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Moduli of stable sheaves, II

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Introduction. Let S be a scheme of finite type over a universally Japanese ring, $f: X \to S$ be a smooth, projective, geometrically integral morphism and let $\mathcal{O}_X(1)$ be an f-very ample invertible sheaf. In this situation, we constructed a moduli scheme $M_{X/S}(H)$ of stable sheaves with Hilbert polynominal H in the preceding paper [12][1]. $M_{X/S}(H)$ is locally of finite type and separated over S. And, moreover, $M_{X/S}(H)$ is quasi-projective over S if and only if the family of classes of stable sheaves with Hilbert polynomial H is bounded. A main aim of this article is, under an assumption, to find a natural projective scheme over S which contains $M_{X/S}(H)$ as an open subscheme. More precisely, we shall construct a "moduli scheme" of semi-stable sheaves with Hilbert polynomial H and show that the moduli scheme is projective if the family of classes of semi-stable sheaves with Hilbert polynomial H is bounded.

As in the case of stable sheaves, our problem is reduced to making a quotient of a suitable open subscheme R of a Quot-scheme Q by a linear group scheme G. For this purpose, we shall use again the projective bundle Z over a finite union of connected components of the Picard scheme of X/S and the morphism μ of Q to Z which were constructed in §4 of [12]. In the case of stable sheaves, we had only to show that μ maps the points of R corresponding to stable sheaves to stable points of Z. But the case of semi-stable sheaves is more difficult than that because semi-stable points do not have, in general, good functorial properties (see [14] Ch. 1, §5). A way to overcome the difficulty is to show that $\mu(R)$ is closed in the open subscheme Z^{ss} of semi-stable points in Z. In fact, when $\dim X/S \leq 2$, this was done by C. S. Seshadri [19] and D. Gieseker [5]. A key result for this was that for a point x of Q corresponding to a torsion free sheaf F, if $\mu(x) \in Z^{ss}$, then F is semi-stable ([5] Theorem 0.7 (iii)). Unfortunately, we can not prove the above in higher dimensional case. We shall adopt, therefore, Seshadri's idea used in [18]. Thus we shall study the structure of orbits of Gieseker spaces (Definition 2.1) in §2. If one reads carefully [18] and [5] and compares products of Grassmann varieties used in

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[1] In [12], S was assumed to be of finite type over a field. Thanks to the results of Seshadri [20], our results in [12] hold good for every S which is of finite type over a universally Japanese ring (see §4 of this article).

[18] with Gieseker spaces, especially, Proposition 4.3 of [14] with Proposition 2.2 and Proposition 2.3 of [5], he would notice that Gieseker spaces are too big for our purpose. This is the motive to introduce the notion of an excellent point of a Gieseker space which is the main idea of this article (Definition 2.9).

§1 is devoted to define an equivalence relation among semi-stable sheaves and introduce a functor $\bar{\Sigma}^H_{X/S}$ of the category of locally noetherian S-scheme to the category of sets. In §3, we shall study strictly e-semi-stable sheaves. Combining the results of §2, §3 and the techniques of [12] §4 and §5, our main theorem of this article (Theorem 4.11) is proved.

In [8], S. G. Langton proved that if a moduli scheme of μ-semi-stable sheaves (Definition 5.1) exists and if it is of finite type, then it is proper. But his result is insufficient for our aim because there exists a μ-semi-stable sheaf which is not semi-stable (Example 5.3). The theorem which we need is proved along the same line as in [8] (Theorem 5.7). Theorem 4.11 and Theorem 5.7 provide us with Corollary 5.9.1 which is the result stated in the first paragraph of this introduction. The results of Seshadri in [18] and Gieseker in [5] are special cases of our Corollary 5.9.1. Therefore, this article supplies an alternative proof of their results.

In §6, we shall study some properties of the moduli schemes; a criterion for smoothness of the moduli schemes, dimensions of the moduli schemes in some very special cases and a criterion for existence of universal families etc.. As an example, the moduli schemes of semi-stable sheaves of rank 2 on $\mathbf{P}^2$ are studied more closely in §7. The main result is that the moduli schemes with fixed Chern classes are irreducible, normal, projective varieties.

Finally, in Appendix we shall show that there exist many stable, locally free sheaves on every smooth, projective variety.

Notation and convention. In addition to the notation and the convention of [12], we shall use the following. For numerical polynomials $f_1(n)$ and $f_2(n)$, $f_1(n)\prec f_2(n)$ (or, $f_1(n)\preceq f_2(n)$) means that $f_1(n)<f_2(n)$ (or, $f_1(n)\leq f_2(n)$, resp.) for all sufficiently large integers n. Let $f\colon X\to S$ be a smooth, projective, geometrically integral morphism and let $\mathcal{O}_X(1)$ be an f-ample invertible sheaf. For a field K, a K-valued point s of S and for a coherent sheaf on the fibre X_s with $r(E)>0$, $P_E(n)$ denotes the numerical polynomial $\chi(E\otimes\mathcal{O}_{X_s}(n))/r(E)$. For a cycle C on X_s, $d(C, \mathcal{O}_X(1))$ denotes the degree of C with respect to $\mathcal{O}_{X_s}(1)$. For a coherent sheaf F on X_s, we shall use the notation $d(F, \mathcal{O}_X(1))$ instead of $d(c_1(F), \mathcal{O}_X(1))$ as in [12], where $c_1(F)$ is the first Chern class of F. If $\dim X/S=1$, then the degree of F is denoted by $d(F)$.

§1. S-equivalence

In this section we shall introduce an equivalence relation among semi-stable sheaves and then define a functor of (Sch/S) to (Sets).

Lemma 1.1. *Let Y be a non-singular projective variety with a very ample invertible sheaf $\mathcal{O}_Y(1)$ and let E_1 (or, E_2) be a stable (or, semi-stable, resp.) sheaf* *on Y. If $P_{E_1}(m)=P_{E_2}(m)$, then every homomorphism ϕ of E_1 to E_2 has one of the following properties;*

1) $\phi=0$
2) *ϕ is injective and $E_2/\phi(E_1)$ is torsion free.*

Proof. Assume that $\phi\neq 0$ and set $F=\ker(\phi)$. Since E_2 is semi-stable, $P_{\phi(E_1)}(m) \preceq P_{E_2}(m)=P_{E_1}(m)$. Thus $P_F(m)\succeq P_{E_1}(m)$. Since E_1 is stable and since $F\neq E_1$, we obtain $F=0$. If $E_2/\phi(E_1)$ has non-trivial torsion, then it is easily seen that $P_{\phi(E_1)}(m)\succ P_{E_2}(m)$. This is not the case by our assumption. q.e.d.

A semi-stable sheaf has a Jordan-Hölder filtration. In fact,

Proposition 1.2. *Let Y be as above and let E be a semi-stable sheaf on Y. Then*

1) *there is a filtration $0=E_0\subset E_1\subset\cdots\subset E_t=E$ by coherent $\mathcal{O}_Y$-modules such that* (a) *E_i/E_{i-1} is stable $(1\leq i\leq t)$ and* (b) *$P_{E_i}(m)=P_E(m)$ $(0<i\leq t)$,*

2) *if $0=E'_0\subset E'_1\subset\cdots\subset E'_s=E$ is another filtration enjoying the properties* (a) *and* (b), *then $t=s$ and there exists a permutation σ of $\{1, 2,\dots, t\}$ such that E_i/E_{i-1} is isomorphic to $E'_{\sigma(i)}/E'_{\sigma(i)-1}$ $(1\leq i\leq t)$.*

Proof. 1) Let us prove our assertion by induction on $r(E)$. Assume that (1) is true for semi-stable sheaves with rank$<r(E)$. If E is stable, there is nothing to prove. Suppose that E is not stable. Then the set $A=\{F|F$ is a proper coherent subsheaf of E with $P_F(m)=P_E(m)\}$ is not empty. Pick a member E_1 of A with the smallest rank. It is obvious that E_1 is stable and E/E_1 is semi-stable. By our induction assumption, E/E_1 has a filtration $0=E_1/E_1\subset E_2/E_1\subset\cdots\subset E_t/E_1= E/E_1$ such that $(E_i/E_1)/(E_{i-1}/E_1)\cong E_i/E_{i-1}$ is stable and that $P_{E_i/E_1}(m)=P_{E/E_1}(m)$. Since $P_{E_1}(m)=P_E(m)$, we know that $P_E(m)=P_{E/E_1}(m)=P_{E_i/E_1}(m)=P_{E_i}(m)$. Hence the filtration $0=E_0\subset E_1\subset E_2\subset\cdots\subset E_t=E$ has the properties (a) and (b).

2) Our proof is by induction on t. If $t=1$, then E is stable, whence our assertion is obvious. Assume that $t>1$. Let r be the smallest integer such that E'_r contains E_1, then the natural homomorphism ϕ: $E_1\to E'_r/E'_{r-1}$ is not zero. By virtue of Lemma 1.1, ϕ is injective, which implies that $E_1\cap E'_{r-1}=0$. Moreover, since E'_r/E'_{r-1} is stable and since $P_{E'_r/E'_{r-1}}(m)=P_E(m)=P_{E_1}(m)$, ϕ should be surjective, that is, E_1 is isomorphic to E'_r/E'_{r-1}. Let us consider $\bar{E}=E/E_1$. Set $\bar{E}_i= E_{i+1}/E_1$ and

$$\bar{E}'_i=\begin{cases} E'_i/(E'_i\cap E_1) & 0\leq i\leq r-1 \\ E'_{i+1}/E_1 & r\leq i\leq s-1 \end{cases}$$

It is clear that $0=\bar{E}_0\subset\bar{E}_1\subset\cdots\subset\bar{E}_{t-1}=\bar{E}$ is a filtration with the properties (a) and (b). On the other hand, $\bar{E}'_i$ is isomorphic to E'_i for $0\leq i\leq r-1$, $\bar{E}'_r/\bar{E}'_{r-1}=E'_{r+1}/(E_1+E'_{r-1})=E'_{r+1}/E'_r$ and $\bar{E}'_j/\bar{E}'_{j-1}=E'_{j+1}/E'_j$ for $r+2\leq j\leq s-1$ because $E'_{r-1}\cap E_1=0$ and $E_1+E'_{r-1}=E'_r$. Thus the filtration $0=\bar{E}'_0\subset\bar{E}'_1\subset\bar{E}'_2\subset\cdots\subset\bar{E}'_{s-1}=\bar{E}$ has the properties (a) and (b). The induction hypothesis implies that $t=s$ and that there exists a permutation τ of $\{1, 2,\dots, t-1\}$ with $\bar{E}_i/\bar{E}_{i-1}=\bar{E}'_{\tau(i)}/\bar{E}'_{\tau(i)-1}$. Define a permutation σ of $\{1, 2,\dots, t\}$ as follows;

$$\sigma(i)=\begin{cases} r & \text{if } i=1 \\ \tau(i-1) & \text{if } 1\leq\tau(i-1)\leq r-1 \\ \tau(i-1)+1 & \text{if } r\leq\tau(i-1)\leq t-1. \end{cases}$$

Then σ is one of the desired permutations. q.e.d.

For convenience' sake, we shall introduce the following definition.

Definition 1.3. Let E be a semi-stable sheaf. A filtration $0=E_0\subset E_1\subset\cdots\subset E_t=E$ enjoying the properties (a) and (b) in Proposition 1.2 is called a Jordan-Hölder filtration of E. For a Jordan-Hölder filtration $0=E_0\subset E_1\subset\cdots\subset E_t=E$, define $\mathrm{gr}(E)$ to be $\bigoplus_{i=1}^{t} E_i/E_{i-1}$. Each E_i/E_{i-1} is called a component of $\mathrm{gr}(E)$.

Proposition 1.2 shows that $\mathrm{gr}(E)$ is independent of the choice of Jordan-Hölder filtrations.

Lemma 1.4. *Assume that*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

is an exact sequence of coherent sheaves with $P_{E'}(m)=P_E(m)=P_{E''}(m)$. E is semi-stable if and only if E' and E'' are semi-stable.

Proof. Assume that E' and E'' are semi-stable. It is clear that E is torsion free. Let F be a coherent subsheaf of E. For $\bar{F}=F/F\cap E'$, $P_{\bar{F}}(m)\preceq P_{E''}(m)$ because of the semi-stability of E''. Similarly, $P_{F\cap E'}(m)\preceq P_{E'}(m)$. Thus

$$\begin{aligned} P_F(m) &= \chi(F(m))/r(F)=\chi((F\cap E')(m))/r(F)+\chi(\bar{F}(m))/r(F) \\ &= r(F\cap E')P_{F\cap E'}(m)/r(F)+r(\bar{F})P_{\bar{F}}(m)/r(F) \\ &\preceq P_E(m)\{r(F\cap E')/r(F)+r(\bar{F})/r(F)\}=P_E(m). \end{aligned}$$

Hence E is semi-stable. Note that if E is semi-stable and if E'' is a coherent quotient sheaf of E with $P_E(m)=P_{E''}(m)$, then E'' is torsion free. Then the proof of the converse is similar to the above and easier, and hence we omit it.

Corollary 1.4.1. *If E is semi-stable, then so is* $\mathrm{gr}(E)$.

The following notion is originally due to C. S. Seshadri ([18] and [5]).

Definition 1.5. Seimi-stable sheaves E_1, E_2 on a non-singular projective variety are said to be S-equivalent if $\mathrm{gr}(E_1)$ is isomorphic to $\mathrm{gr}(E_2)$.

Corollary 1.4.1 implies that every semi-stable sheaf is S-equivalent to one which is isomorphic to a direct sum of stable sheaves.

Remark 1.6. 1) A stable sheaf E_1 is S-equivalent to E_2 if and only if E_1 is isomorphic to E_2.

2) If one takes the results in [1] and the indecomposability of stable sheaves into account, he knows that $\mathrm{gr}(E)=\bigoplus_{i=1}^{t} E_i/E_{i-1}$ is isomorphic to $\mathrm{gr}(E')=\bigoplus_{i=1}^{t'} E'_i/E'_{i-1}$ if and only if $t=t'$ and there exists a permutation σ of $\{1, 2,\dots, t\}$ such that $E_i/E_{i-1} \cong E'_{\sigma(i)}/E'_{\sigma(i)-1}$.

Let $f: X\to S$ be a smooth, projective, geometrically integral morphism of noetherian schemes and fix an f-very ample invertible sheaf $\mathcal{O}_X(1)$. Let (Sch/S) be the category of locally noetherian schemes over S and let $H(m)$ be a numerical polynomial. Our main aim of this article is to study the following functor $\bar{\Sigma}^H_{X/S}$ of (Sch/S) to the category of sets:

For an object T of (Sch/S),

$\bar{\Sigma}^H_{X/S}(T)=\{E|E$ is a T-flat, coherent $\mathcal{O}_{X\times_S T}$-module with the property (1.7.1)$\}/\sim$, where $\sim$ is the equivalence relation defined in (1.7.2).

(1.7.1) For every geometric point t of T, $E\otimes_{\mathcal{O}_T}k(t)$ is semi-stable and its Hilbert polynomial is $H(m)$.

(1.7.2) $E\sim E'$ if and only if (1) $E\cong E'\otimes_{\mathcal{O}_T}L$ or (2) there exist filtrations $0=E_0\subset E_1\subset\cdots\subset E_u=E$ and $0=E'_0\subset E'_1\subset\cdots\subset E'_u=E'$ by coherent $\mathcal{O}_{X\times_S T}$-modules such that for every geometric point t of T, $\{E_i\otimes_{\mathcal{O}_T}k(t)\}$ and $\{E'_i\otimes_{\mathcal{O}_T}k(t)\}$ provide us with Jordan-Hölder filtrations of $E\otimes_{\mathcal{O}_T}k(t)$ and $E'\otimes_{\mathcal{O}_T}k(t)$, respectively, $\bigoplus_{i=1}^{u} E_i/E_{i-1}$ is T-flat and that $\bigoplus_{i=1}^{u} E_i/E_{i-1}\cong(\bigoplus_{i=1}^{u} E'_i/E'_{i-1})\otimes_{\mathcal{O}_T}L$, for some invertible sheaf L on T. The equivalence class of E is denoted by $[E]$.

For a morphism $g: T'\to T$ in (Sch/S), if E has the property (1.7.1), then so does $g^*(E)$ and, moreover, if $E\sim E'$, then $g^*(E)\sim g^*(E')$. Thus we obtain a map g^* of $\bar{\Sigma}^H_{X/S}(T)$ to $\bar{\Sigma}^H_{X/S}(T')$. It is obvious that $\bar{\Sigma}^H_{X/S}$ is a contravariant functor of (Sch/S) to (Sets).

Let s be a geometric point of S. By the definition of $\bar{\Sigma}^H_{X/S}$, we have

(1.7.3) $\bar{\Sigma}^H_{X/S}(\mathrm{Spec}(k(s)))=\{E|E$ is a semi-stable sheaf on X_s whose Hilbert polynomial is $H(m)\}/\sim$, where $E_1\sim E_2$ if and only if E_1 is S-equivalent to E_2.

§2. Semi-stable points of Gieseker spaces

Let V and W be a finite dimensional vector space over a field k and let $\hat{\sigma}_0: V\to V\otimes_k k[G]$ be the dual action of $G=GL(V)$ on V. For a positive integer r, $\hat{\sigma}_0$ provides us with a dual action $\hat{\sigma}$ of G on $\mathrm{Hom}_k(\bigwedge^r V, W)^\vee$. Thus we obtain an action σ of G on $\mathbf{P}(\mathrm{Hom}_k(\bigwedge^r V, W)^\vee)$ and a G-linearized invertible sheaf $\mathcal{O}(1)$.

Definition 2.1. The projective space $\mathbf{P}(\mathrm{Hom}_k(\bigwedge^r V, W)^\vee)$ on which $G=GL(V)$ acts as above[2] and which carries the G-linearized invertible sheaf $\mathcal{O}(1)$ is

[2] The center of $GL(V)$ acts trivially on $P(V, r, W)$. Thus $PGL(V)$ acts on $P(V, r, W)$. Though the $\mathcal{O}(1)$ may not be $PGL(V)$-linearized, $\mathcal{O}(m)$ carries a $PGL(V)$-linearization for some positive integer m.

called a Gieseker space. We denote it by $P(V, r, W)$ (see [5] §2 and [12] §4).

For an over field K of k, a non-zero element T of $\mathrm{Hom}_k(\overset{r}{\wedge} V, W)\otimes_k K = \mathrm{Hom}_K(\overset{r}{\wedge}(V\otimes_k K), W\otimes_k K)$ gives rise to a K-rational point of $P(V, r, W)$, which is denoted by T, too. T is regarded as an alternating multilinear map of $V\otimes_k K$ to $W\otimes_k K$. For $x_1,\dots, x_r$ in $V\otimes_k K$, the value of T at $x_1\wedge\cdots\wedge x_r$ is denoted by $T(x_1,\dots, x_r)$. If $\{e_i\}$ is a basis of V, then x_i can be written in the form $\sum x_{ij}e_j$ and a K-valued point g of G is represented by a square matrix (g_{ij}). For the matrix $X=(x_{ij})$, we shall denote $T(x_1,\dots, x_r)$ by $T(X)$. Then, $\sigma(g, T)(X)=T(X\cdot(g_{ij}))$

An injective homomorphism i: $W\to W'$ of finite dimensional vector spaces yields a surjective homomorphism

$$\mathrm{Hom}_k(\overset{r}{\wedge} V, W') \longrightarrow \mathrm{Hom}_k(\overset{r}{\wedge} V, W) \longrightarrow 0$$

From this, we have a closed immersion i_*: $P(V, r, W)\to P(V, r, W')$ of Gieseker spaces. Clearly i_* is a G-morphism.

Lemma 2.2. *Let G be a reductive algebraic k-group, X and Y be algebraic k-schemes on which G acts and let j: $X\to Y$ be a closed immersion and a G-morphism. Suppose that Y is projective over k and carries a G-linearized ample invertible sheaf $\mathcal{O}_Y(1)$. Then $X^{ss}(j^*(\mathcal{O}_Y(1)))=j^{-1}(Y^{ss}(\mathcal{O}_Y(1)))$ and $X^s(j^*(\mathcal{O}_Y(1)))=j^{-1}(Y^s(\mathcal{O}_Y(1)))$.*

Proof. We may assume that the natural map $R_n=H^0(Y, \mathcal{O}_Y(n))\to R'_n=H^0(X, j^*(\mathcal{O}_Y(n)))$ is surjective for all $n\geq 1$. Consider the surjective homomorphism ϕ: $R=k\oplus(\bigoplus_{n\geq 1} R_n)\to R'=k\oplus(\bigoplus_{n\geq 1} R'_n)$ of graded rings. R and R' have dual G-actions and ϕ is a G-homomorphism. Let x be a geometric point of $X^{ss}(j^*(\mathcal{O}_Y(1)))$. Then there exists an element s of R'^G_n with some $n>0$ such that x is a point of X_s. By virtue of Lemma 5.1.B of [16], there exists a positive integer m such that s^m is contained in $\phi(R^G_{nm})$, say $s^m=\phi(t)$. Since $X_{s^m}=X_s$, x is contained in $j^{-1}(Y_t)=X_{s^m}$. Thus $j(x)$ is in $Y^{ss}(\mathcal{O}_Y(1))$, that is, $X^{ss}(j^*(\mathcal{O}_Y(1)))\subseteq Y^{ss}(\mathcal{O}_Y(1))$. The converse and the assertion on stability are obvious. q.e.d.

Corollary 2.2.1. *Let i: $W\to W'$ be an injective homomorphism of finite dimensional vector spaces. Then, a geometric point T of $P(V, r, W)$ is semi-stable (or, stable) if and only if $i_*(T)$ is semi-stable (or, stable, resp.) in $P(V, r, W')$.*

The above corollary means that we can extend W without disturbing the stability of a point of $P(V, r, W)$.

Definition 2.3. Let $W, W_1,\dots, W_n$ be finite dimensional k-vector spaces. A map ϕ: $W_1\otimes_k W_2\otimes_k\cdots\otimes_k W_n\to W$ is said to be admissible (to extensions) if ϕ is k-linear and for all over fields K of k and for the map ϕ_K: $(W_1\otimes_k K)\otimes_K\cdots\otimes_K(W_n\otimes_k K)\to W\otimes_k K$ induced by ϕ, $\phi_K(x_1\otimes\cdots\otimes x_n)=0$ implies that one of x_i's is zero. When ϕ is admissible, we denote $\phi_K(x_1\otimes\cdots\otimes x_n)$ by $x_1\circ\cdots\circ x_n$.

Definition 2.4. Let K be an over field of k and let T and T' be K-rational points of Gieseker spaces $P(V, r, W)$ and $P(V', r, W')$, respectively. T is isomorphic to T' if $W=W'$ and if there exists an isomorphism j: $V\otimes_k K\to V'\otimes_k K$ such that $T=T'\cdot \overset{r}{\wedge} j$ (as points in $P(V, r, W)$). We shall denote an isomorphism by $T\cong T'$.

In the case of $V=V'$, $T\cong T'$ if and only if there exists a K-rational point g of $GL(V)$ such that $\sigma(g, T')=T$.

Our present aim is to define the notion of extensions of points in Gieseker spaces and study their properties.

Definition 2.5. Let K be an over field of k and let T, T' and T'' be K-rational points of $P(V, r, W)$, $P(V', r', W')$ and $P(V'', r'', W'')$, respectively. Let ϕ: $W'\otimes_k W''\to W$ be an admissible map. T is said to be a ϕ-extension or, simply, an extension of T'' by T' if the following conditions are satisfied;

1) $r=r'+r''$,

2) there exists an exact sequence

$$0\longrightarrow V'\otimes_k K\xrightarrow{f} V\otimes_k K\xrightarrow{g} V''\otimes_k K\longrightarrow 0$$

such that $T(f(x_1),\dots, f(x_{r'}), y_1,\dots, y_{r''})=\phi(T'(x_1,\dots, x_{r'})\otimes T''(g(y_1),\dots, g(y_{r''})))$ for all vectors $x_1,\dots, x_{r'}$ in $V'\otimes_k K$ and $y_1,\dots, y_{r''}$ in $V\otimes_k K$, where both side in the above equality are regarded as points in $\mathbf{P}(\mathrm{Hom}(\overset{r'}{\wedge} V'\otimes_k \overset{r''}{\wedge} V'', W)^{\vee})$.

The exact sequence in (2) is called the underlying exact sequence of the extension. T' (or, T'') is called a subpoint (or, quotient point, resp.) of T.

The following plays a key role in the proof of Theorem 2.13.

Lemma 2.6. *Let V, V' and V'' be finite dimensional k-vector spaces with $\dim_k V=\dim_k V'+\dim_k V''$, r, r' and r'' be positive integers with $r=r'+r''$ and let ϕ: $W'\otimes_k W''\to W$ be an admissible map. Suppose that Z' (or, Z'') is a $GL(V')$ (or, $GL(V'')$, resp.)-invariant closed subset of $P(V', r', W')$ (or, $P(V'', r'', W'')$, resp.). Then there exists a $GL(V)$-invariant closed subset Z of $P(V, r, W)$ such that for all algebraically closed fields K containing k, $Z(K)=\{T\in P(V, r, W)(K)|T$ has one of the properties* (2.6.1), (2.6.2)$\}$.

(2.6.1) *T is ϕ-extension of a T'' in $Z''(K)$ by a T' in $Z'(K)$.*

(2.6.2) *There exists an injection f: $V'\otimes_k K\to V\otimes_k K$ such that $T(f(x_1),\dots, f(x_{r'}), y_1,\dots, y_{r''})=0$ for all vectors $x_1,\dots, x_{r'}$ in $V'\otimes_k K$ and $y_1,\dots, y_{r''}$ in $V\otimes_k K$.*

Proof. Let $n=\dim_k V$, $n'=\dim_k V'$ and $n''=\dim_k V''$. There exists an open set U' (or, U'') of $\mathrm{Hom}(V', V)=\mathbf{A}_k^{nn'}$ (or, $\mathrm{Hom}(V, V'')=\mathbf{A}_k^{nn''}$, resp.) such that for all algebraically closed fields containing k, $U'(K)=\{f\in \mathrm{Hom}_K(V'\otimes_k K, V\otimes_k K)|f$ is injective$\}$ (or, $U''(K)=\{g\in \mathrm{Hom}_K(V\otimes_k K, V''\otimes_k K)|g$ is surjective$\}$, resp.). For these U' and U'', we can find a closed subscheme U_0 of $U'\times_k U''$ such that $U_0(K)=\{(f, g)\in U'(K)\times U''(K)|gf=0\}$. Let us fix a basis $e'_1,\dots, e'_{n'}$ ($e_1,\dots, e_n$ or $e''_1,\dots, e''_{n''}$) of V' (V or V'', resp.). Using these bases, geometric points (f, g) of U_0, $x'_1,\dots, x'_{r'}$ of V' and $y_1,\dots, y_{r''}$ of V are represented by matrices (A, B), $(x'_{11},\dots, x'_{1n'}),\dots, (x'_{r'1},$

$\ldots, x'_{r'n'})$ and $(y_{11},\ldots, y_{1n}),\ldots, (y_{r''1},\ldots, y_{r''n})$, repectively, where A (or, B) is a matrix of $n'\times n$ (or, $n\times n''$, resp.). If we set $X'=(x'_{ij})$, $Y=(y_{ij})$, then $T(f(x'_1),\ldots, f(x'_{r'}), y_1,\ldots, y_{r''})$ is represented by $T(X'A, Y)$ for a geometric point T of $P(V, r, W)$. Similarly, for a geometric point T'' of $P(V'', r'', W'')$, $T''(g(y_1),\ldots, g(y_{r''}))$ is represented by $T''(YB)$.

Sublemma 2.6.3. *There exists a closed subset F of $U_0\times_k Z'\times_k P(V, r, W)\times_k Z''$ such that for all algebraically closed fields K containing k,*

$$F(K)=\{(A, B, T', T, T'')|(1)\ T(X'A, Y)=T'(X')\circ T''(YB)$$

$$\textit{for all } X', Y \textit{ or } (2)\ T(X'A, Y)=0 \qquad \textit{for all}\quad X', Y\}.$$

Proof. Pick K-valued points (A, B) of U_0, T' of Z', T'' of Z'' and T of $P(V, r, W)$. Let $L(a, b)$ be the set of sequences l of integers $l_1,\ldots, l_b$ with $1\le l_1<\cdots<l_b\le a$. For l in $L(n, r)$ (l' in $L(n', r')$ or l'' in $L(n'', r'')$), $e_l^\vee(e_{l'}'^\vee$ or $e_{l''}''^\vee$, resp.) denotes $e_{l_1}^\vee\wedge\ldots\wedge e_{l_r}^\vee$ ($e'^\vee_{l'_1}\wedge\ldots\wedge e'^\vee_{l'_{r'}}$, or $e''^\vee_{l''_1}\wedge\ldots\wedge e''^\vee_{l''_{r''}}$, resp.), where $\{e_1^\vee,\ldots, e_n^\vee\}$ ($\{e_1'^\vee,\ldots, e_{n'}'^\vee\}$ or $\{e_1''^\vee,\ldots, e_{n''}''^\vee\}$, resp.) is the dual basis of $\{e_1,\ldots, e_n\}$ ($\{e'_1,\ldots, e'_{n'}\}$ or $\{e''_1,\ldots, e''_{n''}\}$, resp.). Then, using homogeneous coordinates, we can write

$$T=\sum_{\substack{l\in L(n,r)\\ j}} u(l)_j e_l^\vee\otimes w_j,\quad T'=\sum_{\substack{l'\in L(n',r')\\ j}} u'(l')_j e_{l'}'^\vee\otimes w'_j$$

$$T''=\sum_{\substack{l''\in L(n'',r'')\\ j}} u''(l'')_j e_{l''}''^\vee\otimes w''_j$$

where $\{w_j\}$ ($\{w'_j\}$ or $\{w''_j\}$) is a basis of W (W' or W'', resp.). We have, by Laplace's expansion theorem,

$$T(X'A, Y)=\sum_{\substack{l\in L(n,r)\\ j}} u(l)_j\{\sum_{\substack{l'\in L(n,r')\\ l''\in L(n,r'')\\ l=l'\cup l''}}(-1)^{|l,l'|}(X'A)(l')Y(l'')\}w_j$$

$$=\sum_{\substack{l\in L(n,r)\\ j}} u(l)_j\{\sum_{\substack{l'\in L(n,r')\\ l''\in L(n,r'')\\ l=l'\cup l''}}(-1)^{|l,l'|}(\sum_{k'\in L(n',r')}$$

$$A(k', l')X'(k'))Y(l'')\}w_j$$

$$=\sum_{\substack{k'\in L(n',r')\\ l''\in L(n,r'')\\ j}} R(k', l'', j, T, A)X'(k')Y(l'')w_j;$$

$$R(k', l'', j, T, A)=\sum_{\substack{l'\in L(n,r')\\ l'\cap l''=\phi}}(-1)^{|l'\cup l'', l'|}u(l'\cup l'')_j A(k', l'),$$

where for a matrix M of $a\times b$ and for subsets m, m' of $\{1,\ldots, a\}\{1,\ldots, b\}$ with $\#m=\#m'$, $M(m, m')$ is the minor determinant of M defined by m and m' and if $\#m=a$, $M(m, m')$ is denoted by $M(m')$, and where for a subset $l'=\{l_{i_1}<\cdots<l_{i_{r'}}\}$ of $l=\{l_1<\cdots<l_r\}$, $|l, l'|$ denotes the integer $r'(r'+1)/2+i_1+\cdots+i_{r'}$. On the other hand, we obtain

$$T'(X')=\sum_{\substack{k'\in L(n',r')\\ j}} u'(k')_j X'(k')w'_j \quad\text{and}$$

$$T''(YB)=\sum_{\substack{k''\in L(n'',r'')\\ j}} u''(k'')_j(YB)(k'')w''_j$$

$$=\sum_{\substack{k''\in L(n'',r'')\\ j}} u''(k'')_j\{\sum_{l''\in L(n,r'')} B(l'',k'')Y(l'')\}w''_j.$$

Thus, if $\phi(w'_{j'}\otimes w''_{j''})=\sum_j c(j',j'',j)w_j$ with $c(j',j'',j)\in k$, we have

$$T'(X')\circ T''(YB)=\sum_{\substack{k'\in L(n',r')\\ l''\in L(n,r'')\\ j}} Q(k',l'',j,T',T'',B)X'(k')Y(l'')w_j,$$

where $Q(k',l'',j,T',T'',B)=\sum_{\substack{k''\in L(n'',r'')\\ j',j''}} c(j',j'',j)u'(k')_{j'}u''(k'')_{j''}B(l'',k'')$. Now $T(X'A,Y)=0$ for all X' and Y if and only if $R(k',l'',j,T,A)=0$ for all $k'\in L(n',r')$, $l''\in L(n'',r'')$ and for all j. Note that $T'(X')\circ T''(YB)\neq 0$ for some X' and Y. Therefore, we see that

$$P(k'_1,k'_2,l''_1,l''_2,j_1,j_2,T,T',T'',A,B)=$$

$$Q(k'_1,l''_1,j_1,T',T'',B)R(k'_2,l''_2,j_2,T,A)-$$

$$Q(k'_2,l''_2,j_2,T',T'',B)R(k'_1,l'_1,j_1,T,A)=0$$

for all k'_1, k'_2, l''_1, l''_2, j_1 and j_2 if and only if (1) $T(X'A,Y)=T'(X')\circ T''(YB)$ for all X' and Y or (2) $T(X'A,Y)=0$ for all X' and Y. $P(k'_1,k'_2,l''_1,l''_2,j_1,j_2,T,T',T'',A,B)$ is a polynomial of $v(l,k',k'',j,j',j'')=u(l)_ju'(k')_{j'}u''(k'')_{j''}$, a_{ij} and b_{ij} over k and it is homogeneous with respect to $v(l,k',k'',j,j',j'')$. Thus if F is the closed set defined by the ideal generated by $\{P(k'_1,k'_2,l''_1,l''_2,j_1,j_2,T,T',T'',A,B)\}$, then F is the desired closed set. q.e.d.

Now let us come back to the proof of Lemma 2.6. Let σ (σ' or σ'') be the action of $GL(V)$ ($GL(V')$ or $GL(V'')$, resp.) on $P(V,r,W)$ ($P(V',r',W')$ or $P(V'',r'',W'')$, resp.). Define an action τ' (or, τ'') of $GL(V')$ (or, $GL(V'')$, resp.) on U' (or, U'', resp.) as follows;

for all geometric points g (or, h) of $GL(V')$ (or, $GL(V'')$, resp.) and for all geometric points A (or, B) of U' (or, U'', resp.), $\tau'(g,A)=gA$ (or, $\tau''(h,B)=B(h^{-1})$, resp.).

Then, for $H=GL(V')\times_k GL(V'')$, we have that U_0 is H-invariant with respect to the action $\tau'\times_k\tau''$ and that

$$T(X'\cdot\tau'(g,A),Y)=T(X'g\cdot A,Y)=T'(X'g)\circ T''(YB)$$

$$=\sigma'(g,T')(X')\circ\sigma''(h,T'')(Y\cdot\tau''(h,B))$$

or $$T(X'\cdot\tau'(g,A),Y)=T(X'g\cdot A,Y)=0$$

according as $T(X'A,Y)=T'(X')\circ T''(YB)$ for all X', Y or $T(X'A,Y)=0$ for all X', Y. We see therefore that if (A,B,T',T,T'') is a geometric point of F, then so is $(\tau'(g,A),\tau''(h,B),\sigma'(g,T'),T,\sigma''(h,T''))$ for all geometric points (g,h) of H because Z' (or Z'') is $GL(V')$ (or, $GL(V'')$, resp.) invariant. Let $\tilde{\sigma}$ be the above action of H on F. Then, for this $\tilde{\sigma}$, F is H-invariant and the projection p of F to U_0 is an H-morphism. Moreover, the projection q of F to $P(V, r, W)$ is also an H-morphism with the trivial action of H on $P(V, r, W)$. On the other hand, U_0 is a principal fibre bundle with group H over the Grassmann variety $Gr(n, n')$. Since p is projective and there exists a p-ample invertible sheaf with an H-linearization, we obtain a scheme Q which is projective over $Gr(n, n')$ and over which F is a principal fibre bundle with group H (see [14] Proposition 7.1 and its proof). Thus the following commutative diagram is obtained;

$$\begin{array}{ccccc} P(V, r, W) & \xleftarrow{\ q\ } & F & \xrightarrow{\ p\ } & U_0 \\ & \nwarrow^{q'} & \downarrow & & \downarrow \\ & & Q & \longrightarrow & Gr(n, n') \end{array}$$

It is clear that $Z=q(F)=q'(Q)$ is the desired set. Since Q is projective over $Gr(n, n')$, it is projective over k. Thus Z is closed in $P(V, r, W)$. We have only to show that Z is $GL(V)$-invariant. To do this, pick K-valued geometric points g of $GL(V)$ and T of Z. There exist K-valued geometric points T' of Z', T'' of Z'' and (A, B) of U_0 such that

(1) $T(X'A, Y)=T'(X')\circ T''(YB)$ for all K-valued X' and Y

or (2) $T(X'A, Y)=0$ for all K-valued X' and Y.

In case (1),

$$\sigma(g, T)(X'A(g^{-1}), Y)=T(X'A, Yg)=T'(X')\circ T''(YgB)$$

and we have an exact sequence

$$(*) \qquad 0 \longrightarrow V'\otimes_k K \xrightarrow{Ag^{-1}} V\otimes_k K \xrightarrow{gB} V''\otimes_k K \longrightarrow 0$$

because of $Ag^{-1}gB=AB=0$. Therefore, $\sigma(g, T)$ is a ϕ-extension of T'' by T' with the underlying exact sequence $(*)$. In case (2),

$$\sigma(g, T)(X'Ag^{-1}, Y)=T(X'A, Yg)=0,$$

whence $\sigma(g, T)$ and Ag^{-1} have the property (2). q.e.d.

For the convenience of readers, let us recall some of notions and results in [5] (cf. [12] §4).

Definition 2.7. Let K be an algebraically closed field containing k and let T be a non-zero element of $\mathrm{Hom}_K(\overset{r}{\wedge}(V\otimes_k K), W\otimes_k K)$ or a K-rational point of $P(V, r, W)$. Vectors $x_1,\dots, x_d$ in $V\otimes_k K$ are said to be T-independent if there exist vectors $x_{d+1},\dots, x_r$ in $V\otimes_k K$ such that $T(x_1,\dots, x_r)\neq 0$. A vector x in $V\otimes_k K$ is said to be T-dependent on $x_1,\dots, x_d$ if $T(x_1,\dots, x_d, x, y_{d+2},\dots, y_r)=0$ for all vectors $y_{d+2},\dots, y_r$ in $V\otimes_k K$. The vector subspace of $V\otimes_k K$ formed by vectors which are T-dependent on $x_1,\dots, x_d$ is called the T-span of $x_1,\dots, x_d$ and it is denoted by $\ll x_1$,

$\ldots, x_d \gg_T$.

Using these notions we have

Proposition 2.8. *Let K be an algebraically closed field containing k.*

1) *A point T in $P(V, r, W)(K)$ is properly stable (or semi-stable) with respect to the action $\bar{\sigma}$ of $PGL(V)$ if whenever $x_1, \ldots, x_d$ in $V \otimes_k K$ are T-independent, then $\dim_K \ll x_1, \ldots, x_d \gg_T < (d/r)\dim_k V$ (or, $\dim_K \ll x_1, \ldots, x_d \gg_T \leq (d/r)\dim_k V$, resp.).*

2) *For a point T in $P(V, r, W)(K)$, assume that there exist a vector subspace U of $V \otimes_k K$ and an integer d such that $T(x_1, \ldots, x_{d+1}, y_{d+2}, \ldots, y_r) = 0$ whenever $x_1, \ldots, x_{d+1}$ are in U and that $\dim_K U > (d/r)\dim_k V$ (or, $\dim_K U \geq (d/r)\dim_k V$). Then the T is not semi-stable (or, not stable, resp.).*

Our main idea of this section is the following.

Definition 2.9. Let T be a K-valued geometric point of $P(V, r, W)$. T is said to be excellent if it enjoys the following two properties:

1) For each vector subspace V' of $V \otimes_k K$ and each positive integer s, (a) and (b) are equivalent to each other;

a) $T(y_1, \ldots, y_s, z_{s+1}, \ldots, z_r) = 0$ for all $z_{s+1}, \ldots, z_r$ in $V \otimes_k K$ whenever $y_1, \ldots, y_s$ are contained in V',

b) there exists a set of T-independent vectors $x_1, \ldots, x_d$ in $V \otimes_k K$ such that $s > d$ and $\ll x_1, \ldots, x_d \gg_T \supseteq V'$.

2) For every subpoint T' of T, if x is T'-dependent on T'-independent vectors $x_1, \ldots, x_d$, then $f(x)$ is T-dependent on $f(x_1), \ldots, f(x_d)$, where f is the injection of the underlying exact sequence to define the subpoint T' of T. (Note that $f(x_1), \ldots, f(x_d)$ are T-independent.)

Excellent points have nice properties. In the first place,

Proposition 2.10. *Suppose that T has the property* (1) *in Definition* 2.9. *Then T is semi-stable (or, stable) if and only if*

$$\dim_K \ll x_1, \ldots, x_d \gg_T \leq (d/r)\dim_k V$$

$$(or,\ \dim_K \ll x_1, \ldots, x_d \gg_T < (d/r)\dim_k V,\ resp.),$$

whenever $x_1, \ldots, x_d$ are T-independent.

Proof. "If" part is Proposition 2.8, (1). To show "only if" part assume that there exist T-independent vectors $x_1, \ldots, x_d$ such that $\dim_K \ll x_1, \ldots, x_d \gg_T > (d/r)\dim_k V$ (or, $\dim_K \ll x_1, \ldots, x_d \gg_T \geq (d/r)\dim_k V$). By the property (1) of Definition 2.9, $T(y_1, \ldots, y_{d+1}, z_{d+2}, \ldots, z_r) = 0$ for all $z_{d+2}, \ldots, z_r$ whenever $y_1, \ldots, y_{d+1}$ are in $\ll x_1, \ldots, x_d \gg_T$. By virtue of Proposition 2.8, (2), we know that T is not semi-stable (or, stable, resp.). q.e.d.

In the next place,

Proposition 2.11. *Let T, T' and T'' be K-valued geometric points of $P(V, r,$*

W), P(V', r', W') and P(V'', r'', W''), respectively. Assume that $\dim_k V/r = \dim_k V'/r' = \dim_k V''/r''$, *T is excellent and that T' (or, T'') is a subpoint (or, quotient point, resp.) of T. If T is semi-stable, then so are both T' and T''.*

Proof. We may assume that there exists an admissible map ϕ: $W'\otimes_k W'' \to W$, T is a ϕ-extension of T'' by T' and that T is semi-stable and excellent. Let

$$0 \longrightarrow V'\otimes_k K \xrightarrow{f} V\otimes_k K \xrightarrow{g} V''\otimes_k K \longrightarrow 0$$

be the underlying exact sequence. Pick T''-independent vectors $\bar{y}_1 = g(y_1), \dots, \bar{y}_d = g(y_d)$. Since $T' \neq 0$, we can find vectors $x_1, \dots, x_{r'}$ in $V'\otimes_k K$ with $T'(x_1, \dots, x_{r'}) \neq 0$. Thus there exists vectors $z_{d+1}, \dots, z_{r''}$ in $V\otimes_k K$ such that $T(f(x_1), \dots, f(x_{r'}), y_1, \dots, y_d, z_{d+1}, \dots, z_{r''}) = T'(x_1, \dots, x_{r'}) \circ T''(\bar{y}_1, \dots, \bar{y}_d, g(z_{d+1}), \dots, g(z_{r''})) \neq 0$. Thus $f(x_1), \dots, f(x_{r'}), y_1, \dots, y_d$ are T-independent. If $g(z)$ is contained in $\ll \bar{y}_1, \dots, \bar{y}_d \gg_{T''}$, then $T(f(x_1), \dots, f(x_{r'}), y_1, \dots, y_d, z, w_{d+2}, \dots, w_{r''}) = T'(x_1, \dots, x_{r'}) \circ T''(\bar{y}_1, \dots, \bar{y}_d, g(z), g(w_{d+2}), \dots, g(w_{r''})) = 0$ for all $w_{d+2}, \dots, w_{r''}$ in $V\otimes_k K$. Hence z is an element of $\ll f(x_1), \dots, f(x_{r'}), y_1, \dots, y_d \gg_T$. Therefore $g^{-1}(\ll \bar{y}_1, \dots, \bar{y}_d \gg_{T''})$ is a vector subspace of $\ll f(x_1), \dots, f(x_{r'}), y_1, \dots, y_d \gg_T$. Since T is semi-stable and excellent, we have, by Proposition 2.10,

$$\begin{aligned}
\dim_K \ll \bar{y}_1, \dots, \bar{y}_d \gg_{T''} &= \dim_K g^{-1}(\ll \bar{y}_1, \dots, \bar{y}_d \gg_{T''}) - \dim_k V' \\
&\leq \dim_K \ll f(x_1), \dots, f(x_{r'}), y_1, \dots, y_d \gg_T - \dim_k V' \\
&\leq \{(d+r') \dim_k V\}/r - (\dim_k V - \dim_k V'') \\
&= \{(d+r-r'') \dim_k V''\}/r'' - (r/r'') \dim_k V'' + \dim_k V'' \\
&= (d/r'') \dim_k V''.
\end{aligned}$$

Therefore, T'' is semi-stable by virtue of Proposition 2.8, (1).

Next we shall prove our assertion on T'. Let $x_1, \dots, x_d$ be T'-independent vectors in $V'\otimes_k K$. By virtue of the property (2) of excellent points, we have the inclusion $\ll x_1, \dots, x_d \gg_{T'} \subseteq f^{-1}(\ll f(x_1), \dots, f(x_d) \gg_T)$. This and the fact that T is semi-stable and excellent imply the following;

$$\begin{aligned}
\dim_K \ll x_1, \dots, x_d \gg_{T'} &\leq \dim_K \ll f(x_1), \dots, f(x_d) \gg_T \\
&\leq (d/r) \dim_k V = (d/r') \dim_k V'.
\end{aligned}$$

Hence T' is semi-stable by virtue of Proposition 2.8, (1). q.e.d.

A converse of the above proposition holds good.

Proposition 2.12. *Let T, T' and T'' be K-valued geometric points of P(V, r, W), P(V', r', W') and P(V'', r'', W''), respectively, and let* ϕ: $W'\otimes_k W'' \to W$ *be an admissible map. Assume that* $\dim_k V'/r' = \dim_k V/r = \dim_k V''/r''$, *all of the T, T' and T'' are excellent and that T is a* ϕ*-extension of T'' by T'. If both T' and T'' are semi-stable, then T is semi-stable.*

Proof. Let $x_1, \dots, x_d$ be T-independent vectors in $V\otimes_k K$ and let

$$0 \longrightarrow V' \otimes_k K \xrightarrow{f} V \otimes_k K \xrightarrow{g} V'' \otimes_k K \longrightarrow 0$$

be the underlying exact sequence of the ϕ-extension T of T'' by T'. Set

$$f^{-1}(\ll x_1, \dots, x_d \gg_T) = V'_0$$

$$g(\ll x_1, \dots, x_d \gg_T) = V''_0.$$

Let $\{g(y_1), \dots, g(y_{d''})\}$ be a maximal set of T''-independent vectors in V''_0 and let $\{z_1, \dots, z_{d'}\}$ be a maximal set of T'-independent vectors in V'_0. Then there exist vectors $z_{d'+1}, \dots, z_{r'}$ in $V' \otimes_k K$ and $y_{d''+1}, \dots, y_{r''}$ in $V \otimes_k K$ such that $T'(z_1, \dots, z_{d'}, z_{d'+1}, \dots, z_{r'}) \neq 0$ and $T''(g(y_1), \dots, g(y_{d''}), g(y_{d''+1}), \dots, g(y_{r''})) \neq 0$. Since $T(f(z_1), \dots, f(z_r'), y_1, \dots, y_{r''}) = T'(z_1, \dots, z_{r'}) \circ T''(g(y_1), \dots, g(y_{r''})) \neq 0$, $f(z_1), \dots, f(z_{d'}), y_1, \dots, y_{d''}$ are T-independent. By the property (1) for T being excellent, we get the inequality $d' + d'' \leq d$. On the other hand, if z is in V'_0, then it is T'-dependent on $z_1, \dots, z_{d'}$. Thus $V'_0 \subseteq \ll z_1, \dots, z_{d'} \gg_{T'}$, and hence

$$\dim_K V'_0 \leq (d'/r') \dim_k V'$$

because T' is semi-stable and excellent. Similarly, we have

$$\dim_K V''_0 \leq (d''/r'') \dim_k V''.$$

Therefore, the following inequality is obtained;

$$\begin{aligned} \dim_K \ll x_1, \dots, x_d \gg_T &= \dim_K V'_0 + \dim_K V''_0 \\ &\leq (d'/r') \dim_k V' + (d''/r'') \dim_k V'' = \{(d' + d'') \dim_k V\}/r \\ &\leq (d/r) \dim_k V. \end{aligned}$$

This implies that T is semi-stable by virtue of Proposition 2.8, (1) q.e.d.

The following is one of goals of this section.

Theorem 2.13. *Let ϕ_i: $W_{i-1} \otimes_k W'_i \to W_i$ be admissible maps ($1 \leq i \leq t$, $W_0 = k$), $0 < r_1 < \cdots < r_t = r$ be a sequence of integers and let F_i be a $GL(V_i)$-invariant closed set of $P(V_i, r_i, W_i)$ ($1 \leq i \leq t$). Assume that for every algebraically closed field K containing k, all the points of $F_i(K)$ are excellent and that* $\dim_k V_1/r_1 = \cdots = \dim_k V_t/r_t$. *Let S_i be a stable, excellent point in $P(V'_i, l_i, W'_i)(\bar{k})$ which is k-rational, where $l_i = r_i - r_{i-1}$ and $\bar{k}$ is the algebraic closure of k. Then, there exists a $GL(V_t)$-invariant closed set $Z_t = Z(S_1, \dots, S_t)$ of $F_t^{ss} = F_t^{ss}(\mathcal{O}(1) \otimes \mathcal{O}_{F_t})$ such that for every algebraically closed field K containing k,*

$$Z_t(K) = \{T \in F_t(K) | T \text{ has the following property } (*)_t\}.$$

$(*)_t$: *There exists a K-valued geometric point T_i in each $F_i^{ss} = F_i^{ss}(\mathcal{O}(1) \otimes \mathcal{O}_{F_i})$ such that $T_1 = S_1$, T_i is a ϕ_i-extension of S_i by T_{i-1} ($2 \leq i \leq t$) and $T = T_t$.*

Proof. Our proof is by induction on t. When $t = 1$, then $T = S_1$ and hence there exists a K-valued point g of $GL(V_1)$ such that $\sigma(g, S_1) = T$. Since S_1 is stable, the $GL(V_1)$-orbit Z of S_1 is closed in F_1^{ss}. Clearly, Z is the desired closed set.

Assume that the theorem holds for $t-1$. Then there exists a $GL(V_{t-1})$-invariant closed subset Z_{t-1} of $(F_{t-1})^{ss}$ such that for all algebraically closed fields K containing k, $Z_{t-1}(K)=\{T\in F_{t-1}(K)|T$ has the property $(*)_{t-1}\}$. If $\bar{Z}_{t-1}$ is the closure of Z_{t-1} in F_{t-1}, then it is a $GL(V_{t-1})$-invariant closed subset of F_{t-1}. For the $GL(V'_t)$-orbit A of S_t in $P(V'_t, l_t, W'_t)$, let $\bar{A}$ be the closure of A in $P(V'_t, l_t, W'_t)$. Then $\bar{A}$ is a $GL(V'_t)$-invariant closed subset in $P(V'_t, l_t, W'_t)$. By virtue of Lemma 2.6, there exists a $GL(V_t)$-invariant closed subset $\bar{Z}_t$ in F_t such that $\bar{Z}_t(K)=\{T\in F_t(K)|(1)$ T is a ϕ_t-extension of a T'' in $\bar{A}(K)$ by a T' in $\bar{Z}_{t-1}(K)$ or (2) there exists a injective linear map $f\colon V_{t-1}\otimes_k K\to V_t\otimes_k K$ such that $T(f(x_1),\dots, f(x_{r_{t-1}}), y_1,\dots, y_{l_t})=0$ for all $x_1,\dots, x_{r_{t-1}}$ and $y_1,\dots, y_{l_t}\}$. The $GL(V_t)$-invariant closed subset $Z_t=\bar{Z}_t\cap F_t^{ss}$ is the desired one. In fact, if T is contained in $Z_t(K)$ and if T has the property (2) above, then there exists a set of T-independent vectors $\{x_1,\dots, x_d\}$ in $f(V_{t-1}\otimes_k K)$ with $d<r_{t-1}$ and $\ll x_1,\dots, x_d\gg_T\supseteq f(V_{t-1}\otimes_k K)$ because T is excellent. We have that $\dim_K\ll x_1,\dots, x_d\gg_T\geq \dim_k V_{t-1}=(r_{t-1}/r_t)\dim_k V_t>(d/r_t)\dim_k V_t$, which contradicts the fact that T is semi-stable (see Proposition 2.10). Thus, if T is a point of $Z_t(K)$, then T is excellent, semi-stable and moreover, a ϕ_t-extension of a T'' in $\bar{A}(K)$ by a T' in $\bar{Z}_{t-1}(K)$. Since T is excellent and since $\dim_k V_t/r_t=\dim_k V_{t-1}/r_{t-1}=\dim_k V'_t/l_t$, we know that T' and T'' are semi-stable by virtue of Proposition 2.11. Since $\bar{A}\cap P(V'_t, l_t, W'_t)^{ss}=A$ and $\bar{Z}_{t-1}\cap(F_{t-1})^{ss}=Z_{t-1}$, T' (or, T'') is an element of $Z_{t-1}(K)$ (or, $A(K)$, resp.). Thus T' has the property $(*)_{t-1}$ and $T'\cong S_t$, which implies that T has the property $(*)_t$. Conversely, assume that an element T of $F_t(K)$ has the property $(*)_t$. Then T is a ϕ_t-extension of T'' by T' such that T' has the property $(*)_{t-1}$ and $T''\cong S_t$. Since all the T, T' and T'' are excellent and since T' and T'' are semi-stable, T is semi-stable by virtue of Proposition 2.12. Thus T is contained in $Z_t(K)=\bar{Z}_t(K)\cap F_t^{ss}(K)$. q.e.d.

Our next task is to find typical closed orbits in $P(V, r, W)^{ss}$.

Definition 2.14. Let $\phi\colon W'\otimes_k W''\to W$ be an admissible map and let T, T' and T'' be K-valued geometric points of $P(V, r, W)$, $P(V', r', W')$ and $P(V'', r'', W'')$, respectively. Assume that T is a ϕ-extension of T'' by T' and let

$$0\longrightarrow V'\otimes_k K\xrightarrow{f} V\otimes_k K\xrightarrow{g} V''\otimes_k K\longrightarrow 0$$

be the underlying exact sequence of the extension. T is said to be a ϕ-direct sum of T' and T'' if there exists a linear map $i\colon V''\otimes_k K\to V\otimes_k K$ such that $g\cdot i=id_{V''\otimes K}$ and $T(i(y_1),\dots, i(y_s), w_{s+1},\dots, w_r)=0$ for all $w_{s+1},\dots, w_r$ in $V\otimes_k K$ whenever $s>r''$.

Lemma 2.15. *Let a K-valued geometric point T of $P(V, r, W)$ be a ϕ-extension of a T'' in $P(V'', r'', W'')(K)$ by a T' in $P(V', r', W')(K)$ and let*

$$0\longrightarrow V'\otimes_k K\xrightarrow{f} V\otimes_k K\xrightarrow{g} V''\otimes_k K\longrightarrow 0$$

be the underlying exact sequence of the extension. Then T is a ϕ-direct sum of T' and T'' if and only if the following (2.15.1) *holds*;

(2.15.1) *there exists a linear map h of $V''\otimes_k K$ to $V\otimes_k K$ such that $g\cdot h=\alpha(id_{V''\otimes K})$ for some $\alpha\in K$, $\alpha\neq 0$ and that* $T(f(x_1)+h(y_1),\dots, f(x_r)+h(y_r))=$

$\sum(-1)^{R+i_1+\cdots+i_{r'}}T'(x_{i_1},\dots, x_{i_{r'}})\circ T''(y_{j_1},\dots, y_{j_{r''}})$, *where the sum runs over all indices* $i_1<\cdots<i_{r'}$, $j_1<\cdots<j_{r''}$ *with* $\{i_1,\dots, i_{r'}, j_1,\dots, j_{r''}\}=\{1,\dots, r\}$ *and where* $R=r'(r'+1)/2$.

Proof. Assume that T is a ϕ-direct sum of T' and T''. Then

$$T(f(x_1)+i(y_1),\dots, f(x_r)+i(y_r))$$
$$=\sum(-1)^{s(s+1)/2+i_1+\cdots i_s}T(f(x_{i_1}),\dots, f(x_{i_s}), i(y_{j_1}),\dots, i(y_{j_{r-s}})),$$

where the sum runs over all indices $i_1<\cdots<i_s$, $j_1<\cdots<j_{r-s}$ with $\{i_1,\dots, i_s, j_1,\dots, j_{r-s}\}=\{1,\dots, r\}$. If $s>r'$, then $T(f(x_{i_1}),\dots, f(x_{i_s}), i(y_{j_1}),\dots, i(y_{j_{r-s}}))=T'(x_{i_1},\dots, x_{i_{r'}})\circ T''(0,\dots, 0, y_{j_1},\dots, y_{j_{r-s}})=0$. If $s<r'$, then the assumption that T is a ϕ-direct sum of T' and T'' implies that $T(f(x_{i_1}),\dots, f(x_{i_s}), i(y_{j_1}),\dots, i(y_{j_{r-s}}))=0$. Thus we obtain the equality in (2.15.1). Conversely, assume that (2.15.1) holds. Then, for $i=(1/\alpha)h$, $g\cdot i=id_{V''\otimes K}$. Hence $V\otimes_k K=f(V'\otimes_k K)\oplus h(V''\otimes_k K)$. Thus every vector in $V\otimes_k K$ can be written uniquely in the form $f(x)+h(y)$. By the assumption, we obtain that if $s>r''$,

$$T(i(y_1),\dots, i(y_s), w_{s+1},\dots, w_r)$$
$$=T(h(\alpha^{-1}y_1),\dots, h(\alpha^{-1}y_s), f(x_{s+1})+h(y_{s+1}),\dots, f(x_r)$$
$$+h(y_r))=0.$$

q. e. d.

A direct sum is independent of the choice of extensions.

Lemma 2.16. *Let* T' *be* K*-valued geometric points of* $P(V', r', W')$ *and* $P(V'', r'', W'')$, *respectively, and let* ϕ: $W'\otimes_k W''\to W$ *be an admissible map. If* T_1 *and* T_2 *are* ϕ*-direct sum of* T' *and* T'', *then* $T_1\cong T_2$. *Thus a direct sum of* T' *and* T'' *can be denoted by* $T'\oplus T''$.

Proof. Let

$$0\longrightarrow V'\otimes_k K\xrightarrow{u_i} V\otimes_k K \underset{s_i}{\overset{v_i}{\rightleftarrows}} V''\otimes_k K\longrightarrow 0$$

be the underlying exact sequence for the extension T_i and let s_i be the section of v_i which makes T_i to be a ϕ-direct sum of T' and T''. Fix a basis $e'_1,\dots, e'_{n'}$ (or, e''_1, ..., $e''_{n''}$) of V' (or, V'', resp.). Set

$$a_j^{(i)}=\begin{cases} u_i(e'_j) & \text{if } 1\le j\le n' \\ s_i(e''_{j-n'}) & \text{if } n'<j\le n. \end{cases}$$

Then $\{a_1^{(i)},\dots, a_n^{(i)}\}$ forms a basis of $V\otimes_k K$. There exists a K-valued point g of $GL(V)$ such that $a_j^{(1)}g=a_j^{(2)}$. For vectors $x_1,\dots, x_r$ in $V'\otimes_k K$ and $y_1,\dots, y_r$ in $V''\otimes_k K$, we obtain

$$T_1(u_1(x_1)+s_1(y_1),\dots, u_1(x_r)+s_1(y_r))$$

$$= \sum (-1)^{R+i_1+\cdots+i_{r'}} T'(x_{i_1},\dots, x_{i_{r'}}) \circ T''(y_{j_1},\dots, y_{j_{r''}})$$
$$= T_2(u_2(x_1)+s_2(y_1),\dots, u_2(x_r)+s_2(y_r))$$
$$= T_2((u_1(x_1)+s_1(y_1))g,\dots, (u_1(x_r)+s_1(y_r))g)$$
$$= \sigma(g, T_2)(u_1(x_1)+s_1(y_1),\dots, u_1(x_r)+s_1(y_r)),$$

where the sum in the second line of the above equality runs over all indices $i_1 < \cdots < i_{r'}$, $j_1 < \cdots < j_{r''}$ with $\{i_1,\dots, i_{r'}, j_1,\dots, j_{r''}\} = \{1,\dots, r\}$. Thus we have $T_1 = \sigma(g, T_2)$, that is, $T_1 \cong T_2$. q.e.d.

Let ϕ_i: $W_{i-1} \otimes_k W'_i \to W_i$ be a sequence of admissible maps ($1 \le i \le t$, $W_0 = k$). Then $\phi^{(i)} = \phi_i \cdot (\phi_{i-1} \otimes W'_i) \cdot \cdots \cdot (\phi_1 \otimes W'_2 \otimes \cdots \otimes W'_i)$ defines an admissible map of $W'_1 \otimes_k \cdots \otimes_k W'_i$ to W_i. Let $l_1,\dots, l_t$ be a sequence of positive integers and let V'_i be a k-vector space of dimension m_i. Put $r_i = l_1 + \cdots + l_i$ and $V_i = V'_1 \oplus \cdots \oplus V'_i$, then $\dim_k V_i = \sum_{j=1}^{i} m_j = n_i$ and we have a natural exact sequence with a splitting map s_i:

$$(2.17.1) \qquad 0 \longrightarrow V_{i-1} \otimes_k K \xrightarrow{f_i} V_i \otimes_k K \underset{s_i}{\overset{g_i}{\rightleftarrows}} V'_i \otimes_k K \longrightarrow 0$$

A decomposition I of type $l_1,\dots, l_j$ is a sequence of ordered subsets $I_1,\dots, I_j$ of integers with the following properties:
(1) $I_k \cap I_i = \phi$ if $k \ne i$, (2) $I_1 \cup \cdots \cup I_j = \{1,\dots, r_j\}$, (3) $\# I_i = l_i$. The set of decompositions of type $l_1,\dots, l_j$ is denoted by $D(l_1,\dots, l_j)$. For a decomposition $I = \{I_1,\dots, I_j\}$, $(-1)^I$ denotes the signature of the permutation $\begin{pmatrix} 1 & ,\dots, & l_1 & ,\dots, & l_{j-1}+1, & \cdots, & r_j \\ a_{11}, & \dots, & a_{1l_1}, & \dots, & a_{j1} & ,\cdots, & a_{jl_j} \end{pmatrix}$, where $\{a_{i1} < \cdots < a_{il_i}\}$ is I_i. If $I = \{I_1,\dots, I_j\}$ is a member of $D(l_1,\dots, l_j)$ and if $x_1,\dots, x_{r_j}$ are vectors, then x_{I_k} denotes the sequence of vectors $x_{a_1},\dots, x_{a_{l_k}}$, where $\{a_1 < \cdots < a_{l_k}\}$ is I_k.

Assume that a K-valued point T'_j of $P(V'_j, l_j, W'_j)$ is given for each j. We shall define a K-valued point T_i of $P(V_i, r_i, W_i)$ as follows: Let $\{x_1,\dots, x_{r_i}\}$ be a set of vectors in $V_i \otimes_k K$, then each x_j can be written uniquely in the form $x_j^{(1)} + \cdots + x_j^{(i)}$ with $x_j^{(u)} \in V'_u \otimes_k K$. Then

$$(2.17.2) \quad T_i(x_1,\dots, x_{r_i}) = \sum_{I \in D(l_1,\dots l_i)} (-1)^I \phi^{(i)}(T'_1(x_{I_1}^{(1)}) \otimes T'_2(x_{I_2}^{(2)}) \otimes \cdots \otimes T'_i(x_{I_i}^{(i)})).$$

Remark 2.18. (1) The definition of T_i is independent of the choice of $W_1,\dots, W_{i-1}$. To define T_i, we need only an admissible map $\phi^{(i)}$: $W'_1 \otimes_k \cdots \otimes_k W'_i \to W_i$.

(2) A permutation of $V'_1,\dots, V'_i$ may cause T_i to change $-T_i$. However, as a point of $P(V_i, r_i, W_i)$, it has no influence on T_i.

Lemma 2.19. *The T_i is a $\phi^{(i)}$-extension of T'_i by T_{i-1} with the underlying exact sequence* (2.17.1). *Moreover, T_i is a $\phi^{(i)}$-direct sum of T_{i-1} and T'_i.*

Proof. Let us compute $T_i(f_i(x_1),\dots, f_i(x_{r_{i-1}}), y_1,\dots, y_{l_i})$. If $D'(l_1,\dots, l_i)$ is the set of decompositions of type $l_1,\dots, l_i$ such that $I_i = \{r_{i-1}+1,\dots, r_i\}$, then we have

$$T_i(f_i(x_1),\dots, f_i(x_{r_{i-1}}), y_1,\dots, y_{l_i})$$
$$= \sum_{I\in D'(l_1,\dots,l_i)} (-1)^I \phi^{(i)}(T'_1(x^{(1)}_{I_1})\otimes\cdots\otimes T'_{i-1}(x^{(i-1)}_{I_{i-1}})$$
$$\otimes T'_i(y^{(i)}_1,\dots, y^{(i)}_{l_i}))$$

where $f_i(x_j)=x^{(1)}_j+\cdots+x^{(i-1)}_j$ with $x^{(u)}_j\in V'_u\otimes_k K$ and where $y_j=y^{(1)}_j+\cdots+y^{(i)}_j$ with $y^{(u)}_j\in V'_u\otimes_k K$. Therefore,

$$T_i(f_i(x_1),\dots, f_i(x_{r_{i-1}}), y_1,\dots, y_{l_i})$$
$$= \sum_{I\in D(l_1,\dots,l_{i-1})} (-1)^I \phi_i\{\phi^{(i-1)}(T'_1(x^{(1)}_{I_1})\otimes\cdots\otimes T'_{i-1}(x^{(i-1)}_{I_{i-1}}))$$
$$\otimes T'_i(y^{(i)}_1,\dots, y^{(i)}_{l_i})\}$$
$$=\phi_i\{T_{i-1}(x_1,\dots, x_{r_{i-1}})\otimes T'_i(g_i(y_1),\dots, g_i(y_{l_i}))\}.$$

This shows that T_i is a $\phi^{(i)}$-extension of T'_i by T_{i-1}. By virtue of the definition of T_i, it is obvious that $T_i(s_i(y_1),\dots, s_i(y_t), w_1,\dots, w_{r_i-t})=0$ for all vectors $w_1,\dots, w_{r_i-t}$ in $V_i\otimes_k K$ if $t>l_i$. q.e.d.

Let π be a permutation of $\{0, 1,\dots, t\}$. Assume that another system of admissible maps ϕ'_i: $W''_{\pi(i)-1}\otimes_k W'_{\pi(i)}\to W''_{\pi(i)}$ $(1\le i\le t,\ W''_{\pi(0)}=k)$ is given. Then, as is stated before (2.17.2), they define an admissible map $\phi'^{(t)}$: $W'_{\pi(1)}\otimes_k\cdots\otimes_k W'_{\pi(t)}\to W''_{\pi(t)}$. Since $W'_{\pi(1)}\otimes_k\cdots\otimes_k W'_{\pi(t)}\cong W'_1\otimes_k\cdots\otimes_k W'_t$, $\phi'^{(t)}$ provides us with an admissible map $\psi^{(t)}$ of $W'_1\otimes_k\cdots\otimes_k W'_t$ to $W''_{\pi(t)}$. If $W''_{\pi(t)}=W_t$ and if $\phi^{(t)}=\psi^{(t)}$, then Lemma 2.16, Remark 2.18 and Lemma 2.19 yield

Corollary 2.19.1. *Let V be a k-vector space of dimension n_t. Direct sums $(\cdots((T'_1\oplus T'_2)\oplus T'_3)\oplus\cdots)\oplus T'_t)$ and $(\cdots((T'_{\pi(1)}\oplus T'_{\pi(2)})\oplus T'_{\pi(3)})\oplus\cdots)\oplus T'_{\pi(t)})$ exist in $P(V, r_t, W_t)(K)$. Moreover, both are isomorphic to $T_t(\overset{r_t}{\wedge} h)$, a fortiori, they are isomorphic to each other over K, where h is a K-isomorphism of $V\otimes_k K$ to $V_t\otimes_k K$ and where V_t is defined in* (2.17.2).

The above allows us to employ the following notation.

Definition 2.20. We denote $(\cdots((T'_1\oplus T'_2)\oplus T'_3)\oplus\cdots)\oplus T'_t)$ by $T'_1\oplus T'_2\oplus\cdots\oplus T'_t$.

Every extension can be specialized to a direct sum up to isomorphism. Precisely, we have

Lemma 2.21. *Let V, V' and V'' be k-vector spaces with $\dim_k V=\dim_k V'+\dim_k V''$, ϕ: $W'\otimes_k W''\to W$ be an admissible map and let r, r' and r'' be positive integers with $r=r'+r''$. Let T, T' and T'' be K-valued geometric points of $P(V, r, W)$, $P(V', r', W')$ and $P(V'', r'', W'')$, respectively, and let R be a discrete valuaation ring over K with residue field K. Assume that T is a ϕ-extension of T'' by T'. Then there exists an R-valued point $\tilde{T}$ of $P(V, r, W)$ such that* (1) *$T\cong\tilde{T}$ over L and* (2) *$\tilde{T} \bmod \pi$ is a ϕ-direct sum of T' and T'', where L is the quotient field* *of R and where π is a uniformanizing parameter of R.*

Proof. Let s be a section of the underlying exact sequence

$$0 \longrightarrow V'\otimes_k K \xrightarrow{u} V\otimes_k K \xrightarrow{v} V''\otimes_k K \longrightarrow 0$$

of the extension T. Put $U_1=u(V'\otimes_k K)$ and $U_2=s(V''\otimes_k K)$, then $V\otimes_k K=U_1\oplus U_2$. Fix a basis $\{e_1^{(1)},\dots, e_{n'}^{(1)}, e_1^{(2)},\dots, e_{n''}^{(2)}\}$ of $V\otimes_k K$ with $e_i^{(1)}\in U_1$ and $e_j^{(2)}\in U_2$. Then $\{\tilde{e}_1^{(1)}=e_1^{(1)}\otimes 1,\dots, \tilde{e}_{n'}^{(1)}=e_{n'}^{(1)}\otimes 1, \tilde{e}_1^{(2)}=e_1^{(2)}\otimes 1,\dots, \tilde{e}_{n''}^{(2)}=e_{n''}^{(2)}\otimes 1\}$ forms a basis of $(V\otimes_k K)\otimes_K L=V_L$. Let g be the automorphism of V_L such that $g(x+y)=x+\pi y$ for $x\in U_1\otimes_K L$, $y\in U_2\otimes_K L$. Set $T_1=\sigma(g, T)$. For $z_i=x_i+y_i$, $1\le i\le r$ with $x_i\in U_1\otimes_K L$ and $y_i\in U_2\otimes_K L$,

$$\begin{aligned} T_1(z_1,\dots, z_r) &= T(x_1+\pi y_1,\dots, x_r+\pi y_r)\\ &= \sum_{s=0}^{r}\sum(-1)^{s(s+1)/2+i_1+\cdots+i_s}T(x_{i_1},\dots, x_{i_s}, \pi y_{j_1},\dots, \pi y_{j_{r-s}})\\ &= \sum_{s=0}^{r'}\sum(-1)^{s(s+1)/2+i_1+\cdots+i_s}\pi^{r-s}T(x_{i_1},\dots, x_{i_s}, y_{j_1},\dots, y_{j_{r-s}}) \end{aligned}$$

because T is a ϕ-extension of T'' by T', where the sums of the second and the third equalities run over all indices $i_1<\cdots<i_s$, $j_1<\cdots<j_{r-s}$ with $\{i_1,\dots, i_s, j_1,\dots, j_{r-s}\}=\{1,\dots, r\}$. Thus, as a point of $P(V, r, W)(L)$, $T_1=\tilde{T}$ with

$$\begin{aligned} \tilde{T}(z_1,\dots, z_r) &= \sum(-1)^{r'(r'+1)/2+i_1+\cdots+i_{r'}}T(x_{i_1},\dots, x_{i_{r'}}, y_{j_1},\dots, y_{j_{r''}})\\ &\quad +\pi\Big(\sum_{s=0}^{r'-1}\pi^{r'-s-1}\big(\sum(-1)^{s(s+1)+i_1+\cdots+i_s}T(x_{i_1},\dots, x_{i_s}, y_{j_1},\dots, y_{j_{r-s}})\big)\Big), \end{aligned}$$

where the sum runs over all indices as before. Thus, under the same notation as in the proof of Sublemma 2.6.3,

$$\begin{aligned} \tilde{T} &= \sum_{\substack{l'\in L(n',r')\\ l''\in L(n'',r'')\\ j}} u(l', l'')_j(\tilde{e}_{l'}^{(1)\vee}\wedge \tilde{e}_{l''}^{(2)\vee})\otimes w_j\\ &\quad +\sum_{s=0}^{r'-1}\pi^{r'-s-1}\sum_{\substack{l'\in L(n',s)\\ l''\in L(n'',r-s)\\ j}} u(l', l'')_j(\tilde{e}_{l'}^{(1)\vee}\wedge \tilde{e}_{l''}^{(2)\vee})\otimes w_j, \end{aligned}$$

where all the $u(l', l'')_j$'s are elements of K. Thus $\tilde{T}$ is an R-valued point of $P(V, r, W)$ and

$$\bar{T}=\tilde{T} \bmod \pi = \sum_{\substack{l'\in L(n',r')\\ l''\in L(n'',r'')\\ j}} u(l', l'')_j(e_{l'}^{(1)\vee}\wedge e_{l''}^{(2)\vee})\otimes w_j,$$

which implies that for $z_i'=f(x_i')+s(y_i')$ with $x_i'\in V'\otimes_k K$ and $y_i'\in V''\otimes_k K$,

$$\begin{aligned} \bar{T}(z_1',\dots, z_r') = \sum(-1)^{r'(r'+1)/2+i_1+\cdots+i_{r'}}T(&f(x'_{i_1}),\dots,\\ &f(x'_{i_{r'}}), s(y'_{j_1}),\dots, s(y'_{j_{r''}})). \end{aligned}$$

Since T is a ϕ-extension of T'' by T', we have

$$\bar{T}(z'_1,\dots, z'_r)=\sum(-1)^{r'(r'+1)/2+i_1+\cdots+i_{r'}}T'(x'_{i_1},\dots, x_{i_{r'}})\circ T''(y'_{j_1},\dots, y_{j_{r''}}),$$

where the sums in the above two equations run over all indices $i_1<\cdots<i_{r'}$, $j_1<\cdots<j_{r''}$ with $\{i_1,\dots, i_{r'}, j_1,\dots, j_{r''}\}=\{1,\dots, r\}$. q.e.d.

Now we come to another goal of this section.

Thorem 2.22. *Under the same situation as in Theorem* 2.18, *assume that* $Z(S_1,\dots, S_t)$ *is not empty, then* $GL(V_t)$*-orbit* $o(S_1,\dots, S_t)$ *of* $S_1\oplus\cdots\oplus S_t$ *is a unique closed orbit in* $Z(S_1,\dots, S_t)$.

Proof. First of all, our assumption implies that every $Z(S_1,\dots, S_i)$ is not empty. Let us prove the theorem by induction on t. If $t=1$, then $Z(S_1)=o(S_1)$. Thus we have nothing to prove. Assume that our assertion holds for $t-1$. Let o be a closed orbit in $Z(S_1,\dots, S_t)\otimes_k K$ with K an algebraically closed field K containing k. Since every point of $o(K)$ is an extension S_t by T' in $Z(S_1,\dots, S_{t-1})(K)$, there exists a point of $\tilde{T}$ of o such that a specialization of $\tilde{T}$ is $T'\oplus S_t$ by virtue of Lemma 2.21. Since F_t is proper over k, $T'\oplus S_t$ is a point of $F_t(K)$, whence it is in the set $Z(S_1,\dots, S_t)(K)$. Since o is closed in $Z(S_1,\dots, S_t)\otimes_k K$, $T'\oplus S_t$ is a point of $o(K)$, which implies that $T\cong T'\oplus S_t$. By the induction hypothesis, we can find a point $\tilde{T}$ in $Z(S_1,\dots, S_{t-1})$ such that $\tilde{T}'\cong T'$ and a specialization of $\tilde{T}'$ is $S_1\oplus\cdots\oplus S_{t-1}$. Since $T\cong\tilde{T}'\oplus S_t$ and since $(S_1\oplus\cdots\oplus S_{t-1})\oplus S_t$ is a specialization of $\tilde{T}'\oplus S_t$ (see the proof of Lemma 2.21), we see that $T\cong S_1\oplus\cdots\oplus S_t$ by the same argument as above. q.e.d.

§3. Strictly e-semi-stable sheaves

In this section, we shall introduce the notion of strictly e-semi-stable sheaves and study its property. If the family of the classes of semi-stable sheaves with a fixed Hilbert polynomial on the fibres of X over S is bounded, the results of this section are not necessary in the sequal.

From now on, we shall fix the following situation:

(3.1) Let S be a scheme of finite type over a universally Japanese ring Ξ and let $f\colon X\to S$ be a smooth, projective, geometrically integral morphism such that the dimension of each fibre of X over S is n. Let $\mathcal{O}_X(1)$ be an f-very ample invertible sheaf such that for all points s in S and all integers $i>0$, $H^i(X_s, \mathcal{O}_X(1)\otimes\mathcal{O}_{X_s})=0$.

As is stated in §3 of [12], the last condition in (3.1) is only for convenience' sake.

Definition 3.2. Let e be a non-negative integer and let E be a coherent sheaf of rank r on a geometric fibre X_s of X over S.

1) E is said to be e-stable[3] (or, e-semi-stable) (with respect to $\mathcal{O}_X(1)$) if it is stable (or, semi-stable, resp.) (with respect to $\mathcal{O}_X(1)$) and if for general non-singular curves $C=D_1\cdot\cdots\cdot D_{n-1}$, $D_1,\dots, D_{n-1}\in|\mathcal{O}_{X_s}(1)|$, every coherent subsheaf of $E\otimes_{\mathcal{O}_{X_s}}\mathcal{O}_C$

[3] The definition of e-stable (or, e-semi-stable) sheaves differed from this in [12] Definition 3.1. This definition seems to be better. The results on e-stable (or, e-semi-stable, resp.) sheaves in [12] hold good under this definition, too.

of rank t $(1 \leq t \leq r-1)$ has a degree $\leq t\{d(E, \mathcal{O}_X(1))/r+e\}$.

2) E is said to be strictly e-semi-stable if it is e-semi-stable and if every coherent quotient sheaf E' with $P_{E'}(m)=P_E(m)$ is e-semi-stable.

Remark 3.3. If E is e-stable, then it is strictly e-semi-stable.

As an immediate consequence of the definition of e-semi-stability, we have the following.

Lemma 3.4. *For a geometric point s of S, let*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

be an exact sequence of coherent sheaves on X_s. Assume that $P_{E'}(m)=P_E(m)=P_{E''}(m)$.

1) *If E' and E'' are e-semi-stable, then so is E.*

2) *If E is e-semi-stable, then E' is e-semi-stable and E'' is $r(E)e$-semi-stable, and hence E is strictly $r(E)e$-semi-stable.*

Proof. For semi-stability, our assertions are proved in Lemma 1.4. Choose a non-singular cirve $C=D_1\cdot\dots\cdot D_{n-1}$, $D_i \in |\mathcal{O}_{X_s}(1)|$ so generally that the sequence

$$0 \longrightarrow E'\otimes\mathcal{O}_C \xrightarrow{u} E\otimes\mathcal{O}_C \xrightarrow{v} E''\otimes\mathcal{O}_C \longrightarrow 0$$

is exact and that the condition in Definition 3.2 holds good for E or E', E'' according as E is e-semi-stable or E' and E'' are e-semi-stable.

1) Let F be a coherent subsheaf of rank t $(1 \leq t \leq r(E)-1)$ of $E\otimes\mathcal{O}_C$. Set $F'=u^{-1}(F)$ and $F''=v(F)$. Then we have

$$d(F)=d(F')+d(F''), \quad t=r(F')+r(F'')$$

$$d(F') \leq r(F')d(E', \mathcal{O}_X(1))/r(E')+r(F')e$$

$$d(F'') \leq r(F'')d(E'', \mathcal{O}_X(1))/r(E'')+r(F'')e.$$

Combining these, we obtain

$$d(F) \leq t\{d(E, \mathcal{O}_X(1))/r(E)+e\}$$

because $d(E')/r(E')=d(E)/r(E)=d(E'')/r(E'')$.

2) Let F' be a coherent subsheaf of rank t' $(1 \leq t' \leq r(E')-1)$ of $E'\otimes\mathcal{O}_C$. Then we have

$$d(F') \leq t'\{d(E, \mathcal{O}_X(1))/r(E)+e\}=t'\{d(E', \mathcal{O}_X(1))/r(E')+e\}.$$

Next let F'' be a coherent subsheaf of rank t'' $(1 \leq t'' \leq r(E'')-1)$ of $E''\otimes\mathcal{O}_C$. Set $F=v^{-1}(F'')$. Then

$$d(F)=d(F'')+d(E')$$

$$d(F) \leq (r(E')+t'')\{d(E, \mathcal{O}_X(1))/r(E)+e\}.$$

Thus we obtain

$$
\begin{aligned}
d(F'') &\leq t''d(E'', \mathcal{O}_X(1))/r(E'') + (r(E') + t'')e \\
&= t''\{d(E'', \mathcal{O}_X(1))/r(E'') + (r(E') + t'')e/t''\} \\
&\leq t''\{d(E'', \mathcal{O}_X(1))/r(E'') + r(E)e\}.
\end{aligned}
$$

q.e.d.

As for strict e-semi-stability, we have the following.

Lemma 3.5. *Let E', E and E'' be the same as in Lemma 3.4.*

1) *If E is strictly e-semi-stable, then each component of* $\mathrm{gr}(E)$ *is e-stable.*

2) *E is strictly e-semi-stable if and only if E' and E'' are strictly e-semi-stable.*

Proof. 1) Our proof is by induction on the number α of components of $\mathrm{gr}(E)$. If $\alpha = 1$, then we have nothing to prove. Assume that $\alpha > 1$ and take a Jordan-Hölder filtration $0 \subset E_0 \subset E_1 \subset \cdots \subset E_\alpha = E$ of E. By virtue of Lemma 3.4, E_1 is e-stable. It is easy to see that $\bar{E} = E/E_1$ is strictly e-semi-stable and $0 = \bar{E}_0 \subset \bar{E}_1 = E_2/E_1 \subset \cdots \subset \bar{E}_{\alpha-1} = \bar{E}$ is a Jordan-Hölder filtration of E. Thus $\mathrm{gr}(E) = E_1 \oplus \mathrm{gr}(\bar{E})$ and our induction hypothesis tells us that each component of $\mathrm{gr}(\bar{E})$ is e-stable. We see therefore that each component of $\mathrm{gr}(E)$ is e-stable.

2) It is easy to see that if E is an extension of a semi-stable sheaf E'' by a semi-stable sheaf E' and if $P_{E'}(m) = P_{E''}(m)$, then E is semi-stable, $P_E(m) = P_{E'}(m) = P_{E''}(m)$ and $\mathrm{gr}(E) = \mathrm{gr}(E') \oplus \mathrm{gr}(E'')$. If both E' and E'' are strictly e-semi-stable, then the above remark and (1) of this lemma imply that each component of $\mathrm{gr}(E)$ is e-stable. Let F be a coherent quotient sheaf of E with $P_E(m) = P_F(m)$. Applying the above remark to E, F and $\ker(E \to F)$, we know that $\mathrm{gr}(F)$ is direct summand of $\mathrm{gr}(E)$. By induction on the number of the components of $\mathrm{gr}(F)$, Lemma 3.4, (1) and by the above facts, we see that F is e-semi-stable. The proof of the converse is similar to the above. q.e.d.

Corollary 3.5.1. *E is strictly e-semi-stable if and only if so is* $\mathrm{gr}(E)$.

Now let us show openness of strict e-semi-stability (see Definition 1.4 of [11]).

Lemma 3.6. *Let g: $Y \to T$ be smooth, projective, geometrically integral morphism of locally noetherian scheme, $\mathcal{O}_X(1)$ be a g-very ample invertible sheaf on Y and let F be a T-flat coherent $\mathcal{O}_Y$-module. If $H^i(Y_t, \mathcal{O}_Y(1) \otimes_{\mathcal{O}_T} k(t)) = 0$ for all $i > 0$, $t \in T$, then there exists an open set U of T such that for all algebraically closed fields k,*

$$U(k) = \{t \in T(k) \mid F \otimes_{\mathcal{O}_T} k(t) \text{ is strictly e-semi-stable with respect to } \mathcal{O}_Y(1)\}.$$

Proof. Since the property that a coherent sheaf is e-semi-stable is open under the situation in the lemma ([12] Lemma 3.5), we may assume that for all geometric points t of T, $F \otimes_{\mathcal{O}_T} k(t)$ is e-semi-stable. And, moreover, we may assume that T is noetherian and connected. Then, for every geometric point t of T, $F \otimes_{\mathcal{O}_T} k(t)$ has the same Hilbert polynomial $H(m)$ and rank r. For $H_i(m) = iH(m)/r$, $1 \leq i \leq r-1$, set

$Q_i = \mathrm{Quot}^{H_i(m)}_{F/Y/T}$ and $F_i = (1_Y \times {}_T\pi_i)^*(F)$, where π_i is the natural morphism of Q_i to T. If E_i is the universal quotient sheaf of F_i, then there exists a closed set R_i of Q_i such that for all algebraically closed fields k, $R_i(k) = \{q \in Q_i(k) | E_i \otimes_{\mathcal{O}_{Q_i}} k(q)$ is not e-semi-stable$\}$. If $F \otimes_{\mathcal{O}_T} k(t)$ is not strictly e-semi-stable for some k-valued geometric point t of T, then there exists a coherent quotient sheaf F' of $F \otimes_{\mathcal{O}_T} k(t)$ such that $\chi(F'(m)) = iH(m)/r = H_i(m)$ for some $1 \leq i \leq r-1$ and that F' is not e-semi-stable. Thus there exists a k-valued point q of R_i whose image by π_i is t. We see therefore that $U = T - \bigcup_i \pi_i(R_i)$ is the required set. Since π_i is proper, U is open in T. q.e.d.

§4. Moduli of semi-stable sheaves

Our main aim of this section is to construct a scheme of parametrization of the functor $\bar{\Sigma}^H_{X/S}$ defined in the end of §1.

Let T be a locally noetherian S-scheme and let E_1 and E_2 be T-flat, coherent $\mathcal{O}_{X_T}$-modules. Assume that $E_1 \sim E_2$ by the equivalence relation defined in (1.7.2) and assume that E_1 has the following property;

(4.1.1) for every geometric point t of T, $E_1 \otimes_{\mathcal{O}_T} k(t)$ is strictly e-semi-stable.

Then E_2 has the same property by virtue of Corollary 3.5.1. Thus (4.1.1) is a property of a class $[E]$ in $\bar{\Sigma}^H_{X/S}(T)$. When a class $[E]$ enjoys the property (4.1.1), it is said to be strictly e-semi-stable.

Now let us introduce a subfunctor $\bar{\Sigma}^{H,e}_{X/S}$ of $\bar{\Sigma}^H_{X/S}$ for each non-negative integer e.

(4.1.2) For $T \in (\mathrm{Sch}/S)$, $\bar{\Sigma}^{H,e}_{X/S}(T) = \{[E] \in \bar{\Sigma}^H_{X/S}(T) | [E]$ is strictly e-semi-stable$\}$.

$\bar{\Sigma}^{H,e}_{X/S}$ is an open subfunctor of $\bar{\Sigma}^H_{X/S}$ by virtue of Lemma 3.6 and $\Sigma^{H,e}_{X/S}$ is an open subfunctor of $\bar{\Sigma}^{H,e}_{X/S}$ (see §5 of [12]).

We may assume that S is connected. Set $H^{(i)}(m) = iH(m)/r$ for $1 \leq i \leq r$, where $r = r(E)$ for an E with $[E] \in \bar{\Sigma}^H_{X/S}(\mathrm{Spec}(k(s)))$. Then there exists an integer $m(i, e)$ such that for all integers $m \geq m(i, e)$, all geometric points s of S and for all E in $\Sigma^{H^{(i)},e}_{X/S}(\mathrm{Spec}(k(s)))$,

(4.1.3) $E \otimes \mathcal{O}_{X_s}(m)$ is generated by its global sections and

$$h^j(X_s, E \otimes \mathcal{O}_{X_s}(m)) = 0 \qquad \text{if } j > 0,$$

(4.1.4) for all coherent subsheaves E' of E with $0 \neq E' \neq E$,

$$h^0(X_s, E' \otimes \mathcal{O}_{X_s}(m)) < r(E') h^0(X_s, E \otimes \mathcal{O}_{X_s}(m))/i$$

(see [12] (5.3.1) and (5.3.3)).

Lemma 4.2. *If $m \geq \max_{1 \leq i \leq r}\{m(i, e)\}$, then for all geometric points s of S and for all strictly e-semi-stable sheaves E on X_s of rank i with $\chi(E(m)) = H^{(i)}(m)$,*

(4.2.1) *$E \otimes \mathcal{O}_{X_s}(m)$ is generated by its global sections and $h^j(X_s, E \otimes_{X_s} \mathcal{O}(m)) = 0$ if $j > 0$,* (4.2.2) *for all coherent subsheaves* E' of E *with* $E'\neq 0$, $h^0(X_s, E'\otimes\mathcal{O}_{X_s}(m))\leq r(E')h^0(X_s, E\otimes\mathcal{O}_{X_s}(m))/i$

and, moreover, the equality holds if and only if $P_{E'}(m)=P_E(m)=H(m)/r$.

Proof. We shall prove the lemma by induction on the number α of the components of $\mathrm{gr}(E)$. If $\alpha=1$, we have nothing to prove because of (4.1.3) and (4.1.4). Assume that $\alpha\geq 2$ and that our assertion is true for $\alpha-1$. Pick a Jordan-Hölder filtration $0=E_0\subset E_1\subset\cdots\subset E_\alpha=E$ of E. Then the induction hypothesis implies that our lemma holds for $\bar{E}=E/E_1$. The exact sequence

$$0\longrightarrow E_1\otimes\mathcal{O}_{X_s}(m)\xrightarrow{u} E\otimes\mathcal{O}_{X_s}(m)\xrightarrow{v}\bar{E}\otimes\mathcal{O}_{X_s}(m)\longrightarrow 0$$

and (4.2.1) for E_1 and $\bar{E}$ provide us with an exact sequence

$$0\longrightarrow H^0(X_s, E_1\otimes\mathcal{O}_{X_s}(m))\xrightarrow{\Gamma(u)} H^0(X_s, E\otimes\mathcal{O}_{X_s}(m))\xrightarrow{\Gamma(v)}$$
$$H^0(X_s, \bar{E}\otimes\mathcal{O}_{X_s}(m))\longrightarrow 0$$

and $h^j(X_s, E\otimes\mathcal{O}_{X_s}(m))=0$ for $j>0$. Let a be an element of a stalk of $E\otimes\mathcal{O}_{X_s}(m)$ at x. Then there exist $a_1,\dots, a_t$ in $\mathcal{O}_{X_s,x}$ and $s_1,\dots, s_t$ in $H^0(X_s, E\otimes\mathcal{O}_{X_s}(m))$ such that $a-\Sigma a_i s_{i,x}$ is an element of $u_x((E_1\otimes\mathcal{O}_{X_s}(m))_x)$. Thus we can find $b_1,\dots, b_{t'}$ in $\mathcal{O}_{X_s,x}$ and $s'_1,\dots, s'_{t'}$ in $\Gamma(u)(H^0(X_s, E_1\otimes\mathcal{O}_{X_s}(m)))$ such that $a=\Sigma a_i s_{i,x}+\Sigma b_j s'_{j,x}$. This completes the proof of (4.2.1). For the proof of (4.2.2), let j be the smallest integer such that $E'\subset E_j$. If $j<\alpha$, then E' is a coherent subsheaf of $E_{\alpha-1}$. Since $E_{\alpha-1}$ is strictly e-semi-stable by virtue of Lemma 3.5, the induction hypothesis implies that $h^0(X_s, E'\otimes\mathcal{O}_{X_s}(m))\leq r(E')h^0(X_s, E_{\alpha-1}\otimes\mathcal{O}_{X_s}(m))/r(E_{\alpha-1})$ and the equality holds if and only if $P_{E'}=P_{E_{\alpha-1}}=P_E$. By virtue of (4.2.1) for E and $E_{\alpha-1}$, we know that $h^0(X_s, E_{\alpha-1}\otimes\mathcal{O}_{X_s}(m))/r(E_{\alpha-1})=P_{E_{\alpha-1}}(m)=P_E(m)=h^0(X_s, E\otimes\mathcal{O}_{X_s}(m))/r(E)$. We may assume therefore that $j=\alpha$. Set $E'_{\alpha-1}=E'\cap E_{\alpha-1}$, $\bar{E}'=E'/E'_{\alpha-1}$. Then $\bar{E}'$ is a non-zero subsheaf of $F=E/E_{\alpha-1}$. If $E'_{\alpha-1}=0$, then $h^0(X_s, E'\otimes\mathcal{O}_{X_s}(m))=h^0(X_s, \bar{E}'\otimes\mathcal{O}_{X_s}(m))\leq r(E')h^0(X_s, F\otimes\mathcal{O}_{X_s}(m))/r(F)=r(E')h^0(X_s, E\otimes\mathcal{O}_{X_s}(m))/r(E)$ because of (4.1.3), (4.1.4) for F and (4.2.1) for E. Moreover, the equality holds if and only if $F=\bar{E}'$, that is, $P_E=P_F=P_{\bar{E}'}=P_{E'}$. Assume that $E'_{\alpha-1}\neq 0$. Then,

$$h^0(X_s, E'\otimes\mathcal{O}_{X_s}(m))\leq h^0(X_s, E'_{\alpha-1}\otimes\mathcal{O}_{X_s}(m))+h^0(X_s, \bar{E}'\otimes\mathcal{O}_{X_s}(m))$$
$$\leq r(E'_{\alpha-1})h^0(X_s, E_{\alpha-1}\otimes\mathcal{O}_{X_s}(m))/r(E_{\alpha-1})+r(\bar{E}')h^0(X_s, F\otimes\mathcal{O}_{X_s}(m))/r(F)$$
$$=r(E')h^0(X_s, E\otimes\mathcal{O}_{X_s}(m))/r(E).$$

If the equality holds, then $P_{E'_{\alpha-1}}=P_{E_{\alpha-1}}=P_E$ and $P_{\bar{E}'}=P_F=P_E$, and hence $P_{E'}=P_E$. Conversely, if $P_{E'}=P_E$, then $P_{E'_{\alpha-1}}=P_{\bar{E}'}=P_E$. Thus the equality holds if $h^1(X_s, E'_{\alpha-1}\otimes\mathcal{O}_{X_s}(m))=0$. This follows from (4.2.1) and the fact that $E'_{\alpha-1}$ is a strictly e-semi-stable sheaf with $P_{E'_{\alpha-1}}=P_E$. q.e.d.

Let $\mathfrak{S}'_{X/S}(e, H)$ be the family of classes of coherent sheaves on the fibres of X over S such that E is contained in $\mathfrak{S}'_{X/S}(e, H)$ if and only if E is strictly e-semi-stable and the Hilbert polynomial of E is H. Then, for each e and H, $\mathfrak{S}'_{X/S}(e, H)$ is bounded (Lemma 4.2 or [12] Corollary 3.3.1). Thus there exists an integer $m'(i, e)$ such that for all integers $m \geq m'(i, e)$, all geometric points s of S and for all $\mathcal{O}_{X_s}$-modules E contained in $\mathfrak{S}'_{X/S}(e, H^{(i)})$,

(4.3.1) if an invertible sheaf L on X_s has the same Hilbert polynomial as $\det(E \otimes \mathcal{O}_{X_s}(m)) = c_1(E \otimes \mathcal{O}_{X_s}(m))$, then $h^j(X_s, L) = 0$ for all positive integers j.

Take an integer $m_e \geq \max_{1 \leq i \leq r}\{m(i, e), m'(i, e)\}$. We may assume that $m_e \geq m_{e'}$ if $e \geq e'$. Let $H^{(i,e)}(m) = H^{(i)}(m + m_e)$, then $H^{(i,e)}(m)$ is the Hilbert polynomial of $E \otimes \mathcal{O}_{X_s}(m_e)$ for a coherent sheaf E on X_s with Hilbert polynomial $H^{(i)}(m)$. Set $N^{(i,e)} = H^{(i)}(m_e)$, then (4.2.1) implies that $N^{(i,e)} = h^0(X_s, E \otimes \mathcal{O}_{X_s}(m_e))$ for every $\mathcal{O}_{X_s}$-module contained in $\mathfrak{S}'_{X/S}(e, H^{(i)})$.

Now, $V_{i,e}$ denotes a free Ξ-module of rank $N^{(i,e)}$ and for a Ξ-scheme Y, $V_{i,e}(Y)$ denotes $V_{i,e} \otimes_{\Xi} \mathcal{O}_Y$. Let us consider

$$\tilde{Q}_i = \mathrm{Quot}^{H^{(i,e)}}_{V_{i,e}(X)/X/S}$$

and the universal quotient sheaf $\phi^e_i \colon V_{i,e}(X \times_S \tilde{Q}_i) \to F^e_i$. Then, by virtue of Lemma 3.6, for each integer e' with $0 \leq e' \leq e$, there exists an open set $R^{e,e'}_i$ in $\tilde{Q}_i$ such that a geometric point y of $\tilde{Q}_i$ is contained in $R^{e,e'}_i$ if and only if

(4.3.2) $\Gamma(\phi^e_i \otimes k(y)) \colon V_{i,e} \otimes_{\Xi} k(y) \longrightarrow H^0(X_y, F^e_i \otimes_{\mathcal{O}_{\tilde{Q}_i}} k(y))$ is bijective and

(4.3.3) $F^e_i \otimes_{\mathcal{O}_{\tilde{Q}_i}} k(y)$ is strictly e'-semi-stable.

For every geometric point s of S and for every coherent sheaf E on X_s which is contained in $\mathfrak{S}'_{X/S}(e', H^{(i)})(m_e) = \{F \otimes \mathcal{O}_{X_s}(m_e) | F \in \mathfrak{S}'_{X/S}(e', H^{(i)})\}$, there exists a surjective homomorphism $\alpha \colon V_{i,e}(X_s) \to E$ such that $\Gamma(\alpha) \colon V_{i,e} \otimes_{\Xi} k(s) \to H^0(X_s, E)$ is bijective by virtue of (4.2.1). By the universality of $(\tilde{Q}_i, \phi^e_i, F^e_i)$, α corresponds to a geometric point y of $\tilde{Q}_i$ lying over s. Since y is a geometric point of $R^{e,e'}_i$, we obtain a surjective map $\xi^{e,e'}_i(s)$ for every geometric point s of S;

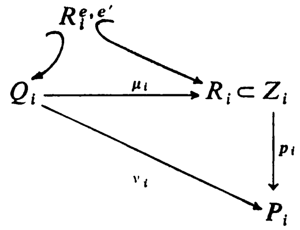
$$\begin{aligned}(4.3.4) \quad \xi^{e,e'}_i(s) \colon R^{e,e'}_i(k(s)) &\longrightarrow \bar{\Sigma}^{H^{(i)},e'}_{X/S}(m_e)(\mathrm{Spec}(k(s))) \\ &= \{[E \otimes \mathcal{O}_{X_s}(m_e)] | [E] \in \bar{\Sigma}^{H^{(i)},e'}_{X/S}(\mathrm{Spec}(k(s)))\}.\end{aligned}$$

On the other hand, for a natural action τ of the Ξ-group scheme $G_i = GL(V_{i,e})$ on $\tilde{Q}_i$, $R^{e,e'}_i$ is G_i-invariant and K-valued geometric points y_1 and y_2 of $R^{e,e'}_i$ are in the same orbit of $G_i(K)$ if and only if $F^e_i \otimes_{\mathcal{O}_{\tilde{Q}_i}} k(y_1) \cong F^e_i \otimes_{\mathcal{O}_{\tilde{Q}_i}} k(y_2)$ ([12] §4 and §5).

Let Q_i be the union of the connected components of $\tilde{Q}_i$ which have a non-empty intersection with $R^{e,e'}_i$. Let ν_i be the morphism Q_i to $\mathrm{Pic}_{X/S}$ defined by $\det(F^e_i|_{X \times_S Q_i})$ and let P_i be the union of connected components which intersect with $\nu_i(Q_i)$. ([12] §4). Then P_i is projective over S. Moreover, by virtue of (4.3.1), we obtain a G_i-morphism μ_i of Q_i to Z_i defined in Proposition 4.10 of [12]:

$$\begin{array}{ccc} Q_i & \xrightarrow{\mu_i} & Z_i \\ & {\scriptstyle p_i}\searrow & \downarrow{\scriptstyle \nu_i} \\ & & P_i \end{array}$$

μ_i induces a closed immersion of $R_i^{e,e'}$ to a G_i-invariant open subscheme of Z_i. This and the fact that μ_i is proper imply that there exists a closed subscheme R_i of Z_i such that R_i is G_i-invariant, $\mu_i(Q_i)=R_i$ as sets and that μ_i induces an open immersion of $R_i^{e,e'}$ to R_i (R_i is the scheme theoretic image of Q_i by μ_i). Therefore we get the following commutative diagram:



For all K-valued geometric points x of P_i, $(Z_i)_x$ is isomorphic to the Gieseker space $P(V_{i,e}\otimes_{\Xi}K, i, W_x)$, where $W_x=H^0(X_y, (\det F_i^e)\otimes k(y))$ with a K-valued geometric point y of Q_i lying over x.

Lemma 4.4. *For all K-valued geometric points x of P_i, every geometric point of $(R_i)_x$ is excellent in $(Z_i)_x=P(V_{i,e}\otimes_{\Xi}K, i, W_x)$.*

Proof. Let T be a geometric point of $(R_i)_x$. We may assume that T is K-rational. Pick a K-valued point y of $(\mu_i)^{-1}(T)$. As a map of $\overset{i}{\wedge} V_{i,e}\otimes_{\Xi}K$ to $H^0(X_y, (\det F_i^e)\otimes k(y))=W_x$, T is defined by $\tilde{\gamma}_y$ (for the definition of $\tilde{\gamma}$, see [12] p. 114). For $a_1,\dots, a_i$ in $V_{i,e}\otimes_{\Xi}K$, put $s_1=\Gamma(\phi_i\otimes k(y))(a_1),\dots, s_i=\Gamma(\phi_i\otimes k(y))(a_i)$. Then $\gamma_y(s_1\wedge\cdots\wedge s_i)$ coincides with $s_1\wedge\cdots\wedge s_i$ on the open set of X_y on which $F_i^e\otimes k(y)$ is locally free. Thus $a_1,\dots, a_j$ in $V_{i,e}\otimes_{\Xi}K$ are T-independent if and only if $(s_1)_z,\dots, (s_j)_z$ are linearly independent in the vector space $(F_i^e)_z$, where z is the generic point of X_y. And a is T-dependent on $a_1,\dots, a_j$ if and only if s_z is linearly dependent on $(s_1)_z,\dots, (s_j)_z$, where $s=\Gamma(\phi_i\otimes k(y))(a)$. These remarks imply that T has the property (1) in Definition 2.9. To show that T enjoys the property (2) in Definition 2.9, assume that T is an extension of T'' by T'. Let ϕ: $W'\otimes_k W''\to W_x$ be the admissible map to define the extension T and let

$$0 \longrightarrow V' \xrightarrow{\ u\ } V_{i,e}\otimes K \xrightarrow{\ v\ } V'' \longrightarrow 0$$

be the underlying exact sequence of the extension. Let E' be the coherent subsheaf of $F_i^e\otimes k(y)$ generated by $\Gamma(\phi_i\otimes k(y))(V')$, E'' be the quotient sheaf E/E' and let $L'=\det E'$, $L''=\det E''$. Since $(\det F_i^e)\otimes k(y)\cong L'\otimes L''$, we have an admissible map ψ: $H'\otimes_k H''\to W_x$, where $H'=H^0(X_y, L')$ and $H''=H^0(X_y, L'')$. Pick vectors $b_1,\dots, b_{r''}$ such that $\beta=T''(v(b_1),\dots, v(b_{r''}))\neq 0$. Let U be the non-empty open set on which E', $F_i^e\otimes k(y)$ and E'' are locally free. Then, for $a_1,\dots, a_{r'}$ in V', $s_1\wedge\cdots\wedge s_{r'}\wedge t_1\wedge \dots\wedge t_{r''}=T(u(a_1),\dots, u(a_{r'}), b_1,\dots, b_{r''})=T'(a_1,\dots, a_{r'})\circ T''(v(b_1),\dots, v(b_{r''}))$ on U, where $s_j=\Gamma(\phi_i\otimes k(y))(a_j)$ and $t_j=\Gamma(\phi_i\otimes k(y))(b_j)$. Since T' is not zero, $t_1\wedge\cdots\wedge t_{r''}$ defines a non-zero element α of H''. If $s_1\wedge\cdots\wedge s_{r'}$ denotes the element of H' which coincides with $s_1\wedge\cdots\wedge s_{r'}$ on U, then $\psi((s_1\wedge\cdots\wedge s_{r'})\otimes\alpha)=T(u(a_1),\dots, u(a_{r'}), b_1,\dots,$

$b_{r''})=\phi(T'(a_1,\dots, a_{r'})\otimes\beta)$. Thus $T'(a_1,\dots, a_{r'})=0$ if and only if $s_1\wedge\cdots\wedge s_{r'}=0$. Assume that $a_1,\dots, a_j$ are T'-independent and that a is T'-dependent on $a_1,\dots, a_j$ then $(s_1)_z,\dots, (s_j)_z$ are linearly independent in the $k(z)$-vector space E'_z and $\Gamma(\phi_i\otimes k(y))(a)_z$ is contained in the vector subspace of E'_z generated by $(s_1)_z,\dots, (s_j)_z$. By the remark made in the first part of this proof, we see that $u(a)$ is T-dependent on $a_1,\dots, a_j$. Therefore, T has the property (2) in Definition 2.9. q.e.d.

From now on, we shall fix a p_i-ample invertible sheaf L_i on Z_i which carries a G_i-linearization. There exist G_i-invariant open subschemes R_i^s and R_i^{ss} of R_i such that for all algebraically closed fields K, $R_i^s(K)=\{x\in R_i(K)|x$ is a properly stable point of $(R_i)_y$ with respect to the pull back of L_i to $(R_i)_y$, where $y=p_i(K)(x)\}$ and $R_i^{ss}(K)=\{x\in R_i(K)|x$ is semi-stable point of $(R_i)_y$ with respect to the pull back of L_i to $(R_i)_y$, where $y=p_i(K)(x)\}$ (see [20] II, §2 and note that R_i is a closed subscheme of $\mathbf{P}(E)$ for some locally free G_i-sheaf E on P_i because L_i is P_i-flat). By virtue of Lemma 2.2 and (4.2.2), the same argument as in Lemma 4.15 of [12] provides us with the following.

Lemma 4.5. *μ_i induces an open immersion of $R_i^{e,e'}$ to R_i^{ss}. Moreover, for a geometric point x of $R_i^{e,e'}$, if $F_i^e\otimes k(x)$ is stable, then $\mu_i(x)$ is in R_i^s.*

Let x be a k-valued geometric point of $R_i^{e,e'}$. Since $E=F_r^e\otimes k(x)$ is strictly e'-semi-stable, we can fined a Jordan-Hölder filtration $0=E_0\subset E_1\subset\cdots\subset E_{\alpha-1}\subset E_\alpha=E$. Set $r_i=r(E_i)$ and $l_i=r_i-r_{i-1}$. By virtue of (4.2.1), the following exact commutative diagram is obtained;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X_x, E_{\alpha-1}) & \xrightarrow{u_\alpha} & H^0(X_x, E) & \xrightarrow{v_\alpha} & H^0(X_x, E/E_{\alpha-1}) & \longrightarrow & 0 \\ & & \eta_{\alpha-1}\uparrow\wr & & \eta_\alpha\uparrow\wr & & \bar\eta_\alpha\uparrow\wr & & \\ & & V_{r_{\alpha-1},e}\otimes_{\Xi}k & \xrightarrow{f_\alpha} & V_{r,e}\otimes_{\Xi}k & \xrightarrow{g_\alpha} & V_{l_\alpha,e}\otimes_{\Xi}k & & \end{array}$$

where $\eta_\alpha=\Gamma(\phi_r\otimes k(x))$. Since $E_{\alpha-1}$ (or, $\bar E=E/E_{\alpha-1}$) is strictly e'-semi-stable (Lemma 3.5), an isomorphism $\eta_{\alpha-1}$ (or, $\bar\eta_\alpha$, resp.) defines a k-rational point $x_{\alpha-1}$ (or, $\bar x_\alpha$, resp.) of $R_{r_{\alpha-1}}^{e,e'}$ (or, $R_{l_\alpha}^{e,e'}$, resp.). If $T_\alpha=\mu_r(k)(x)$, $T_{\alpha-1}=\mu_{r_{\alpha-1}}(k)(x_{\alpha-1})$ and $\bar T_\alpha=\mu_{l_\alpha}(k)(\bar x)$, then $T_\alpha\in P(V_{r,e}\otimes_{\Xi}k, r, W_\alpha)$, $T_{\alpha-1}\in P(V_{r_{\alpha-1},e}\otimes_{\Xi}k, r_{\alpha-1}, W_{\alpha-1})$ and $\bar T_\alpha\in P(V_{l_\alpha,e}\otimes_{\Xi}k, l_\alpha, \bar W)$, where $W_\alpha=H^0(X_x, \det E)$, $W_{\alpha-1}=H^0(X_x, \det E_{\alpha-1})$ and $\bar W_\alpha=H^0(X_x, \det\bar E_\alpha)$. The isomorphism $\det E\cong(\det E_{\alpha-1})\otimes(\det\bar E_\alpha)$ yields an admissible map ψ_α: $W_{\alpha-1}\otimes_k\bar W_\alpha\to W_\alpha$. For $a_1,\dots, a_{r_{\alpha-1}}$ in $V_{r_{\alpha-1},e}\otimes_{\Xi}k$ and for $b_1,\dots, b_{l_\alpha}$ in $V_{r,e}\otimes_{\Xi}k$, put $s_i=\eta_{\alpha-1}(a_i)$ and $t_j=\eta_\alpha(b_j)$. Then,

$$\begin{aligned} T_\alpha(f_\alpha(a_1),\dots, f_\alpha(a_{r_{\alpha-1}}), b_1,\dots, b_{l_\alpha})&=u_\alpha(s_1)\wedge\cdots\wedge u_\alpha(s_{r_{\alpha-1}})\wedge \\ t_1\wedge\cdots\wedge t_{l_\alpha}&=\psi_\alpha((s_1\wedge\cdots\wedge s_{r_{\alpha-1}})\otimes(v_\alpha(t_1)\wedge\cdots\wedge v_\alpha(t_{l_\alpha}))) \\ &=\psi_\alpha(T_{\alpha-1}(a_1,\dots, a_{r_{\alpha-1}})\otimes\bar T_\alpha(g_\alpha(b_1),\dots, g_\alpha(b_{l_\alpha}))) \end{aligned}$$

on a non-empty open set of X_x on which $E_{\alpha-1}$, E and $\bar E_\alpha$ are locally free. Thus, as elements of W_α, $T_\alpha(f_\alpha(a_1),\dots, f_\alpha(a_{r_{\alpha-1}}), b_1,\dots, b_{l_\alpha})=\psi_\alpha(T_{\alpha-1}(a_1,\dots, a_{r_{\alpha-1}})\otimes\bar T_\alpha(g_\alpha(b_1),\dots, g_\alpha(b_{l_\alpha})))$. Therefore, T_α is a ψ_α-extension of $\bar T_\alpha$ by $T_{\alpha-1}$. Let $W_j=H^0(X_x,$

$\det E_j$) and let $\overline{W}_j = H^0(X_x, \det \bar{E}_j)$, where $\bar{E}_j = E_j/E_{j-1}$. We have a sequence of admissible maps ψ_j: $W_{j-1}\otimes_k \overline{W}_j \to W_j$ ($W_0 = k$, $1 \le j \le \alpha$). Repeating the similar argument to the above, we get T_j in $P(V_{r_j,e}\otimes_{\Xi} k, r_j, W_j)$ $(1\le j\le\alpha)$ and $\overline{T}_j$ in $P(V_{l_j,e}\otimes_{\Xi} k, l_j, \overline{W}_j)$ $(1 \le j \le \alpha)$ such that

(4.6.1) $T_j = \mu_{r_j}(k)(x_j)$ for some x_j in $R_{r_j}^{e,e'}(k)$ and $\overline{T}_j = \mu_{l_j}(k)(\bar{x}_j)$ for some $\bar{x}_j$ in $R_{l_j}^{e,e'}(k)$. Moreover, $\overline{T}_j$ is in $R_{l_j}^{s}(k)$.

(4.6.2) T_j is a ψ_j-extension of $\overline{T}_j$ by T_{j-1} and $T_1 \cong \overline{T}_1$.

Lemma 4.7. *$T_j \cong T_{j-1}\oplus \overline{T}_j$ if and only if $E_j \cong E_{j-1}\oplus \bar{E}_j$.*

Proof. It is clear that if $E_j\cong E_{j-1}\oplus\bar{E}_j$, then $T_j\cong T_{j-1}\oplus\overline{T}_j$. Assume that $T_j \cong T_{j-1}\oplus\overline{T}_j$. Then there exists a linear map h_j: $V_{l_j,e}\otimes_{\Xi} k \to V_{r_j,e}\otimes_{\Xi} k$ such that $g_j h_j = id$ and $T_j(h_j(b_1),\dots, h_j(b_t),\dots,) = 0$ if $t > l_j$. Let F_j be the coherent subsheaf of E_j generated by $\eta_j h_j(V_{l_j,e}\otimes_{\Xi} k)$. Since E_j is generated by its global sections and since $u_j\eta_{j-1}(V_{r_{j-1},e}\otimes_{\Xi} k)\oplus\eta_j h_j(V_{l_j,e}\otimes_{\Xi} k) = H^0(X_x, E_j)$, we see that $E_j = E_{j-1} + F_j$. The fact that $T_j(h_j(b_1),\dots, h_j(b_t),\dots) = 0$ if $t > l_j$ implies that $r(F_j) \le l_j$, whence $r(F_j) = l_j$. Thus, at the generic point z of X_x, $(E_j)_z = (E_{j-1})_z \oplus (F_j)_z$, which asserts that $E_{j-1}\cap F_j$ is a torsion sheaf. Since E_j is torsion free, $E_{j-1}\cap F_j = 0$, and hence E_j is a direct sum of E_{j-1} and F_j. The natural projection of E_j to $\bar{E}_j$ induces a surjective homomorphism of F_j to $\bar{E}_j$. Since F_j is torsion free and since $r(F_j) = r(\bar{E}_j)$, F_j is isomorphic to $\bar{E}_j$. q.e.d.

By virtue of Corollary 3.5.1, $\mathrm{gr}(E)$ is strictly e'-semi-stable. Hence $\mathrm{gr}(E)$ corresponds to a point y in $R_r^{e,e'}(k)$.

Corollary 4.7.1. $\mu_r(k)(y) = \overline{T}_1\oplus\cdots\oplus\overline{T}_\alpha$.

Now let us study G_r-orbits in R_r^{ss} and $R_r^{e,e'}$.

Porposition 4.8. *Let y be a k-valued geometric point of P_r and let s be the image of y by the structure morphism $P_r \to S$. Let $\bar{E}_1,\dots,\bar{E}_\alpha$ be e'-stable sheaves on X_s such that $l_i = r(\bar{E}_i)$, $\chi(\bar{E}_i(m)) = H^{(l_i)}(m)$ and $l_1+\cdots+l_\alpha = r$. Then there exists a G_r-invariant closed subset $Z(\bar{E}_1,\dots,\bar{E}_\alpha)$ of $(R_r^{e,e'})_y = (v_r)^{-1}(y)\cap R_r^{e,e'}$ such that*

(4.8.1) *$\mu_r(Z(\bar{E}_1,\dots,\bar{E}_\alpha))$ is closed in $(R_r^{ss})_y$,*

(4.8.2) *for every algebraically closed field K containing k, $Z(\bar{E}_1,\dots,\bar{E}_\alpha)(K) = \{x\in(R_r^{e,e'})(K) \mid \mathrm{gr}(F_r^e\otimes k(x))\cong(\bigoplus_{i=1}^{\alpha}\bar{E}_i)\otimes_k K\}$,*

(4.8.3) *the G_r-orbit of x_0 corresponding to $\bigoplus_{i=1}^{\alpha}\bar{E}_i$ is the unique closed orbit in $Z(\bar{E}_1,\dots,\bar{E}_\alpha)$.*

Proof. Let $\bar{x}_i$ be a k-valued point of $R_{l_i}^{e,e'}$ such that $F_{l_i}^e\otimes k(\bar{x}_i)\cong\bar{E}_i$. If $\mu_{l_i}(k)(\bar{x}_i) = \overline{T}_i$, then $\overline{T}_i$ is a stable point of $(R_{l_i})_{\bar{y}_i}\subset P(V_{l_i,e}\otimes_{\Xi} k, l_i, \overline{W}_i)$, where $\bar{y}_i = v_{l_i}(k)(\bar{x}_i)$ and $\overline{W}_i = H^0(X_s, \det\bar{E}_i)$. Let $W_i = H^0(X_s, (\det\bar{E}_1)\otimes\cdots\otimes(\det\bar{E}_i))$, then there is a natural admissible map ψ_i: $W_{i-1}\otimes_k\overline{W}_i\to W_i$. For $r_i = l_1+\cdots+l_i$, $(R_{r_i})_{y_i}$ is a G_{r_i}-invariant closed set of $P(V_{r_i,e}\otimes_{\mathbf{Z}}k, r_i, W_i)$ whose geometric points are excellent (Lemma 4.4), where y_i is the geometric point of P_{r_i} which corresponds to $(\det\bar{E}_1)\otimes\cdots\otimes(\det\bar{E}_i)$. Applying Theorem 2.13 to the case where $F_i=(R_{r_i})_{y_i}$ and $S_i=\bar{T}_i$, we obtain a G_r-invariant closed set $Z(\bar{T}_1,\dots,\bar{T}_\alpha)$ of R_r^{ss} such that for all algebraically closed fields K containing k, $Z(\bar{T}_1,\dots,\bar{T}_\alpha)(K)=\{T\in R_r(K)|T$ enjoys the property $(*)_\alpha$ in Theorem 2.13$\}$. Set $\tilde{Z}(\bar{T}_1,\dots,\bar{T}_\alpha)=\bigcup_{\delta\in S_\alpha}Z(\bar{T}_{\delta(1)},\dots,\bar{T}_{\delta(\alpha)})$, where S_α is the permutation group of $\{1,\dots,\alpha\}$. By virtue of Theorem 2.22, the G_r-orbit $o(\bar{T}_1,\dots,\bar{T}_\alpha)$ of $\bar{T}_1\oplus\cdots\oplus\bar{T}_\alpha$ is the unique closed orbit in $\tilde{Z}(\bar{T}_1,\dots,\bar{T}_\alpha)$ (see Corollary 2.19.1). Since $C=R_r^{ss}-\mu_r(R_r^{e,e'})$ is a G_r-invariant closed set in R_r^{ss}, $D=C\cap\tilde{Z}(\bar{T}_1,\dots,\bar{T}_\alpha)$ contains $o(\bar{T}_1,\dots,\bar{T}_\alpha)$ if it is non-empty. On the other hand, Corollary 4.7.1 implies that $\bar{T}_1\oplus\cdots\oplus\bar{T}_\alpha$ is contained in $\mu_r(R_r^{e,e'})$, whence so is $o(\bar{T}_1,\dots,\bar{T}_\alpha)$. Thus D is empty, that is, $\tilde{Z}(\bar{T}_1,\dots,\bar{T}_\alpha)$ is a closed subset of $\mu_r(R_r^{e,e'})$. Set $Z(\bar{E}_1,\dots,\bar{E}_\alpha)=(\mu_r)^{-1}(\tilde{Z}(\bar{T}_1,\dots,\bar{T}_\alpha))$. Let us show that this $Z(\bar{E}_1,\dots,\bar{E}_\alpha)$ has the required properties. (4.8.1) is obvious because $\mu_r(Z(\bar{E}_1,\dots,\bar{E}_\alpha))=\tilde{Z}(\bar{T}_1,\dots,\bar{T}_\alpha)$. Let x be in $R_r^{e,e'}(K)$ such that $\mathrm{gr}(F_r^e\otimes k(x))\cong\bigoplus_{i=1}^{\alpha}\bar{E}_i\otimes_k K$. Then (4.6.1) and (4.6.2) imply that $\mu_r(K)(x)$ is contained in $\tilde{Z}(\bar{T}_1,\dots,\bar{T}_\alpha)$, whence x is in $Z(\bar{E}_1,\dots,\bar{E}_\alpha)(K)$. For a x' in $R_r^{e,e'}(K)$, assume that $\mathrm{gr}(F_r^e\otimes k(x'))\not\cong\bigoplus_{i=1}^{\alpha}\bar{E}_i\otimes_k K$. If $\mathrm{gr}(F_r^e\otimes k(x'))\cong\bigoplus_{i=1}^{\beta}\bar{E}'_i$, then a G_r-invariant closed subset $\tilde{Z}(\bar{T}'_1,\dots,\bar{T}'_\beta)$ in $R_r^{ss}\times_{P_r}\mathrm{Spec}(K)$ is obtained as above, where $\bar{T}'_i$ is a K-valued point in a Gieseker space corresponding to $\bar{E}'_i$. $\tilde{Z}(\bar{T}'_1,\dots,\bar{T}'_\beta)$ contains the unique closed orbit $o(\bar{T}'_1,\dots,\bar{T}'_\beta)$. $\bigoplus_{i=1}^{\beta}\bar{E}'_i$ corresponds to a point x'_0 in $R_r^{e,e'}(K)$ and $\mu_r(K)(x_0)$ and $\mu_r(K)(x'_0)$ are in the same G_r-orbit if and only if $\bigoplus_{i=1}^{\beta}\bar{E}'_i$ is isomorphic to $(\bigoplus_{i=1}^{\alpha}\bar{E}_i)\otimes_k K$. Thus the orbit of $\mu_r(K)(x_0)$ differs from that of $\mu_r(K)(x'_0)$. Since $\mu_r(K)(x_0)=(\bar{T}_1\oplus\cdots\oplus\bar{T}_\alpha)\otimes_k K$ and $\mu_r(K)(x'_0)=\bar{T}'_1\oplus\cdots\oplus\bar{T}'_\beta$, $o(\bar{T}_1,\dots,\bar{T}_\alpha)\otimes_k K\neq o(\bar{T}'_1,\dots,\bar{T}'_\beta)$. Thus $\tilde{Z}(\bar{T}_1,\dots,\bar{T}_\alpha)\otimes_k K\cap\tilde{Z}(\bar{T}'_1,\dots,\bar{T}'_\beta)=\phi$. Since $\mu_r(x')$ is a K-valued point of $\tilde{Z}(\bar{T}'_1,\dots,\bar{T}'_\beta)$, we see that $x'\notin Z(\bar{E}_1,\dots,\bar{E}_\alpha)(K)$, which completes the proof of (4.8.2). Since μ_r induces an open immersion of $R_r^{e,e'}$ to R_r^{ss}, $Z(\bar{E}_1,\dots,\bar{E}_\alpha)$ is homeomorphic to $\tilde{Z}(\bar{T}_1,\dots,\bar{T}_\alpha)$ as topological spaces with G_r-action. (4.8.3) follows from this fact. q.e.d.

By virtue of Theorem 4 of [20], there exists a good quotient $\pi\colon R_r^{ss}\to Y$. For $C=R_r^{ss}-\mu_r(R_r^{e,e'})$, set $\bar{M}_{e,e'}=Y-\pi(C)$. Since C is G_r-invariant closed set of R_r^{ss}, $\bar{M}_{e,e'}$ is an open subscheme of Y. Since Y is a categorical quotient of R_r^{ss} and since $p_r\colon R_r\to P_r$ is a G_r-morphism with the trivial action of G_r on P_r, we get a unique morphism $\omega\colon Y\to P_r$ such that $\omega\pi=P_r$:

$$\begin{array}{ccccc} R_r^{e,e'} & \xrightarrow{\sim} & \mu_r(R_r^{e,e'}) & \hookrightarrow & R_r^{ss} \\ & & & & \downarrow{\scriptstyle\pi} \quad \searrow{\scriptstyle p_r} \\ & & \bar{M}_{e,e'} & \hookrightarrow & Y \xrightarrow{\ \omega\ } P_r \end{array}$$

Pick a k-valued geometric point x of P_r. Let y be a k-valued point of $R_r^{e,e'}$ such that $p_r(k)\mu_r(k)(y)=x$ and let $\mathrm{gr}(F_r^e\otimes k(y))\cong\bigoplus_{i=1}^{\alpha}\bar{E}_i$. Then, by virtue of Proposition 4.8, we can find a G_r-invariant closed subset $Z(\bar{E}_1,\dots,\bar{E}_\alpha)$ in $(R_r^{e,e'})_x$ with the properties (4.8.1), (4.8.2) and (4.8.3). By (4.8.2), y is a k-valued point of $Z(\bar{E}_1,\dots,\bar{E}_\alpha)$. (4.8.1), (4.8.3) and [20] Theorem 4, (iii) imply that $z=\pi\mu_r(Z(\bar{E}_1,\dots,\bar{E}_\alpha))$ is a k-valued point of Y. By (4.8.1), we have that z is contained in $\bar{M}_{e,e'}$. Therefore, $\pi^{-1}(\bar{M}_{e,e'})=\mu_r(R_r^{e,e'})$. Moreover, (4.8.3) shows that for k-valued points y_1 and y_2 of $R_r^{e,e'}$, $\pi(k)\mu_r(k)(y_1)=\pi(k)\mu_r(k)(y_2)$ if and only if $\mathrm{gr}(F_r^e\otimes k(y_1))\cong\mathrm{gr}(F_r^e\otimes k(y_2))$. Since S is finite type over a universally Japanese ring Ξ, Y is projective over S, whence $\bar{M}_{e,e'}$ is quasi-projective over S. These and (4.3.4) yields the following.

Porposition 4.9. *$R_r^{e,e'}$ has a good quotient $(\bar{M}_{e,e'},\psi)$ with the following properties;*

(4.9.1) *$\bar{M}_{e,e'}$ is quasi-projective over S,*

(4.9.2) *for each geometric point s of S, there exists a natural bijection $\zeta_{e,e'}(s)$: $\bar{\Sigma}_{X/S}^{H,e'}(m_e)(\mathrm{Spec}(k(s)))\to\bar{M}_{e,e'}(k(s))$.*

From the viewpoint of moduli, we have

Proposition 4.10. *$\bar{M}_{e,e'}$ has the following properties:*

(4.10.1) *For each geometric point s of S, there exists a natural bijection $\bar{\theta}_s$: $\bar{\Sigma}_{X/S}^{H,e'}(\mathrm{Spec}(k(s)))\to\bar{M}_{e,e'}(k(s))$.*

(4.10.2) *For $T\in(\mathrm{Sch}/S)$ and a T-flat coherent sheaf E on $X\times_S T$ with the property* (1.7.1) *and* (4.1.1), *there exists a morphism $\bar{f}_E^{e,e'}$ of T to $\bar{M}_{e,e'}$ such that $\bar{f}_E^{e,e'}(t)=\bar{\theta}_s([E\otimes_{\mathcal{O}_T}k(t)])$ for all points t in $T(k(s))$. Moreover, for a morphism $g\colon T'\to T$ in* (Sch/S),

$$\bar{f}_E^{e,e'}\cdot g=\bar{f}_{(1_X\times_S g)^*(E)}^{e,e'}$$

(4.10.3) *If $\bar{M}'\in(\mathrm{Sch}/S)$ and maps $\bar{\theta}'_s$: $\bar{\Sigma}_{X/S}^{H,e'}(\mathrm{Spec}(k(s)))\to\bar{M}'(k(s))$ have the above property* (4.10.2), *then there exists a unique S-morphism $\bar{\psi}$ of $\bar{M}_{e,e'}$ to $\bar{M}'$ such that $\bar{\psi}(k(s))\cdot\bar{\theta}_s=\bar{\theta}'_s$ and $\bar{\psi}\cdot\bar{f}_E^{e,e'}=\bar{f}'_E$ for all geometric points s of S and for all E, where $\bar{f}'_E$ is the morphism given by the property* (4.10.2) *for $\bar{M}'$ and $\bar{\theta}'_s$.*

Proof. If one uses (4.9.2) and the fact that $\bar{M}_{e,e'}$ is a categorical quotient of $R_r^{e,e'}$, the proof is completely the same as in the proof of [12] Proposition 5.5.

Since both $\bar{M}_{e_1,e'}$ and $\bar{M}_{e_2,e'}$ have the properties (4.10.1), (4.10.2) and (4.10.3), there exists a unique isomorphism $\bar{\psi}_{e_1,e_2}^{e'}$: $\bar{M}_{e_1,e'}\to\bar{M}_{e_2,e'}$ such that $\bar{\psi}_{e_1,e_2}^{e'}\cdot\bar{f}_E^{e_1,e'}=\bar{f}_E^{e_2,e'}$. Since $\bar{M}_{e,e'}$ is an open subscheme of $\bar{M}_{e,e}$, $\bar{M}_{e',e'}$ can be regarded as an open subscheme of $\bar{M}_{e,e}$. Thus $\bar{M}_{X/S}(H)=\varinjlim_e \bar{M}_{e,e}$ is an S-scheme locally of finite type over S. Since each $\bar{M}_{e,e}$ is quasi-projective over S, $\bar{M}_{X/S}$ is separated over S. It is obvious that $\bar{M}_{X/S}(H)$ contains $M_{X/S}(H)$ in [12] as open subscheme. Moreover, by the construction of $\bar{M}_{e,e}$, there exists a natural morphism λ_e: $\bar{M}_{e,e}\to\mathrm{Pic}_{X/S}$ such that for all geometric points t of $\bar{M}_{e,e}$, $\lambda_e(t)=c_1(\bar{\theta}_s^{-1}(t))$, where s is the image of t by the structure morphism of $\bar{M}_{e,e}$ to S and c_1 denotes the first Chern class. Moreover, it is easy to see that $\lambda_e\cdot j_{e,e'}=\lambda_{e'}$ for the open immersion $j_{e,e'}$ of $\bar{M}_{e',e'}$ to $\bar{M}_{e,e}$.

Thus we obtain a natural morphism λ: $\overline{M}_{X/S}(H) \to \mathrm{Pic}_{X/S}$. We have therefore the following theorem whose proof is completely the same as that of Theorem 5.6 of [12].

Theorem 4.11. *In the situation of* (3.1), *there exists an S-scheme* $\overline{M}_{X/S}(H)$ *with the following properties*:

1) $\overline{M}_{X/S}(H)$ *is locally of finite type and separated over S*

2) *A coarse moduli scheme* $M_{X/S}(H)$ *of stable sheaves with Hilbert polynomial H is contained in* $\overline{M}_{X/S}(H)$ *as an open subscheme.*

3) *For each geometric point s of S, there exists a natural bijection* $\bar{\theta}_s$: $\bar{\Sigma}^H_{X/S}$ $(\mathrm{Spec}(k(s))) \to \overline{M}_{X/S}(H)(k(s))$.

4) *For* $T \in (\mathrm{Sch}/S)$ *and for a T-flat coherent sheaf E on* $X \times_S T$ *with the property* (1.7.1), *there exists a morphism* $\bar{f}_E$ *of T to* $\overline{M}_{X/S}(H)$ *such that* $\bar{f}_E(t) = \bar{\theta}_s([E \otimes_{\mathcal{O}_T} k(t)])$ *for all points t in* $T(k(s))$. *Moreover, for all morphism g*: $T' \to T$ *in* (Sch/S),

$$\bar{f}_E \cdot g = \bar{f}_{(1_X \times_S g)^*(E)}.$$

5) *If* $\overline{M}' \in (\mathrm{Sch}/S)$ *and maps* $\bar{\theta}'_s$: $\bar{\Sigma}^H_{X/S}(\mathrm{Spec}(k(s))) \to \overline{M}'(k(s))$ *have the above property* (4), *then there exists a unique S-morphism* $\bar{\psi}$ *of* $\overline{M}_{X/S}(H)$ *to* $\overline{M}'$ *such that* $\bar{\psi}(k(s)) \cdot \bar{\theta}_s = \bar{\theta}'_s$ *and* $\bar{\psi} \cdot \bar{f}_E = \bar{f}'_E$ *for all s and E, where* $\bar{f}'_E$ *is the morphism given by* (4) *for* $\overline{M}'$ *and* $\bar{\theta}'_s$.

6) *There exists a natural morphism* λ: $\overline{M}_{X/S}(H) \to Pic_{X/S}$ *such that for all geometric points t of* $\overline{M}_{X/S}(H)$, $\lambda(t) = c_1(\bar{\theta}_s^{-1}(t))$, *where s is the image of t by the structure morphism of* $\overline{M}_{X/S}(H)$ *to S.*

By the property (5), $\overline{M}_{X/S}(H)$ with the properties (3), (4) and (5) is unique up to isomorphism.

Remerk 4.12. If T is reduced and if $E_1 \sim E_2$ in the sense of (1.7.2), then $\bar{f}_{E_1} = \bar{f}_{E_2}$. Thus, $\overline{M}_{X/S}(H)_{red}$ is a coarse moduli scheme of the functor $\bar{\Sigma}^H_{X/S}$ of $(\mathrm{Sch}/S)_{red}$ to (Sets).

§5. Langton's result and its application

Let us begin with a definition.

Definition 5.1. Let E be a coherent sheaf of rand r on a geometric fibre X_s of X. E is said to be μ-stable (or, μ-semi-stable) (with respect to $\mathcal{O}_X(1)$) if it is torsion free and if for all coherent subsheaves F of E of rank t $(1 \leq t \leq r-1)$,

$$d(F, \mathcal{O}_X(1))/t < d(E, \mathcal{O}_X(1))/r \qquad \text{(or, } \leq \text{, resp.)}.$$

In [21], a μ-stable (or, μ-semi-stable) sheaf is said to be H-stable (or, H-semi-stable, resp.) and in [8] and [10], a μ-stable (or, μ-semi-stable) sheaf is employed for the notion of a stable (or, semi-stable, resp.) sheaf. In [8], S. G. Langton proved the following theorem for μ-semi-stable sheaves.

Theorem 5.2. *Let R be a discrete valuation ring over S, K be the quosient field of R and let k be the residue field of R. Assume that a* μ*-semi-stable sheaf*

E on X_K is given. Then there exists an R-flat coherent sheaf $\tilde{E}$ on $X_R = X \times_S$ Spec(R) *such that $\tilde{E} \otimes_R K \cong E$ and $\tilde{E} \otimes_R k$ is μ-semi-stable.*

It easy to see that if E is μ-stable, then it is stable and that if E is semi-stable, then it is μ-semi-stable.

$$\begin{array}{ccc} \mu\text{-stable} & \Longrightarrow & \text{stable} \\ \Downarrow & & \Downarrow \\ \mu\text{-semi-stable} & \Longleftarrow & \text{semi-stable} \end{array}$$

The semi-stability differs from the μ-semi-stability. In fact,

Example 5.3. Fix a non-singular curve C of degree $2n$ in $\mathbf{P}^2$ and pick two non-zero elements s_1, s_2 in $H^0(C, \mathcal{O}_C(n))$ such that $\{x \in C | s_1(x) = 0\} \cap \{y \in C | s_2(y) = 0\} = \phi$. Then, s_1 and s_2 define a regular vector bundle E of rank 2 on $\mathbf{P}^2$ with $c_1(E) = 2n$ and $c_2(E) = 2n^2$ (see [9] Principle 2.6). $E(-n)$ is the kernel of the surjective homomorphism $\mathcal{O}_{\mathbf{P}^2}^{\oplus 2} \to \mathcal{O}_C(n)$ defined by s_1 and s_2. It is easy to see that E is μ-semi-stable and there exists the following exact sequence;

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(n) \longrightarrow E \longrightarrow L \longrightarrow 0$$

where L is torsion free and rank 1. Since L is a proper subsheaf of $\mathcal{O}_{\mathbf{P}^2}(n)$, for all sufficiently large integers m, $h^0(\mathbf{P}^2, L(m)) < h^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(n+m))$. Thus we see that $\chi(\mathcal{O}_{\mathbf{P}^2}(n)(m)) > \chi(E(m))/2$, which implies that E is not semi-stable. In the category of torsion free sheaves, we have much simpler examples. Let X be a non-singular projective variety with Picard number one. If M is an invertible sheaf on X and if L is a coherent subsheaf of M with Supp$(M/L) \neq \phi$ and codim Supp$(M/L) \geq 2$, then $M \oplus L$ is μ-semi-stable but not semi-stable.

If E is not semi-stable, then for sufficiently large integers m, $E(m)$ defines a point x of a Quot-scheme which has the property (4.3.2), but the point is never mapped to a semi-stable point of Gieseker spaces. Thus the above example shows that Theorem 5.2 is not enough, at least, from the viewpoint of moduli. We shall modify Theorem 5.2 so as to fit our aim.

When F is a coherent subsheaf of a torsion free coherent sheaf E on a non-singular variety Y, $\varepsilon(F)$ denotes the smallest coherent subsheaf of E such that $\varepsilon(F) \supseteq F$ and $E/\varepsilon(F)$ is torsion free. Then there exists a non-empty open set U of Y such that $\varepsilon(F)|_U = F|_U$ as subsheaves of $E|_U$.

Fix a coherent torsion free sheaf E on X_s, where s is a K-valued point of S for some field K. For a field L containing K and a coherent sheaf F on $X_L = X_s \otimes_K L$, set

$$\tilde{\beta}(F, m) = r(E)\chi(F(m)) - r(F)\chi(E(m)).$$

$\tilde{\beta}(F, m)$ is a numerical polynomial of degree n with respect to m. $\tilde{\beta}(F, m)$ has the following properties:

(5.4.1) $\tilde{\beta}(F, m) \preceq \tilde{\beta}(\varepsilon(F), m)$ and the equality holds if and only if $\varepsilon(F) = F$.

(5.4.2) For coherent subsheaf F and G of $E\otimes_K L$, $\tilde{\beta}(F, m)+\tilde{\beta}(G, m)=\tilde{\beta}(F+G, m)+\tilde{\beta}(F\cap G, m)$, whence $\tilde{\beta}(F, m)+\tilde{\beta}(G, m)\preceq\tilde{\beta}(\varepsilon(F+G), m)+\tilde{\beta}(\varepsilon(F\cap G), m)$.

(5.4.3) If $0\to F\to G\to H\to 0$ is an exact sequence of coherent sheaves on X_L, then $\tilde{\beta}(G, m)=\tilde{\beta}(F, m)+\tilde{\beta}(H, m)$.

(5.4.4) $\tilde{\beta}(E, m)=0$ and $\tilde{\beta}(0, m)=0$.

(5.4.5) For an algebraically closed field L containing K, $E\otimes_K L$ is semistable if and only if $\tilde{\beta}(F, m)\preceq 0$ for all coherent subsheaves F of $E\otimes_K L$.

Now let us assume that $\tilde{E}=E\otimes_K L$ is not semi-stable for some algebraically closed field L containing K. Consider proper subsheaves F of $\tilde{E}$ enjoying the following property:

($\tilde{A}$) F is coherent, $\tilde{E}/F$ is torsion free and if G is a coherent subsheaf of F with $G\neq F$, then $\tilde{\beta}(G, m)\prec\tilde{\beta}(F, m)$.

If one uses the polynomials $\tilde{\beta}$ and the order $\prec$ instead of the integers β and $<$ in [8], the same argument as in p 96 of [8] implies that there exists a unique maximal subsheaf $\tilde{B}$ of $\tilde{E}$ having the property ($\tilde{A}$).

Definition 5.5. The above unique maximal subsheaf having the property ($\tilde{A}$) is called the $\tilde{\beta}$-subsheaf of E.

Since $\tilde{\beta}(0, m)=0$, ($\tilde{A}$) provides us with $\tilde{\beta}(\tilde{B}, m)\succ 0$.

Proposition 5.6. *$\tilde{B}$ is defined over K, that is, there exists a coherent subsheaf B of E such that $B\otimes_K L=\tilde{B}$.*

Proof. By using $\tilde{\beta}$ instead of β in the argument in p 96 of [8], we know that $\mathrm{Hom}_{\mathcal{O}_{X_L}}(\tilde{B}, \tilde{E}/\tilde{B})=0$. Then the argument in the proof of Proposition 3 of [8] is applicable to our case without any change.

Corollary 5.6.1. *The property that a coherent sheaf is semi-stable is independent of the choice of the base field. More precisely, for a coherent sheaf E on $X\times_S\mathrm{Spec}(K)$, $E\otimes_K\bar{K}$ is semi-stable if and only if E is torsion free and for all coherent subsheaf F of E with $F\neq 0$, $P_F(m)\preceq P_E(m)$, where $\bar{K}$ is the algebraic closure of K. And, for every over field L of K, $E\otimes_K L$ is semi-stable if and only if so is E.*

Proof. If one notes that $X_K=X\times_S\mathrm{Spec}(K)$ is geometrically integral, then it is easy to see that E is torsion free if and only if so is $E\otimes_K L$ for an over field L of K. Since $\tilde{\beta}(G, m)=\tilde{\beta}(G\otimes_K L, m)$, our assertion follows from Proposition 5.6.

q.e.d.

By virtue of the above corollary, we can use the notion of semi-stable sheaves without assuming that the base field is algebraically closed.

Now, the theorem which we need is the following.

Theorem 5.7. *Let R be a discrete valuation ring over S, K be the quotient field of R and let k be the residue field of R. For a semi-stable sheaf E on $X_K = X \times_S \mathrm{Spec}(K)$, there exists an R-flat coherent sheaf $\tilde{E}$ on $X_R = X \times_S \mathrm{Spec}(R)$ such that $\tilde{E} \otimes_R K \cong E$ and $\tilde{E} \otimes_R k$ is semi-stable.*

First of all, note that for every coherent sheaf F on the fibres of X over S, $n!\tilde{\beta}(F, m)$ is a polynomial with integer coefficients. Thus, if $\{F_i\}_{i\ge 1}$ is an infinite sequence of coherent sheaves on X_k with $\tilde{\beta}(F_1, m) \succeq \tilde{\beta}(F_2, m) \succeq \cdots \succeq 0$, then there exists an integer i_0 such that for all $i, j \ge i_0$, $\tilde{\beta}(F_i, m) = \tilde{\beta}(F_j, m)$. Taking this into account and using $\tilde{\beta}$ and $\tilde{\beta}$-subsheaf instead of β and β-subbundle in the argument in §4 and §5 of [8], we see that all we need are the following (notation is the same as in §5, Lemma 2 of [8])

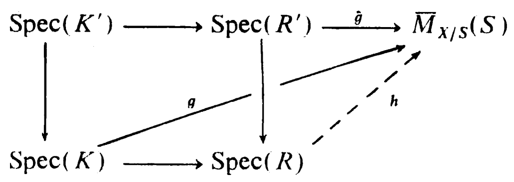
Lemma 5.8. *Assume that the discrete valuation ring R is complete. Let R be an infinite path in the Bruhat-Tits complex S with vertices $[E_\xi]$, $[E'_\xi]$, $[E''_\xi]$,.... Let $\mathrm{Im}(\bar{E}^{(m+1)} \to \bar{E}^{(m)}) = \bar{F}^{(m)}$ $(F' = \bar{F})$. Assume that the canonical homomorphism $\bar{E}^{(m+1)} \to \bar{E}^{(m)}$ maps $\bar{F}^{(m+1)}$ to $\bar{F}^{(m)}$ isomorphically. Then $\chi(\bar{F}(t)) \le r(\bar{F})\chi(E(t))/r(E)$.*

The proof of this lemma is similar to that of Lemma 2 in §5 of [8] and easier than that.

As an application of the above theorem, we have

Theorem 5.9. *Let R, K and k be as in Theorem 5.7. Then the map η: $\mathrm{Hom}_S(\mathrm{Spec}(R), \overline{M}_{X/S}(H)) \to \mathrm{Hom}_S(\mathrm{Spec}(K), \overline{M}_{X/S}(K))$ induced by the injection $R \to K$ is bijective.*

Proof. Since $\overline{M}_{X/S}(H)$ is separated and locally of finite type over the noetherian scheme S, the injectivity of η follows from E. G. A. Ch. II, 7.2.3. Assume that an S-morphism g: $\mathrm{Spec}(K) \to \overline{M}_{X/S}(H)$ is given. Let $\bar{K}$ be the algebraic closure of K. If the geometric point $\bar{g}$: $\mathrm{Spec}(\bar{K}) \to \mathrm{Spec}(K) \xrightarrow{g} \overline{M}_{X/S}(H)$ is contained in $\overline{M}_{e,e}$, then there exists a finite extension K' of K and a K'-valued point x of $R_r^{e,e}$ such that $\pi(x)$ is the K'-valued point g': $\mathrm{Spec}(K') \to \mathrm{Spec}(K) \xrightarrow{g} \overline{M}_{e,e}$. Let R' be an extension of R whose quotient field is K'. For $E = F_r^e \otimes k(x)$, $E \otimes_{K'} \bar{K}$ is e-semi-stable and hence E is semi-stable on $X \times_S \mathrm{Spec}(K')$ (see Corollary 5.6.1). By the natural morphism $\mathrm{Spec}(R') \to \mathrm{Spec}(R) \to S$, $\mathrm{Spec}(R')$ is regarded as an S-scheme. Then, Theorem 5.7 shows that there exists an R'-flat coherent sheaf $\tilde{E}$ on $X \times_S \mathrm{Spec}(R')$ such that $\tilde{E} \otimes_{R'} K' \cong E$ and $\tilde{E} \otimes_{R'} k'$ is semi-stable, where k' is the residue field of R'. The property (4) in Theorem 4.11 gives rise to a morphism $\tilde{g}$: $\mathrm{Spec}(R') \to \overline{M}_{X/S}(H)$. By the construction of $\tilde{g}$, we know that the morphism $\mathrm{Spec}(K') \to \mathrm{Spec}(R') \xrightarrow{\tilde{g}} \overline{M}_{X/S}(H)$ is just g':



Since $R' \cap K = R$, $\tilde{g}$ and g yield a morphism h of $\mathrm{Spec}(R)$ to $\overline{M}_{X/S}(H)$ which extends g. q.e.d.

Let $\mathfrak{S}_{X/S}(H)$ be the family of classes of coherent sheaves on the fibres of X over S such that E is contained in $\mathfrak{S}_{X/S}(H)$ if and only if E is semi-stable and the Hilbert polynomial of E is H.

Corollary 5.9.1. *If $\mathfrak{S}_{X/S}(H)$ is bounded, then $\overline{M}_{X/S}(H)$ is projective over S.*

Proof. If $\mathfrak{S}_{X/S}(H)$ is bounded, $\overline{M}_{X/S}(H) = \overline{M}_{e,e}$ for some positive integer e. Thus $\overline{M}_{X/S}(H)$ is quasi-projective over S. Then, Theorem 5.9 and E. G. A. Ch. II, 7.3.8 imply our assertion. q.e.d.

§6. Some properties of the moduli

To study local properties of $\overline{M}_{X/S}$, we shall investigate the action of $PGL(V_{r,e})$ on $R_r^{e,e'}$.

Lemma 6.1. *Let A be an artin local ring with maximal ideal $\mathfrak{m}$ and residue field k and let E be an A-flat coherent sheaf on $X_A = X \times_S \mathrm{Spec}(A)$. Assume that $E_k = E \otimes_A k$ is torsion free and the natural injection $k \to \mathrm{Hom}_{\mathcal{O}_{X_k}}(E_k, E_k)$ is an isomorphism. Then the natural homomorphism $A \to \mathrm{Hom}_{\mathcal{O}_{X_A}}(E, E)$ is an isomorphism.*

Proof. We shall prove this by induction on $l(A) = \mathrm{length}(A)$. If $l(A) = 1$, then $A = k$, and hence there is nothing to prove. Assume that our assertion is true if $l(A) < l$. If $l(A) = l$, then there exists a principal ideal εA such that $\varepsilon A \cong k$ as A-modules. Since for $\bar{A} = A/\varepsilon A$, $l(\bar{A}) = l(A) - 1$, our assumption says that $\mathrm{Hom}_{\mathcal{O}_{X_{\bar{A}}}}(\bar{E}, \bar{E}) = \bar{A}$, where $\bar{E} = E \otimes_A \bar{A}$. Pick an element ϕ of $\mathrm{Hom}_{\mathcal{O}_{X_A}}(E, E)$. If $\bar{\phi}$ is the member of $\mathrm{Hom}_{\mathcal{O}_{X_{\bar{A}}}}(\bar{E}, \bar{E})$ induced by ϕ, then $\bar{\phi}$ is the multiplication of an element $\bar{a}$ of $\bar{A}$. Lift the $\bar{a}$ to an element a of A and set $\psi = \phi - a \cdot id_E$. Then $\psi(E)$ is contained in $\varepsilon E = E \otimes_A \varepsilon A$. If x is contained in $\mathfrak{m}E = E \otimes_A \mathfrak{m}$, then $\psi(x) = 0$ because $\varepsilon\mathfrak{m} = 0$. Thus ψ induces a homomorphism $\bar{\psi}$: $E_k = E/\mathfrak{m}E \to E \otimes_A \varepsilon A \cong E_k$. By the assumption on E_k, we can find a $\bar{b}$ in k such that $\bar{\psi} = \bar{b} \cdot id_{E_k}$. Lift $\bar{b}$ to a b in A. The definition of $\bar{\psi}$ shows that $\psi = (\varepsilon b) id_E$. Thus we obtain that $\phi = (a + \varepsilon b) id_E$. Pick a non-zero element c in A. The image of $c \cdot id_E$ is cE. Since E is flat over A, $cE = cA \otimes_A E \neq 0$. Therefore, $A \to \mathrm{Hom}_{\mathcal{O}_{X_A}}(E, E)$ is an isomorphism. q.e.d.

The following is a general remark (cf. [14] Lemma 0.5).

Lemm 6.2. *Let S be a scheme of finite type over a universally Japanese ring, X be a flat, projective scheme over S, τ be an action of $\boldsymbol{G}_{m,S} = \mathrm{Spec}(\mathcal{O}_S[T, T^{-1}])$ on X and let L be a $\boldsymbol{G}_{m,S}$-linearized invertible sheaf which is ample over S. If U is a $\boldsymbol{G}_{m,S}$-invariant subscheme of $X^s(L)$, then the action τ on U is proper.*

Proof. We have to prove that $\Phi = (\tau, p_2)$: $\boldsymbol{G}_{m,S} \times_S U \to U \times_S U$ is proper. First of all, note that the image of Φ is closed because U has a geometric quotient by $G_{m,S}$ which is separated over S (see [20]). Let R be a discrete valuation ring over S and let K (k or π) be the quotient field (residue field or uniformanizing parameter, resp.) of R. We may assume that k is algebraically closed. Suppose that (x, y) is an R-valued point of $U\times_S U$ and (g, y) is a K-valued point such that $\Phi(K)(g, y)=(\tau(g, y), y)=(x, y)$. For $\bar{x}=x \bmod \pi$ and $\bar{y}=y \bmod \pi$, we can find a k-valued point $\bar{h}$ of $G_{m,S}$ such that $\bar{x}=\tau(k)(\bar{h}, \bar{y})$ because of the above remark. It is clear that $\bar{h}$ can be lifted to an R-valued point h of $G_{m,S}$. By replacing g by $h^{-1}g$, we may assume that $\bar{x}=\bar{y}$. Since L is ample and $G_{m,S}$-linearized and since X is flat over S, there exist an R-free module V of finite rank, closed immersion ϕ: $X_R=X\times_S \mathrm{Spec}(R)\to \mathbf{P}(V)$ and a representation ρ: $G_{m,R}=G_{m,S}\times_S \mathrm{Spec}(R)\to GL(V)$ such that τ is induced by the action of $GL(V)$ on $\mathbf{P}(V)$. Moreover, there exists a basis $\{e_i\}$ of V such that the dual action $e_i\to e_i\otimes T^{b_i}$ defines the action of $\rho(G_{m,R})$. Then, for an affine open set $X_0=\mathbf{P}(V)-$ a hyper-plane and a suitable system of coordinates $x_1,\dots, x_n$, $\phi(R)(y)$ is contained in $X_0(R)$ and the action of $\rho(G_{m,R})$ is defined by $x_i\to\alpha^{r_i}x_i$. If $\sigma(g, y)_i$ and y_i is the i-th coordinate of $\phi(K)\tau(K)(g, y)$ and y, respectively, then $\sigma(g, y)_i=\beta^{r_i}y_i$, where β is the image of T by the map $R[T, T^{-1}]\to K$ corresponding to the K-valued point g of $G_{m,R}$. $\beta=\beta_0\pi^r$ for a unit β_0 in R. Since $r_i\neq 0$ for some i, $\sigma(g, y)_i=\beta_0^{r_i}\pi^s y_i$ or $\beta_0^{-r_i}\pi^s\sigma(g, y)_i=y_i$ with $s=rr_i>0$ or $s=-rr_i>0$. Since $\sigma(g, y)_i$ and y_i are elements of R with $\sigma(g, y)_i\equiv y_i \bmod \pi$, we see that $r=0$, whence β is a unit of R. q.e.d.

Let U be $R_r^{e,e'}\cap R_r^s$. Then U is a $PGL(N, S)$-invariant subscheme of Z_r, where $N=N^{(r,e)}$.

Lemma 6.3. *The action $\bar{\sigma}$ of $\bar{G}=PGL(N, S)$ on U is free, that is, $\Phi=(\bar{\sigma}, p_2)$: $\bar{G}\times_S U\to U\times_S U$ is a closed immersion.*

Proof. In the first place, we shall show that Φ is proper. Since the projection of U to P_r is $\bar{G}$-morphism with the trivial action of $\bar{G}$ on P_r, we have the following commutative diagram:

$$\begin{array}{ccc} \bar{G}\times_S U & \xrightarrow{\ \Phi\ } & U\times_S U \\ \| & & \uparrow j \\ (\bar{G}\times_S P_r)\times_{P_r} U & \xrightarrow{\ \Psi\ } & U\times_{P_r} U \end{array}$$

Since P_r is separated over S, j is a closed immersion. Thus we have only to show that ψ is proper. Let R, K, k and π be the same as in the proof of Lemma 6.2. Let (x, y) be an R-valued point of $U\times_{P_r} U$ and let (g, y) be a K-valued point of $(\bar{G}\times_S P_r)\times_{P_r} U$ such that $\psi(K)(g, y)=(x, y)$. Since R is a discrete valuation ring, there exists R-valued point g_1 and g_2 of $\bar{G}$ such that $g=g_1(b_{ij})g_2$, where (b_{ij}) is a diagonal matrix with $b_{ii}=\pi^{a_i}$. Let λ: $G_{m,P_r}=\mathrm{Spec}(\mathcal{O}_{P_r}[T, T^{-1}])\to GL(N, P_r)=\mathrm{Spec}(\mathcal{O}_{P_r}[T_{ij}, \det(T_{ij})^{-1}])$ be the homomorphism defined by the $\mathcal{O}_{P_r}$-algebra homomorphism $T_{ij}\to\delta_{ij}T^{a_i}$, where δ_{ij} is Kronecker's delta. Let $\bar{\lambda}$ be the composition $G_{m,P_r}\xrightarrow{\lambda} GL(N, P_r)\longrightarrow PGL(N, P_r)$ and let t be the K-valued point of G_{m,P_r} defined by $T\to\pi$. Then $\bar{\sigma}(\bar{\lambda}(t), \bar{\sigma}(g_2, y))=\bar{\sigma}(g_1^{-1}gg_2^{-1}, \bar{\sigma}(g_2, y))=\bar{\sigma}(g_1^{-1}, \bar{\sigma}(g, y))$ and $\bar{\sigma}(g_2, y)$

are R-valued points of U. It is clear that U is contained in the open set of stable points of Z_r with respect to the action $\bar{\sigma}(\bar{\lambda}(*), *)$ of G_{m,P_r}. Since Z_r is flat and projective over P_r, Lemma 6.2 can be applied to this case. Hence there exists an R-valued point t' of G_{m,P_r} such that $\bar{\sigma}(\bar{\lambda}(t'), \bar{\sigma}(g_2, y))=\bar{\sigma}(\bar{\lambda}(t), \bar{\sigma}(g_2, y))$. Then,

$$x=\bar{\sigma}(g_1\bar{\lambda}(t)g_2, y)=\bar{\sigma}(g_1, \bar{\sigma}(\bar{\lambda}(t), \bar{\sigma}(g_2, y)))$$
$$=\bar{\sigma}(g_1, \bar{\sigma}(\bar{\lambda}(t'), \bar{\sigma}(g_2, y)))=\bar{\sigma}(g_1\bar{\lambda}(t')g_2, y).$$

Therefore (x, y) is the image of the R-valued point $(g_1\bar{\lambda}(t')g_2, y)$, which completes the proof of properness of ψ.

Let A be an artin local ring over S with residue field k. Assume that k is algebraically closed. We claim

(6.3.1) $\Phi(A)$: $\bar{G}(A)\times_{S(A)}U(A) \longrightarrow U(A)\times_{S(A)}U(A)$ is injective.

In fact, if $\Phi(A)(g_1, x)=\Phi(A)(g_2, x)$ for some A-valued points (g_1, x) and (g_2, x) of $\bar{G}\times_S U$, then $\Phi(A)(e, x)=\Phi(A)(g_1^{-1}g_2, x)$. Thus we have only to show that if $x=\bar{\sigma}(A)(g, x)$, then $g=e$. To give a point x in $U(A)$ is just to do an exact sequence $V_{r,e}\otimes_{\Xi}\mathcal{O}_{X_A}\xrightarrow{\phi}E\to 0$ on $X_A=X\times_S\mathrm{Spec}(A)$ such that E is A-flat, $E\otimes_A k$ is stable and $\Gamma(\phi)$: $V_{r,e}\otimes_{\Xi}A\to H^0(X_A, E)$ is bijective. Let h be an A-valued point of $G=GL(N, S)$ whose image by the natural homomorphism $G\to\bar{G}$ is g. $x=\bar{\sigma}(g, x)$ means that there exists an isomorphism f of E which makes the following diagram commutative;

$$\begin{array}{ccc} V_{r,e}\otimes_{\Xi}\mathcal{O}_{X_A} & \xrightarrow{\phi} & E \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ V_{r,e}\otimes_{\Xi}\mathcal{O}_{X_A} & \xrightarrow{\phi} & E \end{array}$$

Since $\mathrm{Hom}_{\mathcal{O}_{X_k}}(E\otimes_A k, E\otimes_A k)=k$ (see Lemma 1.1 and [17] Proposition 4.3), f is the multiplication of a unit a of A by virtue of Lemma 6.1. Then h is the multiplication of a because $\Gamma(\phi)$ is bijective. We see, therefore, that $g=e$.

Applying (6.3.1) to the case where A is an algebraically closed field, one sees that Φ is radical. Combining (6.3.1), E. G. A. Ch. IV, 17.4.1, 17.7.1 and 17.14.2, we have that Φ is unramified. Thus we know that Φ is finite, radicial and unramified, which implies that Φ is a closed immersion (see the proof of [12] Proposition 4.9). q.e.d.

Let M_e be the coarse moduli scheme of e-stable sheaves with Hilbert polynomial H. Then M_e is a geometric quotient of $R_e=R_r^{e,e}\cap R_r^s$.

Proposition 6.4. *The natural map* π: $R_e\to M_e$ *is a principal fibre bundle with group* $\bar{G}$ (*see* [14] *Definition* 0.10).

Proof. If one notes that $\bar{G}$ is a smooth group scheme over S, then he can prove the above, by using Lemma 6.3, in the same way as Proposition 0.9 of [14].

From the above, we have

Corollary 6.4.1. *If* S' *is an* S*-scheme, then for* $X'=X\times_S S'$, $M_{X'/S'}(H)=M_{X/S}(H)\times_S S'$.

Proof. This follows directly from a general fact: If an S-scheme morphism $f\colon Z\to Y$ is a principal fibre bundle with S-group scheme G, then for every S-scheme S', $f'=f\times_S S'\colon Z\times_S S'\to Y\times_S S'$ is a principal fibre bundle with S'-group scheme $G\times_S S'$.

Corollary 6.4.2. $M_{X/S}(H)$ *is smooth over* S *if and only if so is* R_e *for all* $e\geq 0$.

Proof. By virtue of E. G. A. Ch. IV, 17.3.3 and 17.7.10, we have the above immediately from Proposition 6.4. q. e. d.

Our next aim is to give a sufficient condition for smoothness of $M_{X/S}(H)$.

Lemma 6.5. *Let* A *be a noetherian local ring,* B *be a noetherian* A*-algebra and let* I *be an ideal of* A *such that* IB *is contained in the Jacobson radical of* B*. Assume that an exact sequence of finite* B*-modules*

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

enjoys the following properties;

1) M *is* A*-flat and* $M''\otimes_A A/I$ *is* A/I*-flat,*

2) *the map* $u\otimes_A 1\colon M'\otimes_A A/I\to M\otimes_A A/I$ *is injective.*

Then, M'' *is* A*-flat and* u *is injective.*

Proof. Let $\overline{M}'$ be the image of u. The property (2) implies that the map $\overline{M}'\to M\to M\otimes_A A/I$ induces a homomorphism $\overline{M}'\to M'\otimes_A A/I$, whence $\alpha\colon \overline{M}'\otimes_A A/I\to M'\otimes_A A/I$. It is easy to see that α is bijective. Thus we have the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & \mathrm{Tor}_1^A(M, A/I) & \longrightarrow & \mathrm{Tor}_1^A(M'', A/I) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & \overline{M}'\otimes_A I & \longrightarrow & M\otimes_A I & \longrightarrow & M''\otimes_A I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{M}' & \xrightarrow{\bar{u}} & M & \xrightarrow{v} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{M}'\otimes_A A/I & \xrightarrow{\bar{u}\otimes 1} & M\otimes_A A/I & \longrightarrow & M''\otimes_A A/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By the fact that $\bar{u}\otimes_A 1$ is injective and the snake lemma, we have $\mathrm{Tor}_1^A(M'', A/I)=0$. Since $M''\otimes_A A/I$ is A/I-flat, we see that M'' is A-flat (E. G. A. Ch. 0_{III}, 10.2.2). Since both M and M'' are A-flat, so is $\overline{M}'$. Hence, for the kernel K of $M'\to\overline{M}'$, we have the exact sequence

$$0 \longrightarrow K/IK \longrightarrow M'/IM' \xrightarrow{\alpha} \overline{M}'/I\overline{M}' \longrightarrow 0.$$

Thus $K=IK$ because α is an isomorphism. By virtue of Nakayama's lemma, $K=0$. Therefore, u is injective. q.e.d.

As a corollary to the above, we obtain

Lemma 6.6. *Let A be a noetherian local ring over S with residue field k and let I be a nilpotent ideal of A. Let $T=\mathrm{Spec}(A)$ and let $T_0=\mathrm{Spec}(A/I)$. Suppose that there exist a T_0-flat coherent $\mathcal{O}_{X_{T_0}}$-module E_0 and an exact sequence*

$$(6.6.1) \qquad 0 \longrightarrow E_0' \longrightarrow \mathcal{O}_{X_{T_0}}^{\oplus N} \longrightarrow E_0 \longrightarrow 0.$$

If for $\bar{E}=E_0\otimes_{A/I}k$ and for all point x of $X_k=X\times_S\mathrm{Spec}(k)$, depth $\bar{E}_x\geq\min\{\dim(\mathcal{O}_{X_k,x}), n-1\}$, then (6.6.1) is locally liftable to X_T, that is, there exists an open covering $\{U_i\}$ of X_T, a T-flat coherent sheaf E_i on U_i and an exact sequence

$$0 \longrightarrow E_i' \longrightarrow \mathcal{O}_{U_i}^{\oplus N} \longrightarrow E_i \longrightarrow 0,$$

whose inverse image by the natural closed immersion $U_i\times_T T_0\to U_i$ is isomorphic to the restriction of (6.6.1) to $U_i\times_T T_0$.

Proof. Since depth $\bar{E}_x\geq\min\{\dim(\mathcal{O}_{X_k,x}), n-1\}$ for all points x of X_k, $\bar{E}'=E_0'\otimes_{A/I}k$ is locally free on X_k (see [12] p 115). Thus E_0' is locally free on X_{T_0} because E_0' is flat over T_0 (see [11] Lemma 1.3). We can find an affine open covering $\{U_i\}$ of X_T such that $E_0'|_{U_i\times_T T_0}$ is a free module. Let $U_i=\mathrm{Spec}(B)$ and let $B_0=B/IB$. The sequence (6.6.1) provides us with the following exact sequence

$$0 \longrightarrow B_0^{\oplus r} \xrightarrow{u_0} B_0^{\oplus N} \xrightarrow{v_0} M_0 \longrightarrow 0$$

where M_0 is A/I-flat. We have only to lift the above sequence to an exact sequence of A-flat B-modules. Let α: $B^{\oplus r}\to B_0^{\oplus r}$ and β: $B^{\oplus N}\to B_0^{\oplus N}$ be the natural homomorphisms. Then we can lift u_0 to u: $B^{\oplus r}\to B^{\oplus N}$, $u_0\alpha=\beta u$. If one sets $M=\mathrm{coker}(u)$, then he obtains

$$M\otimes_A A/I=\mathrm{coker}(u)\otimes_A A/I\cong\mathrm{coker}(u_0)\cong M_0.$$

Lemma 6.5 can be applied to this case and we see that M ia A-flat and u is injective. q.e.d.

Proposition 6.7. *Let E be a stable sheaf on a geometric fibre X_s of X with Hilbert polynomial H. If depth $E_x\geq\min\{\dim(\mathcal{O}_{X_s,x}), n-1\}$ for all points x of X_s and if $\mathrm{Ext}^2_{\mathcal{O}_{X_s}}(E, E)=0$, then $M_{X/S}(H)$ is smooth over S at the point corresponding to E. In particular, if $\dim X/S=1$, then $M_{X/S}(H)$ is smooth over S. If $\dim X/S=2$, then $\mathrm{Ext}^2_{\mathcal{O}_{X_s}}(E, E)=0$ is sufficient for smoothness of $M_{X/S}(H)$ at the point corresponding to E.*

Proof. Assume that E is e-stable. Since $\mathrm{Ext}^2_{\mathcal{O}_{X_s}}(E(m), E(m))=\mathrm{Ext}^2_{\mathcal{O}_{X_s}}(E, E)$, we may assume that $h^j(X_s, E)=0$ for $j>0$ and that there exist a principal fibre bundle $R_e\to M_e$ with group $\bar{G}=PGL(N, S)$ and the universal quotient sheaf on $X\times_S R_e$

$$0 \longrightarrow F' \longrightarrow \mathcal{O}^{\oplus N}_{X \times_S R_e} \longrightarrow F \longrightarrow 0$$

such that for some $k(s)$-valued point x of R_e, $F \otimes k(x) = E$. We have only to show that R_e is smooth over S at x (see Corollary 6.4.2). To do this, take an artin local ring A over $\mathcal{O}_{S,s_0}$ and an ideal I of A, where s_0 is the scheme point of S which is the image of s: $\mathrm{Spec}(k(s)) \to S$. For $A_0 = A/I$, suppose that the following commutative diagram is given

$$\begin{array}{ccccc} & & & & R_e \\ & & \nearrow^{\eta_0} & & \downarrow q \\ T_0 = \mathrm{Spec}(A_0) & \xrightarrow{i} & T = \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

where $x_0 = \eta_0(T_0)$ for the scheme point x_0 of R_e which is the image of x: $\mathrm{Spec}(k(s)) \to R_e$. What we have to show is to find an S-morphism η: $T \to R_e$ with $\eta i = \eta_0$. Using induction on the length of I, we can reduce the problem to the case where $I = \varepsilon A$ and the length of I is one. The T_0-valued point η_0 gives us an exact sequence of T_0-flat, coherent $\mathcal{O}_{X_{T_0}}$-modules;

(6.7.1) $$0 \longrightarrow E_0' \longrightarrow \mathcal{O}^{\oplus N}_{X_{T_0}} \longrightarrow E_0 \longrightarrow 0,$$

where $E_0 = F \otimes_{\mathcal{O}_{R_e}} \mathcal{O}_{T_0}$ and $E_0' = F' \otimes_{\mathcal{O}_{R_e}} \mathcal{O}_{T_0}$. Note that $E_0 \otimes_{\mathcal{O}_{T_0}} k(s) = E$ and $E' = E_0' \otimes_{\mathcal{O}_{T_0}} k(s) \cong F' \otimes_{\mathcal{O}_{R_e}} k(s)$. By virtue of Lemma 6.6, the sequence (6.7.1) is locally liftable to X_T. Then, a class of obstraction for global lifting of (6.7.1) to X_T is in $H^1(X_s, \mathcal{H}om_{\mathcal{O}_{X_s}}(E', E))$ (see [6] Corollary 5.2). On the other hand, from the exact sequence

$$0 \longrightarrow E' \longrightarrow \mathcal{O}^{\oplus N}_{X_s} \longrightarrow E \longrightarrow 0$$

we obtain the following exact sequence:

$$\mathrm{Ext}^1_{\mathcal{O}_{X_s}}(\mathcal{O}^{\oplus N}_{X_s}, E) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{X_s}}(E', E) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{X_s}}(E, E).$$

Since $\mathrm{Ext}^1_{\mathcal{O}_{X_s}}(\mathcal{O}^{\oplus N}_{X_s}, E) = H^1(X_s, E^{\oplus N}) = 0$, our assumption that $\mathrm{Ext}^2_{\mathcal{O}_{X_s}}(E, E) = 0$ shows that $\mathrm{Ext}^1_{\mathcal{O}_{X_s}}(E', E) = 0$. Since E' is locally free, we have $H^1(X_s, \mathcal{H}om_{\mathcal{O}_{X_s}}(E', E)) = \mathrm{Ext}^1_{\mathcal{O}_{X_s}}(E', E) = 0$. Thus the sequence (6.7.1) is globally liftable to X_T;

(6.7.2) $$0 \longrightarrow \tilde{E}' \longrightarrow \mathcal{O}^{\oplus N}_{X_T} \longrightarrow \tilde{E} \longrightarrow 0$$

This sequence gives rise to a T-valued point η of R_e. Since the inverse image of (6.7.2) by the closed immersion of X_{T_0} to X_T is (6.7.1), ηi is equal to η_0. q.e.d.

As a special case of the above proposition, we have

Corollary 6.7.3. *Suppose that* $\dim X/S = 2$. *If* $d(\overset{2}{\wedge} \Omega_{X_s}, \mathcal{O}_X(1)) < 0$ *for a geometric point* s *of* S, *then* $M_{X/S}(H)$ *is smooth at every point of* $M_{X/S}(H) \times_S \mathrm{Spec}(k(s))$. *Moreover, if* $S = \mathrm{Spec}(k)$ *for a field* k, $\overline{M}_{X/S}(H)$ *is normal.*

Proof. As in the proof of the preceding proposition, we may assume that

$h^j(X_s, E)=0$ for $j>0$ and there exists an open subscheme R of a Quot-scheme and the universal quotient sheaf $\mathcal{O}_{X\times_S R}^{\oplus N}\to F$ such that $\overline{M}_{X/S}(H)$ is a categorical quotient of R by the group scheme $PGL(G, S)$, $\pi^{-1}(M_{X/S}(H))\to M_{X/S}(H)$ is a principal fibre bundle with group $PGL(N, S)$ and F parametrizes all the semi-stable sheaves with Hilbert polynomial H, where π: $R\to\overline{M}_{X/S}(H)$ is the morphism of quotient (note that in this case, $\mathfrak{S}_{X/S}(H)$ in Corollary 5.9.1 is bounded). We have only to show that R is smooth over S. For this, it is enough to prove that $\mathrm{Ext}^2_{\mathcal{O}_{X_s}}(E, E)=0$ for every semi-stable sheaf E (in the proof of Proposition 6.7, we did not use the stability of E to show smoothness of R_ω). Let E be a semi-stable sheaf on a geometric fibre X_s and let $\{x_1,\dots, x_t\}$ be the set of pinch points of E (i.e. x_i is a point where E is not locally free). For the open immersion i: $U=X_s-\{x_1,\dots, x_t\}\to X_s$. $\tilde{E}=i_*i^*(E)$ is a locally free $\mathcal{O}_{X_s}$-module and $G=\tilde{E}/E$ is a torsion sheaf with support $\{x_1,\dots, x_t\}$. We have the following exact sequence

$$\mathrm{Ext}^2_{\mathcal{O}_{X_s}}(G, E)\longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{X_s}}(\tilde{E}, E)\longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{X_s}}(E, E)\longrightarrow 0.$$

(Note that for all coherent $\mathcal{O}_{X_s}$-module H with $\dim\mathrm{Supp}(H)=0$ and for all $i>2$, $\mathrm{Ext}^i_{\mathcal{O}_{X_s}}(H, E)=0$ because X_s is a non-singular projective surface.) Moreover, since E is locally free, $\mathrm{Ext}^i_{\mathcal{O}_{X_s}}(\tilde{E}, G)=H^i(X_s, \tilde{E}^\vee\otimes G)=0$ for $i=1, 2$. Thus $\mathrm{Ext}^2_{\mathcal{O}_{X_s}}(\tilde{E}, E)$ is isomorphic to $\mathrm{Ext}^2_{\mathcal{O}_{X_s}}(\tilde{E}, \tilde{E})=H^2(X_s, \tilde{E}^\vee\otimes\tilde{E})$ which is a dual space of $\mathrm{Hom}_{\mathcal{O}_{X_s}}(\tilde{E}, \tilde{E}\otimes\overset{2}{\wedge}\Omega_{X_s}))$. On the other hand, since E is semi-stable, it is μ-semi-stable, and then $\tilde{E}$ is μ-semi-stable, too. Thus, if $\eta\in\mathrm{Hom}_{\mathcal{O}_{X_s}}(\tilde{E}, \tilde{E}\otimes\overset{2}{\wedge}\Omega_{X_s}))$ is not zero, then $d(\tilde{E}, \mathcal{O}_X(1))/r(\tilde{E})\le d(\eta(\tilde{E}), \mathcal{O}_X(1))/r(\eta(\tilde{E}))\le d(\tilde{E}\otimes(\overset{2}{\wedge}\Omega_{X_s}), \mathcal{O}_X(1))/r(\tilde{E})$. Our assumption implies that $d(\tilde{E}\otimes(\overset{2}{\wedge}\Omega_{X_s}), \mathcal{O}_X(1))=d(\tilde{E}, \mathcal{O}_X(1))+r(\tilde{E})d(\overset{2}{\wedge}\Omega_{X_s}, \mathcal{O}_X(1))<d(\tilde{E}, \mathcal{O}_X(1))$. This is a contradiction. Therefore, we see that $\mathrm{Hom}_{\mathcal{O}_{X_s}}(\tilde{E}, \tilde{E}\otimes(\overset{2}{\wedge}\Omega_{X_s}))=0$. Then the above argument shows that $\mathrm{Ext}^2_{\mathcal{O}_{X_s}}(\tilde{E}, E)=0$, whence $\mathrm{Ext}^2_{\mathcal{O}_{X_s}}(E, E)=0$. q.e.d.

Example 6.8. If X is $\mathbf{P}^2$ or a rational ruled surface over a field k, then Corollary 6.7.3 says that every $M_{X/S}(H)$ is smooth, quasi-projective over k and every $\overline{M}_{X/S}(H)$ is normal, projective over k. It is easy to see that for a ruled surface X, there exists a very ample invertible sheaf $\mathcal{O}_X(1)$ on X such that $d(\overset{2}{\wedge}\Omega_{X/k}, \mathcal{O}_X(1))<0$. If one fixes this $\mathcal{O}_X(1)$, then every $M_{X/S}(H)$ (or, $\overline{M}_{X/S}(H)$) with respect to the $\mathcal{O}_X(1)$ is smooth, quasi-projective (or, normal, projective, resp.) over k.

As for the dimension of $M_{X/S}(H)$, we have

Propositoin 6.9. *Suppose that* $\dim X/S=2$. *Let E be a stable sheaf on a geometric fibre X_s with Hilbert polynomial H and let x be the geometric point of $M_{X/S}(H)$ which corresponds to E. If* $\mathrm{Ext}^2_{\mathcal{O}_{X_s}}(E, E)=0$, *then the relative dimension of $M_{X/S}(H)$ over S at x is*

$$(1-r(E))c_1(E)^2+2r(E)c_2(E)-r(E)^2\chi(\mathcal{O}_{X_s})+1,$$

where $c_i(E)$ is the i-th Chern class of E.

Proof. Both the assumption and the conclusion are independent of twisting

E by $\mathcal{O}_{X_s}(m)$. Thus we may assume that $H^j(X_s, E)=0$ for $j>0$ and that we have a principal fibre bundle $q: R \to M_{X/S}(H)$ with group $PGL(N, S)$ $(N=h^0(X_s, E))$ and the universal quoteint sheaf on $X \times_S R$;

$$0 \longrightarrow F' \longrightarrow \mathcal{O}_{X\times_S R}^{\oplus N} \longrightarrow F \longrightarrow 0.$$

There exists a point y in $R(k(s))$ such that $F \otimes_{\mathcal{O}_R} k(y) \cong E$. Set $E_0 = \mathcal{O}_{X_s}^{\oplus N}$ and $E_1 = F' \otimes_{\mathcal{O}_R} k(y)$. From the above exact sequence we get

$$(6.9.1) \qquad 0 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E \longrightarrow 0.$$

Note that E_0 and E_1 are locally free. (6.9.1) provides us with the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X_s}}(E, E) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X_s}}(E_0, E) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X_s}}(E_1, E)$$
$$\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{X_s}}(E, E) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{X_s}}(E_0, E).$$

Since $\mathrm{Hom}_{\mathcal{O}_{X_s}}(E, E) = \mathrm{End}_{\mathcal{O}_{X_s}}(E) \cong k(s)$, $\dim_{k(s)} \mathrm{Hom}_{\mathcal{O}_{X_s}}(E_0, E) = h^0(X_s, E^{\oplus N}) = N^2$ and since $\mathrm{Ext}^1_{\mathcal{O}_{X_s}}(E_0, E) \cong H^1(X_s, \mathcal{H}om_{\mathcal{O}_{X_s}}(E_0, E)) \cong H^1(X_s, E^{\oplus N})=0$, $\dim_{k(s)} \mathrm{Ext}^1_{\mathcal{O}_{X_s}} (E, E) = \dim_{k(s)} \mathrm{Hom}_{\mathcal{O}_{X_s}}(E_1, E) - N^2 + 1$. On the other hand, $\mathrm{Hom}_{\mathcal{O}_{X_s}}(E_1, E)$ is the tangent space of R_s at y (see [6] Corollary 5.3) and $M_{X/S}(H)$ is smooth over S at x by the assumption and Proposition 6.7. We see therefore that $\dim_x M_{X/S}(H)_s = \dim_{k(s)} \mathrm{Ext}^1_{\mathcal{O}_{X_s}}(E, E)$. By virtue of the spectral sequence $E_2^{p,q} = H^p(X_s, \mathcal{E}xt^q_{\mathcal{O}_{X_s}}(E, E)) \Rightarrow E^{p+q} = \mathrm{Ext}^{p+q}_{\mathcal{O}_{X_s}}(E, E)$, the following exact sequence is obtained;

$$0 \longrightarrow H^1(X_s, \mathcal{H}om_{\mathcal{O}_{X_s}}(E, E)) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{X_s}}(E, E) \longrightarrow$$
$$H^0(X_s, \mathcal{E}xt^1_{\mathcal{O}_{X_s}}(E, E)) \longrightarrow H^2(X_s, \mathcal{H}om_{\mathcal{O}_{X_s}}(E, E)) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{X_s}}(E, E)=0.$$

Since E is locally free outside the set of pinch points of E, $\mathcal{E}xt^1_{\mathcal{O}_{X_s}}(E, E)$ is a sky-scraper sheaf. Hence we have

$$(6.9.2) \qquad \dim_{k(s)} \mathrm{Ext}^1_{\mathcal{O}_{X_s}}(E, E) = \chi(\mathcal{E}xt^1_{\mathcal{O}_{X_s}}(E, E)) - \chi(\mathcal{H}om_{\mathcal{O}_{X_s}}(E, E)) + 1.$$

Now, from the exact sequence (6.9.1), we have an exact complex

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_{X_s}}(E, E) \longrightarrow \mathcal{H}om_{\mathcal{O}_{X_s}}(E_0, E) \stackrel{d}{\longrightarrow} \mathcal{H}om_{\mathcal{O}_{X_s}}(E_1, E)$$

Since $\mathcal{E}xt^1_{\mathcal{O}_{X_s}}(E, E) \cong \mathcal{H}om_{\mathcal{O}_{X_s}}(E_1, E)/\mathrm{im}(d)$, we have

$$(6.9.3) \qquad \chi(\mathcal{E}xt^1_{\mathcal{O}_{X_s}}(E, E)) - \chi(\mathcal{H}om_{\mathcal{O}_{X_s}}(E, E)) = \chi(\mathcal{H}om_{\mathcal{O}_{X_s}}(E_1, E))$$
$$- \chi(\mathcal{H}om_{\mathcal{O}_{X_s}}(E_0, E)).$$

Using the fact that $\mathcal{H}om_{\mathcal{O}_{X_s}}(E_0, E) \cong E^{\oplus N}$, $\mathcal{H}om_{\mathcal{O}_{X_s}}(E_1, E) \cong E \otimes E_1^\vee$, we obtain

$$c_1(\mathcal{H}om_{\mathcal{O}_{X_s}}(E_0, E)) = Nc_1(E)$$
$$c_2(\mathcal{H}om_{\mathcal{O}_{X_s}}(E_0, E)) = Nc_2(E) + N(N-1)c_1(E)^2/2$$
$$c_1(\mathcal{H}om_{\mathcal{O}_{X_s}}(E_1, E)) = Nc_1(E)$$

$$c_2(\mathscr{H}om_{\mathcal{O}_{X_s}}(E_1, E))=(N^2-N+2r-2)c_1(E)^2/2+(N-2r)c_2(E).$$

These, (6.9.2), (6.9.3) and Riemann-Roch theorem imply our assertion. q.e.d.

Our next topic is on universal families. Let $M_{X/S}(H, e)$ be a moduli scheme of e-stable sheaves with Hilbert polynomial H which was constructed in [12] Proposition 5.5.

Definition 6.10. A universal family of $M_{X/S}(H, e)$ is a coherent sheaf F on $X\times_S M_{X/S}(H, e)$ with the following properties:

1) F is flat over $M_{X/S}(H, e)$.

2) For each geometric point s of S and for all $t\in M_{X/S}(H, e)(k(s))$, $F\otimes k(t)=\theta_s^{-1}(t)$, where θ_s is the map of $\Sigma_{X/S}^{H,e}(\mathrm{Spec}(k(s)))$ to $M_{X/S}(H, e)(k(s))$ defined in [12] Proposition 5.5, (i).

A universal family is not necessarily unique. For instance, if F is a universal family of $M_{X/S}(H, e)$, then so is $F\otimes p_2^*(L)$ for every invertible sheaf L on $M_{X/S}(H, e)$.

As is well-known, $H(m)$ can be written in the form $\sum_{i=0}^{n} a_i\binom{m+i}{i}$ for some integers $a_0,\dots, a_n$. Set

$$\delta(H)=G.\,C.\,D.\,\{a_0,\dots, a_n\}.$$

Theorem 6.11. *If $\delta(H)=1$, then $M_{X/S}(H, e)$ has a universal family.*

Proof. One finds an idea to prove this theorem in [15]. Our proof proceeds along the line. There exist a principal fibre bundle $q\colon R\to M=M_{X/S}(H, e)$ with group $PGL(N, S)$ and the universal quotient sheaf F on $X\times_S R$. F parametrizes all the e-stable sheaves with Hilbert polynomial $H_{m_0}(m)=H(m+m_0)$ for some m_0. We may assume that for all $m\geq m_0$ and for all e-stable sheaves E with Hilbert polynomial H, $h^j(E(m))=0$ if $j>0$. For an invertible sheaf L on R, if one can descend $F\otimes p_2^*(L)$ to a coherent sheaf F' on $X\times_S M$, then $F'\otimes p_1^*(\mathcal{O}_X(-m_0))$ is a universal family of $M_{X/S}(H, e)$. Since $\tilde{q}=1_X\times_S q\colon X\times_S R\to X\times_S M$ is a principal fibre bundle with group $\bar{G}=PGL(N, S)$, descent data for $F\otimes p_2^*(L)$ is nothing but a $\bar{G}$-linearization of $F\otimes p_2^*(L)$. On the other hand, F carries a $G=GL(N, S)$-linearization ([12] §4). Thus our task is to find an invertible sheaf L on R and a G-linearization ψ on L such that $p_2^*(\psi)$ cancels the action of the center $C\cong \boldsymbol{G}_{m,S}$ of G on F.

Now, it is easy to see that for the m_0,

$$\delta(H)=G.\,C.\,D.\,\{H(m)|m\geq m_0\}.$$

By our assumption on $\delta(H)$, we can find integers $m_1,\dots, m_t$ such that $m_i\geq m_0$ and $\sum_{i=1}^{t} a_iH(m_i)=-1$ for some integers $a_1,\dots, a_t$. By virtue of the choice of m_0, $p_{2*}(F\otimes p_1^*(\mathcal{O}_X(m_i-m_0)))=E_i$ is a locally free $\mathcal{O}_R$-module of rank $H(m_i)$. Since each $F\otimes p_1^*(\mathcal{O}_X(m_i-m_0))$ is G-linearized, so is E_i by virtue of the base change theorem. And, moreover, the action of C on E_i is the multiplication of constants. Thus the invertible sheaf $L_i=\overset{H(m_i)}{\wedge}E_i$ carries a G-linearization and the action of C on L_i is the multiplication of $H(m_i)$-th power of constants. Then, for $L=L_1^{\otimes a_1}\otimes\cdots\otimes$

$L_i^{\otimes a_i}$, the action of C on L is the multiplication of the inverse of constants. Therefore, $F \otimes p_2^*(L)$ is G-linearized and the action of C on it is canceled. Hence we get a $\bar{G}$-linearization on $F \otimes p_2^*(L)$. q.e.d.

Corollary 6.11.1. *If $\mathfrak{S}_{X/S}(H)$ is bounded and if $\delta(H)=1$, then $M_{X/S}(H)$ has a universal family.*

Remark 6.12. 1) If $S=\mathrm{Spec}(k)$ for a field k, then $M_{X/S}(H, e)$ is a disjoint union of $M_{X/S}(c_1, \dots, c_n, r, e)$, where $M_{X/S}(c_1, \dots, c_n, r, e)$ is a moduli scheme of e-stable sheaves of rank r on X with Chern classes $c_1, \dots, c_n$ (numerical equivalence). For an e-stable sheaf E of rank r with Chern classes $c_1, \dots, c_n$ and for an invertible sheaf L on X, set

$$H_L(m) = \chi((E \otimes_{\mathcal{O}_X} L)(m)).$$

$H_L(m)$ is independent of the choice of E. For $\Delta(H) = G.\,C.\,D.\ \{\delta(H_L) | L \in \mathrm{Pic}(X)\}$, if $\Delta(H)=1$, then $M_{X/S}(c_1, \dots, c_n, r, e)$ has a universal family.

2) Let L be an invertible sheaf on X such that $L^{\otimes \alpha} \cong \mathcal{O}_X(1)$ for some positive integer α. Set $H'(m) = \chi(E \otimes_{\mathcal{O}_{X_s}} L^{\otimes m})$ for an e-stable sheaf on X_s with Hilbert polynomial H. Then $H'(\alpha m) = H(m)$. If $\delta(H')=1$, then $M_{X/S}(H, e)$ has a universal family.

3) If $M_{X/S}(H, e)$ has a universal family, then $M_{X/S}(H, e)$ represents the sheafification in Zariski topology of the functor $\Sigma_{X/S}^{H, e}$.

§7 An example

As an example, let us investigate more closely the moduli schemes of stable sheaves in the case where the base space is $\mathbf{P}_k^2$ and the rank is 2.

Until Theorem 7.17, X denotes $\mathbf{P}_k^2$ and $\mathcal{O}_X(1)$ denotes the invertible sheaf corresponding to lines in X. For $i=0$ or 1, let $M_i(n)$ (or, $\overline{M}_i(n)$) be a moduli scheme of stable (or, semi-stable, resp.) sheaves of rank 2 on X with the first Chern class i and the second Chern class n. Since for a torsion free coherent sheaf E of rank 2 on X, $c_1(E \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)) = 0$ or 1 for a suitable m, every moduli scheme of stable (or, semi-stable) sheaves of rank 2 is isomorphic to one of $M_i(n)$ (or, $\overline{M}_i(n)$, resp.). Let $M_i(n)_0$ denote the open subscheme of $M_i(n)$ whose points correspond to locally free sheaves.

Lemma 7.1. 1) $M_1(n) = \overline{M}_1(n)$. *If n is odd, then $M_0(n) = \overline{M}_0(n)$.*

2) *$M_1(n) \neq \phi$ if and only if $n > 0$. $\overline{M}_0(n) = \phi$ unless $n \geq 0$.*

3) *$M_i(n)$ is smooth and $\dim_x M_i(n) = 4n-3-i$ at every point x of $M_i(n)$.*

4) *If a semi-stable sheaf E of rank 2 on X is locally free and not μ-stable, then $E = \mathcal{O}_X(m)^{\oplus 2}$ for some integer m.*

Proof. 1) If the degree and the rank of a semi-stable sheaf E are coprime, then E is μ-stable, a fortiori, stable. Hence $M_1(n) = \overline{M}_1(n)$. If n is odd, then the constant term of the Hilbert polynomial of $M_0(n)$ is odd. Thus $M_0(n) = \overline{M}_0(n)$ if n is odd.

2) If $d(E, \mathcal{O}_X(1))=1$, $r(E)=2$ and E is stable, then E is μ-stable. Thus $\tilde{E}=(E^{\vee})^{\vee}$ is also μ-stable and locally free. Then $\tilde{E}$ is simple, and hence $c_2(\tilde{E})>0$ (see [9] Theorem 4.6). Since $c_2(E)=c_2(\tilde{E})-c_2(\tilde{E}/E)\geq c_2(\tilde{E})>0$, we know that $M_1(n)=\phi$ unless $n>0$. Conversely, there exists a simple vector bundle of rank 2 on X with Chern classes $c_1=1$ and $c_2=n$ for all positive integer n (see [9] Theorem 4.6). Since every simple vector bundle of rank 2 on X is stable ([10] Appendix Proposition A.1), the first assertion of (2) is proved. If $d(E, \mathcal{O}_X(1))=0$, $r(E)=2$ and E is semi-stable, then $\tilde{E}=(E^{\vee})^{\vee}$ is μ-semi-stable. If $\tilde{E}$ is stable, then $c_2(\tilde{E})>0$, whence $c_2(E)>0$ as above. If $\tilde{E}$ is not stable, then $\tilde{E}$ contains $\mathcal{O}_X$ so that $\tilde{E}/\mathcal{O}_X$ is torsion free. Then $c_2(\tilde{E}/\mathcal{O}_X)\geq 0$. Thus $c_2(E)\geq c_2(\tilde{E})=c_2(\tilde{E}/\mathcal{O}_X)\geq 0$.

3) is a special case of Corollary 6.7.1 and Proposition 6.9.

4) Since E is not μ-stable, $d(E, \mathcal{O}_X(1))$ is even. Thus we may assume that $d(E, \mathcal{O}_X(1))=0$. Then our assumption says that E contains $\mathcal{O}_X$ so that $E/\mathcal{O}_X$ is torsion free. Since $c_1(E/\mathcal{O}_X)=0$, $E/\mathcal{O}_X$ can be regarded as an ideal sheaf of $\mathcal{O}_X$. Hence $h^0((E/\mathcal{O}_X)(m))\leq h^0(\mathcal{O}_X(m))$ and the equality holds if and only if $E/\mathcal{O}_X\cong\mathcal{O}_X$. Therefore, E is an extension of $\mathcal{O}_X$ by $\mathcal{O}_X$ because E is semi-stable. Hence $E\cong\mathcal{O}_X^{\oplus 2}$.

q.e.d.

Let T be a reduced, locally noetherian scheme and let I be a coherent ideal on $Y=\mathbf{P}^2_T$. Assume that $\mathcal{O}_Y/I$ is T-flat and $\dim \operatorname{Supp}(\mathcal{O}_Y/I\otimes_{\mathcal{O}_T}k(t))=0$ for all points t of T. $\mathcal{O}_Y(1)$ denotes an invertible sheaf on Y such that $\mathcal{O}_Y(1)\otimes k(t)\cong\mathcal{O}_{\mathbf{P}^2_{k(t)}}(1)$ for all points t of T. For $a=\min\{h^1(Y_t, I(m)\otimes k(t))|t\in T\}$, set $U=\{t\in T|h^1(Y_t, I(m)\otimes k(t))=a\}$, where $I(m)=I\otimes_{\mathcal{O}_Y}\mathcal{O}_Y(m)$. Then U is a non-empty open set of T and it is easy to see that $h^0(Y_t, I(m)\otimes k(t))$ and $h^2(Y_t, I(m)\otimes k(t))$ are independent of $t\in U$. Thus $R^ip_*(I(m))|_U$ is locally free for all i because T is locally noetherian and reduced, where p is the projection of Y to T. Moreover, for all morphism $g\colon T'\to U$, $g^*R^1p_*(I(m))=R^1(p\times_T 1_{T'})_*(I(m)\otimes_{\mathcal{O}_T}\mathcal{O}_{T'})$. Set $E=R^1p_*(I(m))$, $V=\mathbf{V}(E)=\operatorname{Spec}(S(E))$ and $\tilde{E}=E\otimes_{\mathcal{O}_U}\mathcal{O}_V$. Then there exists a universal homomorphism $\zeta\colon \tilde{E}\to\mathcal{O}_V$.

Let $W=\operatorname{Spec}(A)$ be an affine open subscheme of U and let g be a morphism of $W'=\operatorname{Spec}(A')$ to W. We obtain the following commutative diagram;

$$\begin{array}{ccc}
\operatorname{Ext}^1_{\mathcal{O}_{Y_W}}(I(m)\otimes_{\mathcal{O}_T}\mathcal{O}_W, \overset{2}{\wedge}\Omega_{Y_W/W})\otimes_A A' & \xrightarrow{\xi\otimes_A A'} & \operatorname{Hom}_{\mathcal{O}_W}(R^1p_*(I(m))|_W, \mathcal{O}_W)\otimes_A A' \\
\Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} \\
\operatorname{Ext}^1_{\mathcal{O}_{Y_{W'}}}(I(m)\otimes_{\mathcal{O}_T}\mathcal{O}_{W'}, \overset{2}{\wedge}\Omega_{Y_{W'}/W'}) & \xrightarrow{\xi'} & \operatorname{Hom}_{\mathcal{O}_{W'}}(R^1p'_*(I(m)\otimes_{\mathcal{O}_T}\mathcal{O}_{W'}), \mathcal{O}_{W'})
\end{array}$$

where $p'=p\times_T 1_{W'}$, α and β are canonical functorial homomorphisms and where ξ and ξ' are the canonical isomorphisms defined by the duality morphisms ([7] Ch. III, Corollary 5.2). Since β is an isomorphism, so is α. Applying these to $W'=V\times_U W$, we know that ζ provides us with an element $\zeta_{W'}$ of $\operatorname{Ext}^1_{\mathcal{O}_{Y_{W'}}}(I(m)\otimes_{\mathcal{O}_{T'}}\mathcal{O}_{W'}, \overset{2}{\wedge}\Omega_{Y_{W'}/W'})$. For a point t of W, $E\otimes_{\mathcal{O}_U}k(t)=H^1(Y_t, I(m)\otimes_{\mathcal{O}_T}k(t))$ which is a dual space of $\operatorname{Ext}^1_{\mathcal{O}_{Y_t}}(I(m)\otimes_{\mathcal{O}_T}k(t), \overset{2}{\wedge}\Omega_{Y_t/k(t)})$. Thus the set of $k(t)$-valued points of W'_t is $\operatorname{Ext}^1_{\mathcal{O}_{Y_t}}(I(m)\otimes_{\mathcal{O}_t}k(t), \overset{2}{\wedge}\Omega_{Y_t/k(t)})$. Moreover, for each point s in $W'_t(k(t))$, $\zeta_{W'}\otimes k(s)$ is just the element of $\operatorname{Ext}^1_{\mathcal{O}_{Y_t}}(I(m)\otimes_{\mathcal{O}_T}k(t), \overset{2}{\wedge}\Omega_{Y_t/k(t)})=\operatorname{Ext}^1_{\mathcal{O}_{Y_{W'}}}(I(m)\otimes_{\mathcal{O}_T}\mathcal{O}_{W'}, \overset{2}{\wedge}\Omega_{Y_{W'}/W'})\otimes_{A'}k(s)$ which corresponds to s.

On the other hand, $\zeta_{W'}$ defines an extension ([3] p 292)

$$(7.2)\qquad 0 \longrightarrow \overset{2}{\wedge}\,\Omega_{Y_{W'}/W'} \longrightarrow F_{W'} \longrightarrow I(m)\otimes_{\mathcal{O}_T}\mathcal{O}_{W'} \longrightarrow 0.$$

The above observation shows that for each point s of W', $\zeta_{W'}\otimes k(s)$ is an element of $\mathrm{Ext}^1_{\mathcal{O}_{Y_s}}(I(m)\otimes_{\mathcal{O}_T}k(s), \Omega\overset{2}{\wedge}{}_{Y_s/k(s)})$ defined by the extension

$$(7.2)\otimes k(s)\quad 0 \longrightarrow \overset{2}{\wedge}\,\Omega_{Y_s/k(s)} \longrightarrow F_{W'}\otimes_{\mathcal{O}_{W'}}k(s) \longrightarrow I(m)\otimes_{\mathcal{O}_T}k(s) \longrightarrow 0.$$

Therefore, we obtain a W'-flat coherent sheaf $F_{W'}$ on $Y_{W'}$ which parametrizes all the extensions of $I(m)\otimes_{\mathcal{O}_T}k(t)$ by $\overset{2}{\wedge}\,\Omega_{Y_t/k(t)}$ for all $t\in W$.

Lemma 7.3. *Let E be a stable, locally free sheaf of rank 2 on X_K, where K is a field containing k. If $c_1(E)=i=0$ (or, 1) and $c_2(E)=c_2$, then there exist an integer l and an exact sequence*

$$0 \longrightarrow \overset{2}{\wedge}\,\Omega_{X_K/K} \longrightarrow E(l-3) \longrightarrow J(2l-3+i) \longrightarrow 0$$

with the following properties;

a) $(\sqrt{4c_2+1}-1)/2\geq l>0$ (*or*, $\sqrt{c_2}-1\geq l\geq 0$, *resp.*),
b) *J is a coherent ideal of $\mathcal{O}_{X_K}$ such that* $\dim\mathrm{Supp}(\mathcal{O}_{X_K}/J)=0$,
c) $h^0(X_K, J(2l-3+i))=0$.

Proof. Let us prove the case where $c_1=1$. The proof of another case is similar to that. By Riemann-Roch theorem,

$$\chi(E(m))=m^2+4m+4-c_2.$$

Thus if $m>\sqrt{c_2}-2$, then $\chi(E(m))>0$. Since E is stable, $\sqrt{c_2}-2\geq -1$ by Lemma 7.1, and hence $h^2(E(m))=h^0(E(-m-4))=0$ if $m>\sqrt{c_2}-2$. Thus, for the integer m_1 with $\sqrt{c_2}-1\geq m_1>\sqrt{c_2}-2$, $h^0(E(m_1))\geq m_1^2+4m_1+4-c_2>0$. For a non-zero element a of $H^0(X_K, E(m_1))$, we obtain the following exact sequence

$$0 \longrightarrow \mathcal{O}_{X_K} \xrightarrow{\otimes a} E(m_1) \xrightarrow{u} L \longrightarrow 0.$$

For the torsion part T of L, $u^{-1}(T)$ is locally free and rank 1 because $L/T=E/u^{-1}(T)$ is torsion free and rank 1. Thus we have an exact sequence

$$(7.3.1)\qquad 0 \longrightarrow \mathcal{O}_{X_K}(e) \longrightarrow E(m_1) \longrightarrow M \longrightarrow 0$$

for some e with $m_1\geq e\geq 0$ and for some torsion free coherent sheaf M of rank 1. Let e_1 is the maximum among the integers e such that $\mathcal{O}_{X_K}(e)$ is a subsheaf with $E(m_1)/\mathcal{O}_{X_K}(e)$ torsion free. Then $l=m_1-e_1$ and the exact sequence obtained by tensoring $\overset{2}{\wedge}\,\Omega_{X_K/K}(-e_1)$ to the above sequence with $e=e_1$

$$0 \longrightarrow \overset{2}{\wedge}\,\Omega_{X_K/K} \longrightarrow E(l-3) \longrightarrow M_1 \longrightarrow 0$$

meet our requirement. In fact, (a) is obvious. For $J=M_1(-2l+2)$, the natural injection $J\to(J^\vee)^\vee=\mathcal{O}_{X_K}$ makes J an ideal of $\mathcal{O}_{X_K}$ such that $\dim\mathrm{Supp}(\mathcal{O}_{X_K}/J)=0$ because $c_1(J)=0$ and J is torsion free. If $h^0(J(2l-2))=h^0(M_1)\neq 0$, then $h^0(E(l-3))$

$\neq 0$ because $h^1(\overset{2}{\wedge}\Omega_{X_K/K})=0$. Thus, by a similar argument to the above, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{X_K}(e_2) \longrightarrow E(l-3) \longrightarrow M_2 \longrightarrow 0$$

for some $e_2 \geq 0$ and some torsion free coherent sheaf M_2. After tensoring $\mathcal{O}_{X_K}(e_1+3)$ to the above, we get an exact sequence of type (7.3.1) with e greater than e_1. This contradicts the maximality of e_1. q.e.d.

Let E be as in the above lemma. Then we have l and J. The Hilbert polynomial of $\mathcal{O}_{X_K}/J$ is $\alpha_i(l)=l^2+il+c_2$. Thus, by the exact sequence

$$0 \longrightarrow J(2l-3+i) \longrightarrow \mathcal{O}_{X_K}(2l-3+i) \longrightarrow \mathcal{O}_{X_K}/J \longrightarrow 0$$

and by the fact that $h^0(J(2l-3+i))=h^1(\mathcal{O}_{X_K}(2l-3+i))=0$, we have

$$\begin{aligned} h^1(J(2l-3+i)) &= \alpha_i(l)-(2l-1+i)(2l-2+i)/2 \\ &= -l^2+(3-i)l+i-1+c_2. \end{aligned}$$

We denote the right hand side of the above equality by $\beta_i(l)$. Then, $\beta_i(l)>0$ if l satisfies the inequality in (a) of Lemma 7.3. Let $T_{l,i}=\mathrm{Hilb}^{\alpha_i(l)}_{X/k}$ and let $I_{l,i}$ be the universal family of ideals on $X\times_k T_{l,i}$. For a general point t of $T_{l,i}$, $h^0((I_{l,i}\otimes k(t))(2l-3+i))=\max\{-\beta_i(l), 0\}=0$, and hence $h^1((I_{l,i}\otimes k(t))(2l-3+i))=\alpha_i(l)-h^0(\mathcal{O}_{X_t}(2l-3+i))=\beta_i(l)$. Thus, $U_{l,i}=\{t\in T_{l,i}|h^1((I_{l,i}\otimes k(t))(2l-3+i))=\beta_i(l)\}$ is a non-empty open set of $T_{l,i}$ and for all $t\in U_{l,i}$, $h^0((I_{l,i}\otimes k(t)(2l-3+i))=0$. By the definition of $U_{l,i}$, $J\cong I_{l,i}\otimes k(t)$ as ideals of $\mathcal{O}_{X_K}$ for some K-valued point t of $U_{l,i}$. It is known that $T_{l,i}$, a fortiori, $U_{l,i}$ is a smooth and rational variety ([4] and [13]). By virtue of the results before Lemma 7.3, for an affine open covering $\{W_j\}$ of $U_{l,i}$, there exists a family of coherent sheaves $\{F_{W'_j}\}$, where $W'_j=V(G_{l,i})\times_{U_{l,i}}W_j$ for $G_{l,i}=R^1p_{2*}(I_{l,i})|_{U_{l,i}}$. Each $F_{W'_j}$ is W'_j-flat and it parametrizes all the extensions of $(I_{l,i}\otimes k(t))(2l-3+i)$ by $\overset{2}{\wedge}\Omega_{X_t/k(t)}$ for every $t\in W_j$. Thus there exists a K-valued point x of a W'_j such that $E\cong F_{W'_j}\otimes k(x)$. Moreover, $F_{W'_j}|_{W'_j\cap W'_{j'}}$ is isomorphic to $F_{W'_{j'}}|_{W'_j\cap W'_{j'}}$. Let $V_{l,i}$ be the open subscheme of $\mathbf{V}(G_{l,i})$ such that for all algebraically closed field L,

$$V_{l,i}(L)=\{x\in V(G_{l,i})(L)|F_{W'_j}\otimes k(x) \text{ is stable and locally free, where } x\in W'_j(L)\}.$$

Then, $F_{W'_j}(-l+3)=F_{W'_j}\otimes\mathcal{O}_X(-l+3)$ defines a morphism $f^{(j)}_{l,i}$ of $W'_j\cap V_{l,i}$ to $M_i(c_2)_0$. It is clear that $f^{(j)}_{l,i}=f^{(j')}_{l,i}$ on $W'_j\cap W'_{j}\cap V_{l,i}$. Thus we obtain a morphism f of $V_{l,i}$ to $M_i(c_2)_0$. Since $V_{l,i}$ does not intersect with the zero section of $\mathbf{V}(G_{l,i})$ and since for $t\in U_{l,i}$, $x\in(V_{l,i})_t$ and $\alpha\in\mathbf{G}_{m,k(t)}$, αx is contained in $V_{l,i}$ and $f'_{l,i}(x)=f'_{l,i}(\alpha x)$, $f'_{l,i}$ induces a morphism $f_{l,i}$ of $P_{l,i}$ to $M_i(c_2)_0$, where $P_{l,i}$ is the open subscheme $V_{l,i}/G_m$ of $\mathbf{P}(G_{l,i})$. By the construction of $P_{l,i}$, $\dim P_{l,i}=2\alpha_i(l)+\beta_i(l)-1=l^2+(3+i)l+i+3c_2-2$.

Combining the above results and Lemma 7.3, the following is obtained.

Lemma 7.4. *For each integer l with $(\sqrt{4c_2+1}-1)/2\geq l>0$ (or, $\sqrt{c_2}-1\geq l\geq 0$),* *there exist a non-singular rational variety* $P_{l,0}$ *(or,* $P_{l,1}$*, resp.) of dimension* $l^2+3l+3c_2-2$ *(or,* $l^2+4l+3c_2-1$*, resp.) and a morphism* $f_{l,0}$ *(or,* $f_{l,1}$*, resp.) of* $P_{l,0}$ *(or,* $P_{l,1}$*, resp.) to* $M_0(c_2)_0$ *(or,* $M_1(c_2)_0$*, resp.). Moreover,* $\bigcup_l f_{l,i}(P_{l,i})=M_i(c_2)_0$.

From this lemma, we have

Proposition 7.5. *All the* $M_i(c_2)_0$ *are geometrically integral and non-singular. Moreover, they are unirational over* k.

Proof. $M_0(c_2)_0$ (or, $M_1(c_2)_0$) is smooth and pure dimension $4c_2-3$ (or, $4c_2-4$, resp.). It is easy to see that $l^2+3l+3c_2-2<4c_2-3$ (or, $l^2+4l+3c_2-1<4c_2-4$, resp.) unless $l=l_0$ (or, l_1, resp.) with $(\sqrt{4c_2+1}-1)/2\geq l_0>(\sqrt{4c_2+1}-3)/2$ (or, $\sqrt{c_2}-1\geq l_1>\sqrt{c_2}-2$, resp.). If a connected component C of $M_i(c_2)_0$ does not contain $f_{l_i,i}(P_{l_i,i})$, then C is covered by some of $f_{l,i}(P_{l,i})$'s with $l\neq l_i$ because every $P_{l,i}$ is connected. Then $\dim C\leq\max_{l\neq l_i}\{\dim P_{l,i}\}<4c_2-3-i$, which contradict to the fact $\dim C=4c_2-3-i$. Thus every connected component of $M_i(c_2)_0$ contains $f_{l_i,i}(P_{l_i,i})$, that is, $M_i(c_2)_0\otimes_k K$ is connected for all over fields K of k. Thus $M_i(c_2)_0$ is geometrically integral. Since $P_{l_i,i}$ is rational, $M_i(c_2)_0$ is unirational. q.e.d.

As a corollary to the above, we have

Corrollary 7.5.1. *If* $c_2=a^2-1$ *for an integer* a*, then* $M_1(c_2)_0$ *is a rational variety. If* $c_2=a^2+3a+1$ *for some integer* a*, then* $M_0(c_2)_0$ *is a rational variety.*

Proof. $H(m)=2\binom{m+2}{2}+c_1\binom{m+1}{1}+c_1(c_1+1)/2-c_2$ is the Hilbert polynomial of a coherent sheaf of rank 2 with Chern classes c_1, c_2 on $\mathbf{P}^2_k$. Thus $\delta(H)=1$ if $c_1=1$ or if $c_1=0$ and c_2 is odd. Since a^2+3a+1 is odd, $M_i(c_2)$ has a universal family $\tilde{E}_i$ in both case by virtue of Corollary 6.11.1. We shall prove our assertion in the case of $i=0$ because another case can be proved similarly. Let x be the generic point of $M_0(c_2)_0$ and let r be the integer l_0 in the proof of Proposition 7.5. Set $E=\tilde{E}_0\otimes k(x)$. Then E is a stable sheaf on X_x. Let y be the generic point of $P_{r,0}$. Then $f_{r,0}(y)=x$. Let z be a point of $V_{r,0}$ lying over y. Since for a non-empty open set W' of W'_j, $F_{W'_j}|_{W'}$ is the pull back of $\tilde{E}_0$ by the morphism $W'\longrightarrow P_{r,0}\xrightarrow{f_{r,0}}M_0(c_2)_0$, we have an exact sequence

$$0\longrightarrow\mathcal{O}_{X_z}\longrightarrow(E\otimes_{k(x)}k(z))(r)\longrightarrow J(2r)\longrightarrow 0,\tag{7.5.2}$$

where J is a coherent ideal of $\mathcal{O}_{X_z}$ with $\dim\operatorname{Supp}(\mathcal{O}_{X_z}/J)=0$ and $h^0(\mathcal{O}_{X_z}/J)=\alpha_0(r)$. Since the image of z to $T_{r,0}$ is the generic point of it,

$$\begin{aligned}h^0(J(2r))&=h^0(\mathcal{O}_{X_z}(2r))-h^0(\mathcal{O}_{X_z}/J)\\&=r^2+3r+1-c_2.\end{aligned}$$

If $c_2=a^2+3a+1$, then $r=a$ and $h^0(J(2r))=0$. Thus $\dim_{k(x)}H^0(X_x, E(r))=\dim_{k(z)}H^0(X_z, (E\otimes_{k(x)}k(z))(r))=1$. Hence, for a non-zero element s of $H^0(X_x, E(r))$, the following exact sequence on X_x is obtained;

(7.5.3.) $$0 \longrightarrow \mathcal{O}_{X_x} \longrightarrow E(r) \longrightarrow I(2r) \longrightarrow 0,$$

where I is a coherent ideal of $\mathcal{O}_{X_x}$ with $\dim \operatorname{Supp}(\mathcal{O}_{X_x}/I)=0$ and $h^0(\mathcal{O}_{X_x}/I)=\alpha_0(r)$. Therefore, there exists a morphism g: $\operatorname{Spec}(k(x)) \to T_{r,0}$ such that $I \cong (1_X \times_k g)^*(I_{r,0})$ and, moreover, the extension (7.5.3) defines a non-zero element η of

$$\operatorname{Hom}_{k(x)}(g^*(Rp_{2*}(I_{r,0}(2r-3))), k(x)) \cong$$
$$\operatorname{Hom}_{k(x)}(H^1(X_x, I(2r-3)), k(x)) \cong \operatorname{Ext}^1_{\mathcal{O}_{X_x}}(I(2r-3), \overset{2}{\wedge} \Omega_{X_x/k(x)}).$$

η gives rise to a morphism h of $\operatorname{Spec}(k(x))$ to $\mathbf{V}(G_{r,0})$. It is clear that $h(\operatorname{Spec}(k(x))) \in V_{r,0}$, and hence h induces a morphism $\bar{h}$ of $\operatorname{Spec}(k(x))$ to $P_{r,0}$. Since $h^0(E \otimes_{k(x)} k(z))=1$, $\alpha\xi=\eta$ for some $\alpha \in \mathbf{G}_m(k(z))$, where ξ is the extension class of (7.5.2). Thus $\bar{h}(\operatorname{Spec}(k(x)))=y$. Now, since $(f_{r,0} \cdot \bar{h})^*(\tilde{E}_0)=E$, $f_{r,0} \cdot \bar{h}$ is just the natural morphism of $\operatorname{Spec}(k(x))$ to $M_0(c_2)_0$ (see Remark 6.12, (3)). This means that $k(x) \cong k(y)$. On the other hand, $k(y)$ is the function field of $P_{r,0}$ which is a rational function field over k ([13]). Thus the function field $k(x)$ of $M_0(c_2)_0$ is also rational.

q.e.d.

Corollary 7.5.4. *If E is a stable sheaf of rank 2 on $X=\mathbf{P}^2_k$ with Chern classes c_1, c_2, then E contains a coherent subsheaf L of rank 1 such that $d(E, \mathcal{O}_X(1))/2 - d(L, \mathcal{O}_X(1)) \leq l_0$ or $(2l_1+1)/2$ according as c_1 is even or odd, where l_0 (or, l_1) is the integer with $(\sqrt{4c_2-c_1^2+1}-1)/2 \geq l_0 > (\sqrt{4c_2-c_1^2+1}-3)/2$ (or, $(\sqrt{4c_2-c_1^2+1}-2)/2 \geq l_1 > (\sqrt{4c_2-c_1^2+1}+4)/2$, resp.). Moreover, there exists a stable locally free sheaf of rank 2 such that for all coherent subsheaves L of rank 1, $d(E, \mathcal{O}_X(1))/2 - d(L, \mathcal{O}_X(1)) \geq l_0$ or $(2l_1+1)/2$.*

Proof. We may assume that $c_1=0$ or 1. The first assertion can be proved by a similar way to Lemma 7.3. If the second assertion is not true, then $f_{l,i}$ is generically surjective for some $l<l_i$. This is not the case as was shown in the proof of Proposition 7.5. q.e.d.

Our present aim is to show that $M_1(n)$ and $\bar{M}_0(n)$ are connected. For an algebraic closure $\bar{k}$ of k, $M_1(n) \otimes_k \bar{k} = M_{\mathbf{P}^2_{\bar{k}}}(1, n)$ and $\bar{M}_0(n) \otimes_k \bar{k}$ is homeomorphic to $\bar{M}_{\mathbf{P}^2_{\bar{k}}}(0, n)$. Thus, to prove the connectedness of $M_1(n)$ and $\bar{M}_0(n)$, we may assume that k is algebraically closed.

Lemma 7.6. *If E is a coherent, torsion free sheaf on a non-singular surface Y over k and if E is not locally free, then for a pinch point y of E, there exists an exact sequence*

$$0 \longrightarrow E \longrightarrow E' \longrightarrow k(y) \longrightarrow 0,$$

where E' is coherent and torsion free.

Proof. Since Y is a non-singular surface, $\tilde{E}=(E^\vee)^\vee$ is locally free and $\operatorname{Supp}(\tilde{E}/E)$ is the set of pinch points of E. Hence, $G=(\tilde{E}/E)_y$ is an artinian $\mathcal{O}_{Y,y}$-module. Let G' be a submodule of G which is isomorphic to $k(y)$. Then $u^{-1}(G')$ is the desired sheaf, where u is the natural homomorphism of $\tilde{E}$ to G. q.e.d.

Let T be a k-scheme and let F be a quasi-coherent sheaf on $X\times_k T$. For $Z=\mathbf{P}(F)$ and the projection $q\colon Z\to X\times_k T$, we obtain a natural homomorphism $v\colon q^*(F)\to\mathcal{O}_Z(1)$. Let p_1 (or, p_2) be the projection of Z to X (or, T, resp.). The morphism $p_1q\colon Z\to X$ defines a closed immersion $\Gamma_{p_1q}\colon Z\to X\times_k Z$. It is easy to see that for $g=1_X\times_k p_2q\colon X\times_k Z\to X\times_k T$, $g\cdot\Gamma_{p_1q}=q$. For $W=\Gamma_{p_1q}(Z)$ and $\tilde{F}=g^*(F)$, we have a natural homomorphism

$$\tilde{v}\colon \tilde{F}\longrightarrow \tilde{F}\otimes\mathcal{O}_W\longrightarrow(\Gamma_{p_1q})_*(\mathcal{O}_Z(1))=\tilde{L}.$$

For a geometric point z of Z, $\tilde{v}\otimes k(z)$ is a homomorphism of $F\otimes k(y)$ to $k(x)$, where $x=p_1q(z)$ and $y=p_2q(z)$. By the universality of the couple (Z, v) (E. G. A. Ch. II, 4.2.3), $(Z, \tilde{v})$ parametrizes all the surjective homomorphisms $F\otimes k(t)\to k(x)$ for geometric points t of T and $k(t)$-valued points x of X_t.

On the other hand, there exists an étale covering $T_i(n-1)$ of $M_i(n-1)$ and a $T_i(n-1)$-flat coherent sheaf F on $X\times_k T_i(n-1)$ which parametrizes all the stable sheaves of rank 2 with Chern classes i, $n-1$ (see the proof of Theorem 6.11 and E. G. A. Ch. IV, 17.16.3). Applying the above observation to $T=T_i(n-1)$, we have an exact sequence of coherent sheaves on $X\times_k\mathbf{P}(F)$;

$$0\longrightarrow F'\longrightarrow\tilde{F}\xrightarrow{\ \tilde{v}\ }\tilde{L}\longrightarrow 0.$$

Since both $\tilde{L}$ and $\tilde{F}$ are flat over $\mathbf{P}(F)$, so is F'.

Proposition 7.7. *Let $M_0(n)_1$ be the open subscheme of $M_0(n)$ whose points correspond to μ-stable sheaves. Then $M_0(n)_1$ and $M_1(n)$ are connected.*

Proof. Let $M_1(n)_1$ be the open subscheme of $M_1(n)$ whose points correspond to μ-stable sheaves. Then $M_1(n)_1=M_1(n)$. Thus we have only to show that $M_i(n)_1$ is connected for each i, n. Let $Z=h^{-1}(M_i(n-1)_1)$ and let $T_i(n-1)_1=g^{-1}(M_i(n-1)_1)$, where h (or, g) is the natural morphism of $\mathbf{P}(F)$ (or, $T_i(n-1)$, resp.) to $M_i(n-1)$ for the above $T_i(n-1)$ and $\mathbf{P}(F)$. Lemma 7.6 and the property of $\tilde{v}$ stated above imply that $F'|_{X\times_k Z}$ parametrizes all the μ-stable sheaves of rank 2 with Chern class i, n which are not locally free. Hence we have a morphism ζ of Z to $M_i(n)_1$ such that $\zeta(Z)=M_i(n-1)_1-M_i(n)_0$.

Let us prove our assertion on $M_i(n)_1$ by induction on n. We know that $i=1$, $n\geq 1$ or $i=0$, $n\geq 2$ (see Lemma 7.1). Thus, $M_i(n)_1-M_i(n)_0\neq\phi$ if and only if $i=1$, $n\geq 2$ or $i=0$, $n\geq 3$ because $c_2((E^\vee)^\vee)=c_2(E)+h^0((E^\vee)^\vee/E)$ and because E is μ-stable if and only if so is $(E^\vee)^\vee$. Therefore, if $M_i(n)_1-M_i(n)_0\neq\phi$, then $M_i(n-1)_0\neq\phi$. This and Proposition 7.5 imply that our assertion is true for $i=1$, $n=1$ or $i=0$, $n=2$. Assume that $i=1$, $n>1$ or $i=0$, $n>2$ and that $M_i(n-1)_1$ is connected. Then, $Z_0=h^{-1}(M_i(n-1)_0)\neq\phi$ and, moreover, $Z_0(k)=\{z\in Z(k)|$ for $E=F'\otimes k(z)$, $h^0((E^\vee)^\vee/E)=1\}$. By this property of Z_0, no points of $\zeta(Z_0)$ are specializations of points of $\zeta(Z-Z_0)$.

Lemma 7.8. *Let Y be a noetherian, reduced, irreducible scheme and let F be a coherent $\mathcal{O}_Y$-module. Assume that for the generic point y of Y, $F_y\neq 0$. Then $\mathbf{P}(F)$ is connected.*

Proof. Let p be the projection of $\mathbf{P}(F)$ to Y and let Y_0 be the the largest open set of Y over which F is locally free. By our assumption, Y_0 is not empty and $\mathbf{P}(F)_{Y_0}$ is irreducible. If W is the closure of $\mathbf{P}(F)_{Y_0}$ in $\mathbf{P}(F)$, then $p(W)$ is closed in Y and contains the generic point y. Thus $p(W)=Y$. For a point z of Y, $\mathbf{P}(F)_z$ is connected because $\mathbf{P}(F)_z=\mathbf{P}(F\otimes_{\mathcal{O}_Y}k(z))$ is a finite dimensional projective space. The fact that $p(W)=Y$ implies that $\phi\neq W_z\subset\mathbf{P}(F)_z$. Since W is irreducible, $\mathbf{P}(F)$ is connected. q.e.d.

Now, let us come back to the proof of Proposition 7.7. For a connected component T of $T_i(n-1)_1$, Z_T is connected and $(Z_0)_T$ is irreducible by the above lemma because $T_i(n-1)_1$ is smooth. Since $M_i(n-1)_1$ is irreducible and T is flat over $M_i(n-1)_1$, the image of T to $M_i(n-1)_1$ contains a non-empty open set of $M_i(n-1)_0$. Therefore, $(Z_0)_T$ is not empty. These and the results before Lemma 7.8 show that the closure of $\zeta((Z_0)_T)$ in $M_i(n)_1$ is an irreducible component of $M_i(n)_1-M_i(n)_0$. Let C be the connected component of $M_i(n)_1$ which contains $\zeta(Z_T)$. Since $\dim(Z_0)_T=1+\dim X+\dim T=3+4(n-1)-3-i=4n-3-i-1<4n-3-i=\dim C$, $C\cap M_i(n)_0\neq\phi$. By virtue of Proposition 7.5, $M_i(n)_0$ is connected. Therefore, $M_i(n)_1$ is connected. q.e.d.

Our next step is to show that $M_0(n)$ is connected. Let $T_d=\mathrm{Hilb}^d_{X/k}$ and let I_d be the universal family of ideals on $X\times_k T_d$. Then, as in the proof of Lemma 7.4, we can construct a universal family of extensions

$$0\longrightarrow\mathcal{O}_{X_{W'}}\longrightarrow E^d_{W'}\longrightarrow I_d\otimes_{\mathcal{O}_W}\mathcal{O}_{W'}\longrightarrow 0 \tag{7.9}$$

on $W'=\mathbf{V}(R^1p_*(I_d(-3)))_W$, where W is an affine open of T_d and p is the projection of $X\times_k T_d$ to T_d.

The following is proved in the same way as Lemma 2.5 of [4].

Lemma 7.10. *Let E be a locally free sheaf of rank 2 on a non-singular surface Y. If E' is a coherent subsheaf with* $\dim\mathrm{Supp}(E/E')=0$, *then* $\dim\mathrm{Hom}_{\mathcal{O}_Y}(E', E/E')\leq 4h^0(E/E')$.

By virtue of Corollary 5.3 of [6] and the above. $\dim_y(Q^{n;d}_{W'})\leq\dim W'+4n-4d=4n-d$ at every point y of the open subscheme $Q^{n;d}_{W'}$ of $\mathrm{Quot}^{n-d}_{E_{W'}/X_{W'}/W'}$ such that x is a point of $Q^{n;d}_{W'}$ if and only if it lies over a point z of W' with $E_{W'}\otimes k(z)$ locally free. For the universal subsheaf $E^{n;d}_{W'}$ on $X\times_k Q^{n;d}_{W'}$, we can find an open subscheme $U^{n;d}_{W'}$ of $Q^{n;d}_{W'}$ such that for all algebraically closed fields K,

$$U^{n;d}_{W'}(K)=\{y\in Q^{n;d}_{W'}(K)\mid E^{n;d}_{W'}\otimes k(y)\text{ is stable}\}$$

For every $y\in U^{n;d}_{W'}(K)$, $c_1(E^{n;d}_{W'}\otimes k(y))=0$, $c_2(E^{n;d}_{W'}\otimes k(y))=n$ and $E^{n;d}_{W'}\otimes k(y)$ is not μ-stable. Therefore, $E^{n;d}_{W'}$ defines a morphism $g^{n;d}_{W'}$ of $U^{n;d}_{W'}$ to $M_0(n)$ such that $g^{n;d}_{W'}(U^{n;d}_{W'})$ is contained in $M_0(n)-M_0(n)_1$.

Lemma 7.11. *If $d\geq 2$, then* $\dim g^{n;d}_{W'}(U^{n;d}_{W'})<\dim M_0(n)$.

Proof. It is easy to see that $\dim\mathrm{Aut}(E_{W'}\otimes k(z))=\dim\mathrm{End}(E_{W'}\otimes k(z))\geq 2$ for all $z \in W'$. On the other hand, $\mathrm{Aut}(E_{W'}^{n_1,d} \otimes k(y)) \cong \boldsymbol{G}_m$ for all geometric points y of $U_{W'}^{n_1,d}$. Hence, for all K-valued geometric points y of $U_{W'}^{n_1,d}$, $\{x \in (U_{W'}^{n_1,d})_z(K) | E_{W'}^{n_1,d} \otimes k(x) \cong E_{W'}^{n_1,d} \otimes k(y)\}$ is the set of K-valued points of a subscheme with positive dimension in $(U_{W'}^{n_1,d})_z$, where z is the image of y in W'. Moreover, for the natural action σ of $\boldsymbol{G}_m$ on W', $E_{W'} \otimes k(w) \cong E_{W'} \otimes k(\sigma(\alpha, w))$. Therefore, for every point x of $U_{W'}^{n_1,d}$, $\dim (g_{W'}^{n_1,d})^{-1}(g_{W'}^{n_1,d}(x)) \geq 2$, and hence $\dim g_{W'}^{n_1,d}(U_{W'}^{n_1,d}) \leq 4n-d-2 < 4n-3 = \dim M_0(n)$. q.e.d.

There exists a reduced closed subscheme S_e of $T_1 \times_k T_e$ $(e \geq 1)$ with the following properties:

(7.12.1) For the projections $p\colon S_e \to T_1$ and $q\colon S_e \to T_e$, $J_1 = (1_X \times_k p)^*(I_1)$ contains $J_e = (1_X \times_k q)^*(I_e)$ as ideal sheaves of $\mathcal{O}_{X \times_k S_e}$.

(7.12.2) q is a finite surjective morphism.

(7.12.3) For geometric points t_1 of T_1 and t_2 of T_e, there exists a geometric point s of S_e lying over (t_1, t_2) if and only if $I_1 \otimes k(t_1)$ contains $I_e \otimes k(t_2)$ as subsheaves of $\mathcal{O}_{X_{t_1}} = \mathcal{O}_{X_{t_2}}$.

Let us consider the following exact commutative diagram;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_e & \xrightarrow{\alpha} & \mathcal{O}_{X \times_k S_e} & \longrightarrow & M_1 & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle\beta} & & & & \\ 0 & \longrightarrow & J_e & \xrightarrow{\gamma} & J_1 & \xrightarrow{\delta} & M_2 & \longrightarrow & 0 \end{array}$$

For a point s of S_e, since $\alpha \otimes k(s)$ is injective, so is $\gamma \otimes k(s)$. This and Lemma 6.5 imply that M_2 is flat over S_e. On the other hand, for an affine open set W of T_1, we get an affine scheme W' and a universal family of extensions

$$0 \longrightarrow \mathcal{O}_{X_{W'}} \longrightarrow E_{W'}^1 \xrightarrow{\theta} I_1 \otimes_{\mathcal{O}_{T_1}} \mathcal{O}_{W'} \longrightarrow 0$$

as in (7.9). Since $W_t'(k(t)) = \mathrm{Ext}^1_{\mathcal{O}_{X_t}}(I_1 \otimes k(t), \overset{2}{\wedge}\Omega_{X_t/k(t)}) \cong H^0(X_t, \mathcal{E}xt^1_{\mathcal{O}_{X_t}}(I_1 \otimes k(t), \overset{2}{\wedge}\Omega_{X_t/k(t)})) \cong k(t)$ for all $t \in W$, $E_{W'}^1 \otimes k(y)$ is locally free for all $y \in W'' = W' - 0$-section. Set $V_1^e = W'' \times_W S_e$, $E_{V_1^e} = E_{W'}^1 \otimes_{\mathcal{O}_{W'}} \mathcal{O}_{V_1^e}$ and $\tilde{\theta} = \theta \otimes_{\mathcal{O}_{W'}} \mathcal{O}_{V_1^e}$. Then we have a surjective homomorphism

$$\psi\colon E_{V_1^e} \xrightarrow{\tilde{\theta}} I_1 \otimes_{\mathcal{O}_{T_1}} \mathcal{O}_{V_1^e} = J_1 \otimes_{\mathcal{O}_{S_e}} \mathcal{O}_{V_1^e} \xrightarrow{\delta \otimes \mathcal{O}_{V_1^e}} M_2 \otimes_{\mathcal{O}_{S_e}} \mathcal{O}_{V_1^e}.$$

Set $F_{V_1^e} = \ker(\psi)$. Since $E_{V_1^e}$ and $M_2 \otimes_{\mathcal{O}_{S_e}} \mathcal{O}_{V_1^e}$ are flat over V_1^e, so is $F_{V_1^e}$.

Applying the same argument as above to the case where $S_e = T_e$, $W' = \mathrm{Spec}(k)$, $J_1 = \mathcal{O}_{X \times_k T_e}$ and $E_{W'}^1 = \mathcal{O}_X^{\oplus 2}$, we obtain a V_0^e-flat coherent sheaf $F_{V_0^e}$ on $X \times_k V_0^e$, where $V_0^e = T_e$.

The set of couples $(V_i^e, F_{V_i^e})$ parametrizes all the coherent, torsion free sheaf E of rank 2 on X with the following exact commutative diagram;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & E' & \longrightarrow & L' & \longrightarrow & 0, \end{array}$$

where L and L' are torsion free sheaf of rank 1 with $c_1(L)=c_1(L')=0$, $c_2(L)=e$ and $c_2(L')=i$ and where $E'=(E^\vee)^\vee$.

Let $V_i^{n,e}$ be the open subscheme of $Q_{V_i^e}^n=\mathrm{Quot}^{n-e}{}_{F_{V_i^e}/X_{V_i^e}/V_i^e}$ such that for all algebraically closed fields K,

$V_i^{n,e}(K)=\{y\in Q_{V_i^e}^n(K)\,|\,G_i\otimes k(y)$ is stable and $\dim_y\pi^{-1}\pi(y)\leq 3n-2e\}$, where G_i is the universal subsheaf on $X\times_k Q_{V_i^e}^n$ and $\pi\colon Q_{V_i^e}^n\to V_i^e$ is the structure morphism. Let $F_{V_i^{n,e}}$ be the universal subsheaf on $X\times_k V_i^{n,e}$. Note that $F_{V_i^{n,e}}$ is flat over $V_i^{n,e}$ and $\dim V_i^{n,e}\leq 3n-2e+\dim V_i^e=3n+i$.

Let $Z_n=(\coprod_{\substack{W'\\ [n/2]\geq d\geq 2}} U_{W'}^{n,d})\amalg(\coprod_{\substack{V_1^e\\ [n/2]\geq e\geq 1}} V_1^{n,e})\amalg(\coprod_{[n/2]\geq e\geq 0} V_0^{n,e})$ and let F_n be the coherent sheaf on $X\times_k Z_n$ such that $F_n|_{X\times_k U_{W'}^{n,d}}=E_{W'}^{n,d}|_{X\times_k U_{W'}^{n,d}}$ and $F_n|_{X\times_k V_i^{n,e}}=F_{V_i^{n,e}}$. Then F_n is flat over Z_n.

Lemma 7.13. *Let K be an algebraically closed field containing k and let E be a coherent sheaf of rank 2 on X_K with $c_1(E)=0$ and $c_2(E)=n$. If E is stable but not μ-stable, then there exists a K-valued geometric point y of Z_n such that $F_n\otimes k(y)\cong E$.*

Proof. Since E is stable but not μ-stable, the following exact sequence is obtained;

$$0\longrightarrow L_1\longrightarrow E\longrightarrow L_2\longrightarrow 0,$$

where L_1 and L_2 are coherent ideal sheaves of $\mathcal{O}_{X_K}$ with $\dim\mathrm{Supp}(\mathcal{O}_{X_K}/L_i)=0$ and $c_2(L_1)>c_2(L_2)$. Since $(L_1{}^\vee)^\vee=\mathcal{O}_{X_K}$, this gives rise to an exact sequence

$$0\longrightarrow\mathcal{O}_{X_K}\longrightarrow E'\longrightarrow L_2'\longrightarrow 0,$$

where $E'=(E^\vee)^\vee$ and $c_2(L_2)\geq c_2(L_2')=d$. If $d\geq 2$, then the above exact sequence provides us with a K-valued point x of W' such that $E_{W'}^d\otimes k(x)\cong E'$. By the definition of $Q_{W'}^{n,d}$, E corresponds to a K-valued point y of $Q_{W'}^{n,d}$ lying over x because $h^0(E'/E)=c_2(E)-c_2(E')=n-d$. Since E is stable, y is contained in $U_{W'}^{n,d}(K)$, and hence $F_n\otimes k(y)=E$. Now assume that $d=i=0$ or 1. For the natural homomorphism $\lambda\colon E'\to E'/E$, set $E''=\lambda^{-1}(\mathcal{O}_{X_K}/L_1)$. Then the following exact commutative diagrams are obtained:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{X_K} & \longrightarrow & E'' & \longrightarrow & L_2 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{X_K} & \longrightarrow & E' & \longrightarrow & L_2' & \longrightarrow & 0 \end{array} \tag{7.13.1}$$

(7.13.2)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_1 & \longrightarrow & E & \longrightarrow & L_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{X_K} & \longrightarrow & E'' & \longrightarrow & L_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{X_K}/L_1 & = & \mathcal{O}_{X_K}/L_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

If $e=c_2(L_2)$, then (7.13.1) yields a K-valued point x of V_i^e such that $F_{V_i^e}\otimes k(x)=E''$. Since $h^0(\mathcal{O}_{X_K}/L_1)=c_2(L_1)=n-e$, (7.13.2) defines a K-valued point y of $Q_{V_i^e}^n$. On the other hand, $\dim_K \mathrm{Hom}_{\mathcal{O}_{X_K}}(L_1, \mathcal{O}_{X_K}/L_1)\leq 2(n-e)$ and $\dim_K \mathrm{Hom}_{\mathcal{O}_{X_K}}(L_2, \mathcal{O}_{X_K}/L_1)\leq(n-e)+e=n$ by the same argument as in the proof of Lemma 2.5 of [4]. Thus $\dim_K \mathrm{Hom}_{\mathcal{O}_{X_K}}(E, \mathcal{O}_{X_K}/L_1)\leq \dim_K \mathrm{Hom}_{\mathcal{O}_{X_K}}(L_1, \mathcal{O}_{X_K}/L_1)+\dim_K \mathrm{Hom}_{\mathcal{O}_{X_K}}(L_2, \mathcal{O}_{X_K}/L_1)\leq 3n-2e$. Since $\mathrm{Hom}_{\mathcal{O}_{X_K}}(E, \mathcal{O}_{X_K}/L_1)$ is the Zariski tangent space of $\pi^{-1}\pi(y)$ at y, $\dim_y \pi^{-1}\pi(y)\leq 3n-2e$. This and the fact that E is stable imply that y is a K-valued point of $V_i^{n,e}$ with $F_n\otimes k(y)\cong E$. q.e.d.

Proposition 7.14. *$M_0(n)$ is connected.*

Proof. F_n defines a morphism g_n of Z_n to $M_0(n)$. By the construction of Z_n, $g_n(Z_n)\subseteq M_0(n)-M_0(n)_1$. Lemma 7.13 means that $g_n(Z_n)=M_0(n)-M_0(n)_1$. By a similar argument to the proof of Lemma 7.11, we see that $\dim g_n(V_i^{n,e})\leq 3n-3+2i<4n-3=\dim M_0(n)$ because $\dim V_i^{n,e}=3n+i$, $\dim \mathrm{Aut}(E_{W'}^1\otimes k(t))\geq 2$ for all $t\in W'$, $\dim \mathrm{Aut}(\mathcal{O}_X^{\oplus 2})=4$ and because $n\geq i+1$. This and Lemma 7.11 show that $\dim g_n(Z_n)<\dim M_0(n)$. Then, by the same argument as in the proof of Proposition 7.7, we know that $M_0(n)$ is connected. q.e.d.

Finally let us show that $\overline{M}_0(n)$ is connected.

Lemma 7.15. *Let K be an algebraically closed field containing k and let E be a coherent sheaf of rank 2 on X_K with $c_1(E)=0$ and $c_2(E)=n$. If E is semi-stable but not stable, then n is a non-negative even integer and $\mathrm{gr}(E)=L_1\oplus L_2$, where L_i is an ideal sheaf of $\mathcal{O}_{X_K}$ with $c_1(L_i)=0$ and $c_2(L_i)=n/2$.*

The proof is easy and we omit it.

Let $n=2m$, $T_m=\mathrm{Hilb}_{X/k}^m$ and let I_m be the universal family of ideal sheaves on $X\times_k T_m$. Then, on $X\times_k T_m\times_k T_m$ we have a coherent sheaf $F=(1_X\times_k p_1)^*(I_m)\oplus(1_X\times_k p_2)^*(I_m)$ which is flat over $T_m\times_k T_m$. F defines a morphism f_n: $T_m\times_k T_m\to\overline{M}_0(n)$. Lemma 7.15 implies that $f_n(T_m\times_k T_m)=\overline{M}_0(n)-M_0(n)$, whence $\overline{M}_0(n)-M_0(n)$ is connected. Assume that $A(n)=\overline{M}_0(n)-M_0(n)$ is a connected component of $\overline{M}_0(n)$. Then, there exist a subscheme R of a Quot-scheme and a morphism h: $R\to A(n)$. $(A(n), h)$ is a good quotient by $PGL(N)$, R is smooth and $\dim R=4n-3+N^2-1$ (cf. proofs of Corollary 6.7.3 and Proposition 6.9). Let L_1 and L_2 be ideals of $\mathcal{O}_{X_K}$ such that $\mathcal{O}_{X_K}/L_1=\bigoplus_{i=1}^{m} k(x_i)$ and $\mathcal{O}_{X_K}/L_2=\bigoplus_{j=1}^{m} k(y_j)$ with $x_1,\ldots, x_m$,

$y_1, \ldots, y_m$ mutually distinct. Since $\mathscr{H}om_{\mathcal{O}_{X_K}}(L_2, L_1) \cong L_1$ and $\mathscr{E}xt^1_{\mathcal{O}_{X_K}}(L_2, L_1) \cong \bigoplus_{j=1}^{m} k(y_j)$, for the spectral sequence $E_2^{p,q} = H^p(X_K, \mathscr{E}xt^q_{\mathcal{O}_{X_K}}(L_2, L_1)) \Rightarrow E^{q+q} = \mathrm{Ext}^{p+q}_{\mathcal{O}_{X_K}}(L_2, L_1)$, $E_2^{2,0} = 0$, $\dim_K E_2^{1,0} = m-1$ and $\dim_K E_2^{1,0} = m$. Thus $\dim_K \mathrm{Ext}^1_{\mathcal{O}_{X_K}}(L_2, L_1) = 2m-1$. For a K-algebra B, using the spectral sequence and the fact that $H^1(X_K, \mathscr{H}om_{\mathcal{O}_{X_K}}(L_2, L_1)) \otimes_K B \cong H^1(X_B, \mathscr{H}om_{\mathcal{O}_{X_B}}(L_2 \otimes_K B, L_1 \otimes_K B))$, $H^0(X_K, \mathscr{E}xt^1_{\mathcal{O}_{X_K}}(L_2, L_1)) \otimes_K B \cong H^0(X_B, \mathscr{E}xt^1_{\mathcal{O}_{X_B}}(L_2 \otimes_K B, L_1 \otimes_K B))$, we see that the natural homomorphism $\mathrm{Ext}^1_{\mathcal{O}_{X_K}}(L_2, L_1) \otimes_K B \to \mathrm{Ext}^1_{\mathcal{O}_{X_B}}(L_2 \otimes_K B, L_1 \otimes_K B)$ is an isomorphism. Thus, on $V = \mathbf{V}(\mathrm{Ext}^1_{\mathcal{O}_{X_K}}(L_2, L_1)^\vee)$, a universal element ζ of $\mathrm{Ext}^1_{\mathcal{O}_{X_V}}(L_2 \otimes_K \mathcal{O}_V, L_1 \otimes_K \mathcal{O}_V)$ is given. We can construct, therefore, a universal family of extensions

$$0 \longrightarrow L_1 \otimes_K \mathcal{O}_V \longrightarrow E_V \longrightarrow L_2 \otimes_K \mathcal{O}_V \longrightarrow 0$$

on $X \times_k V$. E_V is flat over V and $\dim V = 2m-1$.

Similarly we have a universal family of extensions

$$0 \longrightarrow L_2 \otimes_K \mathcal{O}_{V'} \longrightarrow E_{V'} \longrightarrow L_1 \otimes_K \mathcal{O}_{V'} \longrightarrow 0.$$

Let $W = V \amalg V'$ and let E_W be the coherent sheaf on X_W such that $E_W|_{X_V} = E_V$ and $E_W|_{X_{V'}} = E_{V'}$. By the universality of R, there exist an open covering $\{U_i\}$ of W and morphisms g_i of U_i to R such that $E_W|_{X_{U_i}}$ is the pull back of the universal quotient sheaf on $X \times_k R$ by g_i. Let g be the morphism

$$g : (\amalg U_i) \times_k PGL(N) \xrightarrow{(\amalg g_i) \times 1} R \times_k PGL(N) \xrightarrow{\bar{\sigma}} R,$$

where $\bar{\sigma}$ is the action of $PGL(N)$ on R. If y is the point of $A(n)$ which corresponds to semi-stable sheaves E with $\mathrm{gr}(E) = L_1 \oplus L_2$, then the image of g is just $h^{-1}(y)$. By a similar argument as before, we see that $\dim(\operatorname{im} g) \leq 2m - 2 + N^2 - 1 = n - 2 + N^2 - 1$. These results show that there exists a non-empty open set W of $A(n)$ such that for all points y of W, $\dim h^{-1}(y) \leq n - 2 + N^2 - 1$. We have therefore that $\dim A(n) \geq 3n - 1$. On the other hand, $\dim(T_m \times_k T_m) = 2n$. This is a contradiction if $n > 0$, whence $\overline{M}_0(n) - M_0(n)$ is not a connected component of $\overline{M}_0(n)$. This and Proposition 7.14 imply that $\overline{M}_0(n)$ is connected if $n > 0$. If $n = 0$, every semi-stable sheaf is isomorphic to $\mathcal{O}_X^{\oplus 2}$ by Lemma 7.15 and the fact that $M_0(0) = \phi$. Thus we obtain

Proposition 7.16 *$\overline{M}_0(n)$ is connected.*

Summarizing the above results, we have

Theorem 7.17. *Let $M(c_1, c_2)$ (or, $\overline{M}(c_1, c_2)$) be the moduli scheme of stable (or, semi-stable, resp.) sheaves of rank 2 on $\mathbf{P}_k^2$ with Chern classes c_1, c_2.*

1) *$M(c_1, c_2)$ is a non-singular, irreducible, unirational variety over k and $\dim M(c_1, c_2) = 4c_2 - c_1^2 - 3$.*

2) *$\overline{M}(c_1, c_2)$ is a normal, irreducible, projective variety over k which contains $M(c_1, c_2)$ as an open subscheme.*

3) *$M(c_1, c_2) \neq \phi$ if and only if $4c_2 - c_1^2 > 0$, $\neq 4$. $\overline{M}(c_1, c_2) \neq \phi$ if and only if $4c_2 - c_1^2 \geq 0$, $\neq 4$. If $4c_2 - c_1^2 = 0$, then $\overline{M}(c_1, c_2) = \mathrm{Spec}(k)$.*

4) *$M(c_1, c_2) \neq \overline{M}(c_1, c_2)$ if and only if $4c_2 - c_1^2 \equiv 0 \bmod 8$.*

5) *$M(c_1, c_2)$ has a universal family if $4c_2 - c_1^2 \not\equiv 0 \bmod 8$.*

6) *If $c_2 = (c_1^2 - 1)/4 + a^2 - 1$ or $c_1^2/4 + a^2 + 3a + 1$ for an integer a, then $M(c_1, c_2)$ is rational.*

Let us close this section with the following questions.

Question 7.18. Is every $\mathfrak{S}_{X/S}(H)$ bounded or every $\overline{M}_{X/S}(H)$ projective?

Question 7.19. What is the closure of $M_{X/S}(H)$ in $\overline{M}_{X/S}(H)$?

Question 7.20. Let $S = \mathrm{Spec}(k)$ for a field k and let $\overline{M}'_{X/S}(c_1, \dots, c_n; r)$ be the moduli scheme of semi-stable sheaves of rank r on X with Chern classes $c_1, \dots, c_n$ (algebraic equivalence). When is $\overline{M}'_{X/S}(c_1, \dots, c_n; r)$ connected?

Question 7.21. Under the notation of Theorem 7.17, is $M(c_1, c_2)$ rational? By virtue of Barth's results in [2], $M(c_1, c_2)$ is rational if c_1 is even.

Appendix.

To show that our results are not trivial on every smooth, projective variety, we shall prove the following.

Proposition A.1. *Let X be a smooth, projective variety over an algebraically closed field k with very ample invertible sheaf $\mathcal{O}_X(1)$, let D be a divisor on X and let r be an integer with $r \geq \dim X$. Assume that $\dim X > 0$ and $X \not\cong \mathbf{P}_k^1$. Then, for every integer s, there exists a locally free μ-stable sheaf E on X with respect to $\mathcal{O}_X(1)$ (see Definition 5.1) such that $r(E) = r$, $c_1(E) = D$ (rational equivalence) and $d(c_2(E), \mathcal{O}_X(1)) \geq s$ if $\dim X \geq 2$.*

First of all, let us prove the following lemma.

Lemma A.2. *Let Y be a smooth projective variety over k with ample invertible sheaf $\mathcal{O}_Y(1)$, let E be a locally free coherent sheaf on Y with $r(E) > \dim Y$ and let B be a bounded family of coherent subsheaves of E such that for all $G \in B$, $r(G) < r(E)$. Then there exists an integer n_0 such that for all integers $n \geq n_0$, E contains $\mathcal{O}_Y(-n)$ as a subsheaf with the following properties;*

1) *$E/\mathcal{O}_Y(-n)$ is locally free,*

2) *$\mathcal{O}_Y(-n) \cap G = 0$ for all $G \in B$.*

Proof. Since B is bounded, the set $\{d(G, \mathcal{O}_Y(1)) | G \in B\}$ is bounded. Thus, the set $\{d(\varepsilon(G), \mathcal{O}_Y(1)) | G \in B\}$ is bounded below, where $\varepsilon(G)$ is the coherent subsheaf of E such that $\varepsilon(G) \supseteq G$, $r(\varepsilon(G)) = r(G)$ and $E/\varepsilon(G)$ is torsion free. By virtue of Corollary 1.2.1 of [11], $\bar{B} = \{\varepsilon(G) | G \in B\}$ is a bounded family. We have only to show the lemma for $\bar{B}$ instead of B. Therefore, replacing B by $\bar{B}$, we may assume that E/G is torsion free for all $G \in B$. Suppose that we can find a subsheaf $\mathcal{O}_Y(-n)$ of E which enjoys the property (1) and (2') $\mathcal{O}_Y(-n) \not\subset G$ for all $G \in B$. If $I = \mathcal{O}_Y(-n)$

$\cap G \neq 0$, then $r(I)=1$ because I is a subsheaf of the torsion free sheaf E. Hence the subsheaf $\mathcal{O}_Y(-n)/I$ of E/G is a torsion sheaf, which contradicts the fact that E/G is torsion free. Thus we have only to find n_0 for the properties (1) and (2′).

Since B is bounded, there exists a k-scheme of finite type T and a coherent subsheaf F of $E'=E\otimes_k\mathcal{O}_T$ with the following three properties; (a) E'/F is flat over T, (b) $r(F\otimes k(s))<r(E)$ for all $s\in T$ and (c) for all $G\in B$, we can find a k-valued point t of T such that $F\otimes k(t)=G$ as subsheaves of E. By the property (b) and the fact that T is finite type, there exists an integer n_0 such that for all $n\geq n_0$ and all $t\in T$, $(F\otimes k(t))(n)$ is generated by its global sections, $h^i((F\otimes k(t))(n))=0$ for all $i>0$ and $h^0(E(n))>h^0((F\otimes k(t))(n))+\dim T$. These and (a) imply that for every $n\geq n_0$, $\tilde{F}=p_*(F(n))$ is a locally free, coherent subsheaf of $\tilde{E}=p_*(E'(n))\cong H^0(Y, E(n))\otimes_k\mathcal{O}_T$, where p is the projection of $X\times_k T$ to T, $F(n)=F\otimes_k\mathcal{O}_Y(n)$ and $E'(n)=E'\otimes_k\mathcal{O}_Y(n)=E(n)\otimes_k\mathcal{O}_T$. For $Z=\mathbf{P}(H^0(Y, E(n))^\vee)$, $\mathbf{P}(\tilde{F}^\vee)$ is a closed subsheme of $Z\times_k T=\mathbf{P}(\tilde{E}^\vee)$. By virtue of the choice of n_0, $\dim \mathbf{P}(\tilde{F}^\vee)<\dim Z$. Thus, for the projection q of $Z\times_k T$ to Z, the closure Z_0 of $q(\mathbf{P}(\tilde{F}^\vee))$ in Z is a proper closed subset of Z. Then, for a k-valued point z of $V=Z-Z_0$, s_z is not contained in $\bigcup_{G\in B} H^0(Y, G(n))$ by virtue of (c) and so $s_z\mathcal{O}_Y\not\subset G(n)$ for all $G\in B$, where s_z is an element of $H^0(Y, E(n))$ such that ks_z corresponds to z. On the other hand, there exists a non-empty open set U of Z such that for all k-valued points u of U, $E(n)/s_u\mathcal{O}_Y$ is locally free because $r(E)>\dim Y$ and $E(n)$ is generated by its global sections. Now, for a k-valued point x of $U\cap V$, the subsheaf $s_x\mathcal{O}_Y\otimes\mathcal{O}_Y(-n)$ of E meets our requirement. q.e.d.

The following is well-known and proved easily.

Lemma A.3. *Let X be a smooth projective variety over k, let Y be an irreducible subvariety of codimension 1 and let G be a coherent $\mathcal{O}_Y$-module of rank r. Then rY is the first Chern class of G as an $\mathcal{O}_X$-module.*

Now we can prove our proposition.

Proof of Proposition A.1. If $\dim X=1$, our assertion is well-known. Thus we assume that $\dim X\geq 2$. Replacing D by $D+rH_m$ with an $H_m\in|\mathcal{O}_X(m)|$, $m\gg 0$, we may assume that $|D|$ contains a smooth irreducible member. Pick a smooth, irreducible, k-rational member Y of $|D|$. Let $F=\mathcal{O}_X^{\oplus r}$ and let B_0 be the set of torsion free, coherent, quotient sheaves G of F with $d(G, \mathcal{O}_X(1))\leq r(G)d(Y, \mathcal{O}_X(1))/r$. By virtue of Corollary 1.2.1 of [11], B_0 is a bounded family. For a coherent quotient sheaf G of F, set $\kappa(G)=\ker(F\otimes\mathcal{O}_Y\to G\otimes\mathcal{O}_Y)$. Then, $B=\{\kappa(G)|G\in B_0\}$ is bounded. For an $\mathcal{O}_Y$-module H, $r(H)$ denotes the rank of H as an $\mathcal{O}_Y$-module. Since every member G of B_0 is torsion free and Y is an irreducible divisor, $r(\kappa(G))=r-r(G)$. Applying Lemma A.2 to the situation that $Y=Y$, $E=F\otimes\mathcal{O}_Y$, $B=B$ and $\mathcal{O}_Y(1)=\mathcal{O}_X(1)\otimes\mathcal{O}_Y$, we obtain the integer n_0. Fix an integer n such that $n\geq n_0$ and $d(T, \mathcal{O}_X(1))\geq s$ for a $T\in|\mathcal{O}_Y(n)|$. $\mathcal{O}_Y(-n)$ is contained in $F\otimes\mathcal{O}_Y$ so that the properties (1) and (2) of Lemma A.2 are enjoyed. Set $F_0=(F\otimes\mathcal{O}_Y)/\mathcal{O}_Y(-n)$ and $E=\ker(F\otimes\mathcal{O}_X(Y)\to F_0\otimes\mathcal{O}_X(Y))$. Let us show that E has the required properties. First of all, E is a regular vector bundle defined by some members $u_1,\dots, u_r$ of $H^0(Y,$

$\mathcal{O}_Y(n)$) ([9] p 112). Thus E is locally free, $r(E)=r$, $c_1(E)=Y$ and $c_2(E)=T$ for a $T\in|\mathcal{O}_Y(n)|$, whence $c_1(E)=D$ and $d(c_2(E), \mathcal{O}_X(1))\geq s$ (see [9] Ch. II). Let K be a coherent subsheaf of E such that $0<r(K)<r$ and $E/K=G$ is torsion free. Then we can find a torsion free, coherent, quotient sheaf G' of F such that K is contained in $K'=\ker(F\otimes\mathcal{O}_X(Y)\to G'\otimes\mathcal{O}_X(Y))$ and $r(K)=r(K')$. Since G is torsion free, the natural homomorphism α of G to $G'\otimes\mathcal{O}_X(Y)$ is injective. Set $H=\operatorname{coker}(\alpha)$ and $I=(\kappa(G')/\kappa(G')\cap\mathcal{O}_Y(-n))\otimes\mathcal{O}_X(Y)$, then we have the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & G & \xrightarrow{\alpha} & G'\otimes\mathcal{O}_X(Y) & \longrightarrow & H & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & E & \longrightarrow & F\otimes\mathcal{O}_X(Y) & \longrightarrow & F_0\otimes\mathcal{O}_X(Y) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K & \longrightarrow & K' & \longrightarrow & I & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Assume that $\kappa(G')\cap\mathcal{O}_Y(-n)\neq 0$, that is, $r(I)=r-r(G')-1$. Since $r(F_0)=r-1$, $r(H)=r(G')$. By Lemma A.3, $c_1(G)=c_1(G')+r(G')Y-r(G')Y=c_1(G')$. By the property (2) for n_0, $G'\notin B_0$, whence $d(G, \mathcal{O}_X(1))=d(G', \mathcal{O}_X(1))>r(G')d(Y, \mathcal{O}_X(1))/r=r(G)d(E, \mathcal{O}_X(1))/r$. Next assume that $\kappa(G')\cap\mathcal{O}_Y(-n)=0$. Then $r(I)=r-r(G')$ and so $r(H)=r(G')-1$. By Lemma A.3 again, $c_1(G)=c_1(G')+Y$ which implies that $d(G, \mathcal{O}_X(1))=d(G', \mathcal{O}_X(1))+d(Y, \mathcal{O}_X(1))$. Since F is semi-stable, $d(G', \mathcal{O}_X(1))\geq 0$. We see therefore that $d(G, \mathcal{O}_X(1))\geq d(Y, \mathcal{O}_X(1))>r(G)d(Y, \mathcal{O}_X(1))/r=r(G)d(E, \mathcal{O}_X(1))/r$. Thus E is μ-stable. q.e.d.

Remark A.4. If $\dim X=3$, then Proposition A.1 holds for $r=2$.

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