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Growth properties of solutions of second order elliptic differential equations

By

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Introduction

In this paper we consider the equation

$$-\sum_{j,k=1}^{n} D_j a_{jk}(x) D_k u - q(x)u + p(x)u = 0 \tag{1}$$

in an exterior domain $\Omega \subset \mathbf{R}^n$, where $D_j = \partial_j + ib_j(x)$ with $\partial_j = \partial/\partial x_j$ and $i = \sqrt{-1}$, and the matrix $(a_{jk}(x))$ is uniformly positive definite in $x \in \Omega$ (the precise condition on the coefficients will be given later). We assume that $a_{jk}(x) \to \delta_{jk}$ (Kronecker's delta) as $|x| \to \infty$, that $\partial_k b_j(x) - \partial_j b_k(x)$ and $p(x)$ behave like $o(|x|^{-1})$ as $|x| \to \infty$ and that there exist some constants $0 < \gamma_0 < 1$, $\lambda_0 > 0$ and $r_0 > 0$ such that the domain $B(r_0) = \{x;\ |x| > r_0\}$ is included in Ω and

$$2\gamma_0 \Big(\sum_{j,k} a_{jk}(x)\tilde{x}_j \tilde{x}_k\Big) q(x) + |x| \sum_{j,k} \tilde{x}_j a_{jk}(x) \partial_k q(x) \geq \lambda_0 \quad \text{for} \quad x \in B(r_0), \tag{2}$$

where $\tilde{x} = x/|x|$. The main purpose of the present paper is to derive a growth estimate at infinity of solutions $u(x)$ of equation (1), from which will follow the uniqueness of L^2-solutions of (1).

Equations of the form (1) appear frequently in applications. In particular, if we assume that $a_{jk}(x) = \delta_{jk}$ and $q(x) = \lambda - c(x)$, where $\lambda > 0$, then (1) becomes

$$-\sum_{j=1}^{n} D_j^2 u + (c(x) + p(x))u - \lambda u = 0, \tag{3}$$

which is the eigenvalue problem of the Schrödinger operator (in this case $n=3s$, s being the number of particles), where $c(x)+p(x)$ represents the potentials of interaction between the particles. By the diminishing condition at infinity $(b_j(x))$ and $p(x)$ represent especially the magnetic vector potentials and the short-range scalar potentials in the two-body problem, respectively. $q(x)=\lambda-c(x)$ satisfies inequality (2) with $0<\gamma_0<1$, $0<\lambda_0\leq 2\gamma_0\lambda$ and $r_0\gg 0$ if $c(x)$ satisfies one of the following three conditions:

(c.1) $c(x)=0$;

(c.2) $c(x)$ and $|x|\sum_j \tilde{x}_j\partial_j c(x)$ tend to zero as $|x|\to\infty$;

(c.3) $-2\gamma_0 c(x)\geq |x|\sum_j \tilde{x}_j\partial_j c(x)$ for $x\in B(r_0)$.

By (c.2) is given a class of long-range potentials in the two-body problem. (c.3) holds if $c(x)$ is a homogeneous function of degree $-2\gamma_0$. Thus some potentials appearing in the manybody problem satisfy (c.3). Thus, from the physical point of view, our problem of the present paper may be said to show the non-existence of the positive eigenvalues of the Schrödinger operator.

In the case where $c(x)$ satisfies (c.1) or (c.2), growth properties of solutions, hence the uniqueness of L^2-solutions, of equation (3), or some of its variants, have been investigated by many authors ([1] ~[4], [6], [8], [10], [11]). There are also several works ([1], [2], [12]~[14]) investigating the Schrödinger equation

$$-\Delta u+c(x)u-\lambda u=0 \quad (\lambda>0) \qquad \text{in } \Omega \tag{4}$$

with $c(x)$ satisfying (c.3). Among them, Uchiyama [14] gave an explicit formulation of the asymptotic estimates at infinity of solutions of (4) assuming that γ_0 in (c.3) satisfies the condition $\frac{1}{3}<\gamma_0\leq 1$. He also proved, as a special problem, that if $\Omega=\mathbf{R}^n$ and (c.3) is satisfied for all $x\in\mathbf{R}^n$, then the condition on γ_0 can be weakened to the condition $0<\gamma_0\leq 1$.

We shall deal with the "*exterior*" problem with γ_0 satisfying the condition $0<\gamma_0<1$. Our results will not exclude the case of $\Omega=\mathbf{R}^n$. Since equation (1) is rather general, it seems difficult to develop a theory including the case of $\gamma_0=1$. However, as we shall show later, for the simpler equation (4) we can obtain a result including the case of $\gamma_0=1$ (Theorem 1.3).

Our method is very close to those developed by Roze [10] and Éidus [3] for equations with short-range potentials. We use two equalities which follow from equation (1). Roughly speaking, the first equality is a consequence of the integration by parts of (1) multiplied by $|x|^\alpha \bar{u}$ and the second equality is similarly given from (1) multiplied by $|x|^\beta \sum_{j,k} \tilde{x}_j a_{jk}(x)\overline{D_k u}$. Combining these equalities, we obtain some convenient apriori estimates. In our proof a difficulty occurs in the existence of singularities of $q(x)$ which may spread out to infinity. In order to remove the influence of the singularities, we use not only condition (2) but also the first equality which includes the term

$$\int |x|^\alpha \Big(\sum_{j,k} a_{jk}(x) D_j u \overline{D_k u} - q(x)|u|^2\Big) dx.$$

One can find similar treatments also in Weidmann [12, 13], Agmon [1, 2] and Uchiyama [14].

§1. Notation and Results

First we shall list the notation which will be used freely in the sequel:

$\langle f\cdot g\rangle = \sum_{j=1}^{n} f_j g_j$ for $f=(f_1,\dots, f_n)$ and $g=(g_1,\dots, g_n)$;

$|f| = \langle f\cdot \bar{f}\rangle^{1/2}$ for $f\in \mathbf{C}^n$;

$[f]_j$ denotes the j-th component of $f\in \mathbf{C}^n$;

$x=(x_1,\dots, x_n)$ is a position vector in $\mathbf{R}^n$;

$r=|x|$ and $\tilde{x}=x/|x|$;

$S(t)=\{x;\ |x|=t\}$ for $t>0$;

$B(s, t)=\{x;\ s<|x|<t\}$ for $0<s<t$;

$B(t)=\{x;\ |x|>t\}$ for $t>0$;

$\partial_j=\partial/\partial x_j$, $\operatorname{grad} f=(\partial_1 f,\dots, \partial_n f)$ for scalar functions f and $\operatorname{div} g$

$=\partial_1 g_1+\cdots+\partial_n g_n$ for vector functions g;
$D_j=\partial_j+ib_j(x)$ and $D=(D_1,\dots, D_n)$, where $i=\sqrt{-1}$;
$A=(a_{jk}(x))$ and $A'_l=(\partial_l a_{jk}(x))$ $(j, k=1,\dots, n)$;
$B'_j=(b'_{jk}(x))$ $(k=1,\dots, n)$, where $b'_{jk}=\partial_j b_k(x)-\partial_k b_j(x)$;
$\Phi=<\tilde{x}\cdot A\tilde{x}>$;
$\varepsilon(r)$ ro $\varepsilon_j(r)$ $(j=1, 2,\dots)$ denotes a positive function which tends to zero as $r\to\infty$;
$L^2(G)$ denotes the class of square integrable functions in the domain G of $\mathbf{R}^n$;
$H^j(G)$ $(j=1, 2)$ denotes the class of L^2-functions in G such that all distribution derivatives up to j belongs to $L^2(G)$;
L^2_{loc} and H^j_{loc} denote the class of locally L^2- and H^j-functions in Ω, respectively;
C^j denotes the class of j-times continuously differentiable functions;
Q_μ $(\mu>0)$ denotes the class of functions $f(x)$ satisfying the "Stummel condition":

$$\begin{cases} \sup_{x\in\Omega}\int_{|x-y|<1} |f(y)|^2|x-y|^{-n+4-\mu}\,dy<\infty & \text{if } n\geq 4 \\ \sup_{x\in\Omega}\int_{|x-y|<1} |f(y)|^2 dy<\infty & \text{if } n\leq 3, \end{cases}$$

where $|x-y|<1$ means the domain $\Omega\cap\{y;\ |x-y|<1\}$.

Next we shall state the conditions required on the coefficients of the differential equation (1).

(A1) The $a_{jk}(x)$ are real-valued C^2-functions in Ω; the $b_j(x)$ are real-valued C^1-functions in Ω; $q(x)$ is a real-valued function belonging to Q_μ for some $\mu>0$; $p(x)$ is a complex-valued function belonging to Q_μ.

(A2) $a_{jk}(x)=a_{kj}(x)$; there exists a constant $C>1$ such that

$$C^{-1}|\xi|^2\leq \sum_{j,k} a_{jk}(x)\xi_j\xi_k\leq C|\xi|^2 \qquad \text{for } x\in\Omega,\ \xi\in\mathbf{R}^n.$$

(A3) (i) $a_{jk}(x)\to\delta_{jk}$ as $|x|\to\infty$;
(ii) $\partial_l a_{jk}(x)$, $b'_{jk}(x)$ and $p(x)$ behave like $o(|x|^{-1})$ as $|x|\to\infty$;

(iii) $\partial_m \partial_l a_{jk}(x)$ behaves like $o(|x|^{-1})$ as $|x| \to \infty$.

(A4) There exist some constants $0 < \gamma_0 < 1$, $\lambda_0 > 0$ and $r_0 > 0$ such that $B(r_0) \subset \Omega$ and for $x \in B(r_0)$

$$2\gamma_0 q(x) + |x| \Phi^{-1} < \tilde{x} \cdot A \operatorname{grad} q(x) > \geq \lambda_0.$$

(A5) The unique continuation property holds.

Remark 1.1. If $p(x)$ and $q(x)$ satisfy a Hölder condition except at their singularities and if each connected component of the domain where $p(x)$ and $q(x)$ are regular extends to infinity, then (A5) is satisfied e.g., by a theorem of Landis [7] or Protter [9].

Remark 1.2. If $q(x) \to \lambda > 0$ and $< \tilde{x} \cdot A \operatorname{grad} q(x) > = o(|x|^{-1})$ as $|x| \to \infty$, then (A4) follows. In this case, the following all results can be obtained without (A3, iii).

Now our main purpose is to show the following theorem under the above conditions on the coefficients:

Theorem 1.1. *If u is a not identically vanishing solution of* (1) *in Ω, then we have for any $\gamma > \gamma_0$*

$$\liminf_{t \to \infty} t^{\gamma} \int_{S(t)} \{|q(x)||u|^2 + |< \tilde{x} \cdot ADu >|^2\} dS = \infty. \tag{1.1}$$

As a consequence of Theorem 1.1, we can prove the following theorem which is sometimes more convenient for applications:

Theorem 1.2. *Let u be a solution of* (1) *which also satisfies the inequality*

$$\int_{\Omega} (1 + |x|)^{\gamma - 1} |u(x)|^2 dx < \infty \quad \textit{for some} \quad \gamma > \gamma_0. \tag{1.2}$$

Then u must identically vanish in Ω.

According to this theorem the uniqueness of L^2-solutions of (1) follows. Namely, choosing $\gamma = 1$ in (1.2), we have the

Corollary 1.1. *Let u be a solution of* (1) *which also belongs to*

$L^2(\Omega)$. *Then* $u=0$ *in* Ω.

Remark 1.3. In this paper, by a solution u of equation (1) is meant an H^2_{loc}-, hence L^2_{loc}-function which satisfies (1) in the distribution sense in Ω.

Next we consider the following special equation

$$(1.3) \qquad -\Delta u - q(x)u = 0 \quad \text{in} \quad \Omega,$$

where Δ is the Laplacian and $q(x) \in Q_\mu$ and satisfies the following condition:

(A4)′ There exist constants $0<\gamma_0 \le 1$, $\lambda_0>0$ and $r_0>0$ such that $B(r_0) \subset \Omega$ and for $x \in B(r_0)$

$$2\gamma_0 q(x) + |x| < \tilde{x} \cdot \operatorname{grad} q(x) > \ge \lambda_0.$$

In this case we can change the result of Theorem 1.1 as follows:

Theorem 1.3. *If* u *is a not identically vanishing solution of* (1.3) *in* Ω, *then we have*

$$(1.4) \qquad \liminf_{t \to \infty} t^{\gamma_0} \int_{S(t)} \{(1+|q(x)|)|u|^2 + |<\tilde{x} \cdot \operatorname{grad} u>|^2\} dS > 0.$$

Remark 1.4. Making use of this theorem, we can also obtain a result corresponding to Theorem 1.2: *Let* u *be a solution of* (1.3) *which also satisfies the inequality*

$$(1.5) \qquad \int_\Omega (1+|x|)^{\gamma_0 - 1} |u(x)|^2 dx < \infty.$$

Then $u=0$ *in* Ω (cf. Uchiyama [14], Theorem 2.1).

We shall prove Theorem 1.1 in §3, Theorem 1.2 in §4 and Theorem 1.3 in §5. §2 will be devoted to obtain apriori estimates related to equation (1).

§2. Preliminaries

Let u be a solution of (1):

$$(2.1)\qquad -<D\cdot ADu>-qu+pu=0.$$

Let $\rho(r)$ be a real-valued, non-decreasing C^3-function of $r>r_0$, and put

$$(2.2)\qquad v(x)=e^{\rho(r)}u(x)\qquad (r=|x|).$$

Then it readily follows from (2.1) that v satisfies in $B(r_0)$ the equation

$$(2.3)\qquad -<D\cdot ADv>+2\rho'<\tilde{x}\cdot ADv>-\tilde{q}v+\tilde{p}v=0;$$

$$(2.4)\qquad \tilde{q}=q+\Phi\Big(\rho'^2-\rho''-\frac{n-1}{r}\rho'\Big),$$

$$(2.5)\qquad \tilde{p}=p+\Big(\mathrm{div}\,(A\tilde{x})-\frac{n-1}{r}\Phi\Big)\rho',$$

where $\rho'=d\rho/dr$ and $\rho''=d^2\rho/dr^2$.

We shall first prove two equalities which are satisfied for solutions v of equation (2.3).

Proposition 2.1. *Let $\psi(x)$ be a real-valued C^1-function of $x\in B(r_0)$. Then we have for $t>s>r_0$*

$$(2.6)\qquad \int_{B(s,t)}\psi(<ADv\cdot\overline{Dv}>-\tilde{q}|v|^2)dx$$
$$=\Big[\int_{S(t)}-\int_{S(s)}\Big]\psi\,\mathrm{Re}[<\tilde{x}\cdot ADv>\bar{v}]dS$$
$$-\int_{B(s,t)}\mathrm{Re}[<(\mathrm{grad}\,\psi+2\rho'\psi\tilde{x})\cdot ADv>\bar{v}-\psi\tilde{p}|v|^2]dx,$$

where $\mathrm{Re}[f]$ *means the real part of* f.

Proof. (2.6) can be obtained by the integration by parts of (2.3) multiplied by $\psi\bar{v}$. q.e.d.

Proposition 2.2. *Let γ be a real number. Then we have for $t>s>r_0$*

(2.7) $$\left[\int_{S(t)}-\int_{S(s)}\right] r^{\gamma}\left\{\Phi^{-1}|<\tilde{x}\cdot ADv>|^2-\frac{1}{2}(<ADv\cdot\overline{Dv}>-\tilde{q}|v|^2)\right\}ds$$

$$=\int_{B(s,t)} r^{\gamma-1}\{(1-\gamma_0)<ADv\cdot\overline{Dv}>-(1-\gamma)\Phi^{-1}|<\tilde{x}\cdot ADv>|^2\}dx$$

$$+\frac{1}{2}\int_{B(s,t)} r^{\gamma-1}(2\gamma_0\tilde{q}+r\Phi^{-1}<\tilde{x}\cdot A\operatorname{grad}\tilde{q}>)|v|^2dx$$

$$-\frac{1}{2}\int_{B(s,t)}\Psi(<ADv\cdot\overline{Dv}>-\tilde{q}|v|^2)dx$$

$$+\int_{B(s,t)} 2\rho' r^{\gamma}\Phi^{-1}|<\tilde{x}\cdot ADv>|^2dx$$

$$+\int_{B(s,t)} r^{\gamma}\Phi^{-1}(K_1+K_2+K_3+K_4)dx,$$

where

(2.8) $$\Psi=r^{\gamma-1}\{\gamma-2\gamma_0+r\Phi^{-1}\operatorname{div}(A\tilde{x})+r<\tilde{x}\cdot A\operatorname{grad}\Phi^{-1}>\},$$

(2.9) $$K_1=\operatorname{Re}\sum_l [ADv]_l(<\tilde{x}\cdot A'_l\overline{Dv}>-i<\tilde{x}\cdot AB'_l\bar{v}>)$$

$$-\frac{1}{2}\operatorname{Re}\sum_l [A\tilde{x}]_l<A'_l\overline{Dv}\cdot Dv>,$$

(2.10) $$K_2=r^{-1}(|ADv|^2-\Phi<ADv\cdot\overline{Dv}>),$$

(2.11) $$K_3=\operatorname{Re}[\Phi<\operatorname{grad}\Phi^{-1}\cdot ADv><\tilde{x}\cdot A\overline{Dv}>],$$

(2.12) $$K_4=\operatorname{Re}[\tilde{p}v<\tilde{x}\cdot A\overline{Dv}>].$$

Proof. We multiply the both sides of (2.3) by $<\tilde{x}\cdot ADv>$ and take the real parts. Then we have

(2.13) $$-\operatorname{Re}[(<D\cdot ADv>+\tilde{q}v)<\tilde{x}\cdot A\overline{Dv}>]+2\rho'|<\tilde{x}\cdot ADv>|^2+K_4=0.$$

The first term of the left side becomes

$$-\operatorname{Re}[(<D\cdot ADv>+\tilde{q}v)<\tilde{x}\cdot A\overline{Dv}>]$$

$$=-\operatorname{Re}\operatorname{div}[ADv<\tilde{x}\cdot A\overline{Dv}>-\frac{1}{2}A\tilde{x}(<ADv\cdot\overline{Dv}>-\tilde{q}|v|^2)]$$

$$+(1-\gamma_0)r^{-1}\Phi<ADv\cdot\overline{Dv}>-r^{-1}|<\tilde{x}\cdot ADv>|^2+K_2$$

$$+\frac{1}{2}(2\gamma_0 r^{-1}\Phi\tilde{q}+<\tilde{x}\cdot A\,\mathrm{grad}\,\tilde{q}>)|v|^2$$

$$-\frac{1}{2}\{\mathrm{div}\,(A\tilde{x})-2\gamma_0 r^{-1}\Phi\}(<ADv\cdot\overline{Dv}>-\tilde{q}|v|^2)+K_1.$$

Thus, multiplying (2.13) by $r^\gamma\Phi^{-1}$ and integrating on $B(s, t)$, we have (2.7). q.e.d.

Next we shall estimate the right side of (2.7). By (A3, i, ii) it follows that there exists a function $\varepsilon(r)$ verifying

(2.14) $$|\partial_j\Phi|+|\partial_j\Phi^{-1}|+|\mathrm{div}\,(A\tilde{x})-\frac{n-1}{r}\Phi|\le\varepsilon(r)r^{-1}.$$

Further it follows from (A3, i) that

(2.15) $$||ADv|^2-\Phi<ADv\cdot\overline{Dv}>|\le\varepsilon(r)\Phi<ADv\cdot\overline{Dv}>.$$

In view of these inequalities, using again (A3, i, ii) and noting the inequality

(2.16) $$\Phi^{-1}|<\tilde{x}\cdot ADv>|^2\le<ADv\cdot\overline{Dv}>,$$

we obtain the following

Lemma 2.1. *There exists a function* $\varepsilon_1(r)$ *such that*

(2.17) $$|K_1|+|K_2|+|K_3|\le\varepsilon_1(r)r^{-1}\Phi(<ADv\cdot\overline{Dv}>+|v|^2).$$

Since the inequality

(2.18) $$|\tilde{p}|\le\varepsilon(r)r^{-1}(1+\rho')$$

follows from (2.5), (A3, ii) and (2.14), we have similarly the

Lemma 2.2. *There exists a function* $\varepsilon_2(r)$ *such that*

(2.19) $$|K_4|\le\varepsilon_2(r)r^{-1}(|<\tilde{x}\cdot ADv>|^2+\Phi|v|^2)$$
$$+\varepsilon_2(r)\rho'(|<\tilde{x}\cdot ADv>|^2+r^{-2}\Phi|v|^2).$$

Lemma 2.3. *There exist a constant $C_1>0$ and a function $\varepsilon_3(r)$ such that*

(2.20) $$|\Psi|\leq C_1 r^{\gamma-1},$$

(2.21) $$|\mathrm{grad}\,\Psi|\leq\varepsilon_3(r)r^{\gamma-1}.$$

Proof. (2.20) follows (2.8) and (2.14). (2.21) follows if we note (A3, iii) also. q.e.d.

Applying the inequalities of this lemma to equality (2.6) of proposition 2.1 with $\psi=\Psi$, we obtain the following

Lemma 2.4. *There exist functions $\varepsilon_4(r)$ and $\varepsilon_5(r)$ such that*

(2.22) $$\int_{B(s,t)}\Psi(<ADv\cdot\overline{Dv}>-\tilde{q}|v|^2)dx$$

$$\leq\left[\int_{S(t)}-\int_{S(s)}\right]\Psi\,\mathrm{Re}\,[<\tilde{x}\cdot ADv>\bar{v}]dS$$

$$+\int_{B(s,t)}\varepsilon_4(r)r^{\gamma-1}(<ADv\cdot\overline{Dv}>+|v|^2)dx$$

$$+\int_{B(s,t)}\rho' r^{\gamma}\{\Phi^{-1}|<\tilde{x}\cdot ADv>|^2+(C_1^2+\varepsilon_5(r))r^{-2}|v|^2\}dx.$$

Lemma 2.5. *Let α and β be non-negative constants. Then there exists a function $\varepsilon_6(r)$ such that*

(2.23) $$\left[\int_{S(t)}-\int_{S(s)}\right](\alpha r^{\gamma-1}-\beta r^{\gamma-2})\Phi|v|^2 dS$$

$$\geq-\int_{B(s,t)}\{\alpha+(\alpha+\beta)\varepsilon_6(r)\}r^{\gamma-1}(\Phi^{-1}|<\tilde{x}\cdot ADv>|^2+|v|^2)dx.$$

Proof. We have for any $v\in H^1_{loc}$

$$\mathrm{div}\,[(\alpha r^{\gamma-1}-\beta r^{\gamma-2})A\tilde{x}|v|^2]=\mathrm{Re}\,[2(\alpha r^{\gamma-1}-\beta r^{\gamma-2})<\tilde{x}\cdot ADv>\bar{v}]$$

$$+\left\{\frac{\alpha(\gamma-1)}{r}-\frac{\beta(\gamma-2)}{r^2}+\left(\alpha-\frac{\beta}{r}\right)\Phi^{-1}\mathrm{div}\,(A\tilde{x})\right\}r^{\gamma-1}\Phi|a|^2.$$

Thus the integration on $B(s, t)$ and the Schwarz inequality yield (2.23). q.e.d.

Now, summarizing the above results, we can prove the following proposition:

Proposition 2.3. *Let γ be a constant such that $\gamma_0<\gamma\leq 1$. Then we have for $t>s>r_0$*

$$(2.24)\quad \left[\int_{S(t)}-\int_{S(s)}\right] r^{\gamma}\left\{\Phi^{-1}|<\tilde{x}\cdot ADv>|^2-\frac{1}{2}(<ADv\cdot\overline{Dv}>-\tilde{q}|v|^2)\right\}ds$$

$$+\frac{1}{2}\left[\int_{S(t)}-\int_{S(s)}\right]\{\Psi \operatorname{Re}[<\tilde{x}\cdot ADv>\bar{v}]+(\alpha r^{\gamma-1}-\beta r^{\gamma-2})\Phi|v|^2\}ds$$

$$\geq\int_{B(s,t)}(\gamma-\gamma_0-\varepsilon_7(r))r^{\gamma-1}<ADv\cdot\overline{Dv}>dx$$

$$+\int_{B(s,t)}\left\{\left(\frac{3}{2}-\varepsilon_2(r)\right)\rho'-\frac{\alpha}{2r}\right\}r^{\gamma}\Phi^{-1}|<\tilde{x}\cdot ADv>|^2dx$$

$$+\frac{1}{2}\int_{B(s,t)}r^{\gamma-1}(2\gamma_0\tilde{q}+r\Phi^{-1}<\tilde{x}\cdot A\operatorname{grad}\tilde{q}>)|v|^2dx$$

$$-\int_{B(s,t)}r^{\gamma-1}\left\{\left(\varepsilon_7(r)+\frac{\alpha}{2}\right)+\left(\frac{1}{2}C_1^2+\varepsilon_8(r)\right)\rho' r^{-1}\right\}|v|^2dx,$$

where $\varepsilon_7=\varepsilon_1+\varepsilon_2+\frac{1}{2}\varepsilon_4+\frac{1}{2}(\alpha+\beta)\varepsilon_6$ *and* $\varepsilon_8=\varepsilon_2+\frac{1}{2}\varepsilon_5$.

Proof. It follows from the condition on γ and (2.16) that the first term of the right of equality (2.7) is estimated from below by the integral

$$\int_{B(s,t)}(\gamma-\gamma_0)r^{\gamma-1}<ADv\cdot\overline{Dv}>dx$$

Then, applying Lemmas 2.1, 2, 4 and 5 to (2.7), we have (2.24). q.e.d.

Next, we shall derive another inequality which follows from the ellipticity of the differential operator $<D\cdot AD>$ and the condition that $\tilde{p}$, $\tilde{q}\in Q_{\mu}$ $(\mu>0)$.

Proposition 2.4. *Suppose that* ρ' *and* ρ'' *in equation* (2.3) *are bounded in* $r>r_0$. *Let* v *be a solution of* (2.3) *which also belongs to* $L^2(B(r_0))$. *Then we have*

$$\int_{B(r_0+1)}\{<ADv\cdot\overline{Dv}>+(|\tilde{q}|+|\tilde{p}|)|v|^2\}dx\leq C_2\int_{B(r_0)}|v|^2dx \tag{2.25}$$

for some $C_2>0$.

Proof. The assertion for equation (2.3) with $\rho'=0$ has been essentially proved in Ikebe-Kato [5]. The existence of the term $2\rho'<\tilde{x}\cdot ADv>$ causes no serious difficulty since we have for any $\varepsilon>0$

$$\int_{B(r_0+1)}2\rho'|<\tilde{x}\cdot ADv>\bar{v}|dx$$

$$\leq\varepsilon\int_{B(r_0+1)}<ADv\cdot\overline{Dv}>dx+C(\varepsilon)\int_{B(r_0+1)}|v|^2dx,$$

where $C(\varepsilon)$ is a positive constant depending on ε (cf., [5]; Lemmas 2 and 5). q.e.d.

§3. Proof of Theorem 1.1

In this section the proof of Theorem 1.1 will be given by means of a series of lemmas (cf., Roze [10] or Éidus [3]; §2).

Lemma 3.1. *Let* u *be a solution of* (1) *satisfying also the condition*

$$\liminf_{t\to\infty}t^{\gamma}\int_{S(t)}(|q||u|^2+|<\tilde{x}\cdot ADu>|^2)dS=0 \tag{3.1}$$

for some $\gamma>\gamma_0$. *Then we have for any* $m>0$

$$\int_{B(r_0)}r^m(|u|^2+|Du|^2)dx<\infty. \tag{3.2}$$

Proof. Without loss of generality we can assume that $\gamma_0<\gamma\leq 1$. We put $\rho(r)=0$, $\alpha=0$ and $\beta=1$ in inequality (2.24). Then $v=u$ and $\tilde{q}=q$. Since

$$\int_{B(s,t)} r^{\gamma-1}(2\gamma_0 q + r\Phi^{-1} <\tilde{x}\cdot A\,\mathrm{grad}\, q>)|u|^2 dx$$

$$\geq \lambda_0 \int_{B(s,t)} r^{\gamma-1}|u|^2 dx$$

by (A4), it follows from (2.16), (2.20) and (2.24) that

$$(3.3)\qquad \frac{1}{2}\int_{S(t)} r^{\gamma}(\Phi^{-1}|<\tilde{x}\cdot ADu>|^2 + |q||u|^2$$

$$+C_1 r^{-1}|<\tilde{x}\cdot ADu>\bar{u}| - r^{-2}|u|^2)dS$$

$$+\frac{1}{2}\int_{S(s)} r^{\gamma}(<ADu\cdot\overline{Du}> - q|u|^2$$

$$+C_1 r^{-1}|<\tilde{x}\cdot ADu>\bar{u}| + r^{-2}|u|^2)dS$$

$$\geq \int_{B(s,t)} r^{\gamma-1}(\gamma-\gamma_0-\varepsilon_7(r))<ADu\cdot\overline{Du}>dx$$

$$+\frac{1}{2}\int_{B(s,t)} r^{\gamma-1}(\lambda_0 - 2\varepsilon_7(r))|u|^2 dx.$$

By the Schwarz inequality

$$(3.4)\qquad C_1 r^{-1}|<\tilde{x}\cdot ADu>\bar{u}| - r^{-2}|u|^2 \leq \frac{1}{4}C_1^2|<\tilde{x}\cdot ADu>|^2.$$

We choose $r_1\geq r_0$ large so that for $r>r_1$

$$\gamma-\gamma_0-\varepsilon_7(r)\geq(\gamma-\gamma_0)/2,\quad \lambda_0-2\varepsilon_7(r)\geq\lambda_0/2.$$

Then, letting $t\to\infty$ in (3.3), we see by (3.1) and (3.4) that for $s>r_1$

$$(3.5)\qquad \int_{S(s)} r^{\gamma}(<ADu\cdot\overline{Du}> - q|u|^2)dS$$

$$+\int_{S(s)}(C_1 r^{\gamma-1}|<\tilde{x}\cdot ADu>\bar{u}| + r^{\gamma-2}|u|^2)dS$$

$$\geq \int_{B(s)} r^{\gamma-1}\left\{(\gamma-\gamma_0)<ADu\cdot\overline{Du}> + \frac{1}{2}\lambda_0|u|^2\right\}dx.$$

Integrating this inequality with respect to s from t to t_1, where $r_1 < t < t_1$, and using equality (2.6) with $\psi = r^\gamma$, we obtain

$$\int_t^{t_1} ds \int_{B(s)} r^{\gamma-1}\left\{(\gamma-\gamma_0) < ADu \cdot \overline{Du} > + \frac{1}{2}\lambda_0 |u|^2\right\} dx$$

$$\leq \left[\int_{S(t_1)} + \int_{S(t)}\right] r^\gamma | < \tilde{x} \cdot ADu > \bar{u} | dS$$

$$+ \int_{B(t,t_1)} r^{\gamma-1}\{(C_1+\gamma)| < \tilde{x} \cdot ADu > \bar{u}| + (r^{-1} + r|p|)|u|^2\} dx.$$

Inequality (3.5) implies that

$$\liminf_{t_1 \to \infty} t_1^\gamma \int_{S(t_1)} | < \tilde{x} \cdot ADu > \bar{u}| dS = 0.$$

Hence

$$\int_{B(t)} (r-t) r^{\gamma-1}\left\{(\gamma-\gamma_0) < ADu \cdot \overline{Du} > + \frac{\lambda_0}{2} |u|^2\right\} dx$$

$$\leq \int_{S(t)} r^\gamma (| < \tilde{x} \cdot ADu > |^2 + |u|^2) dS$$

$$+ (C_1 + \gamma + \varepsilon(t)) \int_{B(t)} r^{\gamma-1}(| < \tilde{x} \cdot ADu > |^2 + |u|^2) dx < \infty.$$

Repeating the integration with respect to t, we see that the assertion of the lemma is valid for arbitrary $m > 0$. q.e.d.

Lemma 3.2. *Let u be a solution of* (1) *satisfying also condition* (3.1) *for some $\gamma > \gamma_0$. Then we have for any $k > 0$*

(3.6) $$\int_{B(r_0)} e^{kr^{1-\nu}} |u|^2 dx < \infty,$$

where ν is a constant such that $0 < \nu < 1$.

Proof. Let $\rho(r) = m \log r$ $(m \geq n)$, $0 < \alpha \leq \min\{\lambda_0/3, 1\}$ and $\beta = 0$ in inequality (2.24). Then $v = r^m u$, $\rho' = \frac{m}{r}$ and $\tilde{q} = q + \Phi \frac{m(m-n+2)}{r^2}$. It follows that there exists a function $\varepsilon_9(r)$ verifying

$$2\gamma_0\tilde{q}+r\Phi^{-1}<\tilde{x}\cdot A\,\mathrm{grad}\,\tilde{q}>\geq\lambda_0-2(1-\gamma_0+\varepsilon_9(r))\frac{m(m-n+2)}{r^2}.$$

If we choose $r_2\geq r_0$ sufficiently large, then for $r>r_2$

$$\begin{cases}\gamma-\gamma_0-\varepsilon_7(r)\geq 0,\\[4pt] \left(\dfrac{3}{2}-\varepsilon_2(r)\right)\dfrac{m}{r}-\dfrac{\alpha}{2r}\geq 0,\\[4pt] \lambda_0-2\varepsilon_7(r)-\alpha\geq\lambda_0/2,\\[4pt] 1-\gamma_0+\varepsilon_9(r)\leq 2(1-\gamma_0),\\[4pt] \dfrac{1}{2}C_1^2+\varepsilon_8(r)\leq C_1^2.\end{cases}$$

Further, since $r^{\gamma/2}v\in L^2(B(r_0))$ by (3.2), it follows from (2.25) that

$$\liminf_{t\to\infty} t^{\gamma}\int_{S(t)}\{(1+|\tilde{q}|)|v|^2+|<\tilde{x}\cdot ADv>|^2\}dS=0. \tag{3.7}$$

Thus, in view of the above inequalities, letting $t\to\infty$ in (2.24), we have

$$\begin{aligned}(3.8)\quad &-\int_{S(s)}r^{\gamma}\left(2\Phi^{-1}|<\tilde{x}\cdot ADv>|^2-\frac{C_1}{r}|<\tilde{x}\cdot ADv>\bar{v}|+\frac{\alpha}{r}\Phi|v|^2\right)dS\\ &+\int_{S(s)}r^{\gamma}(<ADv\cdot\overline{Dv}>-\tilde{q}|v|^2)dS\\ &\geq\int_{B(s)}r^{\gamma-1}\left\{\frac{\lambda_0}{2}-4(1-\gamma_0)\frac{m(m-n+2)}{r^2}-2C_1^2\frac{m}{r^2}\right\}|x|^2dx\end{aligned}$$

for $s>r_2$. Multiply the both sides of (3.8) by s^{-2m} and integrate with respect to s from t to ∞, where $t>r_2$. Then we obtain

$$\begin{aligned}(3.9)\quad &-\int_{B(t)}r^{\gamma-2m}\left\{\Phi^{-1}|<\tilde{x}\cdot ADv>|^2+\left(\frac{\alpha}{r}-\left(\frac{C_1}{2r}\right)^2\right)\Phi|v|^2\right\}dx\\ &+\int_{B(t)}r^{\gamma-2m}(<ADv\cdot\overline{Dv}>-\tilde{q}|v|^2)dx\\ &\geq\int_t^{\infty}\left\{\frac{\lambda_0}{2}-4(1-\gamma_0)\frac{m(m-n+2)}{s^2}-2C_1^2\frac{m}{s^2}\right\}s^{-2m}ds\int_{B(s)}r^{\gamma-1}|v|^2dx.\end{aligned}$$

By (2.6)

$$(3.10)\qquad \int_{B(t)} r^{\gamma-2m}(<ADv\cdot\overline{Dv}> - \tilde{q}|v|^2)dx$$

$$= -\int_{S(t)} r^{\gamma-2m}\,\mathrm{Re}\,[<\tilde{x}\cdot ADv>\bar{v}]dS$$

$$-\int_{B(t)} r^{\gamma-2m-1}(\gamma\,\mathrm{Re}\,[<\tilde{x}\cdot ADv>\bar{v}] - r\,\mathrm{Re}\,[\tilde{p}]|v|^2)dx,$$

where

$$-\int_{S(t)} r^{\gamma-2m}\,\mathrm{Re}\,[<\tilde{x}\cdot ADv>\bar{v}]dS$$

$$= -\frac{1}{2}\frac{d}{dt}\int_{S(t)} r^{\gamma-2m}\Phi|v|^2 dS$$

$$+\frac{1}{2}\int_{S(t)} (\gamma-2m+r\Phi^{-1}\,\mathrm{div}\,(A\tilde{x}))r^{\gamma-2m-1}\Phi|v|^2 dS.$$

Hence, by (3.10),

$$\int_{B(t)} r^{\gamma-2m}(<ADv\cdot\overline{Dv}> - \tilde{q}|v|^2)dx$$

$$\leq -\frac{1}{2}\frac{d}{dt}\int_{S(t)} r^{\gamma-2m}\Phi|v|^2 dS$$

$$-\frac{1}{2}\int_{S(t)} (2m-\gamma-n+1-\varepsilon_{10}(r))r^{\gamma-2m-1}\Phi|v|^2 dS$$

$$+\frac{1}{2}\int_{B(t)} r^{\gamma-2m}\left\{\Phi^{-1}<|\tilde{x}\cdot ADv|>^2 + \left(\left(\frac{\gamma}{r}\right)^2 + \frac{1}{r}\varepsilon_{11}(r)\right)\Phi|v|^2\right\}dx.$$

Substituting this in (3.9), we obtain

$$(3.11)\qquad -\frac{1}{2}\int_{B(t)} r^{\gamma-2m}\left\{\Phi^{-1}|<\tilde{x}\cdot ADv>|^2\right.$$

$$\left. + \left(\frac{2\alpha}{r} - \left(\frac{C_1}{r}\right)^2 - \left(\frac{\gamma}{r}\right)^2 - \frac{1}{r}\varepsilon_{11}(r)\right)\Phi|v|^2\right\}dx$$

$$-\frac{1}{2}\left[\frac{d}{dt}\int_{S(t)} r^{\gamma-2m}\Phi|v|^2 dS + \frac{m}{t}\int_{S(t)} r^{-2m}\Phi|v|^2 ds\right] -$$

$$-\frac{1}{2}\int_{S(t)}(m-\gamma-n+1-\varepsilon_{10}(r))r^{\gamma-2m-1}\Phi|v|^2dS$$

$$\geq\int_t^{\infty}\left\{\frac{\lambda_0}{2}-4(1-\gamma_0)\frac{m(m-n+2)}{s^2}-2C_1^2\frac{m}{s^2}\right\}s^{-2m}dS\times$$

$$\times\int_{B(s)}r^{\gamma-1}|v|^2dx.$$

We fix arbitrary $k>0$ and $0<\nu<1$, and set $m=k(1-\nu)t^{1-\nu}$. Then for $r\geq t>r_3$, where $r_3\geq r_2$ is sufficiently large, we have

$$\begin{cases}\dfrac{\alpha}{r}-\left(\dfrac{C_1}{r}\right)^2-\left(\dfrac{\gamma}{r}\right)^2-\dfrac{1}{r}\varepsilon_{11}(r)\geq 0,\\ m-\gamma-n+1-\varepsilon_{10}(r)\geq 0,\\ \dfrac{\lambda_0}{2}-4(1-\gamma_0)\dfrac{m(m-n+2)}{r^2}-2C_1^2\dfrac{m}{r^2}\geq 0.\end{cases}$$

Hence, putting

$$F(t)=\int_{S(t)}r^{\gamma-2m}\Phi|v|^2dS=\int_{S(t)}r^{\gamma}\Phi|u|^2dS, \tag{3.12}$$

we have from (3.11)

$$\frac{d}{dt}F(t)+k(1-\nu)t^{-\nu}F(t)\leq 0 \quad \text{for} \quad t>r_3.$$

Therefore, for any $k>0$ and $0<\nu<1$

$$F(t)\leq Ce^{-kt^{1-\nu}},$$

where $C>0$ is independent of t. This implies (3.6) and the proof is completed. q.e.d.

Proof of Theorem 1.1. We shall prove the following assertion which is equivalent to Theorem 1.1; *Let u be a solution of* (1) *satisfying also condition* (3.1) *for some $\gamma>\gamma_0$. Then u must identically vanish in Ω.*

We return once more to inequality (2.24). We put $\rho(r)=kr^{1-\nu}$

and $\alpha=\beta=0$. Then $v=e^{kr^{1-\nu}}u$ and $\rho'=k(1-\nu)/r^{\nu}$. Since

$$\tilde{q}=q+\Phi\left\{\frac{k^2(1-\nu)^2}{r^{2\nu}}-\frac{k(1-\nu)(n-1-\nu)}{r^{1+\nu}}\right\},$$

it follows from (A4) that

$$2\gamma_0\tilde{q}+r\Phi^{-1}<\tilde{x}\cdot A\,\mathrm{grad}\,\tilde{q}>$$
$$\geq\lambda_0+2(\gamma_0-\nu-\varepsilon(r))\frac{k^2(1-\nu)^2}{r^{2\nu}}$$
$$+(1+\nu-2\gamma_0-\varepsilon(r))\frac{k(1-\nu)(n-1-\nu)}{r^{1+\nu}}.$$

If we choose $r_4\geq r_0$ sufficiently large, then for $r>r_4$

$$\begin{cases}\gamma-\gamma_0-\varepsilon_7(r)\geq 0,\\ \dfrac{3}{2}-\varepsilon_2(r)\geq 0,\\ \lambda_0-2\varepsilon_7(r)\geq\dfrac{\lambda_0}{2},\\ \dfrac{1}{2}C_1^2+\varepsilon_8(r)\leq C_1^2.\end{cases}$$

By Lemma 3.2 $r^{\gamma/2}v\in L^2(B(r_0))$. Thus, by means of inequality (2.25), we see

$$\text{(3.13)}\qquad \liminf_{t\to\infty} t^{\gamma}\int_{S(t)}\{(1+|\tilde{q}|)|v|^2+|Dv|^2\}dS=0.$$

Using the above inequalities and letting $t\to\infty$ in (2.24), we then have for $s>r_4$

$$\text{(3.14)}\quad -\frac{s^{\gamma}}{2}\int_{S(s)}\{2\Phi^{-1}|<\tilde{x}\cdot ADv>|^2-<ADv\cdot\overline{Dv}>$$
$$+\tilde{q}|v|^2-|\Psi<\tilde{x}\cdot ADv>\bar{v}|\}dS$$
$$\geq\frac{1}{2}\int_{B(s)}r^{\gamma-1}\left\{\frac{\lambda_0}{2}+2(\gamma_0-\nu-\varepsilon(r))\frac{k^2(1-\nu)^2}{r^{2\nu}}+\right.$$

$$+(1+\nu-2\gamma_0-\varepsilon(r))\frac{k(1-\nu)(n-1-\nu)}{r^{1+\nu}}-C_1^2\frac{k(1-\nu)}{r^{1+\nu}}\Big\}|v|^2dx.$$

Now we choose ν less than γ_0: $0<\nu<\gamma_0$. Then, obviously, there exists an $r_5\geq r_4$ such that for any $k\geq 1$ and $s>r_5$

$$\text{(3.15)}\qquad \int_{S(s)}\{2\Phi^{-1}|<\tilde{x}\cdot ADv>|^2-<ADv\cdot\overline{Dv}>$$
$$+\tilde{q}|v|^2-|\Psi<\tilde{x}\cdot ADv>\bar{v}|\}dS\leq 0.$$

Since

$$|<\tilde{x}\cdot ADv>|^2=e^{2kr^{1-\nu}}\Big\{|<\tilde{x}\cdot ADu>|^2$$
$$+\frac{2k(1-\nu)}{r^\nu}\Phi\,\mathrm{Re}\,[<\tilde{x}\cdot ADu>\bar{u}]+\frac{k^2(1-\nu)^2}{r^{2\nu}}\Phi^2|u|^2\Big\}$$

and

$$<ADv\cdot\overline{Dv}>=e^{2kr^{1-\nu}}\Big\{<ADu\cdot\overline{Du}>$$
$$+\frac{2k(1-\nu)}{r^\nu}\mathrm{Re}\,[<\tilde{x}\cdot ADu>\bar{u}]+\frac{k^2(1-\nu)^2}{r^{2\nu}}\Phi|u|^2\Big\},$$

We can write the left side of (3.15) in the form

$$e^{2ks^{1-\nu}}\{k^2M_1(s)+kM_2(s)+M_3(s)\},$$

where

$$M_1(s)=\frac{(1-\nu)^2}{s^{2\nu}}\int_{S(s)}\Phi|u|^2dS,$$

and $M_2(s)$ and $M_3(s)$ are independent of k. Suppose that $M_1(s)>0$ for some $s>r_5$. Then k can be chosen so large that (3.15) is nolonger valid. Hence $u=0$ in $B(r_5)$. By the unique continuation property (A5), it follows that $u=0$ in Ω. This concludes the proof.

q.e.d.

§4. Proof of Theorem 1.2

Suppose that a solution u of (1) satisfies condition (1.2) of Theorem 1.2. Then, by (2.25) of Proposition 2.4, it follows that

$$\liminf_{t\to\infty} t^{\gamma}\int_{S(t)}\{(1+|q|)|u|^2+|Du|^2\}dS=0.$$

This contradicts (1.1) of Theorem 1.1 and hence we see that $u=0$ in Ω. This concludes the proof.

§5. Proof of Theorem 1.3

Let u be a solution of the simpler equation (1.3). We put $\gamma=\gamma_0$ in Proposition 2.2. Then, noting that $K_1=K_2=K_3=K_4=0$ and $\Psi=(n-1-\gamma_0)r^{\gamma_0-1}$, and applying Proposition 2.1 with $\psi=(n-1-\gamma_0)r^{\gamma_0-1}$, we have

$$(5.1)\quad \left[\int_{S(t)}-\int_{S(s)}\right]r^{\gamma_0}\left\{|<\tilde{x}\cdot\text{grad}\, v>|^2-\frac{1}{2}(|\text{grad}\, v|^2-\tilde{q}|v|^2)\right\}dS$$
$$+\frac{n-1-\gamma_0}{2}\left[\int_{S(t)}-\int_{S(s)}\right]r^{\gamma_0-1}\,\text{Re}\,[<\tilde{x}\cdot\text{grad}\, v>\bar{v}]dS$$
$$\geq\frac{1}{2}\int_{B(s,t)}r^{\gamma_0-1}(2\gamma_0\tilde{q}+r<\tilde{x}\cdot\text{grad}\,\tilde{q}>)|v|^2dx$$
$$+\int_{B(s,t)}2\rho' r^{\gamma_0}|<\tilde{x}\cdot\text{grad}\, v>|^2dx$$
$$+\frac{n-1-\gamma_0}{2}\int_{B(s,t)}\left(\frac{\gamma_0-1}{r}+2\rho'\right)r^{\gamma_0-1}\,\text{Re}\,[<\tilde{x}\cdot\text{grad}\, v>\bar{v}]dx.$$

Our aim is to show that *any solution u of (1.3) which also satisfies the condition*

$$(5.2)\qquad \liminf_{t\to\infty} t^{\gamma_0}\int_{S(t)}\{(1+|q|)|u|^2+|<\tilde{x}\cdot\text{grad}\, u>|^2\}dS=0$$

must identically vanish in Ω.

First we put $\rho=0$ in (5.1). Then $v=u$, $\tilde{q}=q$ and by (A4)′ $2\gamma_0 q$

$+r<\tilde{x}\cdot\operatorname{grad} q>\geq\lambda_0$. Thus, by means of (5.2), we can let $t\to\infty$ to obtain

$$(5.3)\qquad \int_{S(s)} r^{\gamma_0}(|\operatorname{grad} u|^2-q|u|^2)dS$$

$$-(n-1-\gamma_0)\int_{S(s)} r^{\gamma_0-1}\operatorname{Re}[<\tilde{x}\cdot\operatorname{grad} u>\bar{u}]dS$$

$$\geq\lambda_0\int_{B(s)} r^{\gamma_0-1}|u|^2dx$$

$$+(n-1-\gamma_0)(\gamma_0-1)\int_{B(s)} r^{\gamma_0-2}\operatorname{Re}[<\tilde{x}\cdot\operatorname{grad} u>\bar{u}]dx.$$

Since

$$\int_{B(s)} r^{\gamma_0-2}\operatorname{Re}[<\tilde{x}\cdot\operatorname{grad} u>\bar{u}]dx$$

$$=-\frac{1}{2}\int_{S(s)} r^{\gamma_0-2}|u|^2dS-\frac{n-3+\gamma_0}{2}\int_{B(s)} r^{\gamma_0-3}|u|^2dx,$$

it follows from (5.3) that

$$(5.4)\qquad \int_{S(s)} r^{\gamma_0}(|\operatorname{grad} u|^2-q|u|^2)dS$$

$$-(n-1-\gamma_0)\int_{S(s)} r^{\gamma_0-1}\left\{\operatorname{Re}[<\tilde{x}\cdot\operatorname{grad} u>\bar{u}]-\frac{\gamma_0-1}{2r}|u|^2\right\}dS$$

$$\geq\int_{B(s)} r^{\gamma_0-1}\left\{\lambda_0-\frac{(n-1-\gamma_0)(\gamma_0-1)(n-3+\gamma_0)}{2r^2}\right\}|u|^2dx.$$

This inequality corresponds to (3.5). Hence we see following the line of proof of Lemma 3.1

$$(5.5)\qquad \int_{B(r_0)} r^m|u|^2dx<\infty \qquad \text{for any } m>0.$$

Next we put $\rho(r)=m\log r$ in (5.1). Then by (5.5) and (2.25)

$$(5.6)\qquad -\int_{S(s)} r^{\gamma_0}\left\{2|<\tilde{x}\cdot\operatorname{grad} v>|^2-\frac{n-1-\gamma_0}{r}|<\tilde{x}\cdot\operatorname{grad} v>\bar{v}|\right\}dS+$$

$$+\int_{S(s)} r^{\gamma_0}(|\mathrm{grad} v|^2-\tilde{q}|v|^2)dS$$

$$\geq \int_{B(s)} r^{\gamma_0-1}\left\{\lambda_0-2(1-\gamma_0)\frac{m(m-n+2)}{r^2}\right\}|v|^2 dx$$

$$+\int_{B(s)} 2mr^{\gamma_0-1}|<\tilde{x}\cdot\mathrm{grad}\, v>|^2 dx$$

$$+(n-1-\gamma_0)\int_{B(s)}\frac{\gamma_0-1+2m}{r}r^{\gamma_0-1}\mathrm{Re}[<\tilde{x}\cdot\mathrm{grad}\, v>\bar{v}]dx,$$

where the last term of the right can be estimated from below by

$$-\int_{B(s)}\frac{2m+\gamma_0-1}{2}r^{\gamma_0-1}|<\tilde{x}\cdot\mathrm{grad}\, v>|^2 dx$$

$$-\int_{B(s)}(n-1-\gamma_0)^2\frac{2m+\gamma_0-1}{2r^2}r^{\gamma_0-1}|v|^2 dx.$$

On the other hand, for $v\in H^1(B(r_0))$ (cf., Lemma 2.5)

$$(5.7)\qquad -\int_{S(s)} t^{\gamma_0-1}|v|^2 dS$$

$$\geq -\int_{B(s)} r^{\gamma_0-1}\left\{|<\tilde{x}\cdot\mathrm{grad}\, v>|^2+\left(1+\frac{n-2-\gamma_0}{r}\right)|v|^2\right\}dx.$$

Combining (5.6) and (5.7), and choosing $r_6\geq r_0$ sufficiently large, we have for $s>r_6$, $m>n$ and $0<\alpha\leq\min\{\lambda_0/3, 1\}$

$$(5.8)\qquad -\int_{S(s)} r^{\gamma_0}\left(2|<\tilde{x}\cdot\mathrm{grad}\, v>|^2-\frac{n-1-\gamma_0}{r}|<\tilde{x}\cdot\mathrm{grad}\, v>\bar{v}|\right.$$

$$\left.+\frac{\alpha}{r}|v|^2\right)dS+\int_{S(s)} r^{\gamma_0}(|\mathrm{grad}\, v|^2-\tilde{q}|v|^2)dS$$

$$\geq\int_{B(s)} r^{\gamma_0-1}\left\{\frac{\lambda_0}{2}-2(1-\gamma_0)\frac{m(m-n+2)}{r^2}\right.$$

$$\left.-(n-1-\gamma_0)^2\frac{2m+\gamma_0-1}{2r^2}\right\}|v|^2 dx,$$

which corresponds to (3.8). Hence we can follow the line of proof of Lemma 3.2 to conclude that for any $k>0$ and ν such that $0<\nu<1$

$$\int_{B(r_0)} e^{kr^{1-\nu}}|u|^2 dx < \infty. \tag{5.9}$$

Finally we return once more to inequality (5.1) with $\rho(r) = kr^{1-\nu}$. Choosing ν less than γ_0 and following the line of proof of Theorem 1.1, we have $u=0$ in Ω. This concludes the proof.

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