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# A complex associated with a symmetric matrix

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## Introduction.

In this paper let $R$ be a commutative ring in which 2 is invertible and let $X$ be a symmetric $n\times n$ matrix with entries in $R(n\geqq 2)$. We denote by $\mathfrak{a}$ the ideal of $R$ generated by all the $(n-1)\times(n-1)$ minors of $X$. It is known that depth $\mathfrak{a}\leqq 3$ if $R$ is Noetherian (c.f. Theorem 1, [4]). (Here depth $\mathfrak{a}$ denotes the common length of maximal $R$-sequences contained in $\mathfrak{a}$.)

The purpose of this note is to construct a complex $C^X$ associated with $X$ of finitely generated free $R$-modules of length 3 and to show that $C^X$ provides the resolution of $R/\mathfrak{a}$ if $R$ is Noetherian and if depth $\mathfrak{a}=3$.

T. H. Gulliksen and O. G. Negård [3] constructed an elegant complex $\tilde{C}^X$ associated with $X$ but this complex is too big to provide the resolution of $R/\mathfrak{a}$ in the present case. Nevertheless we will find that $C^X$ is a direct summand of $\tilde{C}^X$ as a complex and this is available to study some properties of $C^X$.

## 1. Construction of $C^X$.

In the following let $M_n(R)$ denote the free $R$-module of all the $n\times n$ matrices with entries in $R$ and we put

$A=\{S\in M_n(R)/{}^tS=S\}$,

$B=\{T\in M_n(R)/{}^tT=-T\}$,

$C=\{M\in M_n(R)/\mathrm{Trace}\,M=0\}$.

Note that $A, B$ and $C$ are finitely generated free $R$-modules of rank $\binom{n+1}{2}$, $\binom{n}{2}$ and $n^2-1$ respectively. We denote by $Y$ the matrix of cofactors of $X$.

*Construction of* $C^X$.

We put $C_3{}^X=B$, $C_2{}^X=C$, $C_1{}^X=A$, $C_0{}^X=R$, and $C_i{}^X=(0)$ if $i<0$ or $i>3$. For the differential $d$ we define

$d_3(T)=XT$ for $T\in B$,

$d_2(M)=MX+{}^t(MX)$ for $M\in C$,

$d_1(S)=\mathrm{Trace}\,YS$ for $S\in A$.

It is easy to check that $C^X$ is actually a complex with $H_0(C^X)=R/\mathfrak{a}$.

First we note

**Lemma 1.** *Suppose that $f:R\to R'$ is a homomorphism of rings and let $X'$ denote the matrix obtained by applying $f$ to the entries of $X$. Then*

$C^{X'} \cong C^X \otimes_R R'$.

## 2. Relation between $C^X$ and $\tilde{C}^X$.

We recall the construction of $\tilde{C}^X$ ([3]): $\tilde{C}_4{}^X = \tilde{C}_0{}^X = R$, $\tilde{C}_3{}^X = \tilde{C}_1{}^X = M_n(R)$, and $\tilde{C}_2{}^X$ is given by the homology of the complex $R \xrightarrow{i} M_n(R) \oplus M_n(R) \xrightarrow{j} R$ where $i(r) = (rI, rI)$ for $r \in R$ and $j(M, N) = \mathrm{Trace}(M - N)$ for $M, N \in M_n(R)$. (Here $I$ denotes the $n \times n$ unit matrix.) The differential $d$ is defined as follows:

$d_4(r) = rY$ for $r \in R$,

$d_3(M) =$ the class of $(XM, MX)$ for $M \in M_n(R)$,

$d_2$ (the class of $(M, N)) = MX - XN$ for $M, N \in M_n(R)$ with $\mathrm{Trace}\, M = \mathrm{Trace}\, N$,

$d_1(M) = \mathrm{Trace}\, YM$ for $M \in M_n(R)$.

The following is the key lemma of this paper:

**Lemma 2.** *$C^X$ is a direct summand of $\tilde{C}^X$ as a complex.*

*Proof.* If we define the $R$-linear maps $p: M_n(R) \to B$, $q: \tilde{C}_2{}^X \to C$, $r: M_n(R) \to A$ and $s: C \to \tilde{C}_2{}^X$ by

$p(M) = (M - {}^tM)/2$ for $M \in M_n(R)$,

$q$ (the class of $(M, N)) = (M - {}^tN)/2$ for $M, N \in M_n(R)$ with $\mathrm{Trace}\, M = \mathrm{Trace}\, N$,

$r(M) = (M + {}^tM)/2$ for $M \in M_n(R)$,

$s(M) =$ the class of $(M, -{}^tM)$ for $M \in C$,

then some straightforward calculation shows that the following diagram is commutative:

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & R & \longrightarrow & 0 \\
 & & | & & \downarrow{\scriptstyle i_B} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle i_A} & & \| & & \\
0 & \longrightarrow & R & \to & M_n(R) & \to & \tilde{C}_2{}^X & \to & M_n(R) & \to & R & \longrightarrow & 0 \\
 & & | & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} & & \| & & \\
0 & \longrightarrow & 0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & R & \longrightarrow & 0
\end{array}
$$

where $i_B: B \to M_n(R)$ and $i_A: A \to M_n(R)$ denote the inclusion maps. Of course $p \circ i_B = 1_B$, $q \circ s = 1_C$ and $r \circ i_A = 1_A$.

**Corollary.** *Suppose that $E$ is an arbitrary $R$-module. Then* $\mathfrak{a}^2 H(C^X \otimes_R E) = (0)$.

*Proof.* $H(C^X \otimes_R E)$ is an $R$-submodule of $H(\tilde{C}^X \otimes_R E)$ by Lemma 2 and hence the assertion is obvious since $\mathfrak{a}^2 H(\tilde{C}^X \otimes_R E) = (0)$ (c.f. Lemma 2 and 4, [3]).

**Corllary.** *Let $E$ be an $R$-module and suppose that* $\mathrm{Supp}_R E \cap \mathrm{Spec}\, R/\mathfrak{a} = \phi$. *Then we have* $H(C^X \otimes_R E) = (0)$.

**3. Main results.**

**Theorem.** *Suppose that* $R$ *is Noetherian and let* $E$ *be a finitely generated* $R$*-module such that* $E\neq\mathfrak{a}E$. *Then we have*

$$\operatorname{depth}(\mathfrak{a}, E)+\max\{i\in \boldsymbol{Z}/H_i(C^X\otimes_R E)\neq(0)\}=3$$

*where* $\operatorname{depth}(\mathfrak{a}, E)$ *denotes the common length of maximal* $E$*-sequences contained in* $\mathfrak{a}$.

*Proof.* We will prove by induction on $t=\operatorname{depth}(\mathfrak{a}, E)$.

($t=0$) We must show that $H_3(C^X\otimes_R E)\neq(0)$. By the assumption, $\mathfrak{a}e=(0)$ for some $0\neq e\in E$. Since $H_3(C^X\otimes_R E)$ contains $H_3(C^X\otimes_R Re)$ as an $R$-submodule, we may assume that $E=Re$. Moreover, because $C^X\otimes_R E\cong(C^X\otimes_R\bar{R})\otimes_{\bar{R}}E$ where $R=\bar{R}/(0):E$, we may assume further that $(0):E=(0)$. Thus it suffices to prove in case $E=R$ and $\mathfrak{a}=(0)$. Now suppose that $H_3(C^X)=(0)$. Then $H_3(C^X\otimes_R R_\mathfrak{p})=(0)$ for every $\mathfrak{p}\in\operatorname{Spec}R$ and so, to obtain a contradiction, we can assume that $(R, \mathfrak{m})$ is a local ring of $\operatorname{depth}R=0$ after localization at $\mathfrak{p}$ for some $\mathfrak{p}\in\operatorname{Ass}R$. In this situation, $M=\operatorname{Coker}d_3$ is a free $R$-module since $\operatorname{hd}_R M\leqq 1$ and consequently $d_3$ is a split monomorphism. Thus, applying $\otimes_R R/\mathfrak{m}$, we can assume that $R$ is a field and so we have ${}^tPXP=\begin{pmatrix} D & 0\\ 0 & 0\end{pmatrix}$ for some invertible $n\times n$ matrix $P$ where $D$ is an $s\times s$ diagonal matrix ($s=\operatorname{rank}X$). Now consider an $n\times n$ matrix $T=\begin{pmatrix} 0 & 0 & \\ 0 & 0 & 1\\ & -1 & 0\end{pmatrix}$. Then $T\in B$ and $({}^tPXP)T=0$ since $s\leqq n-2$. (Recall that $\mathfrak{a}=(0)$.) Therefore $X(PT{}^tP)=0$ with $0\neq PT{}^tP\in B$ —— this is the required contradiction.

($t>0$) Let $f$ be an $E$-regular element contained in $\mathfrak{a}^2$. Applying $C^X\otimes_R$ to the exact sequence $0\to E\xrightarrow{f} E\to E/fE\to 0$, we have an exact sequence

$$0\to C^X\otimes_R E\xrightarrow{f} C^X\otimes_R E\to C^X\otimes_R E/fE\to 0$$

of complexes. This yields an exact sequence of homologies

$$(*)\qquad 0\to H_i(C^X\otimes_R E)\to H_i(C^X\otimes_R E/fE)\to H_{i-1}(C^X\otimes_R E)\to 0$$

for every $i\in\boldsymbol{Z}$ by the corollary of Lemma 2, since $f\in\mathfrak{a}^2$. By the hypothesis of induction, we have known that $(t-1)+\max\{i\in\boldsymbol{Z}/H_i(C^X\otimes_R E/fE)\neq(0)\}=3$ and so the result follows from (*).

**Corollary.** *Suppose that* $R$ *is Noetherian and assume that* $\operatorname{depth}\mathfrak{a}=3$. *Then* $C^X$ *provides a resolution of* $R/\mathfrak{a}$.

For a Macaulay local ring $(R, \mathfrak{m})$ we put $r(R)=\dim_{R/\mathfrak{m}}\operatorname{Ext}^s_R(R/\mathfrak{m}, R)$ ($s=\dim R$) and call it the *type* of $R$. It is known that $R$ is a *Gorenstein* ring if and only if $r(R)=1$ (c.f. [**2**]).

**Corollary.** *Let* $k$ *be a field of* $\operatorname{ch}k\neq 2$ *and let* $R=k[\{X_{ij}\}_{1\leqq i\leqq j\leqq n}]$ *be* *a polynomial ring. We put* $X=(X_{ij})_{1\leq i,j\leq n}(X_{ij}=X_{ji}$ *if* $i>j)$. *Then* $C^X$ *provides a resolution of* $R/\mathfrak{a}$ *with* $r(R_{\mathfrak{m}}/\mathfrak{a}R_{\mathfrak{m}})=\binom{n}{2}$ *where* $\mathfrak{m}$ *denotes the maximal ideal* $(\{X_{ij}\}_{1\leq i\leq j\leq n})$ *of* $R$. *In particular* $R/\mathfrak{a}$ *is a Gorenstein ring if and only if* $n=2$.

*Proof.* By [4], we know that $R/\mathfrak{a}$ is a Macaulay ring of depth $\mathfrak{a}=3$. Thus $C^X\otimes_R R_{\mathfrak{m}}$ is a minimal free resolution of $R_{\mathfrak{m}}/\mathfrak{a}R_{\mathfrak{m}}$ in this case and consequently we know that $r(R_{\mathfrak{m}}/\mathfrak{a}R_{\mathfrak{m}})=\binom{n}{2}$ (c. f. Lemma 3. 5, [1]). Therefore $R_{\mathfrak{m}}/\mathfrak{a}R_{\mathfrak{m}}$ is a Gorenstein ring if and only if $n=2$ and it is known that $R/\mathfrak{a}$ is a Gorenstein ring *globally* if $R_{\mathfrak{m}}/\mathfrak{a}R_{\mathfrak{m}}$ is a Gorenstein local ring (c.f. Theorem, [1]).

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