian matrix $(\partial f_i / \partial y_j)$.

The *rank* of a proper ideal $\mathfrak{I}$ denoted by $\text{rk } \mathfrak{I}$, is defined as the dimension of $\mathfrak{M}^2 + \mathfrak{I}/\mathfrak{M}^2$. When $\text{rk } \mathfrak{I}$ is s , we write $\Delta^{s+1} \mathfrak{I}$ by $\delta \mathfrak{I}$.

Let $z_1, \dots, z_p$ be a system of coordinates of $\mathbf{R}^p$. Let $z \in J^2(n, p)$ and $f: \mathbf{R}^n \rightarrow \mathbf{R}^p$ be any representative of z . Then we write $f_i = z_i \circ f$ ($i = 1, \dots, p$). We define $\mathfrak{I}(z)$ to be the ideal generated by the images in $\mathfrak{M}/\mathfrak{M}^3$ of $f_1, \dots, f_p$. Clearly the ideal $\mathfrak{I}(z)$ depends only on z .

Definition 1.1. By the Boardman symbol of $\mathfrak{I}(z)$ we mean the pair of integers (i, j) , where $i = n - \text{rk } \mathfrak{I}(z)$ and $j = n - \text{rk } \delta \mathfrak{I}(z)$.

Definition 1.2. Let (i, j) be a pair of integers. Let $\Sigma^{i,j}$ denote the set of $z \in J^2(n, p)$ whose Boardman symbol is (i, j) . We call the set $\Sigma^{i,j}$ Thom-Boardman singularity of order two

Theorem 1.3. (Boardman [1]) i) The set $\Sigma^{i,j}$ is nonempty if and only if

a) $n \geq i \geq j$,

b) $i \geq n - p$,

and c) if $i = n - p$, then $i = j$.

ii) $\Sigma^{i,j}$ is locally closed in Zariski topology and nonsingular.

iii) The codimension of $\Sigma^{i,j}$ in $J^2(n, p)$ is

$$(p - n + i) \left\{ \frac{1}{2} i(i+1) + i - \frac{1}{2} (i-j)(i-j+1) \right\} - j(i-j).$$

Now we define the modified a -Jacobian extension of an ideal $\mathfrak{I}(z)$ associated with an element z of $J^2(n, p)$. Let $f = (f_1, \dots, f_p)$ be any representative of z . Let a be $(a_1, \dots, a_p)$ of $\mathbb{R}^p$. Consider the $(p, n+1)$ matrix,

$$D(f, x, a) = \begin{pmatrix} \partial f_1 / \partial x_1, \dots, \partial f_1 / \partial x_n & a_1 \\ \vdots & \vdots & \vdots \\ \partial f_p / \partial x_1, \dots, \partial f_p / \partial x_n & a_p \end{pmatrix}.$$

Definition 1.4. We define the a -Jacobian extension $\Delta_a^s \mathfrak{I}(z)$ of $\mathfrak{I}(z)$ to be the ideal generated by $\mathfrak{I}(z)$ and all $s \times s$ determinants of the matrix $D(f, x, a)$. When $i = n - \text{rk } \mathfrak{I}(z)$, we put $\delta_a \mathfrak{I}(z) = \Delta_a^{n-i+2} \mathfrak{I}(z)$.

Clearly $\Delta_a^s \mathfrak{I}(z)$ does not depend on the representative f of z and a system of coordinates of $\mathbb{R}^p$. By the following lemma $\Delta_a^s \mathfrak{I}(z)$ does not depend on the choice of a system of coordinates for $\mathfrak{M}/\mathfrak{M}^3$.

Lemma 1.5. Let $y_1, \dots, y_n$ be a system of coordinates for $\mathfrak{M}/\mathfrak{M}^3$. Let f be a representative of z and $f = (f_1, \dots, f_p)$. Then the a -Jacobian extension $\Delta_a^s \mathfrak{I}(z)$ is generated by $f_1, \dots, f_p$ and all $s \times s$ minor determinants of the Jacobian matrix $D(f, y, a)$.

Proof. Since each of systems of coordinates $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ spans $\mathfrak{M}$, we can write $x_i = x_i(y)$, $y_i = y_i(x)$ modulo $\mathfrak{M}^3$. Hence we have

$$D(f, x, a) = (D(f, y, a) |_{y_i = y_i(x)}) \begin{pmatrix} \partial y_1 / \partial x_1, \dots, \partial y_1 / \partial x_n & 0 \\ \vdots & \vdots & \vdots \\ \partial y_n / \partial x_1, \dots, \partial y_n / \partial x_n & 0 \\ 0 & \dots, & 0 & 1 \end{pmatrix}$$

By the linear algebra it is well known that if A, B are $(s, n+1)$ matrix and $(n+1, s)$ matrix respectively, then $\det AB = \sum \det A' \det B'$ where A' and B' run over all (s, s) minor matrices of A and B . Since the matrix $(\partial y/\partial x)$ is nonsingular, the ideal generated by all $s \times s$ minor determinants of the matrices $D(f, x, a)$ is the same as that of $D(f, y, a)$ at $y_i = y_i(x)$. Q. E. D.

Let $L'(m)$ be the group of invertible elements of r -jet space $J^r(m, m)$. Then we have a canonical action of $L'(n) \times L'(p)$ on $J^r(n, p)$, $\mu: L'(n) \times L'(p) \times J^r(n, p) \rightarrow J^r(n, p)$ defined by $\mu((h, h', z)) = h' \circ z \circ h$ where $h \in L'(n)$, $h' \in L'(p)$ and $z \in J^r(n, p)$.

An element $h \in L^2(n)$ induces a ring-isomorphism $h^*: \mathfrak{M}/\mathfrak{M}^3 \rightarrow \mathfrak{M}/\mathfrak{M}^3$ by considering a composition of a polynomial and h . The above lemma can be restated in other words as follows.

Lemma 1.6. *If $h \in L^2(n)$, then we have $\delta_a \mathfrak{I}(z \circ h) = (h^*) \delta_a \mathfrak{I}(z)$.*

Proof. Let $h(x) = (h_1, \dots, h_n(x))$ and $f(x) = (f_1, \dots, f_p(x))$. If we put $y_i(x) = h_i(x)$, then

$$D(f \circ h(x), x, a) = D(f, x, a)(h(x)) \begin{pmatrix} \partial h_i / \partial x_j & 0 \\ \dots & \dots \\ 0 & 1 \end{pmatrix}.$$

The rest of the proof is similar to that of Lemma 1.5. Q. E. D.

Lemma 1.7. *Let $h \in L^2(p)$. Then the ideal $\delta_a \mathfrak{I}(z)$ is equal to the ideal $\delta_b \mathfrak{I}(h \circ z)$ modulo $\mathfrak{M}^2$ where b is defined by*

$$\begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix} = \begin{pmatrix} \partial h_1 / \partial z_1(0), \dots, \partial h_1 / \partial z_p(0) \\ \vdots \\ \partial h_p / \partial z_1(0), \dots, \partial h_p / \partial z_p(0) \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix}.$$

Proof. Let $b_1(x), \dots, b_p(x)$ be defined by

$$a_j = \sum_k \partial h_j / \partial z_k(z(x)) b_k(x) \quad (j=1, \dots, p).$$

Then we have $D(h \circ z, x, a) = (\partial h_j / \partial z_k(z(x))) D(z, x, b(x))$. Let $s = \text{rk } \mathfrak{I}(z) + 2$. Since the matrix $(\partial h_j / \partial z_k)$ is nonsingular, the ideal generated by all $s \times s$ determinants of $D(h \circ z, x, a)$ is equal to that of $D(z, x, b(x))$ by the similar arguments in the proof of Lemma 1.5. Let $b_i = b_i(0)$ ($i=1, \dots, p$). Then the last ideal is equal to the ideal constructed similarly from $D(z, x, b)$ modulo $\mathfrak{M}^2$. Q. E. D.

Remark 1.8. Let z be an element of $J^2(n, p)$. Then the ideal $\delta_a \mathfrak{I}(z)$ is contained in the ideal $\delta \mathfrak{I}(z)$ for any $a \in \mathbf{R}^p$.

§2. A decomposition of $\Sigma^{i,j}$ into manifolds

We will give a decomposition of $\Sigma^{i,j}$ into a finite set of nonsingular submanifolds in $J^2(n, p)$ which are locally closed in Zariski topology.

Definition 2.1. By the symbol for decomposition for an element z of $\Sigma^{i,j}$ we mean a pair of integers (t, c) defined as follows. Let t be an integer such that $\text{rk } \delta_a \mathfrak{Z}(z) \leq n - i + t$ for any element $a \in \mathbf{R}^p$ and that for some element b $\text{rk } \delta_b \mathfrak{Z}(z) = n - i + t$. Let $U(z) = \{a \in \mathbf{R}^p \mid \text{rk } \delta_a \mathfrak{Z}(z) = n - i + t\}$. Then the integer c is defined to be the rank of the ideal $\bigcap_{a \in U(z)} \delta_a \mathfrak{Z}(z)$.

Remark 2.2. For an element z of the singularity $\Sigma^{i,j}$, it follows from Remark 1.8 that $t \leq i - j$. It is clear that $c \leq t$. The set $U(z)$ is a Zariski open set in $\mathbf{R}^p$ since the inequality $\text{rk } \delta_a \mathfrak{Z}(z) \leq n - i + t$ is an algebraic condition.

Definition 2.3. Let (t, c) be a pair of nonnegative integers. Let $\Sigma^{i,j}(t, c)$ denote the set of those elements $z \in \Sigma^{i,j}$ whose symbol for decomposition is (t, c) .

Remark 2.4. $\Sigma^{i,j}(t, c)$ is $L^2(n) \times L^2(p)$ invariant; In fact it is known that $\Sigma^{i,j}$ is $L^2(n) \times L^2(p)$ invariant. Let $h \in L^2(n)$. Then it follows from Lemma 1.6 that the symbols for decomposition of z and $z \circ h$ are equal. Hence $\Sigma^{i,j}(t, c)$ is $L^2(n)$ invariant. Let $h' \in L^2(p)$. By Lemma 1.7 we have $\bigcap_{U(z) \ni a} \delta_a \mathfrak{Z}(z) = \bigcap_{U(h' \circ z) \ni b} \delta_b \mathfrak{Z}(h' \circ z)$. Hence $\Sigma^{i,j}(t, c)$ is also $L^2(p)$ invariant.

Let $n - i \leq n - i + c \leq n - j$. Let $GL_{n-i, n-i+c, n-j, j}(\mathfrak{M}/\mathfrak{M}^2)$ be the set of the following series of three vector subspaces of $\mathfrak{M}/\mathfrak{M}^2$, $\{(\alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \text{ are subspaces of } \mathfrak{M}/\mathfrak{M}^2, \dim \alpha = n - i, \dim \beta = n - i + c \text{ and } \dim \gamma = n - j, \alpha \subseteq \beta \subseteq \gamma\}$. We denote this grassmannian manifold by GL_* for simplicity. Let $GL_{n-i, p-n+i}(\mathbf{R}^p)$ be the grassmannian manifold of subspaces of dimension $n - i$ in $\mathbf{R}^p$.

We will define a map $\varphi: \Sigma^{i,j}(t, c) \rightarrow GL_* \times GL_{n-i, p-n+i}(\mathbf{R}^p)$. Let z be an element of $\Sigma^{i,j}(t, c)$. Then we obtain a triple of vector subspaces (α, β, γ) where $\alpha = \mathfrak{Z}(z)$, $\beta = \bigcap_{U(z) \ni a} \delta_a \mathfrak{Z}(z)$, and $\gamma = \delta \mathfrak{Z}(z)$ modulo $\mathfrak{M}^2$. By definition of $\Sigma^{i,j}(t, c)$, we have $\dim \alpha = n - i$, $\dim \beta = n - i + c$ and $\dim \gamma = n - j$ with $\alpha \subseteq \beta \subseteq \gamma$. Hence $(\alpha, \beta, \gamma) \in GL_*$. Consider the differential map at the origin of $\mathbf{R}^n$, $df_0: T_0 \mathbf{R}^n \rightarrow T_0 \mathbf{R}^p$. Then we put $= \text{Im } df_0(T_0 \mathbf{R}^n)$ which is in $GL_{n-i, p-n+i}$.

Lemma 2.5. $(\Sigma^{i,j}(t, c), \varphi, GL_* \times GL_{n-i, p-n+i}(\mathbf{R}^p))$ is a fiber bundle.

Proof. In Remark 2.4 we have defined the action of $L^2(n) \times L^2(p)$ on $\Sigma^{i,j}(t, c)$. We restrict this action to $L^1(n) \times L^1(p)$. Then it follows from Lemma 1.6 and Lemma 1.7 that the following diagram commutes.

$$\begin{array}{ccc} L^2(n) \times L^2(p) \times \Sigma^{i,j}(t, c) & \xrightarrow{\quad} & \Sigma^{i,j}(t, c) \\ \downarrow pr \times \varphi & & \downarrow \varphi \\ L^1(n) \times L^1(p) \times GL_* \times GL_{n-i, p-n+i} & \xrightarrow{\quad} & GL_* \times GL_{n-i, p-n+i} \end{array}$$

Since $L^1(n)$ and $L^1(p)$ act transitively on GL_* and $GL_{n-i, p-n+i}$ respectively, $GL_* \times GL_{n-i, p-n+i}$ is a homogeneous space of $L^1(n) \times L^1(p)$. Hence there is a local cross section of $GL_* \times GL_{n-i, p-n+i}$ into $L^1(n) \times L^1(p)$. It follows from Lemma 1 of Cerf [2, Appendix] that φ is a fiber bundle. Q. E. D.

The next purpose is to show that φ gives a differentiable fiber bundle. At

first we will show that the fibre of the bundle φ is a differentiable manifold.

We fix $((\alpha, \beta, \gamma), \delta)$ for a while. Let $(x_1, \dots, x_n)$ be a system of coordinates such that α, β and γ are spanned by the images of $(x_1, \dots, x_{n-i})$, $(x_1, \dots, x_{n-i+c})$ and $(x_1, \dots, x_{n-j})$ by the natural map into $\mathfrak{M}/\mathfrak{M}^2$, respectively. Let $y_1, \dots, y_p$ be a system of coordinates such that $\text{Im } df_0$ of the tangent space $T_0(\mathbf{R}^n)$ is spanned by $\partial/\partial y_1, \dots, \partial/\partial y_{n-i}$. Under these systems of coordinates we can identify an element of $z \in J^2(n, p)$ with a polynomial of order 2 ($b_{st} = b_{ts}$),

$$(2.6) \quad f^k = \sum a_i^k x_i + \sum b_{ij}^k x_i x_j \quad (k=1, \dots, p).$$

We now search for the explicit conditions for z so that z is contained in $\varphi^{-1}((\alpha, \beta, \gamma), \delta)$. Let z be in $\varphi^{-1}((\alpha, \beta, \gamma), \delta)$. Then it is easy to see that $a_i^s = 0$ when $s > n-i$ or $t > n-i$, since $\mathfrak{I}(z) = (x_1, \dots, x_{n-i})$ modulo $\mathfrak{M}^2$ and the image of $df: T_0 \mathbf{R}^n \rightarrow T_0 \mathbf{R}^n$ is spanned by $\partial/\partial y_1, \dots, \partial/\partial y_{n-i}$. The $(n-i, n-i)$ minor matrix $(a_i^s)_{1 \leq s, t \leq n-i}$ is nonsingular.

Under (2.6), $\delta_a \mathfrak{I}(z)$ is described more explicitly. We note that $D(f, x, a)$ is the following matrix;

$$\begin{pmatrix} a_1^1 + g_1^1, & \dots, & a_{n-i}^1 + g_{n-i}^1, & g_{n-i+1}^1, \dots, & g_n^1, & a^1 \\ \vdots & & \vdots & & \vdots & \vdots \\ a_1^{n-i} + g_1^{n-i}, & \dots, & a_{n-i}^{n-i} + g_{n-i}^{n-i}, & g_{n-i+1}^{n-i}, \dots, & g_n^{n-i}, & a^{n-i} \\ g_1^{n-i+1}, & \dots, & g_{n-i}^{n-i+1}, & g_{n-i+1}^{n-i+1}, \dots, & g_n^{n-i+1}, & a^{n-i+1} \\ \vdots & & \vdots & & \vdots & \vdots \\ g_1^p, & \dots, & g_{n-i}^p, & g_{n-i+1}^p, \dots, & g_n^p, & a^p \end{pmatrix}$$

where $g_i^s = \partial(\sum b_{ij}^s x_i x_j) / \partial x_i = \sum_{j=1}^n (b_{ij}^s + b_{ji}^s) x_j$. By definition of $\delta_a \mathfrak{I}(z)$, it is an ideal generated by $\mathfrak{I}(z)$ and all $(n-i+2, n-i+2)$ determinants of $D(f, x, a)$. It follows from the form of matrix (a_i^s) that the image of $\delta_a \mathfrak{I}(z)$ into $\mathfrak{M}/\mathfrak{M}^2$ is generated by $\sum_{i=1}^{n-1} a_i^s x_i$ ($1 \leq s \leq n-i$) and $a^s g_u^t - a^t g_u^s$ ($n-i+1 \leq s, t \leq p$ and $n-i+1 \leq u \leq p$).

The module $\mathfrak{M}/\mathfrak{M}^2$ is isomorphic to the vector space $\mathbf{R}^n$ by mapping $\sum \lambda_i x_i$ into the row vector $(\lambda_1, \lambda_2, \dots, \lambda_n)$. Let e_i correspond to x_i ($i=1, \dots, n$). Let A be the matrix $(a_i^s)_{1 \leq s \leq p, 1 \leq i \leq n}$, $\bar{B}^j, (b_{st}^j)_{1 \leq s, t \leq n}$ ($j=1, \dots, p$) and B^j the (i, i) minor matrix of $\bar{B}^j, (b_{st}^j)_{n-i+1 \leq s, t \leq n}$. By the notation $V(X)$ we shall mean the vector subspace spanned by all row vectors of a matrix X . The image of $\delta_a \mathfrak{I}(z)$ onto $\mathfrak{M}/\mathfrak{M}^2$ is identified with the sum of vector subspaces, $V(A)$ and $V(a^l \bar{B}^m - a^m \bar{B}^l)$, $n-i+1 \leq l, m \leq p$ of $\mathbf{R}^n$. It follows from the form of A that the above space is generated by $e_1, \dots, e_{n-i}$ and $V(a^l B^m - a^m B^l)$, $n-i+1 \leq l, m \leq p$. We note that $\text{rk } \delta_a \mathfrak{I}(z) = n-i+t$ if and only if the dimension of the subspace spanned by $V(a^l B^m - a^m B^l)$, $n-i+1 \leq l, m \leq p$ (which will be denoted for convenience by $V(B, a)$ hereafter) is t . Let $U(B) = \{a \in \mathbf{R}^{p-n+i} \mid \dim V(B, a) = t\}$. Similarly $\delta \mathfrak{I}(z)$ is identified with the subspace generated by $e_1, \dots, e_{n-j}$ and $V(B^j)$, $n-i+1 \leq j \leq p$ of $\mathbf{R}^n$. Hence we have the following lemma.

Lemma 2.7. Let z be an element of $\Sigma^{i,j}(t, c)$ and represented as in 2.6.

Then $\mathfrak{Z}(z) = (x_1, \dots, x_{n-i})$, $\delta\mathfrak{Z}(z) = (x_1, \dots, x_{n-j})$ and $\bigcap_{U(z) \ni a} \delta_a \mathfrak{Z}(z) = (x_1, \dots, x_{n-i+c})$ modulo $\mathfrak{M}^2$ if and only if

i) the $(n-i, n-i)$ minor matrix $(a_t^s)_{1 \leq s, t \leq n-i}$ is nonsingular and $a_t^s = 0$ if s or $t > n-i$,

ii) $\sum_{j=n-i+1}^p V(B^j) = \{e_{n-i+1}, \dots, e_{n-j}\}$,

iii) $\dim V(B, a) \leq t$ and $U(B) \neq \emptyset$,

iv) $\bigcap_{U(B) \ni a} V(B, a) = \{e_{n-i+1}, \dots, e_{n-i+c}\}$.

In the rest of this section we assume $p-n+i=2$. We note that $J^2(n, p)$ is identified with the product of the space of (n, p) matrices and p powers of the space A_n of (n, n) symmetric matrices. Now we give a proof of Theorem except for v).

Proof of Theorem. The fibre $\varphi^{-1}((\alpha, \beta, \gamma), \delta)$, which is denoted by F here after, is identified with the space of those elements of $J^2(n, p)$ whose representations as in (2.6) satisfy the conditions i), ii), iii) and iv) of Lemma 2.7. Since $p-n+i=2$, F is identified with the product of the space of nonsingular $(n-i, n-i)$ matrices, $(A_n)^{n-i}$ and $\tilde{\Sigma}$ where $\tilde{\Sigma}$ denotes the following subspace of $A_n \times A_n$. Let $\bar{B} = (b_{ij})$ be n -th symmetric matrices and B i -th symmetric matrices $(b_{ij})_{n-i+1 \leq i, j \leq n}$. We define a map $\pi: A_n \rightarrow A_i$ by $\pi(\bar{B}) = B$. We let Σ' be the subspace of all (B^1, B^2) of $(A_i)^2$ such that the conditions ii), iii) and iv) holds for (B^1, B^2) . Then $\tilde{\Sigma}$ is $\pi^{-1}(\Sigma')$. Therefore it follows from Lemma 2.11 that F is nonsingular in $J^2(n, p)$. It is easy to see that $\dim F = (n-i)^2 + \frac{1}{2}n(n-i)(n+1) + 2\{i(n-i) + \frac{1}{2}(n-i)(n-i+1)\} + (t-c)(2c-t) + 2\{\frac{1}{2}c(c+1) + (t-c)(i-j-c)\}$.

We consider the differentiable map $\mu: L^1(n) \times L^1(p) \times F \rightarrow J^2(n, p)$ defined by $\mu(h, h')(z) = h' \circ z \circ h$. At first we show that the image of μ is $\Sigma^{i,j}(t, c)$. It is clear that the image of μ is contained in $\Sigma^{i,j}(t, c)$ since F is a subset of $\Sigma^{i,j}(t, c)$ (Remark 2.4). Consider the following diagram,

$$\begin{array}{ccc} L^1(n) \times L^1(p) \times F & \longrightarrow & \Sigma^{i,j}(t, c) \\ \downarrow id \times \varphi & & \downarrow \varphi \\ L^1(n) \times L^1(p) \times (*) & \longrightarrow & G_* \times GL_{n-i, p-n+i} \end{array}$$

where $(*)$ means $((\alpha, \beta, \gamma), \delta)$. Since $L^1(n) \times L^1(p)$ acts on $G_* \times GL_{n-i, p-n+i}$ transitively, the image of μ is $\Sigma^{i,j}(t, c)$. Next we show that μ is of constant rank. It is enough to show that μ is of constant rank at the unit element 1 of $L^1(n) \times L^1(p)$. We put $\mu(v, z) = \mu_v(z) = \mu_z(v)$. Let $(d\mu)_{1,z}(x, y) = 0$ where $x \in T_1(L^1(n) \times L^1(p))$ and $y \in T_z F$. Since $(d\mu)_{1,z} = d(\mu_z)_1 + d(\mu_x)_z$, we have $d(\mu_z)_1(x) + d(\mu_1)_z(y) = 0$. As $\mu_1: F \rightarrow J^2(n, p)$ is an embedding, $d(\mu_z)_1(x)$ determines y . Since $d\varphi \circ d(\mu_1)_z(y) = d(\varphi \circ \mu_1)_z(y) = 0$, x is in $\text{Ker } d\varphi \circ d(\mu_z)_1$. Hence $\dim \text{Ker } d(\varphi \circ \mu_z)_1$ is equal to $\dim \text{Ker } (d\mu)_{1,z}$. Now we have

$$\begin{aligned} \text{rk}_{1,z} d\mu &= \dim T_1(L^1(n) \times L^1(p)) + \dim T_z F - \dim \text{Ker } (d\mu)_{1,z} \\ &= \dim T_1(L^1(n) \times L^1(p)) - \dim \text{Ker } d(\varphi \circ \mu_z)_1 + \dim T_z F \end{aligned}$$

$$= \dim G_* \times GL_{n-i, p-n+i} + \dim F.$$

Finally we assert that $\mu: L^1(n) \times L^1(p) \times F \rightarrow \Sigma^{i,j}(t, c)$ is an open map. This follows from Lemma 2.5. Hence by the constant rank theorem ([3, Chapter X, Section 3]), $\Sigma^{i,j}(t, c)$ is nonsingular with $\dim G_* \times GL_{n-i, p-n+i} + \dim F$. Moreover since μ is a proper map, $\Sigma^{i,j}(t, c)$ is a regular submanifold of $J^2(n, p)$. Hence $\Sigma^{i,j}(t, c)$ is a regular submanifold of $\Sigma^{i,j}$.

Now we determine the dimension of $\Sigma^{i,j}(t, c)$ in $J^2(n, p)$. Consider canonical bundles over grassmannian manifolds, $GL_{n-i, n-i+c, n-j, j}(\mathbf{R}^n) \rightarrow GL_{n-j, j}(\mathbf{R}^n)$ with fibre $GL_{n-i, n-i+c, i-c-j}(\mathbf{R}^{n-j})$ and $GL_{n-i, n-i+c, i-c-j}(\mathbf{R}^{n-j}) \rightarrow GL_{n-i+c, i-c-j}(\mathbf{R}^{n-j})$ with fibre $GL_{n-i, c}(\mathbf{R}^{n-i+c})$. Then we have that $\dim G_* = \dim GL_{n-i, c} + \dim GL_{n-i+c, i-c-j} + \dim GL_{n-j, j} = (n-i)c + (n-i+c)(i-c-j) + (n-j)j$. Hence

$$\begin{aligned} \dim \Sigma^{i,j}(t, c) &= (n-i) \left\{ (n-i) + \frac{1}{2}n(n+1) \right\} + 2i(n-i) + 2(t-c) \\ &\quad \times (i-j-c) + (n-i)(n-i+1) + c(c+1) + (t-c)(2c-t) \\ &\quad + (n-i)c + (n-i+c)(i-c-j) + (n-j)j + 2(n-i). \end{aligned}$$

The codimension of $\Sigma^{i,j}(t, c)$ follows from $\dim J^2(n, p) = p\{n + \frac{1}{2}n(n+1)\}$.

Since $J^2(N, P)$ is a bundle over $N \times P$ with fibre $J^2(N, P)_{x,y}$ over (x, y) which is identified with $J^2(n, p)$, we can define $\Sigma^{i,j}(t, c)_{x,y}$ in $J^2(N, P)_{x,y}$. Let $\Sigma^{i,j}(t, c) = \bigcup_{x \in N, y \in P} \Sigma^{i,j}(t, c)_{x,y}$. It follows from Remark 2.4 that $\Sigma^{i,j}(t, c)$ is a submanifold of $J^2(N, P)$. Q. E. D.

Remark 2.8. $\Sigma^{i,j}(t, c)$ is locally closed in Zariski topology. For $\Sigma^{i,j}$ is locally closed in Zariski topology. The set S_t of all z so that $\text{rk } \delta_a \mathfrak{F}(z) \leq n-i+t$ is closed in Zariski topology. The set T_c of all z so that $\text{rk } \bigcap_{U(z) \ni a} \delta_a \mathfrak{F}(z) \leq n-i+c$ is also closed. Hence $S_t - S_{t-1}$ and $T_c - T_{c-1}$ are locally closed in Zariski topology of $\Sigma^{i,j}$. Then $\Sigma^{i,j}(t, c) = (S_t - S_{t-1}) \cap (T_c - T_{c-1})$. This is what we want.

Lemma 2.9. Let B^1 and B^2 be $i \times i$ symmetric matrices. Let $\dim V(B, a) \leq t$, $U(B) \ni a$, $\bigcap_{U(B) \ni a} V(B, a) = \{e_1, \dots, e_c\}$, and $V(B^1) + V(B^2) = \{e_1, \dots, e_i\}$. Then there exists an element T with the form $\begin{pmatrix} S_1 & 0 \\ * & S_2 \end{pmatrix}$, where $S_1 \in GL(c)$ and $S_2 \in GL(i-c)$ as follows.

$$\text{i) } {}^tTB^jT = \begin{pmatrix} * & * & X_1^j \\ * & X_2^j & 0 \\ {}^tX_1^j & 0 & 0 \end{pmatrix} \text{ where } X_2^j \text{ are symmetric } (2c-t, 2c-t) \text{ matrices,}$$

X_1^j are $(i-c, t-c)$ nonsingular matrices and $\dim V(X_1^j) + \dim V(X_2^j) = 2c-t$.

ii) The intersection of all $V(a^2X_1^j - a^1X_2^j)$ is $\{0\}$, where a is in $\{a \in \mathbf{R}^2 \mid \text{rk}(a^2X_1^j - a^1X_2^j) = t-c\}$.

iii) For some $a \in \mathbf{R}^2$, $\text{rk}(a^2X_1^j - a^1X_2^j) = t-c$.

iv) $\dim V(X_1^j) + \dim V(X_2^j) = i-c$.

Proof. In the proof let $V(X)$ denote the subspace generated by all column vectors of X . Since B^1 and B^2 are symmetric, it does not matter. At first we only consider B^2 . Let C be the intersection of all vector spaces $V(B, a)$, $a \in U(B)$. Let d be the dimension of $V(B^2) + C/C$. Then there is an element $T_1 \in GL(i)$ such that $B^2 T_1$ is $(b'_1, \dots, b'_i)$ where $b'_{d+1}, \dots, b'_i$ are contained in $V(B^2) \cap C$, $b'_1, \dots, b'_d$ are not contained in $V(B^2) \cap C$ and linearly independent. Hence if ${}^t T_1 B^2 T_1$ is written as $(b_1^2, \dots, b_i^2)$, then $b_{d+1}^2, \dots, b_i^2$ are contained in $V({}^t T_1 B^2 T_1) \cap {}^t T_1(C)$, $b_1^2, \dots, b_d^2$ are not and $b_1^2, \dots, b_d^2$ are linearly independent. Here ${}^t T_1(C)$ is the intersection of all $V({}^t T_1 B T_1, a)$, $a \in U(B)$. Let ${}^t T_1 B^1 T_1$ be written as $(b_1^1, \dots, b_i^1)$. The vector $a^2 b_j^1 - a^1 b_j^2$ is contained in $V({}^t T_1 B T_1, a)$. If $j > d$ and $a \in U$, then $a^1 b_j^2$ is an element of ${}^t T_1(C)$. Hence $a^2 b_j^1$ is contained in $V({}^t T_1 B T_1, a)$, and so is b_j^1 if $a^2 \neq 0$. Let $(a^1, 0) \in U(B)$. Then $\dim V({}^t T_1 B T_1, a)$ is constantly t for $a \in U(B)$ and there is a sequence of points in $U(B)$ which converges a point $(a^1, 0)$. Therefore, b_j^1 is also an element of $V({}^t T_1 B T_1, a)$ for $a^2 = 0$. Hence $b_j^1 \in {}^t T_1(C)$ if $j > d$.

Similarly we can prove that $b_1^1, \dots, b_d^1$ are not contained in ${}^t T_1(C)$ and linearly independent. In fact let $\sum_{i=1}^d \lambda_i b_i^1 = 0$. Then $a^2(\sum \lambda_i b_i^2) - a^1(\sum \lambda_i b_i^1) \in V({}^t T_1 B T_1, a)$, hence we have $a^1(\sum \lambda_i b_i^1) \in V({}^t T_1 B T_1, a)$. By the similar argument as above $\sum \lambda_i b_i^2$ is contained in ${}^t T_1(C)$. This means $\lambda_1 = \dots = \lambda_d = 0$.

Let π_2 denote the projection of $\mathbf{R}^i$ into $\mathbf{R}^{i-d}$. Let $\mathbf{R}^d$ (or $\mathbf{R}^{i-d}$) is spanned by $e_1, \dots, e_d$ (or $e_{d+1}, \dots, e_i$). Let $\dim {}^t T_1(C) \cap \{e_1, \dots, e_d\}$ is d_1 and $\dim \pi_2({}^t T_1(C))$ is d_2 ($d_1 + d_2 = c$). There exist $S_1 \in GL(d)$, $S_2 \in GL(i-d)$ such that the linear map ${}^t S_1: \mathbf{R}^d \rightarrow \mathbf{R}^d$ (or ${}^t S_2: \mathbf{R}^{i-d} \rightarrow \mathbf{R}^{i-d}$) sends ${}^t T_1(C) \cap \mathbf{R}^d$ (or $\pi_2({}^t T_1(C))$) onto the subspace $\{e_1, \dots, e_{d_1}\}$ (or $\{e_{d+1}, \dots, e_{d+d_2}\}$). If we transform matrices ${}^t T_1 B^j T_1$ by $T_2 = S_1 \times S_2$, then they have the following forms.

$$(*) \quad (x_{st}) = \begin{pmatrix} X_1^j & \cdot & \cdot & \cdot & {}^t X_2^j \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ X_2^j & 0 & 0 & \cdot & 0 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ i-d-d_2 \end{pmatrix} d$$

where $x_{st} = 0$ ($s \geq t$) if $s > d_1$ and $t > d + d_2$. For if we write ${}^t T_1 B^j T_1 = \begin{pmatrix} X_1 & {}^t X_2 \\ X_2 & X_3 \end{pmatrix}$

where X_1 is a (d, d) matrix, X_2 a $(i-d, d)$ matrix, X_3 a $(i-d, i-d)$ matrix, then $V(X_3) \in \pi_2({}^t T(C))$ and $V({}^t S_2 X_3) \subseteq \{e_{d+1}, \dots, e_{d+d_2}\}$. We put $T = T_1 \cdot T_2$.

Now we show $d_1 = d$. Let x_h^j be the h -th column vector of ${}^t T B^j T$. Let $W(a)$ be the space generated by $a^2 x_{d_1+1}^1 - a^1 x_{d_1+1}^2, \dots, a^2 x_d^1 - a^1 x_d^2$ of dimension $d - d_1$. It follows from $(*)$ that $W(a)$ is contained in the space $\{e_1, \dots, e_{d+d_2}\}$. By the above construction $W(a) \cap ({}^t T)(C) = 0$. Otherwise there exists an element $\sum_{h=d_1+1}^d \lambda_h (a^2 x_h^1 - a^1 x_h^2)$ of $({}^t T)(C)$. Hence $t \sum_h \lambda_h (a^2 x_h^1 - a^1 x_h^2) + sa^1 \sum_h \lambda_h x_h^2$ is in $V({}^t T B T, b)$ where $b_1 = sa^1 - ta^1$ and $b_2 = ta^2$. Hence $(\sum \lambda_h x_h^2) \in V({}^t T B T, b)$, $b \in U(B)$. This is a contradiction. Since $({}^t T)(C)$ is a subspace of $\dim d_1 + d_2$ in $\{e_1, \dots, e_{d+d_2}\}$, $W(a)$

$+({}^tT)(C) = \{e_1, \dots, e_{d+d_2}\}$ for $a \in U(B)$. This shows that the subspace $\{e_1, \dots, e_{d+d_2}\}$ is contained in $V({}^tTBT, a)$ for any $a \in U(B)$. That is, $\{e_1, \dots, e_{d+d_2}\}$ is contained in $({}^tT)(C)$. By comparing dimensions of $\{e_1, \dots, e_{d+d_2}\}$ and C , we obtain $d_1 = d$.

We here note that the intersection of vector spaces $V({}^tTBT, a)$, $a \in U(B)$ is $\{e_1, \dots, e_c\}$. This means that the intersection C is preserved by the transformation T . Hence $T(=T_1 \cdot T_2)$ has the following form, $\begin{pmatrix} S_1 & 0 \\ * & S_3 \end{pmatrix}$, where S_1 and S_3 are $c \times c$

matrix and $(i-c) \times (i-c)$ matrix. Since $c = d_1 + d_2$ and $t = d_1 + c$, we have $d_1 = t - c$ and $d_2 = 2c - t$. Q. E. D.

Let Σ denote the space of all pairs (B^1, B^2) of i -th symmetric matrices with $\dim V(B, a) \leq t$, $U(B) \ni \emptyset$, $\bigcap_{U(B) \ni a} V(B, a) = \{e_1, \dots, e_c\}$ and $V(B^1) + V(B^2) = \{e_1, \dots, e_i\}$.

Corollary 2.10. *The space Σ is nonempty if and only if i) $2c - t \geq 0$, $i > t > c$ and $2(t - c) \geq i - c$ or ii) $c = t = i - j$.*

Proof. Necessity follows from Lemma 2.9. We give an example (B^1, B^2) with the conditions of the lemma. Under the notations of Lemma 2.9, let ${}^tX_1 = \begin{pmatrix} E_{t-c} \\ 0 \end{pmatrix}$ and ${}^tX_2 = \begin{pmatrix} 0 & E_{2t-c-i} \\ 0 & 0 \\ E_{i-t} & 0 \end{pmatrix}$. Q. E. D.

We now define a map $\varphi: \Sigma \rightarrow GL_{t-c, 2c-t}(\mathbb{R}^c)$. Let $\varphi((B^1, B^2))$ be the space of the intersection of $V(a^1 B^2 - a^2 B^1)$, $a \in \mathbb{R}^2 - \{0\}$. It is easily seen from Lemma 2.9 that $\varphi((B^1, B^2))$ is equal to the image of $\{e_1, \dots, e_{t-c}\}$ by the linear transformation tS_1 of Lemma 2.9.

Lemma 2.11. *The space Σ is a differentiable submanifold of dimension $(t-c)(2c-t) + 2\{\frac{1}{2}c(c+1) + (t-c)(i-c)\}$ in the space of all pairs of i -th symmetric matrices:*

Proof. The group $GL(c)$ operates Σ naturally. Let $d = t - c$. Then we have the commutative diagram,

$$\begin{array}{ccc} GL(c) \times \Sigma & \longrightarrow & \Sigma \\ \downarrow id \times \varphi & & \downarrow \varphi \\ GL(c) \times GL_{d, c-d} & \longrightarrow & GL_{d, c-d}. \end{array}$$

Hence φ is a fibre bundle ([2, Appendix, Lemma 1]). Let M be the space of matrices with the following form

$$\begin{pmatrix} * & * & * \\ * & * & 0 \\ * & 0 & 0 \end{pmatrix} \begin{matrix} d \\ c-d \\ i-c. \end{matrix}$$

Then the fibre F is an open manifold of M^2 , since any element (B^1, B^2) mapped on the space $\{e_1, \dots, e_{t-c}\}$ in $GL_{t-c, 2c-t}$ by φ has the form given in Lemma 2.9 and ii), iii) and iv) are open conditions in M^2 . Now we consider the differentiable map $\mu: GL(c) \times F \rightarrow A_i \times A_i$ where A_i is the space of i -th symmetric matrices. By the similar argument as in the proof of Theorem 1, we can prove that μ is of constant rank and an open map and that the image of μ is Σ . Hence Σ is a submanifold of dimension $\dim F + \dim GL_{t-c, 2c-t}$. It is easy to see that $\dim F = c(c+1) + 2(t-c)(i-c)$.
Q. E. D.

§3. Extensibility

In [6] Andrew du Plessis introduced the notion of extensibility. This is defined as follows. Let N and P be differentiable manifolds. Let Ω be an open set of the jet space $J^r(N, P)$. Then Ω is called *extensible* if there exists an open subbundle Ω' of $J^r(N \times \mathbf{R}, P) \rightarrow N \times \mathbf{R}$, such that the following conditions hold for Ω' .

i) A local diffeomorphism of $N \times \mathbf{R}, f: U \rightarrow V$ induces a diffeomorphism $j^r f: J^r(U, P) \rightarrow J^r(V, P)$ covering f . Then Ω' is invariant by these induced diffeomorphisms $j^r f$.

ii) Let i be the inclusion: $N = N \times 0 \rightarrow N \times \mathbf{R}$. Let $i^* J^r(N \times \mathbf{R}, P)$ denote the pull back of the canonical bundle: $J^r(N \times \mathbf{R}, P) \rightarrow N \times \mathbf{R}$ by i . Then $i^* \Omega'$ is carried onto Ω by the natural bundle map: $i^* J^r(N \times \mathbf{R}, P) \rightarrow J^r(N, P)$.

These conditions are a little stronger than those given in [6]. Condition i) is called *naturality*.

Theorem 3.1. (du Plessis [6]). *If Ω is extensible, then*

$$j^r: C^\infty_\Omega(N, P) \longrightarrow \Gamma_N(\Omega)$$

is a weak homotopy equivalence.

Remark 3.2. Since Σ^I in $J^r(n, p)$ is invariant by the action of $L^r(n)$, so is $\Omega^I = \bigcup_{K \leq I} \Sigma^K$. Hence Ω^I in $J^r(N, P)$ is a natural open set.

We will denote Σ^I and Ω^I in $J^r(n, p)$ by $\Sigma^I(n, p)$ and $\Omega^I(n, p)$ in this section. We will determine the image of $i^* \Omega^I$ by the natural map: $i^* J^2(N \times \mathbf{R}, P) \rightarrow J^2(N, P)$ in terms of $\Sigma^I(t, c)$ in case of $p - n + i = 2$. For this purpose it is enough to describe the image of $i^* \Omega^I(n+1, p)$. Let $M^{i, j+1}(n, p)$ in $J^2(n, p)$ be the set of z such that $\text{rk } \Im(z) \leq n-i$ and $\text{rk } \Delta^{n-i+1} \Im(z) \leq n-j-1$ ($n-i \leq n-j-1$). Note that $\Delta^{n-i+1} \Im(z)$ is not necessarily $\delta \Im(z)$ in this case. Let I be (i, j) . Then $\Omega^I(n, p) = \bigcup_{K \leq I} \Sigma^K(n, p)$.

Lemma 3.3. $J^2(n, p) - M^{i, j+1}(n, p) = \Omega^I(n, p)$.

Proof. We will prove the equivalent assertion that $M^{i, j+1}(n, p) = \bigcup_{K > I} \Sigma^K$. Let z be in the union of Σ^K , $K > I$. If $\text{rk } \Im(z) < n-i$, then $\Delta^{n-i+1} \Im(z) = \Im(z)$ modulo

$\mathfrak{M}^2$. Hence $\text{rk } \Delta^{n-i+1}\mathfrak{I}(z) < n-i \leq n-j-1$. If $\text{rk } \mathfrak{I}(z) = n-i$, then $\Delta^{n-i+1}\mathfrak{I}(z) = \delta\mathfrak{I}(z)$ by definition. Hence it follows from $\text{rk } \delta\mathfrak{I}(z) \leq n-j-1$ that $z \in M^{i,j+1}(n, p)$. Conversely for any element z of $M^{i,j+1}(n, p)$ we have $\text{rk } \mathfrak{I}(z) \leq n-i$ and $\text{rk } \Delta^{n-i+1}\mathfrak{I}(z) \leq n-j-1$. If $\text{rk } \mathfrak{I}(z) < n-i$, then $z \in \bigcup_{K>I} \Sigma^K$. If $\text{rk } \mathfrak{I}(z) = n-i$, and $\text{rk } \Delta^{n-i+1}\mathfrak{I}(z) = \text{rk } \delta\mathfrak{I}(z) \leq n-j-1$, then z is contained in $\bigcup_{K>I} \Sigma^K$. Q. E. D.

Let $\tilde{z}$ be an element of $J^2(n+1, p)$ and its representative $\tilde{f}: \mathbf{R}^{n+1} \rightarrow \mathbf{R}^p$. Then we let $f(x) = \tilde{f}(x, 0)$, $x \in \mathbf{R}^n$ and z , the element of $J^2(n, p)$ represented by f . A natural map $i: J^2(n+1, p) \rightarrow J^2(n, p)$ is defined by $i(\tilde{z}) = z$. By definition it is easy to see that $M^{i,j+1}(n+1, p)$ is an algebraic set of $J^2(n+1, p)$. Hence $J^2(n, p) - i(\Omega^I(n+1, p))$ is closed in Zariski topology, so it is a finite union of differentiable manifolds ([9]). In this section we show that $J^2(n, p) - i(\Omega^I(n+1, p))$ is explicitly described in terms of $\Sigma^I(t, c)$ when $p-n+i=2$.

Lemma 3.4. *Let $p-n+i>1$. Then z is an element of $J^2(n, p) - i(\Omega^{i,j}(n+1, p))$ if and only if $\text{rk } \mathfrak{I}(z) \leq n-i$ and $\text{rk } \Delta_a^{n-i+2}\mathfrak{I}(z) \leq n-j-1$ for any $a \in \mathbf{R}^p$.*

Proof. By the definition of Δ_a^{n-i+2} , the last statement is equivalent to say that i) $\text{rk } \mathfrak{I}(z) < n-i$ or ii) $\text{rk } \mathfrak{I}(z) = n-i$ and $\text{rk } \delta_a \mathfrak{I}(z) \leq n-j-1$ for any $a \in \mathbf{R}^p$. Note that z is an element of $J^2(n, p) - i(\Omega^I(n+1, p))$ if and only if for any $\tilde{z}$ with $i(\tilde{z}) = z$, $\tilde{z}$ is contained in $M^{i,j+1}(n+1, p)$. By Lemma 3.3 the last statement is equivalent to say that for any $\tilde{z}$ with $i(\tilde{z}) = z$, i) $\text{rk } \mathfrak{I}(\tilde{z}) < n-i+1$ or ii) $\text{rk } \mathfrak{I}(\tilde{z}) = n-i+1$ and $\text{rk } \delta\mathfrak{I}(\tilde{z}) \leq n-j$. Let f and $\tilde{f}$ be representatives of z and $\tilde{z}$. Let f be written in the form of (2.6) and $\tilde{f}$, as follows,

$$\tilde{f}_i(x, x_{n+1}) = \sum_{j=1}^n a_j^i x_j + a^{i,n+1} x_{n+1} + \sum_{s,t \leq n+1} b_{st}^i x_s x_t \quad (i=1, \dots, p).$$

Then the rank of the Jacobian matrix $(\partial \tilde{f}_i / \partial x_j)$ is equal to $\text{rk } D(f, x, a)$ at $x=0$. Let $k < \min(n, p)$. Then $\text{rk } \mathfrak{I}(z) = k$ if and only if $\text{rk } D(f, x, a) \leq k+1$ and $\text{rk } D(f, x, b) = k+1$ for some b at $x=0$. Hence if $p-n+1 > 1$, then i) $\text{rk } \mathfrak{I}(z) < n-i$ if and only if $\text{rk } \mathfrak{I}(\tilde{z}) < n-i+1$ for any $\tilde{z}$ with $i(\tilde{z}) = z$ and ii) $\text{rk } \mathfrak{I}(z) = n-i$ if and only if $\text{rk } \mathfrak{I}(\tilde{z}) \leq n-i+1$ for any z with $i(\tilde{z}) = z$ and $\text{rk } \mathfrak{I}(\tilde{z}_0) = n-i+1$ for some $\tilde{z}_0$. Hence it is enough to prove the case of $\text{rk } \mathfrak{I}(z) = n-i$. If $\text{rk } \mathfrak{I}(\tilde{z}) = n-i+1$, then $\delta\mathfrak{I}(\tilde{z})$ is generated by $\delta_a \mathfrak{I}(z)$ and x_{n+1} modulo $\mathfrak{M}^2$. Hence $\text{rk } \delta\mathfrak{I}(z) = \text{rk } \delta_a \mathfrak{I}(z) + 1$. Q. E. D.

Proposition 3.5. *Let $p-n-i=2$. Then $i(\Omega^{i,j}(n+1, p))$ is $\Omega^{i,j}(n, p) - \bigcup_{\substack{h \leq j \\ j-i+t < 0 \\ c \leq t}} \Sigma^{i,h}(t, c)$.*

Proof. By Lemma 3.4 $i(\Omega^I(n+1, p))$ is the set of all elements z such that i) $\text{rk } \mathfrak{I}(z) > n-i$ or ii) $\text{rk } \mathfrak{I}(z) = n-i$ and $\Delta_a^{n-i+2}\mathfrak{I}(z) \geq n-j$ for some $a \in \mathbf{R}^p$. The maximal rank of $\Delta_a^{n-i+1}\mathfrak{I}(z)$, $a \in \mathbf{R}^p$ is $n-i+t$ for an element $z \in \Sigma^{i,h}(t, c)$. Hence $i(\Omega^I(n+1, p)) = \bigcup_{k < i} \Sigma^{k,h}(n, p) \cup \bigcup_{\substack{h \leq j \\ j-i+t \geq 0 \\ c \leq t}} \Sigma^{i,h}(t, c)$.

Since $t \leq i-h$, we have $j \geq h$. Hence

$$i(\Omega^I(n+1, p)) = \Omega^I(n, p) - \left(\bigcup_{\substack{h \leq j \\ j-i+t < 0 \\ c \leq t}} \Sigma^{i,h}(t, c) \right). \quad \text{Q. E. D.}$$

Proof of (v) of Theorem. Note that $\Sigma^I(t, c) = \bigcup_{x \in N, y \in P} \Sigma^I(t, c)_{x,y}$. It follows from Proposition 3.5 that $i(\Omega^I(N \times R, P))$ is extensible. Hence (v) of Theorem follows from Theorem 3.1. Q. E. D.

§4. Proof of Corollary

Let $\Omega(N)$ (or $\Omega'(N)$) be an open subbundle of $J^r(N, P) \rightarrow N$ such that $\Omega(N) - \Omega'(N)$ (or $\Omega'(N) - \Omega(N)$) is a differentiable subbundle whose fiber over (x, y) is a finite union of submanifolds of $J^r(N, P)_{x,y}$ with codimensions in $J^r(N, P)_{x,y}$ greater than $n + \sigma_1$ (or $n + \sigma_2$). In [5] A. du Plessis has shown the following lemma.

Transversality Lemma ([5]). Let $\Omega(N)$ and $\Omega'(N)$ be as above. Let $\Omega(N)$ be integrable. (i) Let $\sigma_1 \leq \sigma_2 - 1$. Then $(j^r)_*: \pi_i(C_{\Omega(N)}^\infty(N, P)) \rightarrow \pi_i(\Gamma_N(\Omega(N)))$ is isomorphic for $i < \sigma_1$ and surjective for $i = \sigma_1$. (ii) Let $\sigma_2 - 1 < \sigma_1$. Then $(j^r)_*: \pi_i(C_{\Omega(N)}^\infty(N, P)) \rightarrow \pi_i(\Gamma_N(\Omega(N)))$ is isomorphic for $i < \sigma_2$.

We give here two consequences of Transversality lemma.

Theorem 4.1. Let Ω^I be an integrable subset of $J^r(N, P)$. i) Let $K \geq I$. We define σ_1 to be $\sigma_1 + n + 1 = \min \{\text{codim } \Sigma^H \text{ in } J^r(N, P) | K \geq H > I\}$. Then $(j^r)_*: \pi_i(C_{\Omega^K}^\infty(N, P)) \rightarrow \pi_i(\Gamma_N(\Omega^K))$ is isomorphic for $i < \sigma_1$ and surjective for $i = \sigma_1$. ii) Let $K \leq I$. We define σ_2 to be $\sigma_2 + n + 1 = \min \{\text{codim } \Sigma^H \text{ in } J^r(N, P) | I \geq H > K\}$. Then $(j^r)_*: \pi_i(C_{\Omega^K}^\infty(N, P)) \rightarrow \pi_i(\Gamma_N(\Omega^K))$ is isomorphic for $i < \sigma_2$.

Proof. We only prove the first part of the theorem since the proof of the second part is similar. In (i) of Transversality lemma we consider Ω^K and Ω^I instead of $\Omega(N)$ and $\Omega'(N)$. Then $\Omega^K - \Omega^I$ is the union of Σ^H , $I < H \leq K$ with codimensions greater than $\sigma_1 + n$. Q. E. D.

We should note that σ_2 is greater than σ_1 in most cases.

The following corollary follows from Theorem in §0 and includes Corollary in §0 since $\text{codim } \Sigma^{i,1} = 3i + 1$.

Corollary 4.2. Let $r=2$ and $I=(i, j)$. If $p-n+i=2$, then $(j^2)_*: \pi_k(C_{\Omega^I}^\infty(N, P)) \rightarrow \pi_k(\Gamma_N(\Omega^I))$ is isomorphic for $k < \text{codim } \Sigma^{i,j+1} - j - n - 1$ and surjective for $k = \text{codim } \Sigma^{i,j+1} - j - n - 1$.

Proof. In (i) of Transversality lemma we consider Ω^I and $\Omega^I - \bigcup_{\substack{h \leq j \\ j-i+t < 0 \\ c \leq t}} \Sigma^{i,h}(t, c)$ instead of $\Omega(N)$ and $\Omega'(N)$. Then $\Omega(N) - \Omega'(N)$ is the union of $\Sigma^{i,h}(t, c)$ with $h \leq j$, $j-i+t < 0$ and $c \leq t$. It is easy to see that $\text{codim } \Sigma^{i,h}(t, c) \geq \text{codim } \Sigma^{i,j+1} - j$. Q. E. D.

Examples. i) $J^2(3, 2)$; Ω^I for $I=(3, 0)$, $(2, j_2)$ and $(1, 1)$ ($j_2 < 2$) are not

extensible. $\Sigma^{3,0} = \Sigma^{3,0}(3, 3) \cup \Sigma^{3,0}(2, 1)$.

ii) $J^2(4, 2); \Omega^I$ for $I = (4, j_1), (3, j_2)$ and $(2, 2)$ ($j_1 \leq 1, j_2 < 3$) are not extensible. $\Sigma^{4,0} = \Sigma^{4,0}(4, 4) \cup \Sigma^{4,0}(3, 2)$. $\Sigma^{4,1} = \Sigma^{4,1}(3, 3) \cup \Sigma^{4,1}(2, 1)$.

iii) $J^2(5, 2); \Omega^I$ for $I = (5, j_1), (4, j_2)$ and $(3, 3)$ ($j_1 \leq 2, j_2 < 4$) are not extensible. $\Sigma^{5,2} = \Sigma^{5,2}(3, 3) \cup \Sigma^{5,2}(2, 1)$. $\Sigma^{5,1} = \Sigma^{5,1}(4, 4) \cup \Sigma^{5,1}(3, 2)$. $\Sigma^{5,0} = \Sigma^{5,0}(5, 5) \cup \Sigma^{5,0}(4, 3) \cup \Sigma^{5,0}(4, 2)$.

iv) $J^2(6, 2); \Omega^I$ for $I = (6, j_1), (5, j_2)$ and $(4, 4)$ ($j_1 \leq 3, j_2 < 5$) are not extensible. $\Sigma^{6,3} = \Sigma^{6,3}(3, 3) \cup \Sigma^{6,3}(2, 1)$. $\Sigma^{6,2} = \Sigma^{6,2}(4, 4) \cup \Sigma^{6,2}(3, 2)$. $\Sigma^{6,1} = \Sigma^{6,1}(5, 5) \cup \Sigma^{6,1}(4, 3) \cup \Sigma^{6,1}(4, 2)$. $\Sigma^{6,0} = \Sigma^{6,0}(6, 6) \cup \Sigma^{6,0}(5, 4) \cup \Sigma^{6,0}(5, 3) \cup \Sigma^{6,0}(4, 2)$.

We give a table of σ_1, σ_2 and $\xi = \text{codim } \Sigma^{i,j+1} - j - n - 1$ for some cases (In the table $(i, -1)$ means $(i-1, i-1)$).

(n, p)	$J^2(3, 2)$	$J^2(4, 2)$	$J^2(5, 2)$	$J^2(6, 2)$
I	3, 0	4, 1 4, 0	5, 2 5, 1 5, 0	6, 3 6, 2 6, 1 6, 0
ξ	6	8 8	20 15 10	30 24 17 12
σ_1	4	8 8	4 4 4	5 5 5 5
σ_2	6	13 8	26 16 10	33 26 19 12
I	2, j	3, j	4, j	5, j
σ_2	$3j+3$	$4j+5$	$5j+7$	$6j+9$

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