On the associated graded ring of a local Cohen-Macaulay ring

By

Judith D. SALLY*

(Communicated by Prof. Nagata Feb. 21, 1976)

It seldom happens that properties of a local ring $(R, \mathfrak{m})$ are carried over to its associated graded ring $G(R)$. Here we investigate when the property of being Cohen-Macaulay is transferred from R to $G(R)$.

First, we set up some notation. $(R, \mathfrak{m})$ denotes a local Cohen-Macaulay ring of dimension $d > 0$ and multiplicity $e(R)$. $G(R)$ denotes its associated graded ring. $G(R) = R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \cdots$. We denote the embedding dimension of R , that is the number of generators in a minimal basis of $\mathfrak{m}$, by $v(\mathfrak{m})$.

It is true that $d \leq v(\mathfrak{m}) \leq e(R) + d - 1$. The first inequality is the Krull principal ideal theorem and the second inequality is a result of Abhyankar [A]. We investigate whether $G(R)$ is Cohen-Macaulay in terms of $v(\mathfrak{m})$. If $v(\mathfrak{m}) = d$, R is regular and $G(R)$ is also regular. The fact that $G(R)$ is Cohen-Macaulay if $v(\mathfrak{m}) = d + 1$ is also well known (see, for example, [B-S]) but for the convenience of the reader we sketch a proof here. We may assume that R is a complete Cohen-Macaulay local ring and thus a homomorphic image of a $d + 1$ dimensional complete regular local ring S . The kernel C must be a height 1 unmixed, hence principal, ideal of S . Say $C = fS$. Then $G(R) = G(S)/\bar{f}G(S)$, where $\bar{f}$ is the initial form of f in $G(S)$. Since $G(S)$ is a polynomial ring over a field, $G(R)$ is Cohen-Macaulay.

We will show that if $v(\mathfrak{m}) = e(R) + d - 1$ then $G(R)$ is Cohen-Macaulay and that this is the only other case where $G(R)$ is Cohen-Macaulay for all such $(R, \mathfrak{m})$.

Theorem 1. *Assume that $R/\mathfrak{m}$ is an infinite field. Then there exist elements $x_1, x_2, \dots, x_d$ in $\mathfrak{m}$ such that $\mathfrak{m}^2 = (x_1, x_2, \dots, x_d)\mathfrak{m}$ if and only if $v(\mathfrak{m}) = e(R) + d - 1$.*

Proof. By the results of Northcott and Rees [N-R] on minimal reductions of an ideal, there exist elements $x_1, x_2, \dots, x_d$ in $\mathfrak{m}$ with $\mathfrak{m}^n = \mathfrak{x}\mathfrak{m}^{n+1}$ for some positive integer n , where $\mathfrak{x} = (x_1, x_2, \dots, x_d)$. Tensor the exact sequence

---

* The author was partially supported by NSF GP 29815

$$0 \rightarrow \mathfrak{m} \rightarrow R \rightarrow R/\mathfrak{m} \rightarrow 0$$

by $R/\mathfrak{x}$ to obtain the exact sequence:

$$0 \rightarrow \mathrm{Tor}_1^R(R/\mathfrak{m}, R/\mathfrak{x}) \rightarrow \mathfrak{m}/\mathfrak{x}\mathfrak{m} \rightarrow R/\mathfrak{x} \rightarrow R/\mathfrak{m} \rightarrow 0.$$

Let $\lambda_R(A)$ denote length of the R -module A . We have that $\lambda_{R/\mathfrak{m}}(\mathrm{Tor}_1^R(R/\mathfrak{m}, R/\mathfrak{x})) = d$ because $\mathfrak{x}$ is a regular sequence. By the properties of minimal reductions, $e(R) = \lambda_{R/\mathfrak{x}}(R/\mathfrak{x})$. Hence, $\lambda_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{x}\mathfrak{m}) = e(R) + d - 1$. From this it follows immediately that $v(\mathfrak{m}) \leq e(R) + d - 1$, and that $v(\mathfrak{m}) = e(R) + d - 1$ if and only if $\mathfrak{m}^2 = \mathfrak{x}\mathfrak{m}$.

Theorem 2. *If $v(\mathfrak{m}) = e(R) + d - 1$, then $G(R)$ is Cohen-Macaulay.*

Proof. We may assume that $R/\mathfrak{m}$ is an infinite field. By Theorem 1, there exist elements $x_1, x_2, \dots, x_d$ in $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{x}\mathfrak{m}$. We prove by induction on d , that $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_d$, the images of $x_1, x_2, \dots, x_d$ in $G(R)$, form a regular sequence. By [H-R], [M-R] this is sufficient to prove that $G(R)$ is Cohen-Macaulay. If $d=1$, we have $\mathfrak{m}^2 = x_1\mathfrak{m}$. $\bar{x}_1$ is not a zero divisor for $x_1y \in \mathfrak{m}^{t+1} = x_1\mathfrak{m}^t$, for any $t \geq 1$, implies that $y \in \mathfrak{m}^t$ because x_1 is not a zero divisor in R . Assume that $d \geq 1$. We first check that $\bar{x}_1$ is not a zero divisor in $G(R)$. If $x_1y \in \mathfrak{m}^{t+1} = (x_1, \dots, x_d)^t \mathfrak{m}$, where $t \geq 1$, we must show that $y \in \mathfrak{m}^t = (x_1, \dots, x_d)^{t-1} \mathfrak{m}$. $x_1y = g(x_1, \dots, x_d)x_1 + f(x_2, \dots, x_d)$, where $g(x_1, \dots, x_d)$ is a homogeneous polynomial of degree $t-1$ in $x_1, \dots, x_d$ with coefficients in $\mathfrak{m}$ and $f(x_2, \dots, x_d)$ is a homogeneous polynomial of degree t in $x_2, \dots, x_d$ with coefficients in $\mathfrak{m}$. Hence $hx_1 = f(x_2, \dots, x_d)$ with $h = y - g(x_1, \dots, x_d)$. Since $x_1, \dots, x_d$ is a regular sequence in R , the associated graded ring of R with respect to the ideal $\mathfrak{x} = (x_1, \dots, x_d)$ is a polynomial ring in d variables over $R/\mathfrak{x}$. Hence $h \in (x_1, \dots, x_d)^t$ and $y \in (x_1, \dots, x_d)^{t-1} \mathfrak{m}$. Pass to the Cohen-Macaulay ring R/x_1R . $e(R/x_1R) = e(R)$, $\dim R/x_1R = d-1$ and $v(\mathfrak{m}/x_1R) = v(\mathfrak{m}) - 1$. The induction hypothesis applies to R/x_1R so that $\bar{x}_2, \dots, \bar{x}_d$ form a regular sequence in $G(R/x_1R) = G(R)/\bar{x}_1G(R)$. Hence $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_d$ is a regular sequence in $G(R)$.

Remark. The case $d=1$ is also proved in [D].

Corollary 3. *If $e(R) \leq 3$, then $G(R)$ is Cohen-Macaulay.*

To show that $v(\mathfrak{m}) = d, d+1$ and $e(R) + d - 1$ are the only cases where $G(R)$ is Cohen-Macaulay for all local Cohen-Macaulay rings R , we construct a 1-dimensional local Cohen-Macaulay ring $(R, \mathfrak{m})$ with $v(\mathfrak{m}) = d+2 = e(R) + d - 2$ and with $G(R)$ not Cohen-Macaulay. M. Hochster showed me that if k is a field, then $k[[t^5, t^8, t^{27}]]$ is an example of a 1-dimensional complete local domain R with $G(R)$ not Cohen-Macaulay. The following examples show that such a domain can be constructed for any multiplicity $e(R) \geq 4$. Let k be a field. Let e be any integer ≥ 4 . The conductor of the numerical semigroup generated by e and $e+1$ is $(e-1)e$. Let $R = k[[t^e, t^{e+1}, t^{(e-1)e-1}]]$. Then $\mathfrak{m} = (t^e, t^{e+1}, t^{(e-1)e-1})$ and $v(\mathfrak{m}) = 3$. Now $t^{(e-1)e-1}\mathfrak{m} \subseteq \mathfrak{m}^3$; for $t^{(e-1)e-1}t^e = (t^{e+1})^{e-1}$, $t^{(e-1)e-1}t^{e+1} = t^{e^2}$ and $(t^{(e-1)e-1})^2 = (t^e)^{e-1}(t^{e+1})^{e-2}$. Thus the maximal homogeneous ideal of $G(R)$ belongs to 0 in $G(R)$. In particular, if $e=4$, then $R = k[[t^4, t^5,$

$t^{11}]$ has $v(\mathfrak{m}) = 3 = d + 2 = e(R) + d - 2$ and $G(R)$ is not Cohen-Macaulay.

DEPARTMENT OF MATHEMATICS
NORTHWESTERN UNIVERSITY
EVANSTON, ILLINOIS 60201
U.S.A.

References

- [A] S. S. Abhyankar, Local rings of high embedding dimension, *Amer. J. Math.* **89** (1967), 1073-1077.
- [B-S] M. Boratynski and J. Swiecicka, The Hilbert-Samuel function of a Cohen-Macaulay ring, preprint.
- [D] E. D. Davis, On the geometric interpretation of seminormality, preprint.
- [H-R] M. Hochster and L. J. Ratliff, Jr., Five theorems on Macaulay rings, *Pac. J. of Math.* **44** (1973), 147-172.
- [M-R] J. Matijevic and P. Roberts, A conjecture of Nagata on graded Cohen-Macaulay rings, *J. Math. Kyoto Univ.* **14** (1974), 125-128.
- [N-R] D. G. Northcott and D. Rees, Reductions of ideals in local rings, *Proc. Camb. Phil. Soc.* **50** (1954), 145-158.