J. Math. Kyoto Univ. (JMKYAZ)
15-2 (1975) 303-349

Symmetric spaces associated with Siegel domains

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(Received, April 16, 1974)

Introduction

The purpose of the present paper is to give the details of the results announced in [5].

Let D be a Siegel domain of the second kind associated with a convex cone V in a real vector space R and a V-hermitian from F on a complex vector space W. Denote by Aut(D) the Lie group of all holomorphic transformations of the domain D and by $\mathfrak{g}(D)$ the Lie algebra of Aut(D). From Kaup, Matsushima and Ochiai [1], we know that $\mathfrak{g}(D)$ has the following graded structure:

$$\mathfrak{g}(D)=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^{0}+\mathfrak{g}^{1}+\mathfrak{g}^{2},\ \ [\mathfrak{g}^{\lambda},\mathfrak{g}^{\mu}]\subset\mathfrak{g}^{\lambda+\mu},$$

$$\mathfrak{r}=\mathfrak{r}^{-2}+\mathfrak{r}^{-1}+\mathfrak{r}^{0},\ \mathfrak{r}^{\lambda}=\mathfrak{r}\cap\mathfrak{g}^{\lambda},$$

where $\mathfrak{r}$ denotes the radical of $\mathfrak{g}(D)$. With relation to the semi-simple part of $\mathfrak{g}(D)$, we shall construct a symmetric Siegel domain S with $\dim_C S=\dim_R \mathfrak{g}^2+\frac{1}{2}\dim_R \mathfrak{g}^1$ which is invariant under a suitable equivalence. At the same time, we establish structure theorems of the Lie algebra $\mathfrak{g}(D)$.

In § 1 we prepare some algebraic facts. The most important one is the following: For any graded Lie algebra $\mathfrak{g}=\sum_{\lambda}\mathfrak{g}^{\lambda}(\lambda\in \boldsymbol{Z})$, we can choose a graded subalgebra as a semi-simple part of $\mathfrak{g}$ (Theorem 1.1). By using this fact, we can show in § 2 that there exists a semi-simple graded subalgebra $\mathfrak{s}=\sum_{\lambda=-2}^{2}\mathfrak{s}^{\lambda}$ of $\mathfrak{g}(D)$ such that $\mathfrak{s}^1=\mathfrak{g}^1$, $\mathfrak{s}^2=\mathfrak{g}^2$ and the adjoint representation of $\mathfrak{s}^0$ on $\mathfrak{s}^1+\mathfrak{s}^2$ is faithful (Theorem 2.1).

Let $\mathfrak{s}$ be as above. Then we have $\mathfrak{g}^{-2}=\mathfrak{s}^{-2}+\mathfrak{r}^{-2}$ and $\mathfrak{g}^{-1}=\mathfrak{s}^{-1}+\mathfrak{r}^{-1}$. The space $\mathfrak{g}^{-1}$ (resp. $\mathfrak{g}^{-2}$) can be identified with W (resp. with R) in a natural manner. Then $\mathfrak{s}^{-1}$ is a complex subspace of $\mathfrak{g}^{-1}$. Let V_s be the image of V by the projection of $\mathfrak{g}^{-2}$ to $\mathfrak{s}^{-2}$. From the results in § 3 concerning the structure of $\mathfrak{r}$, we can see that V_s is a convex cone in $\mathfrak{s}^{-2}$ and that the restriction F_s of F to $\mathfrak{s}^{-1}\times\mathfrak{s}^{-1}$ is a V_s-hermitian form. We shall prove in § 4 that the Siegel domain S of the second kind associated with V_s and F_s is symmetric and that the Lie algebra $\mathfrak{s}$ is identified with $\mathfrak{g}(S)$ (Theorem 4. 4). Moreover in § 5 we can show the uniqueness of the domain S. From construction, the domain S is contained in $\bar{D}$. Let $\mathfrak{s}'$ be another semi-simple graded subalgebra of $\mathfrak{g}(D)$ having the properties stated before. And let S' be the corresponding symmetric domain. Then there exists $X\in\mathfrak{g}^0$ such that $Ad(\exp X)\,\mathfrak{s}=\mathfrak{s}'$ and $\exp X(S)=S'$ (Theorem 5. 2).

As an application, we investigate the case where V is the cone of all positive definite real symmetric matrices, the cone of all positive definite complex hermitian matrices or the cone of all positive definite quaternion matrices. In § 6, we shall prove that if D is a Siegel domain over a cone of this type and if D is degenerate, then $\mathfrak{g}^1=0$ (Proposition 6. 4). And in § 7 we shall find out the symmetric domain S for any homogeneous Siegel domain constructed in Pyatetski-Shapiro [6] over these cones. In particular we can calculate dim $\mathfrak{g}^1$ and dim $\mathfrak{g}^2$. In these calculations, we partially use an idea, due to T. Tsuji, of considering the case where $W=W_1+W_2$ (direct sum) and $F(W_1, W_2)=0$. Starting from this idea, but by different methods, Tsuji [10][1] also calculated $\mathfrak{g}^1$ and $\mathfrak{g}^2$ in our Theorem 7. 6, Theorem 7. 8 and special cases in Theorem 7. 12.

Throughout this paper we use the following notations. For a real vector space R, denote by R_c its complexification. And for an element z of R_c, denote by Re z (resp. by Im z) its real (resp. imaginary) part. Let f be an endomorphism of a vector space W and let W' be a subspace of W invariant by f. We then denote by $f|_{W'}$ the restriction of f to W'.

Finally the author should express his thanks to N. Tanaka for his constant encouragement.

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1) The author received [10] as a preprint during the preparation of this paper.

§ 1. Levi decompositions of graded Lie algebras and algebraic preliminaries.

1. 1. Let $\mathfrak{g}=\sum_{\lambda}\mathfrak{g}^{\lambda}(\lambda\in\boldsymbol{Z}, [\mathfrak{g}^{\lambda}, \mathfrak{g}^{\mu}]\subset\mathfrak{g}^{\lambda+\mu})$ be a graded Lie algebra over $\boldsymbol{R}$ with $\dim\mathfrak{g}<\infty$. And let $\mathfrak{r}$ be its radical. Being invariant by the derivation of $\mathfrak{g}$ defined by: $X\rightarrow\lambda X$ for $X\in\mathfrak{g}^{\lambda}$, $\mathfrak{r}$ is a graded ideal of $\mathfrak{g}$, i.e., $\mathfrak{r}=\sum_{\lambda}\mathfrak{r}^{\lambda}(\mathfrak{r}^{\lambda}=\mathfrak{r}\cap\mathfrak{g}^{\lambda})$. Concerning Levi decompositions of $\mathfrak{g}$, we can prove the following

Theorem 1.1. *There exists a semi-simple graded subalgebra* $\tilde{\mathfrak{s}}$ *of* $\mathfrak{g}$ *such that* $\mathfrak{g}=\mathfrak{r}+\tilde{\mathfrak{s}}$ *(direct sum).*

Proof. We shall prove this by induction with respect to $\dim\mathfrak{r}$. Suppose that $\mathfrak{r}$ is not abelian. We put $\mathfrak{r}'=[\mathfrak{r}, \mathfrak{r}]$ and $\mathfrak{g}'=\mathfrak{g}/\mathfrak{r}'$. Since $\mathfrak{r}$ is a graded ideal of $\mathfrak{g}$, so is $\mathfrak{r}'$. Therefore $\mathfrak{g}'$ has a natural graded structure such that $\mathfrak{g}'^{\lambda}=\pi(\mathfrak{g}^{\lambda})$, where π is the projection of $\mathfrak{g}$ onto $\mathfrak{g}'$. Clearly the radical of $\mathfrak{g}'$ is $\mathfrak{r}/\mathfrak{r}'$ and $\dim\mathfrak{r}/\mathfrak{r}'<\dim\mathfrak{r}$. Thus from the inductive hypothesis, there exisis a semi-simple graded subalgebra $\mathfrak{s}'$ of $\mathfrak{g}'$ such that $\mathfrak{g}'=\mathfrak{r}/\mathfrak{r}'+\mathfrak{s}'$ (direct sum). We put $\mathfrak{s}^{*}=\pi^{-1}(\mathfrak{s}')$. We assert that $\mathfrak{s}^{*}$ is a graded subalgebra of $\mathfrak{g}$. Indeed, let $X\in\mathfrak{s}^{*}$. We write $X=\sum_{\lambda}X^{\lambda}(X^{\lambda}\in\mathfrak{g}^{\lambda})$. Then $\pi(X)=\sum_{\lambda}\pi(X^{\lambda})$. Since $\pi(X)\in\mathfrak{s}'$ and $\mathfrak{s}'$ is a graded subalgebra of $\mathfrak{g}'$, we have $\pi(X^{\lambda})\in\mathfrak{s}'$, proving our assertion. Now $\mathfrak{r}'$ is the radical of $\mathfrak{s}^{*}$ and $\dim\mathfrak{r}'<\dim\mathfrak{r}$. Thus there exists a semi-simple graded subalgebra $\mathfrak{s}$ of $\mathfrak{s}^{*}$ (and hence a graded subalgebra of $\mathfrak{g}$) such that $\mathfrak{s}^{*}=\mathfrak{r}'+\mathfrak{s}$ (direct sum). Clearly $\mathfrak{g}=\mathfrak{r}+\mathfrak{s}$ (direct sum).

Next we investigate the case where $\mathfrak{r}$ is abelian. Let $\mathfrak{g}=\mathfrak{r}+\mathfrak{s}$ be a Levi decomposition of $\mathfrak{g}$. Denote by π the natural projection of $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{r}$. Being isomorphic to $\mathfrak{g}/\mathfrak{r}$, the semi-simple subalgebra $\mathfrak{s}$ has a graded structure such that $\mathfrak{s}=\sum_{\lambda}\mathfrak{s}^{\lambda}$ and $\pi(\mathfrak{s}^{\lambda})=\pi(\mathfrak{g}^{\lambda})$. Let $X\in\mathfrak{s}^{\lambda}$. We write $X=\sum X^{\nu}(X\in\mathfrak{g}^{\nu})$. Since $\pi(X)\in\pi(\mathfrak{g}^{\lambda})$, we get $X^{\nu}\in\mathfrak{r}^{\nu}$ for $\nu\neq\lambda$. Clearly the correspondence: $X\rightarrow X^{\lambda}$ gives an injective linear mapping ρ_{λ} of $\mathfrak{s}^{\lambda}$ to $\mathfrak{g}^{\lambda}$. We extend the mapping ρ_{λ} to a mapping of $\mathfrak{s}$ to $\mathfrak{g}$ by defining as follows:

$$\rho_{\lambda}(\mathfrak{s}^{\nu})=0 \qquad \text{for} \quad \nu\neq\lambda\,.$$

Then the mapping $\rho=\sum\rho_{\lambda}$ is an injective linear mapping of $\mathfrak{s}$ to $\mathfrak{g}$ such that $\rho(\mathfrak{s}^{\lambda})\subset\mathfrak{g}^{\lambda}$. Let $X\in\mathfrak{s}^{\lambda}$ and $Y\in\mathfrak{s}^{\mu}$. We write $X=\sum_{\nu}X^{\nu}$ and $X=\sum_{\nu}Y^{\nu}(X^{\nu}, Y^{\nu}\in\mathfrak{g}^{\nu})$. Then

$$[X, Y] = [X^{\lambda}, Y^{\mu}] + \sum_{\nu'\neq\mu}[X^{\lambda}, Y^{\nu'}] + \sum_{\nu\neq\lambda}[X^{\nu}, Y^{\mu}] + \sum_{\nu\neq\lambda, \nu'\neq\mu}[X^{\nu}, X^{\nu'}].$$

Since $X^{\nu}\in\mathfrak{r}^{\nu}$ for $\nu\neq\lambda$ and $Y^{\nu'}\in\mathfrak{r}^{\nu'}$ for $\nu'\neq\mu$, we have $[X^{\nu}, Y^{\nu'}]=0$ for $\nu\neq\lambda$ and $\nu'\neq\mu$. Then from the definition of the mapping ρ, we get $\rho([X, Y]) = [X^{\lambda}, Y^{\mu}]$, because $[X, Y]\in\mathfrak{s}^{\lambda+\mu}$. Clearly $[\rho(X), \rho(Y)] = [X^{\lambda}, Y^{\mu}]$. As a result, ρ is an injective homomorphism and the decomposition $\mathfrak{g}=\mathfrak{r}+\rho(\mathfrak{s})$ has the desired properties. q.e.d.

1. 2. Let $\mathfrak{s}$ be a semi-simple graded Lie algebra such that $\mathfrak{s}=\mathfrak{s}^{-2}+\mathfrak{s}^{-1}+\mathfrak{s}^{0}+\mathfrak{s}^{1}+\mathfrak{s}^{2}$. The killing form φ of $\mathfrak{s}$ gives dualities between $\mathfrak{s}^{-1}$ and $\mathfrak{s}^{1}$ and between $\mathfrak{s}^{-2}$ and $\mathfrak{s}^{2}$. Therefore $\dim\mathfrak{s}^{-1}=\dim\mathfrak{s}^{1}$ and $\dim\mathfrak{s}^{-2}=\dim\mathfrak{s}^{2}$.

Lemma 1. 2 (cf. [8]).

(1) *Let* $X^{-2}\in\mathfrak{s}^{-2}$ *(resp.* $Y^{2}\in\mathfrak{s}^{2}$*). Suppose that* $[X^{-2}, \mathfrak{s}^{2}]=0$ *(resp.* $[Y^{2}, \mathfrak{s}^{-2}]=0$*). Then* $X^{-2}=0$ *(resp.* $Y^{2}=0$*).*

(2) *Let* $X^{-1}\in\mathfrak{s}^{-1}$ *(resp.* $Y^{1}\in\mathfrak{s}^{1}$*). Supppose that* $[X^{-1}, \mathfrak{s}^{1}]=0$ *(resp.* $[Y^{1}, \mathfrak{s}^{-1}]=0$*). Then* $X^{-1}=0$ *(resp.* $Y^{1}=0$*).*

Proof. (1) Let $Z\in\mathfrak{s}^{2}$. Clearly $ad\, X^{-2}\circ ad\, Z(\mathfrak{s}^{1}+\mathfrak{s}^{2})=0$. And $ad\, X^{-2}\circ ad\, Z(\mathfrak{s}^{0})\subset ad\, X^{-2}(\mathfrak{s}^{2})=0$. Moreover $ad\, X^{-2}\circ ad\, Z(\mathfrak{s}^{-1}+\mathfrak{s}^{-2})\subset ad([X^{-2}, Z])(\mathfrak{s}^{-1}+\mathfrak{s}^{-2})+ad\, Z\circ ad\, X^{-2}(\mathfrak{s}^{-1}+\mathfrak{s}^{-2})=0$. Therefore $\varphi(X^{-2}, \mathfrak{s}^{2})=0$ and hence $X^{-2}=0$. We can verify the second assertion similarly.

(2) Let $Z\in\mathfrak{s}^{1}$. By the same argument as in Proof of (1) we can show that $ad\, X^{-1}\circ ad\, Z(\mathfrak{s}^{-2}+\mathfrak{s}^{0}+\mathfrak{s}^{1}+\mathfrak{s}^{2})=0$. We set $\mathfrak{s}'^{-1}=\{X\in\mathfrak{s}^{-1}; [X, \mathfrak{s}^{1}]=0\}$. Then $[\mathfrak{s}^{0}, \mathfrak{s}'^{-1}]\subset\mathfrak{s}'^{-1}$. Therefore $ad\, X^{-1}\circ ad\, Z(\mathfrak{s}^{-1})\subset\mathfrak{s}'^{-1}$ and $ad\, X^{-1}\circ ad\, Z(\mathfrak{s}'^{-1})=0$. Thus we get $\varphi(X^{-1}, \mathfrak{s}^{1})=0$ and hence $X^{-1}=0$. Second assertion follows similarly. q.e.d.

Lemma 1. 3 (cf. [8]). *Under the assumption that* $\mathfrak{s}$ *is simple, we have*

(1) *If* $\mathfrak{s}^{1}=0$ *and* $\mathfrak{s}^{2}\neq 0$, *then* $[\mathfrak{s}^{-2}, \mathfrak{s}^{2}]=\mathfrak{s}^{0}$.

(2) *If* $\mathfrak{s}^{1}\neq 0$, *then* $\mathfrak{s}^{-2}=[s^{-1}, \mathfrak{s}^{-1}]$, $\mathfrak{s}^{0}=[\mathfrak{s}^{-1}, \mathfrak{s}^{1}]$ *and* $\mathfrak{s}^{2}=[\mathfrak{s}^{1}, \mathfrak{s}^{1}]$.

(3) *If* $\mathfrak{s}^{2}\neq 0$, *then* $\mathfrak{s}^{1}=[\mathfrak{s}^{2}, \mathfrak{s}^{-1}]$ *and* $\mathfrak{s}^{-1}=[\mathfrak{s}^{-2}, \mathfrak{s}^{1}]$.

Proof. (1) Clearly the subspace $\mathfrak{s}^{-2}+[\mathfrak{s}^{-2}, \mathfrak{s}^{2}]+\mathfrak{s}^{2}$ is an ideal of $\mathfrak{s}$ and hence $\mathfrak{s}^0=[\mathfrak{s}^{-2}, \mathfrak{s}^2]$.

(2) We set $\mathfrak{s}'=[\mathfrak{s}^{-1}, \mathfrak{s}^{-1}]+\mathfrak{s}^{-1}+[\mathfrak{s}^{-1}, \mathfrak{s}^1]+\mathfrak{s}^1+[\mathfrak{s}^1, \mathfrak{s}^1]$. Then $\mathfrak{s}'$ is an ideal of $\mathfrak{s}$, proving our assertion.

(3) It is sufficient to consider the case where $\mathfrak{s}^1\neq 0$. we set $\mathfrak{s}'^{-2}=\mathfrak{s}^{-2}$ and $\mathfrak{s}'^{\lambda+1}=[\mathfrak{s}'^{\lambda}, \mathfrak{s}^1]$ inductively. And put $\mathfrak{s}'=\mathfrak{s}'^{-2}+\mathfrak{s}'^{-1}+\mathfrak{s}'^0+\mathfrak{s}'^1+\mathfrak{s}'^2$. Then $[\mathfrak{s}', \mathfrak{s}^0+\mathfrak{s}^1]\subset\mathfrak{s}'$. Since $\mathfrak{s}^2=[\mathfrak{s}^1, \mathfrak{s}^1]$ by Assertion (2), we get $[\mathfrak{s}'^{-2}, \mathfrak{s}^2]\subset\mathfrak{s}'^0$ and hence $[\mathfrak{s}'^{-2}, \mathfrak{s}]\subset\mathfrak{s}'$. Assume that $[\mathfrak{s}'^{\lambda}, \mathfrak{s}]\subset\mathfrak{s}'$. Then $[\mathfrak{s}'^{\lambda+1}, \mathfrak{s}]=[[\mathfrak{s}'^{\lambda}, \mathfrak{s}^1], \mathfrak{s}]\subset[[\mathfrak{s}'^{\lambda}, \mathfrak{s}], \mathfrak{s}^1]+[\mathfrak{s}'^{\lambda}, \mathfrak{s}]\subset[\mathfrak{s}', \mathfrak{s}^1]+\mathfrak{s}'\subset\mathfrak{s}'$. As a result, $\mathfrak{s}'$ is an ideal of $\mathfrak{s}$ and hence $\mathfrak{s}^{-1}=[\mathfrak{s}^{-2}, \mathfrak{s}^1]$. We can verify the equation $\mathfrak{s}^1=[\mathfrak{s}^2, \mathfrak{s}^{-1}]$ analogously. q.e.d.

The correspondence: $X\to\lambda X$ for $X\in\mathfrak{s}^{\lambda}$ is a derivation of the semi-simple Lie algebra $\mathfrak{s}$. Therefore there exists a unique element E of $\mathfrak{s}$ such that $\mathfrak{s}^{\lambda}=\{X\in\mathfrak{s}; ad\,EX=\lambda X\}$. It is easy to see that E belongs to $\mathfrak{s}^0$. Being invariant by $ad\,E$, each ideal of $\mathfrak{s}$ is a graded ideal.

Colollary 1.4. *Let* $\mathfrak{g}=\sum_{\lambda=-2}^{2}\mathfrak{g}^{\lambda}(\lambda\in\boldsymbol{Z})$ *be a graded Lie algebra whose radical* $\mathfrak{r}$ *is of the form*: $\mathfrak{r}=\mathfrak{r}^{-2}+\mathfrak{r}^{-1}+\mathfrak{r}^0(\mathfrak{r}^{\lambda}=\mathfrak{r}\cap\mathfrak{g}^{\lambda})$. *Suppose that* $\mathfrak{g}^{-2}=[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$. *Then* $\mathfrak{g}^2=[\mathfrak{g}^1, \mathfrak{g}^1]$.

Proof. By considering $\mathfrak{g}/\mathfrak{r}$ instead of $\mathfrak{g}$, we may assume that $\mathfrak{g}$ is semi-simple. And by considering the decomposition into simple ideals, we may assume that $\mathfrak{g}$ is simple. Then our assertion follows immediately from Lemma 1.3. q.e.d.

1.3. Let R be a real vector space with $\dim R<\infty$. A linear endomorphism A of R is called *real-diagonal* if it can be written as a diagonal matrix with respect to some basis of R.

Lemma 1.5. *Let* $\mathfrak{s}$ *be a semi-simple Lie algebra and let* f *be a representation of* $\mathfrak{s}$ *on a real vector space* R. *Assume that* $ad\,E$ $(E\in\mathfrak{s})$ *is real diagonal. Then* $f(E)$ *is real-diagonal.*

Proof. We first assert that all eigenvalues of $f(E)$ are real.[2] In fact, we set for $\alpha\in\boldsymbol{C}$,

$$U_{\alpha}=\{X\in R_c;\ (f(E)-\alpha)^nX=0 \quad \text{for some } n\in\boldsymbol{N}\}.$$

Then $R_c=\sum_{\alpha}U_{\alpha}$. Let λ be an eigenvalue of $f(E)$ and put $U'=\sum'U_{\alpha}$,

[2] This fact is also proved in [12] by different methods.

where $\sum'$ indicates the sum is taken only over the spaces U_α with $\operatorname{Im}\alpha=\operatorname{Im}\lambda$. It is easy to see that U' is $\mathfrak{s}$-invariant. It follows that $\operatorname{Im} Tr f(E)|_{U'}=\dim_C U'\times\operatorname{Im}\lambda$. On the other hand, we know that $Tr f(E)|_{U'}=0$, because $E\in[\mathfrak{s},\mathfrak{s}]$. Therefore we have $\operatorname{Im}\lambda=0$, proving our assertion.

Now we may assume that the representation f is irreducible. Denote by $\mathcal{A}$ the associative algebra of operators generated by $f(\mathfrak{s})$. Then $\mathcal{A}$ is a Lie algebra in the usual braket rule. It is not difficult to show that $ad f(E)$ is a real-diagonal element in $\mathcal{A}$. Therefore we can write $\mathcal{A}=\sum_{\alpha\in R}\mathcal{A}_\alpha$, where $\mathcal{A}_\alpha=\{X\in\mathcal{A};\ [f(E),X]=\alpha X\}$. Put $R_\alpha=\{v\in R;\ (f(E)-\alpha)^n v=0$ for some $n\in N\}$, for each eigenvalue α of $f(E)$. Then $R=\sum R_\alpha$. We consider the case where there exists a non-zero eigenvalue λ of $f(E)$. Let v be an eigenvector corresponding to the eigenvalue λ, i.e., $f(E)v=\lambda v\,(v\neq 0)$. Then the space $\{\mathcal{A}v\}$ coincides with R, because $\{\mathcal{A}v\}$ is $\mathfrak{s}$-invariant. Let $u\in R_\alpha$. There exists $A\in\mathcal{A}$ such that $Av=u$. We can write $A=\sum A_\beta(A_\beta\in\mathcal{A}_\beta)$. Since $A_\beta v\in R_{\beta+\lambda}$, we have $A_{\alpha-\lambda}v=u$. It follows

$$\begin{aligned} f(E)u &= f(E)A_{\alpha-\lambda}v \\ &= [f(E),A_{\alpha-\lambda}]v+A_{\alpha-\lambda}f(E)v \\ &= (\alpha-\lambda)A_{\alpha-\lambda}v+\lambda A_{\alpha-\lambda}v=\alpha u\,. \end{aligned}$$

As a result $R_\alpha=\{v\in R;\ f(E)v=\alpha v\}$. Next we consider the case where all eigenvalues of $f(E)$ are zero. Let $\mathfrak{s}'=\mathfrak{s}/f^{-1}(0)$ and let E' be the image of E in $\mathfrak{s}'$. Since $f(E)$ is nilpotent, so is $ad\,E'$. On the other hand, $ad\,E'$ is a real-diagonal element in $\mathfrak{s}'$. Therefore $[E',\mathfrak{s}']=0$ and hence $E'=0$. This implies that $f(E)=0$ and completes the proof. q.e.d.

§ 2. A Siegal domain D and the Lie algebra of Aut(D).

2.1. Let R (resp. W) be a real (resp. complex) vector space of finite dimension. An open set V of R is called *a convex cone* if it satisfies the following conditions:

1) For any $x\in V$ and for any $t>0$, $tx\in V$.
2) For any $x, x'\in V$, $x+x'\in V$.
3) V contains no entire straight lines.

We say a mapping F of $W\times W$ into R_c is *a V-hermitian form* on W if it satisges the following conditions:

1) $F(w, w')$ is complex linear in w and $F(w, w')=\overline{F(w', w)}$.
2) $F(w, w)\in\overline{V}$, where $\overline{V}$ denotes the closure of V in R.
3) $F(w, w)=0$ implies $w=0$.

We define a domain D in $R_c\times W$ by

$$D=\{(z, w)\in R_c\times W;\ \mathrm{Im}\, z-F(w, w)\in V\}.$$

The domain D is called *a Siegel domain of the second kind.* In the special case where $W=0$, the domain D is called *a Siegel domain of the first kind.*

Denote by $\mathrm{Aut}(D)$ the group of all holomorphic transformations of D and by $\mathfrak{g}(D)$ the Lie algebra of $\mathrm{Aut}(D)$. Define a subgroup $GL(D)$ of $\mathrm{Aut}(D)$ by

$$GL(D)=\mathrm{Aut}(D)\cap GL(R_c\times W).$$

An element $f\in GL(R_c\times W)$ belongs to $GL(D)$ if and only if f satisfies the following conditions (Pyatetski-Shapiro [6]):

$$\text{(2. 1)}\quad \begin{cases} A(R)=R,\ A(W)=W \text{ and } A(V)=V. \\ AF(w, w')=F(Aw, Aw') \text{ for } w\ w'\in W. \end{cases}$$

Let E (resp. I) be the element of $\mathfrak{g}(D)$ induced by the following one parameter group in $GL(D)$ (with parameter t):

$$\text{(2. 2)}\qquad (z, w)\rightarrow(e^{-2t}z, e^{-t}w)$$

$$\text{(resp. (2. 2)}'\qquad (z, w)\rightarrow(z, e^{\sqrt{-1}t}w)\quad).$$

For every $a\in R$ (resp. $c\in W$) we denote by $s(a)$ (resp. by $s(c)$) the element of $\mathfrak{g}(D)$ induced by the following one parameter group (with parameter t):

$$(z, w)\rightarrow(z+ta, w)$$

$$\text{(resp.}\quad (z, w)\rightarrow(z+2\sqrt{-1}F(w, tc)+\sqrt{-1}F(tc, tc), w+tc)).$$

Then s gives an injective linear mapping of $R+W$ to $\mathfrak{g}(D)$. We set for $\lambda=-2,\ -1,\ 0,\ 1$ and 2

$$g^{\lambda}=\{X\in\mathfrak{g}(D);\ [E, X]=\lambda X\}.$$

Kaup, Matsushima and Ochiai [1] showed that the Lie algebra $\mathfrak{g}(D)$ has the graded structure as follows:

1) $\mathfrak{g}(D)=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^{0}+\mathfrak{g}^{1}+\mathfrak{g}^{2}$, $[\mathfrak{g}^{\lambda}, \mathfrak{g}^{\mu}]\subset\mathfrak{g}^{\lambda+\mu}$.

2) $\mathfrak{r}=\mathfrak{r}^{-2}+\mathfrak{r}^{-1}+\mathfrak{r}^{0}(\mathfrak{r}^{\lambda}=\mathfrak{r}\cap\mathfrak{g}^{\lambda})$, where $\mathfrak{r}$ denotes the radical of $\mathfrak{g}(D)$. And $\dim\mathfrak{g}^{-2}=\dim\mathfrak{g}^{2}+\dim\mathfrak{r}^{-2}$, $\dim\mathfrak{g}^{-1}=\dim\mathfrak{g}^{1}+\dim\mathfrak{r}^{-1}$.

3) $\mathfrak{g}^{-2}=\{s(a);\, a\in R\}$, $\mathfrak{g}^{-1}=\{s(c);\, c\in W\}$ and $\mathfrak{g}^{0}$ is the subalgebra corressonding to the subgroup $GL(D)$ of $\mathrm{Aut}(D)$.

From 3) and (2. 1), we know that $\mathfrak{g}^{0}$ consists of all $A\in\mathfrak{gl}(R_c\times W)$ satisfying the following conditions:

$$(2.3)\qquad \begin{cases} A(R)\subset R,\ A(W)\subset W \quad\text{and}\quad \exp tA(V)=V\ \ (t\in\boldsymbol{R}). \\ AF(w, w')=F(Aw, w')+F(w, Aw')\ \ (w, w'\in W). \end{cases}$$

Cleary E and I are in the center of $\mathfrak{g}^{0}$ and the equality $s(\sqrt{-1}c)=[I, s(c)]$ holds for any $c\in W$. In what follows, we identify the space R (resp. W) with $\mathfrak{g}^{-2}$ (resp. with $\mathfrak{g}^{-1}$) by the isomorphism s. Then a complex subspace of $\mathfrak{g}^{-1}$ is an $ad\,I$-invariant subspace and the following equalities hold (cf. [9]):

$$(2.4)\qquad [A, c]=Ac \quad\text{for}\quad A\in\mathfrak{g}^{0} \quad\text{and for}\quad c\in\mathfrak{g}^{-2}+\mathfrak{g}^{-1}.$$

$$(2.5)\qquad F(c, c')=\tfrac{1}{4}([[I, c], c']+\sqrt{-1}[c, c']) \quad\text{for}\quad c, c'\in\mathfrak{g}^{-1}.$$

2. 2. Let $\tilde{\mathfrak{s}}=\sum_{\lambda=-2}^{2}\tilde{\mathfrak{s}}^{\lambda}$ be a semi-simple graded subalgebra of $\mathfrak{g}(D)$ given by Theorem 1. 1. Then $\tilde{\mathfrak{s}}^{1}=\mathfrak{g}^{1}$ and $\tilde{\mathfrak{s}}^{2}=\mathfrak{g}^{2}$. We set $\mathfrak{k}=\{X\in\tilde{\mathfrak{s}}^{0};\ [X, \tilde{\mathfrak{s}}^{1}+\tilde{\mathfrak{s}}^{2}]=0\}$. Let φ be the killing form of $\tilde{\mathfrak{s}}$. Then $\varphi([\mathfrak{k}, \tilde{\mathfrak{s}}^{-1}], \tilde{\mathfrak{s}}^{1})=0$ and $\varphi([\mathfrak{k}, \tilde{\mathfrak{s}}^{-2}], \tilde{\mathfrak{s}}^{2})=0$. Therefore we have $[\mathfrak{k}, \tilde{\mathfrak{s}}^{-1}]=0$ and $[\mathfrak{k}, \tilde{\mathfrak{s}}^{-2}]=0$. Clearly $[\mathfrak{k}, \tilde{\mathfrak{s}}^{0}]\subset\mathfrak{k}$. As a result, $\mathfrak{k}$ is an ideal of $\tilde{\mathfrak{s}}$. Let $\mathfrak{s}$ be the orthogonal complement of $\mathfrak{k}$ with respect to φ. Then $\mathfrak{s}$ is an ideal of $\tilde{\mathfrak{s}}$ such that $\tilde{\mathfrak{s}}=\mathfrak{s}+\mathfrak{k}$ (direct sum). Since $\mathfrak{s}$ is a graded ideal of $\tilde{\mathfrak{s}}$, $\mathfrak{s}$ is a graded subalgebra of $\mathfrak{g}(D)$, i.e., $\mathfrak{s}=\sum_{\lambda=-2}^{2}\mathfrak{s}^{\lambda}$ ($\mathfrak{s}^{\lambda}=\mathfrak{s}\cap\mathfrak{g}^{\lambda}$). Clearly $\mathfrak{s}^{1}=\mathfrak{g}^{1}$, $\mathfrak{s}^{2}=\mathfrak{g}^{2}$ and the adjoint representation of $\mathfrak{s}^{0}$ on $\mathfrak{s}^{1}+\mathfrak{s}^{2}$ is faithful. Therefore we have proved the following

Theorem 2. 1. *There exists a semi-simple graded subalgebra* $\mathfrak{s}=\mathfrak{s}^{-2}+\mathfrak{s}^{-1}+\mathfrak{s}^{0}+\mathfrak{s}^{1}+\mathfrak{s}^{2}$ *having the following properties*:

1) $\mathfrak{s}^{1}=\mathfrak{g}^{1}$ *and* $\mathfrak{s}^{2}=\mathfrak{g}^{2}$.

2) *Let* $A\in\mathfrak{s}^{0}$. *The condition* $[A, \mathfrak{s}^{1}+\mathfrak{s}^{2}]=0$ *implies* $A=0$.

A Siegel domain of the second kind is called *irreducible* if it is an irreducible riemannian manifold with respect to the Bergman metric. (Note that every Siegel domain of the second kind is a connected simply connected complete Kähler manifold with respect to the Bergmann metric and that every irreducible component of a Siegel domain is also a Siegel domain ([3]).) By using Theorem 1.1 we can also prove the following

Proposition 2.2. *Let D be an irreducible Siegel domain of the second kind such that* $\mathfrak{r}^{-1}=0$ *and* $\mathfrak{g}^{1}\neq 0$. *Then D is a symmetric homogeneous domain.*

Proof. Let $\tilde{\mathfrak{s}}=\sum_{\lambda=-2}^{2}\tilde{\mathfrak{s}}^{\lambda}$ be a semi-simple graded subalgebra of $\mathfrak{g}(D)$ given by Theorem 1.1. Then from our hypothesis, we have $[\mathfrak{r}, \tilde{\mathfrak{s}}^{1}]=0$. We set $\mathfrak{h}_1=\{X\in\tilde{\mathfrak{s}};\ [X, \mathfrak{r}]=0\}$. Clearly $\mathfrak{h}_1$ is an ideal of $\tilde{\mathfrak{s}}$. Therefore there exists an ideal $\mathfrak{h}_2$ of $\tilde{\mathfrak{s}}$ such that $\tilde{\mathfrak{s}}=\mathfrak{h}_1+\mathfrak{h}_2$ (direct sum). Then $\mathfrak{g}(D)=\mathfrak{h}_1+\mathfrak{h}_2+\mathfrak{r}$ (direct sum) and both $\mathfrak{h}_1$ and $\mathfrak{h}_2+\mathfrak{r}$ are ideals of $\mathfrak{g}(D)$. Now the irreducibility of D implies $\mathfrak{g}(D)=\mathfrak{h}_1$([3]), and hence $\mathfrak{g}(D)$ is semi-simple. As a result, the domain D is homogeneous ([11]) and hence symmetric. q.e.d.

By using expressions of elements of $\mathfrak{g}(D)$ as polynomial vector fields in [1], we can easily observe the followings:

$$
\begin{cases} ad\, I=0 \quad \text{on} \quad \mathfrak{g}^{-2}+\mathfrak{g}^{0}+\mathfrak{g}^{2}. \\ (ad\, I)^{2}=-id. \quad \text{on} \quad \mathfrak{g}^{-1}+\mathfrak{g}^{1}. \end{cases} \tag{2.6}
$$

Proposition 2.3. *Let* $\mathfrak{s}=\sum_{\lambda=-2}^{2}\mathfrak{s}^{\lambda}$ *be a semi-simple graded subalgebra of* $\mathfrak{g}(D)$ *as in Theorem* 2.1. *Then*

(1) $\mathfrak{g}^{-2}=\mathfrak{s}^{-2}+\mathfrak{r}^{-2}$ *(direct sum)*,
$\mathfrak{g}^{-1}=\mathfrak{s}^{-1}+\mathfrak{r}^{-1}$ *(direct sum)*.

(2) $ad\, I\ \mathfrak{s}\subset\mathfrak{s}$.

(3) $\mathfrak{s}^{-1}=[\mathfrak{s}^{-2}, \mathfrak{s}^{1}]$, $\mathfrak{s}^{1}=[\mathfrak{s}^{2}, \mathfrak{s}^{-1}]$ and $\mathfrak{s}^{0}=[\mathfrak{s}^{-1}, \mathfrak{s}^{1}]+[\mathfrak{s}^{-2}, \mathfrak{s}^{2}]$.

Moreover if the domain D is non-degenerate, i.e., $\mathfrak{g}^{-2}=[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$. *Then* $\mathfrak{s}^{-2}=[\mathfrak{s}^{-1}, \mathfrak{s}^{-1}]$, $\mathfrak{s}^{2}=[\mathfrak{s}^{1}, \mathfrak{s}^{1}]$ *and* $\mathfrak{s}^{0}=[\mathfrak{s}^{-1}, \mathfrak{s}^{1}]$.

Proof. Assertion (1) follows immediately from the equalities $\dim\mathfrak{g}^{-2}=\dim\mathfrak{r}^{-2}+\dim\mathfrak{s}^{-2}$ and $\dim\mathfrak{g}^{-1}=\dim\mathfrak{r}^{-1}+\dim\mathfrak{s}^{-1}$. Let $\mathfrak{s}=\sum_j\mathfrak{s}_j$

be the decomposition of $\mathfrak{s}$ into simple ideals. Then each $\mathfrak{s}_j$ is a graded subalgebra of $\mathfrak{g}(D)$, i.e., $\mathfrak{s}_j=\sum_{\lambda=-2}^{2}\mathfrak{s}_j^{\lambda}(\mathfrak{s}_j^{\lambda}=\mathfrak{s}_j\cap\mathfrak{g}^{\lambda})$. Suppose that $\mathfrak{s}_j^1\neq 0$. Then from [4], we know that $[\mathfrak{s}_j^1, [I, \mathfrak{s}_j^1]]\neq 0$. On the other hand, $[\mathfrak{s}_j^1, [I, \mathfrak{s}_j^1]]\subset[\mathfrak{s}_j^1, \mathfrak{s}^1]\subset\mathfrak{s}_j^2$. As a result $\mathfrak{s}_j^2\neq 0$. Thus each $\mathfrak{s}_j$ is of one of the following two types.

(i) $\mathfrak{s}_j=\mathfrak{s}_j^{-2}+\mathfrak{s}_j^{-1}+\mathfrak{s}_j^0+\mathfrak{s}_j^1+\mathfrak{s}_j^2$ $(\mathfrak{s}_j^1\neq 0, \mathfrak{s}_j^2\neq 0)$.

(ii) $\mathfrak{s}_j=\mathfrak{s}_j^{-2}+\mathfrak{s}_j^0+\mathfrak{s}_j^2$ $(\mathfrak{s}_j^2\neq 0)$.

In the case where $\mathfrak{g}^{-2}=[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$, each ideal $\mathfrak{s}_j$ is clearly of the type (i). Now Assertion (3) follows immediately from Lemma 1.3. Finally by using (2.6), we have $[I, \mathfrak{s}^{-1}]=[I, [\mathfrak{s}^{-2}, \mathfrak{s}^1]]=[\mathfrak{s}^{-2}, [I, \mathfrak{s}^1]]=[\mathfrak{s}^{-2}, \mathfrak{s}^1]=\mathfrak{s}^{-1}$. This implies Assertion (2). q.e.d.

Corollary 2.4. *Let* $X\in\mathfrak{g}^{-1}$. *If* $[\mathfrak{g}^{-2}, [X, \mathfrak{g}^1]]=0$. *Then* $X\in\mathfrak{r}^{-1}$.

Proof. Let $\mathfrak{s}$ be as in Theorem 2.1, and let X_s be the $\mathfrak{s}^{-1}$-component of X with respect to the decomposition $\mathfrak{g}^{-1}=\mathfrak{s}^{-1}+\mathfrak{r}^{-1}$ in Proposition 2.3. Then $[\mathfrak{s}^{-2}, [X_s, \mathfrak{s}^1]]=0$. And hence $[X_s, \mathfrak{s}^{-1}]=0$, because $\mathfrak{s}^{-1}=[\mathfrak{s}^{-2}, \mathfrak{s}^1]$ by Proposion 2.3. In particular, $[[I, X_s], X_s]=0$. Therefore by using (2.5) we have $X_s=0$. q.e.d.

Corollary 2.5. $\mathfrak{r}^{-1}=\{X\in\mathfrak{g}^{-1}; [X, \mathfrak{g}^2]=0\}$.

Proof. Denote by $\mathfrak{r}'^{-1}$ the right side of the above equality. Clearly $\mathfrak{r}^{-1}\subset\mathfrak{r}'^{-1}$. Conversely let $X\in\mathfrak{r}'^{-1}$ and let X_s be the $\mathfrak{s}^{-1}$-component as in Proof of Corollary 2.4. Then $[X_s, \mathfrak{s}^2]=0$. Therefore $[[[I, X_s], X_s], \mathfrak{s}^2]=0$ by (2.6). Since $[[I, X_s], X_s]$ belong to $\mathfrak{s}^{-2}$, we obtain $[[I, X_s], X_s]=0$ by Lemma 1.2 and hence $X_s=0$. q.e.d.

§ 3. The structure of the radical $\mathfrak{r}$.

3.1. Let $\mathfrak{s}=\sum_{\lambda=-2}^{2}\mathfrak{s}^{\lambda}$ be a semi-simple graded subalgebra of $\mathfrak{g}(D)$ as in Theorem 2.1. There exists a unique element E_s of $\mathfrak{s}^0$ such that

$$\mathfrak{s}^{\lambda}=\{X\in\mathfrak{s}; [E_s, X]=\lambda X\}. \tag{3.1}$$

We set

$$(3.2)\qquad \begin{cases} \mathfrak{r}_0^{-2}=\{X\in\mathfrak{r}^{-2};\ [\mathfrak{s},X]=0\},\\ \mathfrak{r}_s^{-2}=\{X\in\mathfrak{r}^{-2};\ [E_s,X]=-X\}.\end{cases}$$

$$(3.3)\qquad \begin{cases} \mathfrak{r}_0^{0}=\{X\in\mathfrak{r}^{0};\ [\mathfrak{s},X]=0\},\\ \mathfrak{r}_s^{0}=\{X\in\mathfrak{r}^{0};\ [E_s,X]=X\}.\end{cases}$$

In the notations as above we shall show the following theorem.

Theorem 3.1. *The radical* $\mathfrak{r}$ *has the following structure.*

(1) $\mathfrak{r}^{-2}=\mathfrak{r}_0^{-2}+\mathfrak{r}_s^{-2}$ *(direct sum),* $\mathfrak{r}_0^{-2}\supset[\mathfrak{r}^{-1},\mathfrak{r}^{-1}]$ *and* $\mathfrak{r}_s^{-2}=[\mathfrak{r}^{-2},\mathfrak{s}^0]=[\mathfrak{r}^0,\mathfrak{s}^{-2}]\supset[\mathfrak{r}^{-1},\mathfrak{s}^{-1}]$.

(2) $\mathfrak{r}^0=\mathfrak{r}_0^0+\mathfrak{r}_s^0$ *(direct sum) and* $\mathfrak{r}_s^0=[\mathfrak{r}^{-2},\mathfrak{s}^2]=[\mathfrak{r}^0,\mathfrak{s}^0]\supset[\mathfrak{r}^{-1},\mathfrak{s}^1]$.

(3) $\dim\mathfrak{r}_s^{-2}=\dim\mathfrak{r}_s^0$.

(4) $ad\,E_s=0$ on $\mathfrak{r}^{-1}$.

(5) $\mathfrak{r}_s^0$ *is an abelian ideal of* $\mathfrak{g}^0$ *satisfying the followings:*

a) $[\mathfrak{r}_s^0,\mathfrak{r}_0^{-2}+\mathfrak{r}^{-1}]=0$

b) $[\mathfrak{r}_s^0,\mathfrak{r}_s^{-2}]\subset\mathfrak{r}_0^{-2}$.

3.2. We first show the following

Lemma 3.2. $[[\mathfrak{r}^{-1},\mathfrak{s}^1],\mathfrak{r}^{-1}]=0$.

Proof. Since $[\mathfrak{s}^2,\mathfrak{r}^{-1}]=0$, we have by (3) of Proposition 2.3 $[[\mathfrak{r}^{-1},\mathfrak{s}^1],\mathfrak{r}^{-1}]=[[\mathfrak{r}^{-1},[\mathfrak{s}^2,\mathfrak{s}^{-1}]],\mathfrak{r}^{-1}]=[[\mathfrak{s}^2,[\mathfrak{r}^{-1},\mathfrak{s}^{-1}]],\mathfrak{r}^{-1}]=[[\mathfrak{s}^2,\mathfrak{r}^{-1}],[\mathfrak{r}^{-1},\mathfrak{s}^{-1}]]=0$. q.e.d.

Next we verify

Lemma 3.3. $[\mathfrak{s},[\mathfrak{r}^{-1},\mathfrak{r}^{-1}]]=0$.

Proof. By Lemma 3.2 $[\mathfrak{s}^1,[\mathfrak{r}^{-1},\mathfrak{r}^{-1}]]=0$. Clearly $[\mathfrak{s}^{-2}+\mathfrak{s}^{-1}+\mathfrak{s}^2,[\mathfrak{r}^{-1},\mathfrak{r}^{-1}]]=0$. Since $\mathfrak{s}^0=[\mathfrak{s}^{-2},\mathfrak{s}^2]+[\mathfrak{s}^{-1},\mathfrak{s}^1]$, we get $[\mathfrak{s}^0,[\mathfrak{r}^{-1},\mathfrak{r}^{-1}]]=0$. q.e.d.

Since $\mathfrak{r}^{-1}$ is a complex subspace of $\mathfrak{g}^{-1}$, the restriction F_r of F to $\mathfrak{r}^{-1}\times\mathfrak{r}^{-1}$ is an $[\mathfrak{r}^{-1},\mathfrak{r}^{-1}]_c$-valued hermitian form on $\mathfrak{r}^{-1}$. Clearly $F(c,c)$ $(c\in\mathfrak{r}^{-1})$ is contained in $[\mathfrak{r}^{-1},\mathfrak{r}^{-1}]\cap\overline{V}$. Therefore there exists a linear coordinate system $z^1,\cdots,z^n$ of $[\mathfrak{r}^{-1},\mathfrak{r}^{-1}]_c$ such that a hermitian form $H(c,c')=\sum_j z^j\circ F_r(c,c')$ on $\mathfrak{r}^{-1}$ is positive definite. From Lemma

3.3, (2.3), (2.4) and (2.5), we have

$$(3.4) \qquad H(ad\, Xc, c') + H(c, ad\, Xc') = 0 \quad (X \in \mathfrak{s}^0),$$

because I is in the center of $\mathfrak{g}^0$.

Lemma 3.4. *The following equalities hold:*

$$ad\, E_s = 0 \quad \text{on} \quad \mathfrak{r}^{-1},$$

$$ad E_s = id. \quad \text{on} \quad [\mathfrak{r}^{-1}, \mathfrak{s}^1],$$

$$ad\, E_s = -id. \quad \text{on} \quad [\mathfrak{r}^{-1}, \mathfrak{s}^{-1}].$$

Proof. By (3.1) we have only to prove the first equation. From (3.4), we know that the endomorphism $ad\, E_s$ of $\mathfrak{r}^{-1}$ is semi-simple and its eigenvalues are purely imaginary. On the other hand, $ad\, E_s$ has only real eigenvalues on $\mathfrak{r}$ by Lemma 1.5, because $\mathfrak{r}$ is an invariant space under the action of the semi-simple Lie algebra $\mathfrak{s}$ and $ad\, E_s$ is real diagonal in $\mathfrak{s}$. Therefore we can conclude that $ad\, E_s = 0$ on $\mathfrak{r}^{-1}$.
q.e.d.

By using Lemma 1.5, we can also see that $ad\, E_s|_{\mathfrak{r}^{-2}}$ and $ad\, E_s|_{\mathfrak{r}^0}$ are real diagonal. Therefore if we set for $\lambda \in \boldsymbol{R}$

$$(3.5) \qquad \begin{cases} \mathfrak{a}_\lambda^{-2} = \{X \in \mathfrak{r}^{-2};\ [E_s, X] = \lambda X\} \\ \mathfrak{a}_\lambda^{0} = \{X \in \mathfrak{r}^{0};\ [E_s, X] = \lambda X\}. \end{cases}$$

Then we have

$$(3.6) \qquad \begin{cases} \mathfrak{r}^{-2} = \sum_\lambda \mathfrak{a}_\lambda^{-2} \\ \mathfrak{r}^{0} = \sum_\lambda \mathfrak{a}_\lambda^{0}. \end{cases}$$

We know from Lemma 3.3 and Lemma 3.4,

$$(3.7) \qquad \begin{cases} \mathfrak{a}_0^{-2} \supset [\mathfrak{r}^{-1}, \mathfrak{r}^{-1}], \\ \mathfrak{a}_{-1}^{-2} \supset [\mathfrak{r}^{-1}, \mathfrak{s}^{-1}], \\ \mathfrak{a}_1^{0} \supset [\mathfrak{r}^{-1}, \mathfrak{s}^{1}]. \end{cases}$$

And by considering the eigenvalues of $ad\, E_s$, we have from Lemma 3.4,

$$(3.8)\quad \begin{cases} [\mathfrak{a}_\lambda^{-2}, \mathfrak{s}^1]=0 \quad \text{for} \quad \lambda\neq -1\,, \\ [\mathfrak{a}_\lambda^{\,0}, \mathfrak{s}^{-1}]=0 \quad \text{for} \quad \lambda\neq 1\,. \end{cases}$$

Lemma 3.5.

(1) $\mathfrak{a}_\lambda^{-2}=0$ *for* $\lambda\leqq -2$ *or* $\lambda>0$,
$\mathfrak{a}_\lambda^{\,0}=0$ *for* $\lambda<0$ *or* $\lambda\geqq 2$.

(2) $[\mathfrak{a}_\lambda^{-2}, \mathfrak{s}^2]=\mathfrak{a}_{\lambda+2}^0$ *for* $-2<\lambda<0$,
$[\mathfrak{a}_\lambda^{\,0}, \mathfrak{s}^{-2}]=\mathfrak{a}_{\lambda-2}^{-2}$ *for* $0<\lambda<2$.

Proof. From (3.1), $[\mathfrak{a}_\lambda^{-2}, \mathfrak{s}^2]\subset\mathfrak{a}_{\lambda+2}^0$ and $[\mathfrak{a}_\lambda^{\,0}, \mathfrak{s}^{-2}]\subset\mathfrak{a}_{\lambda-2}^{-2}$. Clearly $\mathfrak{a}_\lambda^{-2}$ and $\mathfrak{a}_\lambda^{\,0}$ are *ad* $\mathfrak{s}^0$-invariant subspaces, because $[E_s, \mathfrak{s}^0]=0$. Therefore by (3.8) the space $\mathfrak{a}_\lambda^{-2}+\mathfrak{a}_{\lambda+2}^0$ is *ad* $\mathfrak{s}$-invariant for $\lambda\neq -1$. Since $E_s\in[\mathfrak{s}, \mathfrak{s}]$, we have

$$Tr\ ad\ E_s|_{\mathfrak{a}_\lambda^{-2}+\mathfrak{a}_{\lambda+2}^0}=0\,.$$

In the case $\lambda=-1$, the space $\mathfrak{a}_{-1}^{-2}+\mathfrak{r}^{-1}+\mathfrak{a}_1^{\,0}$ is also *ad* $\mathfrak{s}$-invariant by (3.7). And by Lemma 3.4, we get the above equality for $\lambda=-1$. It follows

$$\lambda \dim \mathfrak{a}_\lambda^{-2}+(\lambda+2)\dim \mathfrak{a}_{\lambda+2}^0=0.$$

If $\mathfrak{a}_\lambda^{-2}\neq 0$, then we have

$$(3.9)\qquad \lambda=\frac{-2\dim \mathfrak{a}_{\lambda+2}^0}{\dim \mathfrak{a}_\lambda^{-2}+\dim \mathfrak{a}_{\lambda+2}^0}\,.$$

Therefore we get $-2<\lambda\leqq 0$. We can verify the fact that $\mathfrak{a}_\lambda^{\,0}=0$ for $\lambda<0$ or $\lambda\geqq 2$ by the same way. Thus we obtain Assertion (1). For $\lambda\neq -1$, the space $\mathfrak{a}_\lambda^{-2}+[\mathfrak{a}_\lambda^{-2}, \mathfrak{s}^2]$ is also *ad* $\mathfrak{s}$-invariant. And for $\lambda=-1$, the space $\mathfrak{a}_{-1}^{-2}+\mathfrak{r}^{-1}+[\mathfrak{a}_{-1}^{-2}, \mathfrak{s}^2]$ is *ad* $\mathfrak{s}$-invariant, because by (3.7) we have

$$[\mathfrak{r}^{-1}, \mathfrak{s}^1]=[\mathfrak{r}^{-1}, [\mathfrak{s}^2, \mathfrak{s}^{-1}]]=[[\mathfrak{r}^{-1}, \mathfrak{s}^{-1}], \mathfrak{s}^2]\subset[\mathfrak{a}_{-1}^{-2}, \mathfrak{s}^2].$$

Thus we have

$$\lambda \dim \mathfrak{a}_\lambda^{-2}+(\lambda+2)\dim[\mathfrak{a}_\lambda^{-2}, \mathfrak{s}^2]=0.$$

As a result we get $\dim \mathfrak{a}_{\lambda+2}^0=\dim[\mathfrak{a}_\lambda^{-2}, \mathfrak{s}^2]$ for $-2<\lambda<0$. Similarly we can prove $\dim \mathfrak{a}_{\lambda-2}^{-2}=\dim[\mathfrak{a}_\lambda^{\,0}, \mathfrak{s}^{-2}]$ for $0<\lambda<2$. Therefore we obtain Assertion (2). q.e.d.

Lemma 3.6. $[\mathfrak{a}_0^{-2}+\mathfrak{a}_0^{\,0}, \mathfrak{s}]=0.$

Proof. By (3.8) and Lemma 3.5, we have $[\mathfrak{a}_0^{-2}, \mathfrak{s}^1+\mathfrak{s}^2]=0$. Since $\mathfrak{s}^0=[\mathfrak{s}^{-2}, \mathfrak{s}^2]+[\mathfrak{s}^{-1}, \mathfrak{s}^1]$, we get $[\mathfrak{a}_0^{-2}, \mathfrak{s}]=0$. The fact $[\mathfrak{a}_0^{0}, \mathfrak{s}]=0$ can be verified similarly. q.e.d.

3.3. In the next section, we shall prove the followings:

$$(3.10)\quad \begin{cases} \mathfrak{a}_\lambda^{-2}=0 \quad \text{for} \quad \lambda\neq -1,\ 0. \\ \mathfrak{a}_\lambda^{0}=0 \quad \text{for} \quad \lambda\neq 0,\ 1. \\ \dim \mathfrak{a}_{-1}^{-2}=\dim \mathfrak{a}_1^{0}. \end{cases}$$

We can now prove Theorem 3.1 under the assumption that (3.10) holds. From (3.6), (3.10), Lemma 3.5 and Lemma 3.6, it follows

$$[\mathfrak{r}^{-2}, \mathfrak{s}^2]=[\mathfrak{a}_{-1}^{-2}, \mathfrak{s}^2]=\mathfrak{a}_1^{0}.$$

And $\mathfrak{a}_1^0=ad\,E_s(\mathfrak{a}_1^0)\subset[\mathfrak{s}^0, \mathfrak{r}^0]$. On the other hand,

$$\begin{aligned}[\mathfrak{r}^0, \mathfrak{s}^0] &\subset [\mathfrak{r}^0, [\mathfrak{s}^{-2}, \mathfrak{s}^2]]+[\mathfrak{r}^0, [\mathfrak{s}^{-1}, \mathfrak{s}^1]] \\ &\subset [\mathfrak{r}^{-2}, \mathfrak{s}^2]+[\mathfrak{r}^{-1}, \mathfrak{s}^1] \\ &=\mathfrak{a}_1^0.\end{aligned}$$

Therefore we have

$$(3.11)\qquad \mathfrak{a}_1^0=[\mathfrak{r}^{-2}, \mathfrak{s}^2]=[\mathfrak{r}^0, \mathfrak{s}^0].$$

Similarly

$$(3.12)\qquad \mathfrak{a}_{-1}^{-2}=[\mathfrak{r}^{-2}, \mathfrak{s}^0]=[\mathfrak{r}^0, \mathfrak{s}^{-2}].$$

From (3.2), (3.3), (3.5) and Lemma 3.6, we known $\mathfrak{a}_0^{-2}=\mathfrak{r}_0^{-2}$, $\mathfrak{a}_{-1}^{-2}=\mathfrak{r}_s^{-2}$, $\mathfrak{a}_0^0=\mathfrak{r}_0^0$ and $\mathfrak{a}_1^0=\mathfrak{r}_s^0$. Then Assertions (1), (2) and (3) of Theorem 3.1 follows from (3.7), (3.10), (3.11) and (3.12). Assertion (4) is already proved in Lemma 3.4. Since $\mathfrak{r}_s^0=[\mathfrak{r}^{-2}, \mathfrak{s}^2]=[\mathfrak{r}^{-2}, \mathfrak{g}^2]$, $\mathfrak{r}_s^0$ is clearly an ideal of $\mathfrak{g}^0$. And by considering the eigenvalues of $ad\,E_s$, we get Assertion (5).

Corollary 3.7. *Let* $\mathfrak{s}$ *be as in Theorem* 2.1. *Assume that the domain D is non-degenerate. Then we have*

$$\mathfrak{r}_s^{-2}=[\mathfrak{r}^{-1}, \mathfrak{s}^{-1}],\ \ \mathfrak{r}_0^{-2}=[\mathfrak{r}^{-1}, \mathfrak{r}^{-1}]\ \ \textit{and}\ \ \mathfrak{r}_s^0=[\mathfrak{r}^{-1}, \mathfrak{s}^1].$$

Proof. Since $\mathfrak{g}^{-1}=\mathfrak{r}^{-1}+\mathfrak{s}^{-1}$, we have

$$\mathfrak{g}^{-2}=[\mathfrak{g}^{-1},\mathfrak{g}^{-1}]=[\mathfrak{s}^{-1},\mathfrak{s}^{-1}]+[\mathfrak{s}^{-1},\mathfrak{r}^{-1}]+[\mathfrak{r}^{-1},\mathfrak{r}^{-1}].$$

Now our assertions follow immediately from Theorem 3.1. q.e.d.

Remark 1. We can easily observe

$$\mathfrak{r}_0^{-2}=\{X\in\mathfrak{g}^{-2};\ [X,\mathfrak{g}^2]=0\}.$$

Therefore the space $\mathfrak{r}_0^{-2}$ is independent of the choice of the semi-simple graded subalgebra $\mathfrak{s}$. Clearly so is $\mathfrak{r}_s^0$.

§ 4. The subalgebra $\mathfrak{s}$ and the symmetric domain S.

4.1. Let D be a Siegel domain of the second kind in $R_c\times W$ associated with a convex cone V in R and a V-hermitian form F on W. We use the notations given in § 2 and § 3.

The subalgebra $\mathfrak{g}^0$ may be identified with a subalgebra of the Lie algebra of all graded derivations of the graded Lie algebra $\mathfrak{g}^{-2}+\mathfrak{g}^{-1}$. Let $\hat{\mathfrak{g}}=\sum_{\lambda=-2}^{\infty}\hat{\mathfrak{g}}^{\lambda}$ be the algebraic prolongation of $(\mathfrak{g}^{-2}+\mathfrak{g}^{-1},\mathfrak{g}^0)$ (cf. [9]). In earlier paper [3], the author proved the following theorem which is a generalization of Tanaka's result [9].

Theorem 4.1. *The Lie algebra $\mathfrak{g}(D)$ can be imbedded as a graded subalgebra of $\hat{\mathfrak{g}}$ and $\mathfrak{g}^1$ and $\mathfrak{g}^2$ are determined as follows*:

(1) $\mathfrak{g}^1=\hat{\mathfrak{g}}^1$.

(2) $\mathfrak{g}^2$ *consists of all $X\in\hat{\mathfrak{g}}^2$ such that* $\mathrm{Im}\,ad([X,Y])|_{\mathfrak{g}^{-1}}=0$ *for all $Y\in\mathfrak{g}^{-2}$, where $ad([X,Y])|_{\mathfrak{g}^{-1}}$ is considered as a complex linear endomorphism of $\mathfrak{g}^{-1}$ with the complex structure $ad\ I$.*

4.2. Let $\mathfrak{s}$ be as in Theorem 2.1. Denote by η_s the projection of $\mathfrak{g}_c^{-2}+\mathfrak{g}^{-1}(=R_c\times W)$ onto $\mathfrak{s}_c^{-2}+\mathfrak{s}^{-1}$ corresponding to the direct sum: $\mathfrak{g}_c^{-2}+\mathfrak{g}^{-1}=\mathfrak{s}_c^{-2}+\mathfrak{s}^{-1}+\mathfrak{r}_c^{-2}+\mathfrak{r}^{-1}$. We put

$$V_s=\eta_s(V). \tag{4.1}$$

Lemma 4.2. *The set V_s is a convex cone in $\mathfrak{s}^{-2}$.*

Proof. It is sufficient to prove that V_s contains no entire straight lines. Let $v\in\mathfrak{g}^{-2}$. Then we can wright $v=v_s+v_r$, where $v_s\in\mathfrak{s}^{-2}$ and $v_r\in\mathfrak{r}^{-2}$. We assert

$$\lim_{t\to\infty}\frac{1}{e^{2t}}\exp(-tE_s)v_r=0.$$

In fact, by Lemma 3. 5, we can write

$$v_r=\sum_{\lambda} u_\lambda, \quad u_\lambda \in \mathfrak{a}_\lambda^{-2} \quad (-2<\lambda\leqq 0)$$

Then

$$\lim_{t\to\infty} \frac{1}{e^{2t}} \exp(-tE_s)\, u_\lambda = \lim_{t\to\infty} \frac{1}{e^{t(2+\lambda)}} u_\lambda = 0\,,$$

proving our assertion. As a result

$$\lim_{t\to\infty} \frac{1}{e^{2t}} \exp(-tE_s)\, v = v_s = \eta_s(v)\,.$$

Since $\exp(-tE_s)\, V=V$, V_s is contained in $\overline{V}\cap\mathfrak{s}^{-2}$. This fact implies that V_s contains no entire straight lines. q.e.d.

The restriction F_s of F to the complex subspace $\mathfrak{s}^{-1}\times\mathfrak{s}^{-1}$ of $\mathfrak{g}^{-1}\times\mathfrak{g}^{-1}$ is clearly a V_s-hermitian form. Denote by S the Siegel domain of the second kind in $\mathfrak{s}_c^{-2}+\mathfrak{s}^{-1}$ associated with V_s and F_s.

Proposition 4. 3. *The projection η_s maps D onto S.*

Proof. Let $z+w\in\mathfrak{g}_c^{-2}+\mathfrak{g}^{-1}$ $(z\in\mathfrak{g}_c^{-2}, w\in\mathfrak{g}^{-1})$. Then $\eta_s(\operatorname{Im} z-F(w, w)) = \operatorname{Im} \eta_s(z) - F_s(\eta_s(w), \eta_s(w))$. Therefore $\eta_s(D)\subset S$. Conversely, let $z+w\in S(z\in\mathfrak{s}_c^{-2}, w\in\mathfrak{s}^{-1})$. Then $\operatorname{Im} z-F(w, w)\in V_s$. There exists $y\in\mathfrak{r}^{-2}$ such that $\operatorname{Im} z-F(w, w)+y\in V$. We then have $z+\sqrt{-1}y+w\in D$. As a result $z+w=\eta_s(z+\sqrt{-1}y+w)\in\eta_s(D)$. q.e.d.

Next we shall prove the following

Theorem 4. 4. *The domain S is symmetric and $\mathfrak{s}$ may be identified with the Lie algebra of* Aut(S).

Proof. Let $\mathfrak{g}(S)=\mathfrak{s}'^{-2}+\mathfrak{s}'^{-1}+\mathfrak{s}'^{0}+\mathfrak{s}'^{1}+\mathfrak{s}'^{2}$ be the graded Lie algebra of Aut(S). Then $\mathfrak{s}'^{-2}=\mathfrak{s}^{-2}$ and $\mathfrak{s}'^{-1}=\mathfrak{s}^{-1}$. Since $\exp tA\,(V_s)=V_s$ for any $A\in\mathfrak{s}^0$, $\mathfrak{s}^0$ may be identified with a subalgebra of $\mathfrak{s}'^0$. (Note that the adjoint representation of $\mathfrak{s}^0$ on $\mathfrak{s}^{-2}+\mathfrak{s}^{-1}$ is faithful.) Let $\hat{\mathfrak{s}}'=\sum_{\lambda=-2}^{\infty}\hat{\mathfrak{s}}'^{\lambda}$ be the algebraic prolongation of $(\mathfrak{s}'^{-2}+\mathfrak{s}'^{-1}, \mathfrak{s}'^{0})$. By Lemma 1. 2, the Lie algebra $\mathfrak{s}$ can be imbedded as a graded subalgebra of $\hat{\mathfrak{s}}'$. Therefore by Theorem 4. 1, we know that $\mathfrak{s}^1$ is a subspace of $\mathfrak{s}'^1$. Let $X\in\mathfrak{s}^2$. Then for any $Y\in\mathfrak{s}^{-2}$,

$$\mathrm{Im}\, Tr\, ad([X, Y])|_{\mathfrak{g}^{-1}} = 0 \quad (\text{cf. Theorem 4.1}).$$

Since $ad([X, Y]) = 0$ on $\mathfrak{r}^{-1}$, we have

$$\mathrm{Im}\, Tr\, ad([X, Y]|_{\mathfrak{s}^{-1}} = 0 . \tag{4.2}$$

Let I_s be the element of $\mathfrak{s}'^0$ given by (2.2)′ for the domain S. It is clear that $ad\, I = ad\, I_s$ on $\mathfrak{s}^{-1}$. Thereby from (4.2) and Theorem 4.1, we know $\mathfrak{s}^2 \subset \mathfrak{s}'^2$. Since $\dim \mathfrak{s}'^1 \leqq \dim \mathfrak{s}^{-1} = \dim \mathfrak{s}^1$ and $\dim \mathfrak{s}'^2 \leqq \dim \mathfrak{s}^{-2} = \dim \mathfrak{s}^2$, we have $\mathfrak{s}^1 = \mathfrak{s}'^1$ and $\mathfrak{s}^2 = \mathfrak{s}'^2$. As a result, the radical of $\mathfrak{g}(S)$ is trivial[3] and hence S is a symmetric homogeneous domain. Since the adjoint representation of $\mathfrak{s}'^0$ on $\mathfrak{s}'^{-2} + \mathfrak{s}'^{-1}$ is faithful, by using Lemma 1.3, we have

$$\mathfrak{s}'^0 = [\mathfrak{s}'^{-2}, \mathfrak{s}'^2] + [\mathfrak{s}'^{-1}, \mathfrak{s}'^1] = \mathfrak{s}^0,$$

which completes the proof. q.e.d.

4.3. Let S' be the Siegel domain of the first kind associated with the cone V_s. It is well known that the domain S' is symmetric.

Proposition 4.5.[4] *The subalgebra* $\mathfrak{s}^{-2} + [\mathfrak{s}^{-2}, \mathfrak{s}^2] + \mathfrak{s}^2$ *is semi-simple and may be identified with the Lie algebra of* $\mathrm{Aut}(S')$.

Proof. Let $\mathfrak{g}(S') = \mathfrak{s}'^{-2} + \mathfrak{s}'^0 + \mathfrak{s}'^2$ be the graded Lie algebra of $\mathrm{Aut}(S')$. There exists a natural homomorphism α_s of $\mathfrak{s}^{-2} + \mathfrak{s}^0 + \mathfrak{s}^2$ to $\mathfrak{g}(S')$ as graded Lie algebras ([1]) such that α_s is injective on $\mathfrak{s}^{-2} + \mathfrak{s}^2$ and $\alpha_s(\mathfrak{s}^{-2}) = \mathfrak{s}'^{-2}$ (cf. [4]). Since $\dim \mathfrak{s}'^2 \leqq \dim \mathfrak{s}'^{-2} = \dim \mathfrak{s}^{-2} = \dim \mathfrak{s}^2$, we have $\alpha_s(\mathfrak{s}^2) = \mathfrak{s}'^2$ and $\dim \mathfrak{s}'^2 = \dim \mathfrak{s}'^{-2}$. (From this fact, we known that $\mathfrak{g}(S')$ is semi-simple and that S' is symmetric.) Since the adjoint representation of $\mathfrak{s}'^0$ on $\mathfrak{s}'^{-2}$ is faithful, we obtain $\mathfrak{s}'^0 = [\mathfrak{s}'^{-2}, \mathfrak{s}'^2]$ by Lemma 1.3. As a result, α_s is surjective. Let $\mathfrak{c}$ be the radical of $\mathfrak{s}^{-2} + \mathfrak{s}^0 + \mathfrak{s}^2$. Since α_s is surjective, $\alpha_s(\mathfrak{c})$ is a solvable ideal of $\mathfrak{g}(S')$ and hence is trivial. Therefore $\mathfrak{c}$ is contained in $\mathfrak{s}^0 \cap \alpha_s^{-1}(0)$. By Theorem 1.1, there exists a semi-simple graded subalgebra $\mathfrak{s}_1$ such that

$$\mathfrak{s}^{-2} + \mathfrak{s}^0 + \mathfrak{s}^2 = \mathfrak{s}_1 + \mathfrak{c} \text{ (direct sum)},$$

$$\mathfrak{s}_1 = \mathfrak{s}^{-2} + \mathfrak{s}_1^0 + \mathfrak{s}^2 \quad (\mathfrak{s}_1^0 = \mathfrak{s}_1 \cap \mathfrak{s}^0).$$

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[3] For any graded ideal $\mathfrak{h}$ of $\mathfrak{g}(S)$, the condition $\mathfrak{h}^{-2} = 0$ implies $\mathfrak{h} = 0$.

[4] This proposition holds for any symmetric Siegel domain of the second kind.

We set $\mathfrak{s}_2^0=\{X\in\mathfrak{s}_1^0;\ [X,\mathfrak{s}^{-2}]=0\}$. Then by considering the decomposition of $\mathfrak{s}_1$ into simple ideals and by Lemma 1. 3, we know that $\mathfrak{s}_1=\mathfrak{s}^{-2}+[\mathfrak{s}^{-2},\mathfrak{s}^2]+\mathfrak{s}^2+\mathfrak{s}_2^0$ (direct sum) and that the subalgebra $\mathfrak{s}^{-2}+[\mathfrak{s}^{-2},\mathfrak{s}^2]+\mathfrak{s}^2$ is semi-simple. It follows that $\mathfrak{s}_2^0+\mathfrak{c}=\{X\in\mathfrak{s}^0;\ [X,\mathfrak{s}^{-2}]=0\}=\alpha_s^{-1}(0)$. And α_s is an isomorphism of $\mathfrak{s}^{-2}+[\mathfrak{s}^{-2},\mathfrak{s}^2]+\mathfrak{s}^2$ onto $\mathfrak{g}(S')$. q.e.d.

Corollary 4. 6. *The element E_s belongs to $[\mathfrak{s}^{-2},\mathfrak{s}^2]$.*

Proof. Since the Lie algebra $\mathfrak{s}^{-2}+[\mathfrak{s}^{-2},\mathfrak{s}^2]+\mathfrak{s}^2$ is semi-simple, there exists a unique element E_s' of $[\mathfrak{s}^{-2},\mathfrak{s}^2]$ such that $ad\,E_s'=-2id.$ on $\mathfrak{s}^{-2}$, $ad\,E_s'=0$ on $[\mathfrak{s}^{-2},\mathfrak{s}^2]$ and $ad\,E_s'=2id.$ on $\mathfrak{s}^2$. By Lemma 1.5, the endomorphism $ad\,E_s'$ of $\mathfrak{s}$ is real diagonal. As a result

$$\mathfrak{s}^{-1}=\sum_\alpha \mathfrak{s}_\alpha^{-1},\quad \text{where}\quad \mathfrak{s}_\alpha^{-1}=\{X\in\mathfrak{s}^{-1};\ [E_s',X]=\alpha X\}.$$

Let $X\in\mathfrak{s}_\alpha^{-1}$. Then $[E_s',[[I,X],X]]=2\alpha[[I,X],X]$. It follows that $\mathfrak{s}_\alpha^{-1}=0$ for $\alpha\neq-1$. Therefore $ad(E_s-E_s')=0$ on $\mathfrak{s}^{-2}+\mathfrak{s}^{-1}$ and hence $E_s=E_s'$, because the representation of $\mathfrak{s}^0$ on $\mathfrak{s}^{-2}+\mathfrak{s}^{-1}$ is faithful. q.e.d.

4. 4. We can now prove (3. 10). Let $v\in V_s$. Then the correspondence: $X\to[v,[v,X]]$ $(X\in\mathfrak{s}^2)$ is an injective linear mapping of $\mathfrak{s}^2$ to $\mathfrak{s}^{-2}$ ([11]). Since $\dim\mathfrak{s}^{-2}=\dim\mathfrak{s}^2$, we have

$$\mathfrak{s}^{-2}=[v,[v,\mathfrak{s}^2]]. \tag{4. 4}$$

For $0<\lambda<2$, by Lemma 3. 5

$$\begin{aligned}\mathfrak{a}_{\lambda-2}^{-2}=[\mathfrak{a}_\lambda^0,\mathfrak{s}^{-2}]&=[\mathfrak{a}_\lambda^0,[v,[v,\mathfrak{s}^2]]]\\&\subset[[\mathfrak{a}_\lambda^0,v],[v,\mathfrak{s}^2]]+[v,[\mathfrak{a}_\lambda^0,[v,\mathfrak{s}^2]]]\\&\subset[\mathfrak{a}_{\lambda-2}^{-2},[v,\mathfrak{s}^2]]+[v,\mathfrak{a}_\lambda^0]\\&=[v,[\mathfrak{a}_{\lambda-2}^{-2},\mathfrak{s}^2]]+[v,\mathfrak{a}_\lambda^0]\\&=[v,\mathfrak{a}_\lambda^0].\end{aligned}$$

Therefore $\mathfrak{a}_{\lambda-2}^{-2}=[v,\mathfrak{a}_\lambda^0]$ and hence $\dim\mathfrak{a}_{\lambda-2}^{-2}\leqq\dim\mathfrak{a}_\lambda^0$. It is well known that there exists an involutive automorphism σ of the semi-simple Lie algebra $\mathfrak{s}^{-2}+[\mathfrak{s}^{-2},\mathfrak{s}^2]+\mathfrak{s}^2$ such that $\sigma(\mathfrak{s}^{-2})=\mathfrak{s}^2$ and $\sigma(\mathfrak{s}^2)=\mathfrak{s}^{-2}$.[5] Then

[5] cf. [8].

from (4. 4) we have

$$\mathfrak{s}^2 = [\sigma(v), [\sigma(v), \mathfrak{s}^{-2}]].$$

Therefore we get $\mathfrak{a}_\lambda{}^0 = [\sigma(v), \mathfrak{a}_{\lambda-2}^{-2}]$ similarly and hence $\dim \mathfrak{a}_{\lambda-2}^{-2} \geqq \dim \mathfrak{a}_\lambda{}^0$. Thus we have $\dim \mathfrak{a}_{\lambda-2}^{-2} = \dim \mathfrak{a}_\lambda{}^0$. And by using (3. 9), we get $\lambda = 1$ if $\mathfrak{a}_\lambda{}^0 \neq 0$. This implies (3. 10).

4. 5. The projection η_s gives a "*fibering*" of D in which the bace space is the symmetric space S. We shall show that any two fibers are holomorphically equivalent to each other and are equivalent to a bounded domain.

Let $a = z_s + w_s \in S (z_s \in \mathfrak{s}_c^{-2}, w_s \in \mathfrak{s}^{-1})$. We set

$$V(a) = \{y \in \mathfrak{r}^{-2};\ y + \mathrm{Im}\, z_s - F(w_s, w_s) \in V\}.$$

Clearly $V(a)$ is an open convex set in $\mathfrak{r}^{-2}$ and contains no entire straight lines. And

$$\eta_s^{-1}(a) \cong \{z + w \in \mathfrak{r}_c^{-2} + \mathfrak{r}^{-1};$$
$$\mathrm{Im}\, z - F(w, w) - 2\,\mathrm{Re}\, F(w, w_s) \in V(a)\}.$$

Therefore the fiber $\eta_s^{-1}(a)$ is a domain in $\mathfrak{r}_c^{-2} + \mathfrak{r}^{-1}$. Let $a' = z_s - \sqrt{-1}$ $F(w_s, w_s)$. Then $a' \in S$ and $V(a') = V(a)$. And

$$\eta_s^{-1}(a') \cong \{z + w \in \mathfrak{r}_c^{-2} + \mathfrak{r}^{-1};\ \mathrm{Im}\, z - F(w, w) \in V(a)\}.$$

Lemma 4. 7. (1) *The domains $\eta_s^{-1}(a)$ and $\eta_s^{-1}(a')$ are holomorphically equivalent to each other.*

(2) *The domain $\eta_s^{-1}(a')$ is holomorphically equivalent to a bounded domain.*

Proof. Assertion (2) follows immediately from the fact that $V(a)$ is an open convex set, containing no entire straight lines. We can easily observe that the automorphism of $\mathfrak{r}_c^{-2} + \mathfrak{r}^{-1}$ defiined by

$$z + w \to z - 2\sqrt{-1}\, F(w, w_s) + w \quad (z \in \mathfrak{r}_c^{-2}, w \in \mathfrak{r}^{-1})$$

maps $\eta_s^{-1}(a)$ onto $\eta_s^{-1}(a')$. Thus we get Assertion (1). q.e.d.

Let $b = z_s' + w_s' \in S$ and $b' = z_s' - \sqrt{-1}\, F(w_s', w_s')$. The homogeneity of S implies that V_s is affine homogeneous. Therefore there exist $A_1, \cdots, A_n \in \mathfrak{s}^0$ such that

$$\exp A_1 \circ \cdots \circ \exp A_n (\operatorname{Im} a') = \operatorname{Im} b'.$$

We set $f = \exp A_1 \circ \cdots \circ \exp A_n$. Then the linear transformation of $\mathfrak{r}_c^{-2} + \mathfrak{r}^{-1}$ defined by

$$z + w \to fz + fw \quad (z \in \mathfrak{r}_c^{-2}, w \in \mathfrak{r}^{-1})$$

maps $V(a)$ onto $V(b)$ and hence maps $\eta_s^{-1}(a')$ onto $\eta_s^{-1}(b')$. As a result, by Lemma 4.7, we get $\eta_s^{-1}(a) \cong \eta_s^{-1}(b)$. Thus we have proved

Theorem 4.8. *Let $a, b \in S$. Then the two fibers $\eta_s^{-1}(a)$ and $\eta_s^{-1}(b)$ are holomorphically equivalent to each other. Moreover every fiber is holomorphically equivalent to a bounded domain.*

The domain S is contained in $R_c \times W$ in a natural manner. Let $z + w \in S$. Then $\operatorname{Im} z - F(w, w) \in V_s \subset \overline{V}$ (cf. Proof of Lemma 4.2). Hence $z + w \in \overline{D}$. Thus we know that S is contained in $\overline{D}$. Moreover we can prove the following

Proposition 4.9. *If $\mathfrak{r} = 0$, then $S = D$. And if $\mathfrak{r} \neq 0$, then S is contained in the boundary of D.*

Proof. It is clear from the construction that S coincides with D in the case where $\mathfrak{r} = 0$. We now assume that $\mathfrak{r} \neq 0$. And suppose that there exists $p \in S \cap D (p = z + w)$. Let $E' = E - E_s$. Since $ad\,E' = 0$ on $\mathfrak{s}$, E' is contained in the isotropy subalgebra of $\mathfrak{g}(D)$ at $p \in D$. Hence all eigenvalues of $ad\,E'$ are purely imaginary. On the other hand, $ad\,E' = -id.$ on $\mathfrak{r}_s^{-2}$ and $ad\,E' = -2id.$ on $\mathfrak{r}_0^{-2}$. Therefore by Theorem 3.1 we have $\mathfrak{r}^{-2} = 0$. As a result $\mathfrak{r} = 0$, contradicting the assumption that $\mathfrak{r} \neq 0$. q.e.d.

Remark 2. For every simple ideal of $\mathfrak{s}$, we can construct a symmetric domain in $\overline{D}$, for which similar assertions in Theorem 4.4 and Theorem 4.8 hold.

Remark 3. If $\mathfrak{r} \neq 0$. Then the domain S is contained in the boundary of D by Proposition 4.9. Moreover we can show that S is a regular boundary component of D, i.e., a regular analytic set in

$R_c \times W$ contained in the boundary of D with the property that every analytic curve $\psi(t)$ in the boundary of D which meets S is completely contained in S (cf. [6]).

§ 5. The uniqueness of the domain S.

5. 1. The symmetric domain S is constructed from the semi-simple graded subalgebra $\mathfrak{s}$. Let $\mathfrak{s}' = \sum_{\lambda=-2}^{2} \mathfrak{s}'^{\lambda}$ be another semi-simple graded subalgebra of $\mathfrak{g}(D)$ having the properties 1) and 2) in Theorem 2. 1. And let $E_{s'}$, be corresponding element of $\mathfrak{s}'^{0}$ defined by (3. 1) for the subalgebla $\mathfrak{s}'$.

Lemma 5. 1. $E_{s'} - E_s \in \mathfrak{r}_s^0$.

Proof. It follows from Theorem 3. 1 and Corollary 4. 6

$$E_{s'} \in [\mathfrak{s}'^{-2}, \mathfrak{s}'^{2}] \subset [\mathfrak{r}^{-2} + \mathfrak{s}^{-2}, \mathfrak{s}^{2}] \subset \mathfrak{s}^{0} + \mathfrak{r}_s^0.$$

Thus we can write $E_{s'} = E' + A$, where $E' \in \mathfrak{s}^0$ and $A \in \mathfrak{r}_s^0$. Since $ad\, E' = ad\, E_{s'} = ad\, E_s$ on $\mathfrak{s}^1 + \mathfrak{s}^2$, we have $E' = E_s$ because E' and E_s belong to $\mathfrak{s}^0$. q.e.d.

Now we con prove the following theorem which implies the uniqueness of the symmetric domain S.

Theorem 5. 2. *Let $\mathfrak{s}$ and $\mathfrak{s}'$ be two semi-simple garded subalgebra as in Theorem 2. 1. And let S and S' be the symmetric domains corresponding to $\mathfrak{s}$ and $\mathfrak{s}'$ respectively. Then there exists $A \in \mathfrak{g}^0$ suct that*

1) $Ad(\exp A)\, \mathfrak{s}' = \mathfrak{s}$.

2) $\exp AS = S'$ *and* $\eta_s \circ \exp A = \exp A \circ \eta_{s'}$.

Proof. Let $A = E_{s'} - E_s$. Then by Lemma 5. 1, we have $ad\, E_{s'} A = A$. (Note that $\mathfrak{r}_s^0 = [\mathfrak{r}^{-2}, \mathfrak{g}^2] = \mathfrak{r}_{s'}^0$.) Thus we get $Ad(\exp A) E_{s'} = E_{s'} - A = E_s$. Clearly $Ad(\exp A)\, \mathfrak{s}'^{\lambda} = \mathfrak{s}^{\lambda}$ for $\lambda = 1, 2$. From Theorem 3. 1, we know

$$\mathfrak{s}^{\lambda} = \{X \in \mathfrak{g}^{\lambda};\ [E_s, X] = \lambda X\} \quad \text{for } \lambda = -2,\ -1$$

$$(\text{resp. } \mathfrak{s}'^{\lambda} = \{X \in \mathfrak{g}^{\lambda};\ [E_{s'}, X] = \lambda X\} \quad \text{for } \lambda = -2,\ -1).$$

Therefore $Ad(\exp A) E_{s'} = E_s$ implies $Ad(\exp A)\, \mathfrak{s}'^{\lambda} = \mathfrak{s}^{\lambda}$ for $\lambda = -2$,

-1. It follows

$$Ad(\exp A)\mathfrak{s}'^0 = Ad(\exp A)\ \ ([\mathfrak{s}'^{-2}, \mathfrak{s}'^2] + [\mathfrak{s}'^{-1}, \mathfrak{s}'^1])$$

$$= [\mathfrak{s}^{-2}, \mathfrak{s}^2] + [\mathfrak{s}^{-1}, \mathfrak{s}^1] = \mathfrak{s}^0.$$

Thus we have proved 1). Let $z+w \in \mathfrak{g}_c^{-2} + \mathfrak{g}^{-1}$. We write $z = z_{s'} + z_r$ $(z_{s'} \in \mathfrak{s}_c'^{-2}, z_r \in \mathfrak{r}_c^{-2})$ and $w = w_{s'} + w_r (w_{s'} \in \mathfrak{s}'^{-1}, w_r \in \mathfrak{r}^{-1})$. Then $\eta_s \circ \exp A(z+w) = \eta_s \circ \exp A(z_{s'} + w_{s'}) = \exp A(z_{s'} + w_{s'}) = \exp A \circ \eta_{s'}(z+w)$. Therefore $\eta_s \circ \exp A = \exp A \circ \eta_{s'}$. As a result $S = \eta_s(D) = \eta_s \circ \exp A(D) = \exp A \circ \eta_{s'}(D) = \exp AS'$. Thus we get 2). q.e.d.

By Theorem 5.2, the domain S has an invariant meaning. In what follows we call S *the associated symmetric domain.*

Corollary 5.3. *Let D (resp. D') be a Siegel domain of the second kind in $R_c \times W$ (resp. in $R_c' \times W'$). And let S (resp. S') be the associated symmetric domain corresponding to D (resp. to D'). Assume that the two domains D and D' are holomorphically equivalent. Then there exists a linear isomorphism f of $R_c \times W$ onto $R_c' \times W'$ such that*

$$f(D) = D' \quad and \quad f(S) = S'.$$

Proof. From [1], we know that there exists a linear isomorphism g of $R_c \times W$ onto $R_c' \times W'$ such that $g(R) = R$, $g(W) = W$ and $g(D) = D'$. The isomorphism g induces an isomorphism g_* of $\mathfrak{g}(D)$ $(= \sum_{\lambda=-2}^{2} \mathfrak{g}^\lambda)$ onto $\mathfrak{g}(D')$ $(= \sum_{\lambda=-2}^{2} \mathfrak{g}'^\lambda)$. Clearly $g_*(\mathfrak{g}^{-2}) = \mathfrak{g}'^{-2}$ and $g_*(\mathfrak{g}^{-1}) = \mathfrak{g}'^{-1}$. From this fact we can easily observe that $g_*(\mathfrak{g}^\lambda) = \mathfrak{g}'^\lambda$ for all λ. Let $\mathfrak{s}$ be the semi-simple graded subalgebra corresponding to S. Then $g_*(\mathfrak{s})$ satisfies the properties in Theorem 2.1 and $g(S)$ is just the symmetric domain corresponding to $g_*(\mathfrak{s})$. By Theorem 5.2, there exists $A \in \mathfrak{g}'^0$ such that $\exp A(g(S)) = S'$. Now the mapping $f = \exp A \circ g$ has the desired properties. q.e.d.

5.2. We set

$$\tilde{\mathfrak{k}} = \{X \in \mathfrak{g}^0;\ [X, \mathfrak{g}^1 + \mathfrak{g}^2] = 0\}.$$

Clearly $\tilde{\mathfrak{k}}$ is an ideal of $\mathfrak{g}^0$. The following proposition means that any semi-simple graded subalgebra as in Theorem 2.1 is obtained only by the method in the proof of Theorem 2.1.

Proposition 5.4. *Let* $\mathfrak{s}$ *be as in Theorem* 2.1. *Then there exists a semi-simple part* $\mathfrak{k}$ *of* $\tilde{\mathfrak{k}}$ *such that*

1) $\tilde{\mathfrak{k}}=\mathfrak{k}+\mathfrak{r}^0$ *(direct sum).*

2) *The direct sum* $\mathfrak{k}+\mathfrak{s}$ *is a semi-simple part of* $\mathfrak{g}(D)$ *and* $[\mathfrak{k}, \mathfrak{s}]=0$.

Proof. Let $\tilde{\mathfrak{s}}$ be as in Theorem 1.1. Then from the proof of Theorem 2.1, there exist ideals $\mathfrak{s}'$ and $\mathfrak{k}'$ of $\tilde{\mathfrak{s}}$ such that

i) $\mathfrak{s}'$ satisfies the properties 1) and 2) in Theorem 2.1.

ii) $\tilde{\mathfrak{s}}=\mathfrak{s}'+\mathfrak{k}'$ (direct sum) and hence $\mathfrak{k}'\subset\tilde{\mathfrak{k}}$.

We then have $\tilde{\mathfrak{k}}=\mathfrak{k}'+\mathfrak{r}^0$ (direct sum). Therefore $\mathfrak{k}'$ is a semi-simple part of $\tilde{\mathfrak{k}}$ and $\mathfrak{r}^0$ is the radical of $\tilde{\mathfrak{k}}$. By Theorem 5.2, there exists $A\in\mathfrak{g}^0$ such that $Ad(\exp A)\mathfrak{s}'=\mathfrak{s}$. We put $\mathfrak{k}=Ad(\exp A)\mathfrak{k}'$. Since $\tilde{\mathfrak{k}}$ is invariant by $Ad(\exp A)$, $\mathfrak{k}$ is a semi-simple part of $\tilde{\mathfrak{k}}$ and has the desired properties. q.e.d.

Theorem 2.1, Proposition 2.3, Theorem 3.1 and Proposition 5.4 give structure equations of $\mathfrak{g}(D)$. Note that the spaces $\mathfrak{r}_s^{-2}$ and $\mathfrak{r}_0^0$ are $ad\,\mathfrak{k}$-invariant (cf. Remark 1).

§ 6. Siegel domains over classical cones, I.

6.1. In this and the next paragraphs, $\boldsymbol{F}$ denotes the field $\boldsymbol{R}$ or $\boldsymbol{C}$. We denote by $M(p, q, \boldsymbol{F})$ the vector space of all $p\times q$ matrices over $\boldsymbol{F}$. For a matrix A, denote by A^* the transpose of the conjugate matrix $\bar{A}$ of A. And denote by e_p the unit matrix of degree p. We set

$$H(m, \boldsymbol{F}) = \{A\in M(m, m, \boldsymbol{F});\ A^*=A\},$$

$$H^+(m, \boldsymbol{F}) = \{A\in H(m, \boldsymbol{F});\ A \text{ is positive definite}\}.$$

Then $H^+(m, \boldsymbol{F})$ is a convex cone in the vector space $H(m, \boldsymbol{F})$. Let D be a Siegel domain of the second kind associated with the cone $V=H^+(m, \boldsymbol{F})$ in $H(m, \boldsymbol{F})$ and a V-hermitian form F on some vector space W. And let $\mathfrak{g}(D)=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2$ be the graded Lie algebra of $\mathrm{Aut}(D)$. Denote by D' the Siegel domain of the first kind associated with the cone $H^+(m, \boldsymbol{F})$, and by $\mathfrak{g}(D')=\mathfrak{g}'^{-2}+\mathfrak{g}'^0+\mathfrak{g}'^2$

the graded Lie algebra of $\mathrm{Aut}(D')$. There exists a natural homomorphism α of $\mathfrak{g}^{-2}+\mathfrak{g}^{0}+\mathfrak{g}^{2}$ to $\mathfrak{g}'^{-2}+\mathfrak{g}'^{0}+\mathfrak{g}'^{2}$ as graded Lie algebras such that α is injective on $\mathfrak{g}^{-2}+\mathfrak{g}^{2}$ and $\alpha(\mathfrak{g}^{-2})=\mathfrak{g}'^{-2}$ ([1], [4]). Therefore we identify $\mathfrak{g}^{-2}$ with $\mathfrak{g}'^{-2}$ and $\mathfrak{g}^{2}$ with the subspace $\alpha(\mathfrak{g}^{2})$ of $\mathfrak{g}'^{2}$. Let $A\in GL(m, \boldsymbol{F})$, the group of all non-singular matrices of degree m. Denote by $\theta(A)$ the transformation of $H(m, \boldsymbol{F})$ defined by

$$\theta(A)X=AXA^{*} \quad (X\in H(m, \boldsymbol{F})).$$

Clearly the cone $H^{+}(m, \boldsymbol{F})$ is invariant by $\theta(A)$. Therefore θ defines a homorphism: $GL(m, \boldsymbol{F})\to\mathrm{Aut}(D')$. Denote by θ_{*} the corresponding homomorphism: $\mathfrak{gl}(m, \boldsymbol{F})\to\mathfrak{g}(D')$, where $\mathfrak{gl}(m, \boldsymbol{F})$ is the Lie algebra of $GL(m, \boldsymbol{F})$, i.e., $\mathfrak{gl}(m, \boldsymbol{F})=M(m, m, \boldsymbol{F})$. It is well known that $\theta_{*}(\mathfrak{gl}(m, \boldsymbol{F}))=\mathfrak{g}'^{0}$. The kernel $\mathfrak{z}$ of θ_{*} is trivial if $\boldsymbol{F}=\boldsymbol{R}$ and is $\{\lambda\sqrt{-1}\,e_{m};\ \lambda\in\boldsymbol{R}\}$ if $\boldsymbol{F}=\boldsymbol{C}$. We set

$$\tilde{\mathfrak{g}}=\left\{\begin{pmatrix} A & B \\ C & -A^{*}\end{pmatrix}\in M(2m, 2m, \boldsymbol{F});\ \begin{matrix} B, C\in H(m, \boldsymbol{F}) \\ A\in\mathfrak{gl}(m, \boldsymbol{F})\end{matrix}\right\}.$$

Then $\tilde{\mathfrak{g}}$ is a Lie algebra in a usual bracket rule. The center $\tilde{\mathfrak{z}}$ of $\tilde{\mathfrak{g}}$ is trivial if $\boldsymbol{F}=\boldsymbol{R}$ and is $\{\lambda\sqrt{-1}\,e_{2m};\ \lambda\in\boldsymbol{R}\}$ if $\boldsymbol{F}=\boldsymbol{C}$. We also know that the Lie algebra $\mathfrak{g}(D')$ is isomorphic to $\tilde{\mathfrak{g}}/\tilde{\mathfrak{z}}$ and that

$$\mathfrak{g}'^{-2}=\{B\in H(m, \boldsymbol{F})\}\cong\left\{\begin{pmatrix} 0 & B \\ 0 & 0\end{pmatrix}\in\tilde{\mathfrak{g}}\right\}$$

$$\mathfrak{g}'^{0}=\{\in\mathfrak{gl}(m, \boldsymbol{F})\}/\mathfrak{z}\cong\left\{\begin{pmatrix} A & 0 \\ 0 & -A^{*}\end{pmatrix}\in\tilde{\mathfrak{g}}\right\}/\tilde{\mathfrak{z}}$$

$$\mathfrak{g}'^{2}=\{C\in H(m, \boldsymbol{F})\}\cong\left\{\begin{pmatrix} 0 & 0 \\ C & 0\end{pmatrix}\in\tilde{\mathfrak{g}}\right\}.$$

For $A\in\mathfrak{gl}(m, \boldsymbol{F})$, we shall denote by the same latter A the image of A in $\mathfrak{gl}(m, \boldsymbol{F})/\mathfrak{z}=\mathfrak{g}'^{0}$. Let $A\in\mathfrak{g}'^{0}$ $(=\mathfrak{gl}(m, \boldsymbol{F})/\mathfrak{z})$, $B\in\mathfrak{g}'^{-2}(=H(m, \boldsymbol{F}))$ and $C\in\mathfrak{g}'^{2}$ $(=H(m, \boldsymbol{F}))$. Then

$$(6.1)\qquad \begin{cases} [A, B]=AB+BA^{*}, \quad [A, C]=-(A^{*}C+CA), \\ [B, C]=BC. \end{cases}$$

6.2. Let $A\in GL(m, \boldsymbol{F})$ and put $F_{A}(c, c')=AF(c, c')A^{*}$, $(c, c'\in W)$. Then F_{A} is also a V-hermitian from on W. Let D_{A} be the Siegel domain of the second kind corresponding to $H^{+}(m, \boldsymbol{F})$ and

F_A. The automorphism of $R_c \times W(R=H(m, \boldsymbol{F}))$ defined by the rule: $(z, w) \to (\theta(A)z, w)$ maps D onto D_A. Denote by θ_A the induced isomorphism of $\mathfrak{g}(D)$ onto $\mathfrak{g}(D_A)$ $(=\sum_{\lambda=-2}^{2} \mathfrak{g}_A{}^{\lambda})$. Clearly $\theta_A(\mathfrak{g}^{\lambda}) = \mathfrak{g}_A{}^{\lambda}$ and the following equalities hold:

$$(6.2) \qquad \begin{cases} \theta_A(B) = ABA^* & \text{for } B \in \mathfrak{g}^{-2} = \mathfrak{g}'^{-2} = H(m, \boldsymbol{F}), \\ \alpha_A \circ \theta_A(P) = A\alpha(P)A^{-1} & \text{for } P \in \mathfrak{g}^0, \end{cases}$$

where α_A is the homomorphism of $\mathfrak{g}_A{}^0$ into $\mathfrak{gl}(m, \boldsymbol{F})/\mathfrak{z}$ corresponding to the domain D_A.

Let $\mathfrak{s}$ be a semi-simple graded subalgebra of $\mathfrak{g}(D)$ given in Theorem 2.1 and let V_s be the cone in $\mathfrak{s}^{-2}$ given by (4.1). Then $V_s \subset \overline{V}$. Let $v \in V_s$ and p be the rank of the matrix v. (Note that the rank of each matrix which belongs to V_s is constant, because V_s is homogeneous.) Then $p \neq 0$ if and only if $\mathfrak{s} \neq 0$.

Lemma 6.1. *For a sutable D_A, the following equalities hold under the identification of $\mathfrak{g}_A^{-2}$ with $H(m, \boldsymbol{F})$.*

$$\mathfrak{s}^{-2} = \left\{\begin{pmatrix} 0 & 0 \\ 0 & B_{22} \end{pmatrix} \in H(m, \boldsymbol{F});\ B_{22} \in H(p, \boldsymbol{F})\right\},$$

$$\mathfrak{r}_s^{-2} = \left\{\begin{pmatrix} 0 & B_{12} \\ B_{12}^* & 0 \end{pmatrix} \in H(m, \boldsymbol{F});\ B_{12} \in M(m-p, p, \boldsymbol{F})\right\},$$

$$\mathfrak{r}_0^{-2} = \left\{\begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix} \in H(m, \boldsymbol{F});\ B_{11} \in H(m-p, \boldsymbol{F})\right\}.$$

Proof. Let R_{-2}, R_{-1} and R_0 denote the right sides in the above equations. Let $v \in V_s$. There exists $A \in GL(m, \boldsymbol{F})$ such that $AvA^* = \begin{pmatrix} 0 & 0 \\ 0 & e_p \end{pmatrix} \in H(m, \boldsymbol{F})$. Therefore from (6.2), by considering D_A instead of D, we may assume $v = \begin{pmatrix} 0 & 0 \\ 0 & e_p \end{pmatrix}$. Then by (6.1), we have $[v, [v, X]] \in R_{-2}$ for any $X \in \mathfrak{g}^2$. Since $\mathfrak{s}^{-2} = [v, [v, \mathfrak{s}^2]]$, we know that $\mathfrak{s}^{-2} \subset R_{-2}$. Then by using (6.1) we have

$$\alpha([\mathfrak{s}^{-2}, \mathfrak{s}^2]) \subset \left\{\begin{pmatrix} 0 & 0 \\ A_{21} & A_{22} \end{pmatrix};\ \begin{matrix} A_{21} \in M(p, m-p, \boldsymbol{F}) \\ A_{22} \in \mathfrak{gl}(p, \boldsymbol{F}) \end{matrix}\right\} \pmod{\mathfrak{z}}.$$

Since $E_s \in [\mathfrak{s}^{-2}, \mathfrak{s}^2]$ by Corollary 4.6, we can write

$$\alpha(E_s)=\begin{pmatrix}0 & 0\\ A_{21} & A_{22}\end{pmatrix}.$$

The equation $[E_s, v]=-2v$ implies $A_{22}+A_{22}^*=-2e_p$. Therefore the matrix A_{22} is non-singular. We put

$$A=\begin{pmatrix}e_{m-p} & 0\\ A_{22}^{-1}A_{21} & e_p\end{pmatrix}.$$

We then have $A\,\alpha(E_s)A^{-1}=\begin{pmatrix}0 & 0\\ 0 & A_{22}\end{pmatrix}$. Thus by considering D_A instead of D, we may assume by (6. 2)

$$\alpha(E_s)=\begin{pmatrix}0 & 0\\ 0 & A_{22}\end{pmatrix},$$

$$A_{22}=-e_p+A_{22}',\quad A_{22}'+A_{22}'^*=0.$$

By a direct calculation we can see that $ad\,E_s$ leaves R_{-2} invariant and that the following equality holds:

(6. 3) $\quad [E_s, Z]=-2Z+A_{22}'Z+ZA_{22}'^*\quad$ for $\quad Z\in R_{-2}(=H(p, \boldsymbol{F}))$.

Recall that $ad\,E_s$ has only real eigenvalues. Then from (6. 3), we know that $A_{22}'=0$ if $\boldsymbol{F}=\boldsymbol{R}$ and $A_{22}'=\lambda\sqrt{-1}\,e_p\,(\lambda\in\boldsymbol{R})$ if $\boldsymbol{F}=\boldsymbol{C}$. In the case $\boldsymbol{F}=\boldsymbol{C}$, for any $X\in R_{-1}$, we have

$$ad\,E_sX=-(1+\lambda\sqrt{-1})X.$$

Therefore $\lambda=0$ or $R_{-1}=0$. If $R_{-1}=0$. Then $p=m$ and $ad\,E_s=-2id.$ on $\mathfrak{g}^{-2}$. As a result, $\mathfrak{s}^{-2}=\mathfrak{g}^{-2}$ and hence we have nothing to prove. Thereby we can assume $\alpha(E_s)=\begin{pmatrix}0 & 0\\ 0 & -e_p\end{pmatrix}$ in both cases $\boldsymbol{F}=\boldsymbol{R}$ and $\boldsymbol{F}=\boldsymbol{C}$. If follows

$$ad\,E_s=-2\,id.\quad\text{on}\quad R_{-2},$$

$$ad\,E_s=-id.\quad\text{on}\quad R_{-1},$$

$$ad\,E_s=0\quad\text{on}\quad R_0.$$

Now our assertion follows immediately from Theorem 3. 1. q.e.d.

6. 3. We next investigate domains over cones of another type. We set

$$H(m, \boldsymbol{K})=\{Y\in H(2m, \boldsymbol{C});\ YJ=J\overline{Y}\},$$

where $J=\begin{pmatrix} j & & 0 \\ & j & \\ 0 & & \ddots j \end{pmatrix}$ and $j=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If we write $Y=(y_{kt})$, $k, t=1, \cdots, m$, where y_{kt} is a 2×2 matrix. Then

$$y_{kk}=\begin{pmatrix} u_{kk} & 0 \\ 0 & u_{kk} \end{pmatrix} (u_{kk}\in \boldsymbol{R}),$$

$$y_{kt}=\begin{pmatrix} u_{kt} & v_{kt} \\ -\bar{v}_{kt} & \bar{u}_{kt} \end{pmatrix} (u_{kt}, v_{kt}\in \boldsymbol{C}) \quad \text{for} \quad k\neq t.$$

Put $H^+(m, \boldsymbol{K})=H(m, \boldsymbol{K})\cap H^+(2m, \boldsymbol{C})$. Then $H^+(m, \boldsymbol{K})$ is a convex cone in $H(m, \boldsymbol{K})$. Let D (resp. D') be a (resp. the) Siegel domain of the second kind (resp. of the first kind) with $H^+(m, \boldsymbol{K})$ as a convex cone. We set

$$GL(m, \boldsymbol{K})=\{A\in GL(2m, \boldsymbol{C});\ AJ=J\bar{A}\}.$$

Then $GL(m, \boldsymbol{K})$ is a closed subgroup of $GL(2m, \boldsymbol{C})$. The Lie algebra $\mathfrak{gl}(m, \boldsymbol{K})$ of $GL(m, \boldsymbol{K})$ consists of all $A\in\mathfrak{gl}(2m, \boldsymbol{C})$ such that $AJ=J\bar{A}$. Let $A\in GL(m, \boldsymbol{K})$. The correspondence: $X\to AXA^*$ ($X\in H(m, \boldsymbol{K})$) is a linear transformation of $H(m, \boldsymbol{K})$ leaving the cone $H^+(m, \boldsymbol{K})$ invariant. Therefore there exists a natural homomorphism of $\mathfrak{gl}(m, \boldsymbol{K})$ to $\mathfrak{g}'^0$ which is an isomorphism if $m>1$. We identify $\mathfrak{g}'^{-2}$ and $\mathfrak{g}'^2$ with $H(m, \boldsymbol{K})$ and $\mathfrak{g}'^0$ with $\mathfrak{gl}(m, \boldsymbol{K})$ as before. Then the bracket rule is also given by a similar fashion to (6.1). We can also consider the domain D_A for $A\in GL(m, \boldsymbol{K})$ defined similarly as before. The following lemma is verified analogously to Lemma 6.1.

Lemma 6.2. *Let D be a Siegel domain over the cone $H^+(m, \boldsymbol{K})$. For a suitable D_A, the following equalities hold under the identification of $\mathfrak{g}_A^{-2}$ with $H(m, \boldsymbol{K})$:*

$$\mathfrak{s}^{-2}=\left\{\begin{pmatrix} 0 & 0 \\ 0 & B_{22} \end{pmatrix}\in H(m, \boldsymbol{K}),\ B_{22}\in H(p, \boldsymbol{K})\right\},$$

$$\mathfrak{r}_s^{-2}=\left\{\begin{pmatrix} 0 & B_{12} \\ B_{12}^* & 0 \end{pmatrix}\in H(m, \boldsymbol{K});\ B_{12}\in M(2m-2p, 2p, \boldsymbol{C})\right\}.$$

$$\mathfrak{r}_0^{-2}=\left\{\begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix}\in H(m, \boldsymbol{K});\ B_{11}\in H(m-p, \boldsymbol{K})\right.$$

Proof. We may assume $m>1$. Let $v\in V_s$. By considering D_A for suitable $A\in GL(m, \boldsymbol{K})$, we may assume $v=\begin{pmatrix} 0 & 0 \\ 0 & e_{2p} \end{pmatrix}$. Therefore

$\alpha(E_s)=\begin{pmatrix}0 & 0\\ A_{21} & A_{22}\end{pmatrix}\in\mathfrak{gl}(m,\boldsymbol{K})$, where $A_{22}=-e_{2p}+A'_{22}$, $A'_{22}+A'^{*}_{22}=0$. We can easily observe that the matrix $A=\begin{pmatrix}e_{2m-2p} & 0\\ A_{22}^{-1}A_{21} & e_{2p}\end{pmatrix}$ belongs to $GL(m,\boldsymbol{K})$. We can therefore assume that $\alpha(E_s)=\begin{pmatrix}0 & 0\\ 0 & A_{22}\end{pmatrix}$. Suppose that $p>1$. Then $A'_{22}=0$ because $\alpha(E_s)$ has only real eigenvalues. Hence we get $\alpha(E_s)=\begin{pmatrix}0 & 0\\ 0 & -e_{2p}\end{pmatrix}$. In the case $p=1$, we can write $A'_{22}=\begin{pmatrix}a & b\\ -\bar{b} & \bar{a}\end{pmatrix}$. The fact that $A'_{22}+A'^{*}_{22}=0$ implies that a is purely imaginary. Let

$$R'_{-1}=\left\{\begin{pmatrix}0 & B_{12}\\ B_{12}^{*} & 0\end{pmatrix}\in H(m,\boldsymbol{K});\ B_{12}=\begin{pmatrix}u & v\\ -\bar{v} & \bar{u}\\ 0 & 0\end{pmatrix}\in M(2m-2p,2,\boldsymbol{C})\right\}.$$

Clearly $ad\,E_sR'_{-1}\subset R'_{-1}$. Let $X(\neq 0)$ be an eigenvector for an eigenvalue $\lambda(\in\boldsymbol{R})$ of $ad\,E_s$, where

$$X=\begin{pmatrix}0 & B_{12}\\ B_{12}^{*} & 0\end{pmatrix}\in R'_{-1},\quad B_{12}=\begin{pmatrix}u & v\\ -\bar{v} & \bar{u}\\ 0 & 0\end{pmatrix}.$$

Then $ad\,E_sX=\lambda X$ implies

$$\begin{pmatrix}u & v\\ -\bar{v} & \bar{u}\end{pmatrix}\begin{pmatrix}\bar{a} & -b\\ \bar{b} & a\end{pmatrix}=(\lambda+1)\begin{pmatrix}u & v\\ -\bar{v} & \bar{u}\end{pmatrix}.$$

From this relation we can see that $\lambda=-1$. Therefore $ad\,E_s=-id.$ on R'_{-1} and hence

$$\begin{pmatrix}u & v\\ -\bar{v} & \bar{u}\end{pmatrix}\begin{pmatrix}\bar{a} & -b\\ \bar{b} & a\end{pmatrix}=0\quad\text{for any } u,\ v\in\boldsymbol{C}.$$

As a result $a=b=0$ and $\alpha(E_s)=\begin{pmatrix}0 & 0\\ 0 & -e_{2p}\end{pmatrix}$. Now our assertion follows immediately. q.e.d.

6.4. In this paragraph we consider a domain D over the cone $H^{+}(m,\boldsymbol{F})$, where $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{K}$. As an immediate corollary of Lemma 6.1 and Lemma 6.2, we have

Proposition 6.3. *Let D be a Siegel domain of the second kind with $H^{+}(m,\boldsymbol{F})$ as a convex cone, where $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{K}$. Then the associatd symmetric domain S is an irreducible classical domain.*

Next we shall prove the following

Proposition 6.4. *Let D be a Siegel domain of the second kind with $H^+(m, \boldsymbol{F})$ as a convex cone, where $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{K}$. Assume that $\mathfrak{g}^{-2}\neq[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$. Then $\mathfrak{g}^1=0$.*

Proof. Suppose that $\mathfrak{g}^1\neq 0$. Then $\mathfrak{s}\neq 0$. By using Lemma 6.1 and Lemma 6.2, we may assume, (by considering D_A instead of D)

$$\mathfrak{s}^{-2}=\left\{\begin{pmatrix}0 & 0\\ 0 & B_{22}\end{pmatrix}\in H(m, \boldsymbol{F});\ B_{22}\in H(p, \boldsymbol{F})\right\},$$

$$\mathfrak{r}_s^{-2}=\left\{\begin{pmatrix}0 & B_{12}\\ B_{12}^* & 0\end{pmatrix}\in H(m, \boldsymbol{F})\right\},$$

$$\mathfrak{r}_0^{-2}=\left\{\begin{pmatrix}B_{11} & 0\\ 0 & 0\end{pmatrix}\in H(m, \boldsymbol{F});\ B_{11}\in H(m-p, \boldsymbol{F})\right\}.$$

Let $C=\begin{pmatrix}C_{11} & C_{12}\\ C_{12}^* & C_{22}\end{pmatrix}\in\mathfrak{s}^2(=H(m, \boldsymbol{F}))$ and $B=\begin{pmatrix}B_{11} & 0\\ 0 & 0\end{pmatrix}\in\mathfrak{r}_0^{-2}$.

By (6.1) we have

$$\alpha([B, C])=BC=\begin{pmatrix}B_{11}C_{11} & B_{11}C_{12}\\ 0 & 0\end{pmatrix}.$$

Since $[\mathfrak{r}_0^{-2}, \mathfrak{s}^2]=0$, we have $C_{11}=0$ and $C_{12}=0$. Recalling that $\dim\mathfrak{s}^{-2}=\dim\mathfrak{s}^2$, we get

$$\mathfrak{s}^2=\left\{\begin{pmatrix}0 & 0\\ 0 & C_{22}\end{pmatrix}\in H(m, \boldsymbol{F});\ C_{22}\in H(p, \boldsymbol{F})\right\}.$$

It follows

$$\alpha(\mathfrak{r}_s^0)=\alpha([\mathfrak{r}_s^{-2}, \mathfrak{s}^2])=\left\{\begin{pmatrix}0 & A_{12}\\ 0 & 0\end{pmatrix}\in\mathfrak{gl}(m, \boldsymbol{F})\right\}.$$

Let $B=\begin{pmatrix}0 & B_{12}\\ B_{12}^* & 0\end{pmatrix}\in\mathfrak{r}_s^{-2}$ and $A=\begin{pmatrix}0 & A_{12}\\ 0 & 0\end{pmatrix}\in\alpha(\mathfrak{r}_s^0)$. Then

$$[A, B]=\begin{pmatrix}A_{12}B_{12}^*+B_{12}A_{12}^* & 0\\ 0 & 0\end{pmatrix}.$$

Clearly $\{A_{12}B_{12}^*+B_{12}A_{12}^*;\ A\in\alpha(\mathfrak{r}_s^0),\ B\in\mathfrak{r}_s^{-2}\}$ spans the space $H(m-p, \boldsymbol{F})$. Therefore $\mathfrak{r}_0^{-2}=[\mathfrak{r}_s^0, \mathfrak{r}_s^{-2}]$. On the other hand, the associated symmetric space is irreducible and hence $\mathfrak{s}$ is simple. Therefore $\mathfrak{s}^{-2}$

$=[\mathfrak{s}^{-1}, \mathfrak{s}^{-1}]$ and $\mathfrak{s}^2=[\mathfrak{s}^1, \mathfrak{s}^1]$ by Lemma 1.3. As a consequence,

$$\mathfrak{r}_s^{-2}=[\mathfrak{r}^0, [\mathfrak{s}^{-1}, \mathfrak{s}^{-1}]] \subset [\mathfrak{r}^{-1}, \mathfrak{s}^{-1}] \subset \mathfrak{r}_s^{-2}.$$

Thus $\mathfrak{r}_s^{-2}=[\mathfrak{r}^{-1}, \mathfrak{s}^{-1}]$. Ane hence

$$[\mathfrak{r}_s^{-2}, \mathfrak{r}_s^0]=[[\mathfrak{r}^{-1}, \mathfrak{s}^{-1}], \mathfrak{r}_s^0]=[\mathfrak{r}^{-1}, [\mathfrak{s}^{-1}, \mathfrak{r}_s^0]]$$
$$\subset [\mathfrak{r}^{-1}, \mathfrak{r}^{-1}] \subset \mathfrak{r}_0^{-2}.$$

Therefore $\mathfrak{r}_0^{-2}=[\mathfrak{r}^{-1}, \mathfrak{r}^{-1}]$ and hence $\mathfrak{g}^{-2}=[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$. This contradicts the assumption. q.e.d.

§ 7. Siegel domains over classical cones, II.

7.1. Let D be a Siegel domain of the second kind in $R_c \times W$ associated with a convex cone V in R and a V-hermitian form F on W. We now consider the case where the space W and the form F satisfy the following conditions[6].

1) $W=W_1+W_2$ (direct sum), where W_i is a complex subspace $(i=1,2)$.

2) $F(W_1, W_2)=0$.

Under the identification of W with $\mathfrak{g}^{-1}$, the condition 2) is equivalent to the condition "$[W_1, W_2]=0$".

The restriction F_i of F to $W_i \times W_i$ is a V-hermitian form on W_i $(i=1,2)$. Denote by D_i the Siegel domain of the second kind associated with V and F_i and denote by $\mathfrak{g}(D_i)=\sum_{\lambda=-2}^{2} \mathfrak{g}_i^\lambda$ the graded Lie algebra of $\mathrm{Aut}(D_i)$. Then we can identify $\mathfrak{g}_i^{-2}$ with $R(=\mathfrak{g}^{-2})$ and $\mathfrak{g}_i^{-2}$ with the complex subspace W_i of $W(=\mathfrak{g}^{-1})$. Denote by $\rho_i^{(-1)}$ the projection of $\mathfrak{g}^{-1}$ onto $\mathfrak{g}_i^{-1}$ with respect to the sum: $\mathfrak{g}^{-1}=\mathfrak{g}_1^{-1}+\mathfrak{g}_2^{-1}$. Let $\hat{\mathfrak{g}}=\sum_{\lambda=-2}^{\infty} \hat{\mathfrak{g}}^\lambda$ (resp. $\hat{\mathfrak{g}}_i=\sum_{\lambda=-2}^{\infty} \hat{\mathfrak{g}}_i^\lambda$) be the algebraic prolongation of $(\mathfrak{g}^{-2}+\mathfrak{g}^{-1}, \mathfrak{g}^0)$ (resp. of $(\mathfrak{g}^{-2}+\mathfrak{g}_i^{-1}, \mathfrak{g}_i^0)$). Then we have

Lemma 7.1.[7] *There exists a unique system of linear mappings $\rho_i^{(\lambda)}$ of $\hat{\mathfrak{g}}^\lambda$ to $\hat{\mathfrak{g}}_i^\lambda$ $(\lambda \geqq 0)$ such that*

$$(\sharp)\quad \begin{cases} [\rho_i^{(\lambda)}(A), X]=\rho_i^{(\lambda-2)}([A, X]) & (X \in \mathfrak{g}^{-2}) \\ [\rho_i^{(\lambda)}(A), Y]=\rho_i^{(\lambda-1)}([A, Y]) & (Y \in \mathfrak{g}_i^{-1}), \end{cases}$$

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6) The idea of considering this case is originally due to T. Tsuji (cf. [10]).

7) cf. [10].

where $A\in\hat{\mathfrak{g}}^{\lambda}$ *and* $\rho_i^{(-2)}$ *denotes the identity.*

Proof. From ($\sharp$), the uniqueness is obvious. Let $A\in\mathfrak{g}^0(=\hat{\mathfrak{g}}^0)$. We can define by ($\sharp$) the element $\rho_i^{(0)}(A)$ of $\mathfrak{gl}(\mathfrak{g}^{-2}+\mathfrak{g}_i^{-1})$, i.e., $\rho_i^{(0)}(A)X=[A, X]$ for $X\in\mathfrak{g}^{-2}$ and $\rho_i^{(0)}(A)Y=\rho_i^{(-1)}([A, Y])$. Clearly $\rho_i^{(0)}(E)$ and $\rho_i^{(0)}(I)$ are elements of $\mathfrak{g}_i^0$ obtained from (2. 2) and (2. 2)' for the domain D_i. Since I is in the center of $\mathfrak{g}^0$ and $\mathfrak{g}_i^{-1}$ is a complex subspace, we get for any $Y\in\mathfrak{g}_i^{-1}$

$$\begin{aligned}\rho_i^{(0)}(A)\circ\rho_i^{(0)}(I)Y&=\rho_i^{(-1)}([A, [I, Y]])=\rho_i^{(-1)}([I, [A, Y]])\\&=\rho_i^{(0)}(I)\circ\rho_i^{(0)}(A)Y.\end{aligned}$$

Therefore $\rho_i^{(0)}(A)$ is a complex linear endomorphism of $\mathfrak{g}_i^{-1}$. And for any X, $Y\in\mathfrak{g}_i^{-1}$, from (2. 3) we get (cf. (2. 4))

$$\begin{aligned}\rho_i^{(0)}(A)F(X, Y)&=[A, F(X, Y)]=F([A, X], Y)+F(X, [A, Y])\\&=F(\rho_i^{(-1)}([A, X]), Y)+F(X, \rho_i^{(-1)}([A, Y])),\end{aligned}$$

because $F(\mathfrak{g}_1^{-1}, \mathfrak{g}_2^{-1})=0$. Therefore we have

$$\rho_i^{(0)}(A)F(X, Y)=F_i(\rho_i^{(0)}(A)X, Y)+F_i(X, \rho_i^{(0)}(A)Y).$$

Since $\exp t\rho_i^{(0)}(A)V=\exp tAV=V$ for any $t\in\boldsymbol{R}$, we can conclude by (2. 3) that $\rho_i^{(0)}(A)$ belongs to $\mathfrak{g}_i^0$.

We now assume that there exist mappings $\rho_i^{(\nu)}(0\leqq\nu<\lambda)$ satisfying ($\sharp$). Define the element $\rho_i^{(\lambda)}(A)$ of $\mathrm{Hom}(\mathfrak{g}^{-2}, \hat{\mathfrak{g}}_i^{\lambda-2})+\mathrm{Hom}(\mathfrak{g}_i^{-1}, \hat{\mathfrak{g}}_i^{\lambda-1})$ for $A\in\hat{\mathfrak{g}}^{\lambda}$ by

$$\begin{aligned}&\rho_i^{(\lambda)}(A)X=\rho_i^{(\lambda-2)}([A, X]) \quad X\in\mathfrak{g}^{-2},\\&\rho_i^{(\lambda)}(A)Y=\rho_i^{(\lambda-1)}([A, Y]) \quad Y\in\mathfrak{g}_i^{-1}.\end{aligned}$$

In order to prove that $\rho_i^{(\lambda)}(A)$ belongs to $\hat{\mathfrak{g}}_i^{\lambda}$, we have only to check the following equalities:[8)]

(i) $[\rho_i^{(\lambda)}(A)X, X']+[X, \rho_i^{(\lambda)}(A)X']=0 \quad (X, X'\in\mathfrak{g}^{-2})$

(ii) $[\rho_i^{(\lambda)}(A)X, Y]+[X, \rho_i^{(\lambda)}(A)Y]=0 \quad (X\in\mathfrak{g}^{-2}, Y\in\mathfrak{g}_i^{-1})$

(iii) $\rho_i^{(\lambda)}(A)([Y, Y'])=[\rho_i^{(\lambda)}(A)Y, Y']+[Y, \rho_i^{(\lambda)}(A)Y'] \quad (Y, Y'\in\mathfrak{g}_i^{-1}).$

It follows

8) cf. [9].

$$[\rho_i^{(\lambda)}(A)X, X'] + [X, \rho_i^{(\lambda)}(A)X']$$
$$= [\rho_i^{(\lambda-2)}([A, X]), X'] + [X, \rho_i^{(\lambda-2)}([A, X'])]$$
$$= \rho_i^{(\lambda-4)}([[A, X], X'] + [X, [A, X']]),$$

where we put $\rho_i^{(\lambda-4)}=0$ if $\lambda-4<-2$. Then the equality $[[A, X], X'] + [X, [A, X']] = [A, [X, X']] = 0$ proves (i). The equalities (ii) and (iii) are verified similary. q.e.d.

Let $\rho_i^{(1)}$ and $\rho_i^{(2)}$ be as in Lemma 7.1. Then

Lemma 7.2.
(1) $\mathrm{Ker}\ \rho_1^{(1)} \cap \mathrm{Ker}\ \rho_2^{(1)} = 0$.
(2) $\rho_i^{(2)}$ *is injective on* $\mathfrak{g}^2$ $(i=1, 2)$.

Proof. (1) Let $A \in \mathrm{Ker}\ \rho_1^{(1)} \cap \mathrm{Ker}\ \rho_2^{(1)}$. Then we have $[\mathfrak{g}^{-2}, [\mathfrak{g}^{-1}, A]] = 0$ and hence $[\mathfrak{g}^{-1}, [\mathfrak{g}^{-2}, A]] = 0$. Therefore $[\mathfrak{g}^{-2}, A] = 0$. As a result $A=0$ ([3]).

(2) Let $A \in \mathfrak{g}^2$ such that $\rho_i^{(2)}(A) = 0$. Then $[\mathfrak{g}^{-2}, [\mathfrak{g}^{-2}, A]] = 0$. Therefore we know $A=0$ by Vey's result ([11]). q.e.d.

Now we can prove the following proposition which is convenient to calculate $\dim \mathfrak{g}^1$ and $\dim \mathfrak{g}^2$.

Proposition 7.3. *Assume that* $\mathfrak{g}_1{}^1 = 0$. *Then*
(1) *Under the identification of* W *with* $\mathfrak{g}^{-1}$, W_1 *is contained in* $\mathfrak{r}^{-1}$.
(2) *The mapping* $\rho_2^{(1)}$ (*resp.* $\rho_2^{(2)}$) *is an injective linear mapping of* $\mathfrak{g}^1$ (*resp. of* $\mathfrak{g}^2$) *to* $\mathfrak{g}_2{}^1$ (*resp. to* $\mathfrak{g}_2{}^2$).
(3) $\mathfrak{r}_2^{-1}$ *is contained is* $\mathfrak{r}^{-1}$, *where* $\mathfrak{r}_2^{-1} = \mathfrak{r}_2 \cap \mathfrak{g}_2^{-1}$ *and* $\mathfrak{r}_2$ *is the radical of* $\mathfrak{g}(D_2)$.

Proof. Since $\mathfrak{g}_1{}^1 = 0$, $\rho_1^{(1)}(\mathfrak{g}^1) = 0$. Therefore $\rho_1^{(0)}([W_1, \mathfrak{g}^1]) = 0$ and hence $[\mathfrak{g}^{-2}, [W_1, \mathfrak{g}^1]] = 0$. Now Assertion (1) follows immediately from Corollary 2.4.

In order to prove (2), from Theorem 4.1. and Lemma 7.2. we have only to show that $\rho_2^{(2)}(\mathfrak{g}^2) \subset \mathfrak{g}_2{}^2$. Let $A \in \mathfrak{g}^2$ and $Z \in \mathfrak{g}^{-2}$. Then $[[A, Z], W_1] \subset [[\mathfrak{s}^2, \mathfrak{g}^{-2}], \mathfrak{r}^{-1}] = 0$. Therefore

$$\mathrm{Im}\ Tr\ ad([A, Z])|_{\mathfrak{g}^{-1}} = \mathrm{Im}\ Tr\ ad\ \rho_2^{(0)}([A, Z])|_{\mathfrak{g}_2^{-1}}$$

$$=\operatorname{Im} Tr\, ad([\rho_2^{(2)}(A), Z])|_{\mathfrak{g}_2^{-1}}.$$

By using Theorem 4. 1, we have $\rho_2^{(2)}(A) \in \mathfrak{g}_2{}^2$.

Finally since $[\mathfrak{r}_2^{-1}, \mathfrak{g}_2{}^2]=0$, we have $[\mathfrak{r}_2^{-1}, \rho_2^{(2)}(\mathfrak{g}^2)]=0$ and hence $\rho_2^{(1)}([\mathfrak{r}_2^{-1}, \mathfrak{g}^2])=0$. From (2), we get $[\mathfrak{r}_2^{-1}, \mathfrak{g}^2]=0$. Now Assertion (3) follows from Corollary 2. 5. q.e.d.

Next we put $\mathfrak{h}^{-2}=[\mathfrak{g}_1^{-1}, \mathfrak{g}_1^{-1}]$. By regarding $\mathfrak{h}^{-2}$ as a subspace of $\mathfrak{g}(D_2)$ we set

$$\mathfrak{t}^\lambda=\{X\in \mathfrak{g}_2{}^\lambda;\ [X, \mathfrak{h}^{-2}]=0\},\ \lambda=0,\ 1,\ 2.$$

And put

$$\mathfrak{t}=\mathfrak{g}^{-2}+\mathfrak{g}_2^{-1}+\mathfrak{t}^0+\mathfrak{t}^1+\mathfrak{t}^2.$$

It is easy to see that $\mathfrak{t}$ is a subalgebra of $\mathfrak{g}(D_2)$.

Proposition 7. 4. *The Lie algebra* $\mathfrak{t}$ *can be imbedded as a graded subalgebra of* $\mathfrak{g}(D)$.

Proof. The Lie algebra $\mathfrak{g}^{-2}+\mathfrak{g}_2^{-1}$ is clearly a graded subalgebra of $\mathfrak{g}(D)$. For any $A\in \mathfrak{t}^0$, define an element $\iota^0(A)$ of $\mathfrak{gl}(\mathfrak{g}^{-2}+\mathfrak{g}^{-1})$ by $\iota^0(A)X=[A, X]$ for $X\in\mathfrak{g}^{-2}+\mathfrak{g}_2^{-1}$ and $\iota^0(A)\mathfrak{g}_1^{-1}=0$. Clearly $\iota^0(A)$ is complex linear on $\mathfrak{g}^{-1}$ and by using the fact $F(\mathfrak{g}_1^{-1}, \mathfrak{g}_2^{-1})=0$, we can see that (2. 3) is holds for $\iota^0(A)$. Therefore $\iota^0(A)\in\mathfrak{g}^0$. And the correspondence ι^0: $\mathfrak{t}^0\to\mathfrak{g}^0$ is an injective homomorphism of Lie algebras as is easily observed. Let $A\in\mathfrak{t}^1$. Define an element $\iota^1(A)$ of $\operatorname{Hom}(\mathfrak{g}^{-2}, \mathfrak{g}^{-1})+\operatorname{Hom}(\mathfrak{g}^{-1}, \mathfrak{g}^0)$ by $\iota^1(A)X=[A, X]$ $(\in\mathfrak{g}_2^{-1})$ for $X\in\mathfrak{g}^{-2}$, $\iota^1(A)Y=\iota^0([A, Y])$ for $Y\in\mathfrak{g}_2^{-1}$ and $\iota^1(A)\mathfrak{g}_1^{-1}=0$. We can see that $\iota^1(A)$ belongs to the first prolongation of $(\mathfrak{g}^{-2}+\mathfrak{g}^{-1}, \mathfrak{g}^0)$ and hence $\iota^1(A)$ belongs to $\mathfrak{g}^1$. Clearly the correspondence ι^1 is injective. Finally for any $A\in\mathfrak{t}^2$, we set $\iota^2(A)X=\iota^0([A, X])$ for $X\in\mathfrak{g}^{-2}$, $\iota^2(A)Y=\iota^1([A, Y])$ for $Y\in\mathfrak{g}_2^{-1}$ and $\iota^2(A)\mathfrak{g}_1^{-1}=0$. Then we can see that $\iota^2(A)$ belongs to $\hat{\mathfrak{g}}^2$. And for any $X\in\mathfrak{g}^{-2}$,

$$\operatorname{Im} Tr\, ad(\iota^2(A)X)|_{\mathfrak{g}^{-1}}=\operatorname{Im} Tr\, ad([A, X])|_{\mathfrak{g}_2^{-1}}=0.$$

Therefore $\iota^2(A)\in\mathfrak{g}^2$ by Theorem 4. 1. The injectivity of ι^2 is clear. Thus we have constructed the imbedding ι of $\mathfrak{t}$ into $\mathfrak{g}(D)$. It is not difficult to see that ι is a homomorphism of graded Lie algebras.

q.e.d.

Note that $\rho_2^{(\lambda)}\circ\iota(X)=X$ for $X\in\mathfrak{t}^{\lambda}$.

7.2. In this paragraph, we determine the associated symmetric domain for every homogeneous Siegel domain with $H^+(m, \boldsymbol{R})$ $(m\geqq 2)$ as a convex cone constructed in [6].

Let $r(t)$ be a non-decreasing positive integer valued function on an interval $[1, s]$ $(s\in\boldsymbol{N})$ such that $r(s)\leqq m$. And let W be a complex vector space defined by

$$W=\{(u_{kt})\in M(m, s, \boldsymbol{C});\ u_{kt}=0 \quad \text{for} \quad k>r(t)\}.$$

Define an $H^+(m, \boldsymbol{R})$-hermitian form F on W by

$$F(u, v)=\tfrac{1}{2}(uv^*+\bar{v}{}^tu),$$

$$({}^tu=\text{the transpose of the matrix } u).$$

The Siegel domain D obtained from $H^+(m, \boldsymbol{R})$ and F is homogeneous and non-symmetric (Pyatetski-Shapiro [6]).

Lemma 7.5. *If* $r(t)$ *is constant. Then* $\mathfrak{g}^1=0$.

Proof. Put $n=r(t)\leqq m$. Denote by u_k the k-th row vector of the matrix $u(1\leqq k\leqq m)$. Then $u_k=0$ for $k>n$. Let $A_k(1\leqq k\leqq n)$ be the $m\times m$ matrix such that the (k, k)-component of A_k is 1 and others are zero. Clearly the endomorphism g_k of $H(m, \boldsymbol{R})\times W$ defined by the following equalities is belongs to $\mathfrak{g}^0$ (cf. (2.3)):

$$g_k(X)=A_kX+XA_k \quad \text{for} \quad X\in H(m, \boldsymbol{R})$$

$$g_k(u)=A_ku \quad \text{for} \quad u\in W.$$

Note that $[g_k, u_i]=\delta_{ik}u_i$. Let $B_{ik}(1\leqq i, k\leqq n)$ be the $m\times m$ matrix such that the (i, k), (k, i) and (h, h)-components are 1 $(h\neq i, k)$ and others are zero. Then the following linear transformation f_{ik} of of $H(m, \boldsymbol{R})\times W$ belongs to $GL(D)$ (cf. (2.1)):

$$f_{ik}(X)=B_{ik}XB_{ik} \quad \text{for} \quad X\in H(m, \boldsymbol{R})$$

$$f_{ik}(u)=B_{ik}u \quad \text{for} \quad u\in W.$$

Clearly $Ad\, f_{ik}(u_k)=u_i$, $Ad\, f_{ik}(u_i)=u_k$ and $Ad\, f_{ik}(u_h)=u_h$ for $h\neq i, k$.

Recall that the domain D is non-symmetric and irreducible. Therefore there exists an element $u(\neq 0)$ of $\mathfrak{r}^{-1}$ by Proposition 2.2.

Since $ad\, g_k(\mathfrak{r}^{-1}) \subset \mathfrak{r}^{-1}$ and $Ad\, f_{tk}(\mathfrak{r}^{-1}) = \mathfrak{r}^{-1}$, we may assume that $u_k = 0$ for $k \neq 1$. Then for any $v \in W$ such that $v_k = 0$ for $k \neq 1$, we have $[[I, v], v] = 4F(v, v) = c4F(u, u) = c[[I, u], u]$ $(c \in \boldsymbol{R})$. As a result we get $[[I, v], v] \in \mathfrak{r}^{-2}$ and hence $v \in \mathfrak{r}^{-1}$.[9] Now by considering the transformation $Ad\, f_{tk}$, we have $\mathfrak{g}^{-1} = \mathfrak{r}^{-1}$. q.e.d.

We turn to general cases. We set $W_1 = \{u \in W;\ u_{kt} = 0 \text{ for } t < s\}$ and $W_2 = \{u \in W;\ u_{ks} = 0\}$. Clearly $W = W_1 + W_2$ (direct sum) and $F(W_1, W_2) = 0$. Let D_1 be the domain as in 7.1. Then D_1 is the domain corresponding to $H^+(m, \boldsymbol{R})$ and the function $r(t)$ such that $s = 1$. Therefore by Lemma 7.5 we have $\mathfrak{g}_1{}^1 = 0$. Hence by Proposition 7.3, W_1 is contained in $\mathfrak{r}^{-1}$. Since $[W, W] = [W_1, W_1] \subset \mathfrak{r}^{-2}$, we have $W = \mathfrak{g}^{-1} = \mathfrak{r}^{-1}$. And hence $\mathfrak{g}^1 = 0$. Put $n = r(s)$. Then

$$[\mathfrak{r}^{-1}, \mathfrak{r}^{-1}] = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in H(m, \boldsymbol{R})\ ;\ x \in H(n, \boldsymbol{R}) \right\}.$$

Therefore by Lemma 6.1, we have $\dim \mathfrak{g}^2 \leqq \dim H(m-n, \boldsymbol{R})$, because $[\mathfrak{r}^{-1}, \mathfrak{r}^{-1}]$ is contained in $\mathfrak{r}_0^{-2}$.

Next we change the decomposition of W by putting $W_1 = W$ and $W_2 = 0$. Then the domain D_2 constructed in 7.1 is of the first kind associated with the cone $H^+(m, \boldsymbol{R})$. And the Lie algebra $\mathfrak{g}(D_2)$ is given as follows (cf. §6).

$$\mathfrak{g}(D_2) \cong \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \in \mathfrak{gl}(2m, \boldsymbol{R});\ A \in \mathfrak{gl}(m, \boldsymbol{R}),\ B, C \in H(m, \boldsymbol{R}) \right\}.$$

We put

$$\mathfrak{s} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & c & 0 & -{}^ta \end{pmatrix} \in \mathfrak{gl}(2m, \boldsymbol{R});\ \begin{matrix} a \in \mathfrak{gl}(m-n, \boldsymbol{R}) \\ b, c \in H(m-n, \boldsymbol{R}) \end{matrix} \right\}.$$

Then $\mathfrak{s}$ is a semi-simple graded subalgebra of $\mathfrak{g}(D_2)$ and $[\mathfrak{h}^{-2}, \mathfrak{s}] = 0$, where $\mathfrak{h}^{-2}$ is a subspace of $\mathfrak{g}^{-2}$ given by $\mathfrak{h}^{-2} = [\mathfrak{r}^{-1}, \mathfrak{r}^{-1}]$. Therefore $\mathfrak{s}$ can be imbedded as a graded subalgebra of $\mathfrak{g}(D)$ by Proposition 7.4. As a result $\dim \mathfrak{g}^2 \geqq \dim H(m-n, \boldsymbol{R})$ and hence the equality holds. Now it is clear that the semi-simple graded subalgebra $\mathfrak{s}$ of

[9] If we write $v = v_r + v_s$ $(v_r \in \mathfrak{r}^{-1},\ v_s \in \mathfrak{s}^{-1})$, then $[[I, v_s], v_s] \in \mathfrak{r}^{-2} \cap \mathfrak{s}^{-2} = 0$ and hence $v_s = 0$.

$\mathfrak{g}(D)$ has the properties 1) and 2) in Theorem 2.1 and that the corresponding symmetric domain is of the first kind associated with the cone $H^+(m-n, \boldsymbol{R})$. Thus we have proved the following

Theorem 7.6.[10] *Let D be the Siegel domain corresponding to the cone $H^+(m, \boldsymbol{R})$ $(m \geqq 2)$ and the function $r(t)$ on the interval $[1, s]$. Then $\mathfrak{g}^1=0$, $\mathfrak{g}^2 \cong H(m-n, \boldsymbol{R})$ and the associated symmetric domain is of the first kind corresponding to the cone $H^+(m-n, \boldsymbol{R})$, where $n=r(s)$.*

7.3. Next we investigate domains for the cone $H^+(m, \boldsymbol{C})$ $(m \geqq 2)$. Let $r_1(t)$ (resp. $r_2(t)$) be a function on the interval $[1, s_1]$ (resp. $[1, s_2]$) as before. And let $W^{(1)}$ (resp. $W^{(2)}$) be the vector space corresponding to the function $r_1(t)$ (resp. $r_2(t)$), constructed in 7.2. We set $W=W^{(1)}+W^{(2)}$. Let $R=H(m, \boldsymbol{C})$ and $V=H^+(m, \boldsymbol{C})$. Define a V-hermitian form F on W by

$$F(u, v)=\frac{1}{2}(u^{(1)}v^{(1)*}+\bar{v}^{(2)t}u^{(2)}),$$

where $u=u^{(1)}+u^{(2)}$ and $v=v^{(1)}+v^{(2)}$. Let D be the Siegel domain associated with V and F. We may assume that $r_1(s_1) \geqq r_2(s_2)$. And $W^{(2)}$ may be 0. The domain D is symmetric if and only if $W^{(2)}=0$ and $r_1(1)=m$ (Pyatetski-Shapiro [6]). In what follows we put $r_1(0)=0$ for convenience.

Lemma 7.7. *In the following cases we have $\mathfrak{g}^1=0$.*

(1) $r_1(s_1)=r_2(s_2)$.

(2) $r_1(s_1)<m$.

Proof. In the case (2), $\mathfrak{g}^1=0$ follows immediately from Proposition 6.4 because $\mathfrak{g}^{-2} \neq [\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$. But here we give a simpler proof. We first consider the case where $s_1=s_2=1$ and $r_1(1)=r_2(1)$ or the case where $W^{(2)}=0$, $s_1=1$ and $r_1(1)<m$. In each case there exist g_k $(1 \leqq k \leqq r_1(1))$ of $\mathfrak{g}^0$ and f_{ik} $(1 \leqq i, k \leqq r_1(1))$ of $GL(D)$ such that

$$g_k(X)=A_kX+XA_k \quad \text{for} \quad X \in H(m, \boldsymbol{C})$$

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[10] Tanaka [9] and Murakami [2] calculated $\mathfrak{g}^1$ and $\mathfrak{g}^2$ in the case $s=1$. Sudo [7] calculated $\mathfrak{g}^1$ in the case $s=1$ and $r(1)=m$. And Tsuji [10] obtained the same results for $\mathfrak{g}^1$ and $\mathfrak{g}^2$ of this theorem.

$$g_k(u) = A_k u^{(1)} + A_k u^{(2)} \quad \text{for} \quad u = u^{(1)} + u^{(2)} \in W$$

$$f_{ik}(X) = B_{ik} X B_{ik} \quad \text{for} \quad X \in H(m, \boldsymbol{C})$$

$$f_{ik}(u) = B_{ik} u^{(1)} + B_{ik} u^{(2)} \quad \text{for} \quad u = u^{(1)} + u^{(2)} \in W,$$

where A_k and B_{ik} are $m \times m$ matrices as in Proof of Lemma 7.5. Thus by using the fact that D is non-symmetric, we can see that $\mathfrak{g}^1 = 0$ analogously.

Now in the case (1), we set $W_1 = \{u \in W;\ u_{kt}^{(1)} = 0$ for $t < s_1$ and $u_{kt}^{(2)} = 0$ for $t < s_2\}$, and $W_2 = \{u \in W;\ u_{ks_1}^{(1)} = 0$ and $u_{ks_2}^{(2)} = 0\}$. In the case (2), we put $W_1 = \{u \in W;\ u_{kt}^{(1)} = 0$ for $t < s_1$ and $u^{(2)} = 0\}$ and $W_2 = \{u \in W;\ u_{ks_1}^{(1)} = 0\}$. Then in both cases (1) and (2), $W = W_1 + W_2$ (direct sum) and $F(W_1, W_2) = 0$. And the domain D_1 constructed in 7.1 corresponding to this decomposition is just the domain considered above. Therefore $\mathfrak{g}_1{}^1 = 0$ and hence by Proposition 7.3, we get $W_1 \subset \mathfrak{r}^{-1}$. Since $[W, W] = [W_1, W_1]$, we have $W = \mathfrak{g}^{-1} = \mathfrak{r}^{-1}$ and hence $\mathfrak{g}^1 = 0$. q.e.d.

We shall prove the following

Theorem 7.8.[11] *Let D be the Siegel domain corresponding to the cone $H^+(m, \boldsymbol{C})$ and functions $r_1(t)$ and $r_2(t)$ on the intervals $[1, s_1]$ and $[1, s_2]$ respectively. Assume that $r_1(s_1) \leqq r_2(s_2)$.*

(1) *If $r_1(s_2) = m$. Then $\mathfrak{g}^1 = 0$, $\mathfrak{g}^2 = 0$ and the associated symmetric domain S is trivial, i.e., $S = (0)$.*

(2) *If $r_1(s_1) < m$. Then $\mathfrak{g}^1 = 0$, $\mathfrak{g}^2 \cong H(m - r_1(s_1), \boldsymbol{C})$ and the associated symmetric domain S is of the first kind corresponding to the cone $H^+(m - r_1(s_1), \boldsymbol{C})$.*

(3) *If $r_1(s_1) = m$ and $r_2(s_2) < m$. Let s_1' be the integer $(0 \leqq s_1' < s_1)$ such that $r_1(s_1') < r_1(s_1' + 1) = m$. And put $n = \mathrm{Max}(r_1(s_1'), r_2(s_2))$. Then $\mathfrak{g}^1 \cong M(m - n, s_1 - s_1', \boldsymbol{C})$, $\mathfrak{g}^2 \cong H(m - n, \boldsymbol{C})$ and the associated symmetric domain S is the domain corresponding to the cone $H^+(m - n, \boldsymbol{C})$ and the function $r_1(t)$ on the interval $[1, s_1 - s_1']$ such that $s_1(1) = m - n$.*

Proof. (1) In this case, $r_1(s_1) = r_2(s_2) = m$. Hence by Lemma

[11] Sudo [7] calculated $\mathfrak{g}^1$ in the case $s_1 = 1$, $s_2 = 1$ and $r_1(1) = r_2(1) = 1$. Tsuji [10] calculated $\mathfrak{g}^1$ and $\mathfrak{g}^2$ of this theorem by different methods.

7.7, we get $\mathfrak{g}^1=0$. We then have $\mathfrak{g}^2=0$ by using the fact that $\mathfrak{g}^{-2}=[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$ and by Corollary 1.4. Therefore S is trivial.

(2) Since $\mathfrak{g}^{-2}\neq[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$, we have $\mathfrak{g}^1=0$ by Lemma 7.7 or by Proposition 6.4. Other assertions can be proved by almost similar way as in the proof of Theorem 7.6.

(3) We set $W_1=\{u\in W;\ u_{kt}^{(1)}=0 \text{ for } t>s_1'\}$ and $W_2=\{u\in W;\ u_{kt}^{(1)}=0 \text{ for } t\leqq s_1' \text{ and } u^{(2)}=0\}$. Then $W=W_1+W_2$ (direct sum) and $F(W_1, W_2)=0$. Since $g_1{}^1=0$ by Lemma 7.7 or by Proposition 6.4, we get $W_1\subset\mathfrak{r}^{-1}$ by Proposition 7.3. Let W' be the subspace of W defined by

$$W'=\{u\in W;\ u_{kt}^{(1)}=0 \quad \text{for} \quad k>n\}.$$

Note that $W'\supset W_1\supset W^{(2)}$. Since $[W', W']=[W_1, W_1]$, we have $W'\subset\mathfrak{r}^{-1}$. As a result $\dim\mathfrak{g}^1\leqq\dim M(m-n, s_1-s_1', \boldsymbol{C})$, because $\dim\mathfrak{g}^1=\dim\mathfrak{g}^{-1}-\dim\mathfrak{r}^{-1}$. Let D_2 be the Siegel domain as in 7.1. The domain D_2 is symmetric and the semi-simple Lie algebra $\mathfrak{g}(D_2)$ $(=\sum_{\lambda=-2}^{2}\mathfrak{g}_2{}^{\lambda})$ is expressed as follows. We set

$$\tilde{\mathfrak{g}}=\left\{\begin{pmatrix} A & U & X \\ \sqrt{-1}V^* & C & -\sqrt{-1}U^* \\ Y & V & -A^* \end{pmatrix};\ \begin{matrix} X, Y\in H(m, \boldsymbol{C}),\ A\in\mathfrak{gl}(m, \boldsymbol{C}) \\ C\in\mathfrak{gl}(s_1-s_1', \boldsymbol{C}),\ C+C^*=0, \\ U, V\in M(m, s_1-s_1', \boldsymbol{C}) \end{matrix}\right\}.$$

Then $\tilde{\mathfrak{g}}$ is a subalgebra of $\mathfrak{gl}(2m+s_1-s_1', \boldsymbol{C})$ and its center $\tilde{\mathfrak{z}}$ is one dimensional generated by $\sqrt{-1}\, e_{2m+s_1-s_1'}$. It is well known that the Lie elgebra $\mathfrak{g}(D_2)$ is isomorphic to $\tilde{\mathfrak{g}}/\tilde{\mathfrak{z}}$ and that

$$\mathfrak{g}_2^{-2}\cong\left\{\begin{pmatrix} 0 & 0 & X \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\in\tilde{\mathfrak{g}};\ X\in H(m, \boldsymbol{C})\right\},$$

$$\mathfrak{g}_2^{-1}\cong\left\{\begin{pmatrix} 0 & U & 0 \\ 0 & 0 & -\sqrt{-1}U^* \\ 0 & 0 & 0 \end{pmatrix}\in\tilde{\mathfrak{g}};\ U\in M(m, s_1-s_1', \boldsymbol{C})\right\},$$

$$\mathfrak{g}_2^{0}\cong\left\{\begin{pmatrix} A & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & -A^* \end{pmatrix}\in\tilde{\mathfrak{g}};\ \begin{matrix} A\in\mathfrak{gl}(m, \boldsymbol{C}), \\ C\in\mathfrak{gl}(s_1-s_1', \boldsymbol{C}),\ C+C^*=0 \end{matrix}\right\} \mod \tilde{\mathfrak{z}},$$

$$\mathfrak{g}_2^{1} \cong \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{-1}V^* & 0 & 0 \\ 0 & V & 0 \end{pmatrix} \in \tilde{\mathfrak{g}};\ V \in M(m, s_1 - s_1', \boldsymbol{C}) \right\},$$

$$\mathfrak{g}_2^{2} \cong \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Y & 0 & 0 \end{pmatrix} \in \tilde{\mathfrak{g}};\ Y \in H(m, \boldsymbol{C}) \right\}.$$

Note that if we put $\mathfrak{g} = [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ then $\tilde{\mathfrak{g}} = \mathfrak{g} + \mathfrak{z}$ and that $\mathfrak{g} \cong \mathfrak{g}(D_2)$.

Now we set

$$\tilde{\mathfrak{s}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & a & u & 0 & x \\ 0 & \sqrt{-1}v^* & C & 0 & -\sqrt{-1}u^* \\ 0 & 0 & 0 & 0 & 0 \\ 0 & y & v & 0 & -a^* \end{pmatrix};\ \begin{matrix} x, y \in H(m-n, \boldsymbol{C}) \\ a \in \mathfrak{gl}(m-n, \boldsymbol{C}), \\ C \in \mathfrak{gl}(s_1 - s_1', \boldsymbol{C}),\ C + C^* = 0, \\ u, v \in M(m-n, s_1 - s_1', \boldsymbol{C}) \end{matrix} \right\}.$$

Clearly $\tilde{\mathfrak{s}}$ is a subalgebra of $\tilde{\mathfrak{g}}$. And $\tilde{\mathfrak{s}} = \mathfrak{s} + \mathfrak{c}$ where $\mathfrak{s} = [\tilde{\mathfrak{s}}, \tilde{\mathfrak{s}}]$ and $\mathfrak{c}$ denotes the center of $\mathfrak{s}$. Then the semi-simple Lie algebra $\mathfrak{s}$ has the graded structure ($\mathfrak{s} = \sum_{\lambda=-2}^{2} \mathfrak{s}^{\lambda}$) and can be imbedded as a graded subalgebra of $\mathfrak{g}(D_2)$ in obvious manner. Then we have $[\mathfrak{h}^{-2}, \mathfrak{s}] = 0$, where $\mathfrak{h}^{-2}$ denotes the subspace of $\mathfrak{g}_2^{-2} (= \mathfrak{g}^{-2})$ given by $\mathfrak{h}^{-2} = [W_1, W_1]$. Therefore by Proposition 7. 4, $\mathfrak{s}$ can be imbedded as a graded subalgebra of $\mathfrak{g}(D)$. Consequently, $\dim \mathfrak{g}^1 \geqq \dim M(m-n, s_1 - s_1', \boldsymbol{C})$ and hence the equality holds. We assert that $\mathfrak{s}$ has the properties 1) and 2) in Theorem 2. 1. Since the domain D is non-degenerate, we get $\mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}^1]$ by Corollary 1.4. Clearly $\mathfrak{s}^2 = [\mathfrak{s}^1, \mathfrak{s}^1]$. Thus we have $\mathfrak{g}^2 = \mathfrak{s}^2$, proving 1). The property 3) is obvions. Therefore $\mathfrak{g}^2 \cong H(m-n, \boldsymbol{C})$ and the associated symmetric domain S is given by

$$S = \{(z, u) \in M(m-n, m-n, \boldsymbol{C}) \times M(m-n, s_1 - s_1', \boldsymbol{C});$$
$$\sqrt{-1}(z^* - z) - uu^* \in H(m-n, \boldsymbol{C})\}. \quad \text{q.e.d.}$$

7.4. In this paragraph R denotes the vector space $H(m, \boldsymbol{K})$ $(m \geqq 2)$ and V denotes the cone $H^+(m, \boldsymbol{K})$. Let $r(t)$ be a function, as in 7. 2, on the interval $[1, s]$ such that $1 \leqq r(t) \leqq 2m$. And let W be the corresponding vector space, i.e., $W = \{(u_{kt}) \in M(2m, s, \boldsymbol{C});\ u_{kt} = 0$ for $k > r(t)\}$. Define a V-hermitian form F on W by

$$F(u, v) = \tfrac{1}{2}(uv^* + J\bar{v}^t u^t J).$$

The Siegel domain D associated with V and F is symmetric if and only if $s=1$ and $r(1)=2m$ (Pyatetski-Shapiro [6]).

Lemma 7.9. *The following cases,* $\mathfrak{g}^1=0$.

(1) $r(s)\leqq 2m-2$.

(2) $r(s-1)=2m$ $(s\geqq 2)$.

Proof. In the case (1), $\mathfrak{g}^{-2}\neq[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$. Therefore we have $\mathfrak{g}^1=0$ by Proposition 6.4. In the case (2), it is sufficient to prove our assertion with the assumption that $s=2$ (cf. Proof. of Lemma 7.7). Let A_k $(1\leqq k\leqq m)$ be the $2m\times 2m$ matrix such that the $(2k-1, 2k-1)$ and $(2k, 2k)$-components are 1 and others are zero. Then the following endomorphism g_k of $R\times W$ belongs to $\mathfrak{g}^0$:

$$g_k(X)=A_kX+XA_k \quad \text{for} \quad X\in H(m, \boldsymbol{K})$$

$$g_k(u)=A_ku \quad \text{for} \quad u\in W.$$

Let $B_{ik}(1\leqq i, k\leqq m)$ be the $2m\times 2m$ matrix such that the $(2i-1, 2k-1)$, $(2i, 2k)$, $(2k-1, 2i-1)$, $(2k, 2i)$ and (h, h)-components are 1 $(h\neq 2i-1, 2i, 2k-1, 2k)$ and others are zero. Then the following transformation f_{ik} of $R\times W$ belongs to $GL(D)$:

$$f_{ik}(X)=B_{ik}XB_{ik} \quad \text{for} \quad X\in H(m, \boldsymbol{K})$$

$$f_{ik}(u)=B_{ik}u \quad \text{for} \quad u\in W.$$

For every $u\in W$, denote by $u_i(i=1, \cdots, 2m)$ the i-th row vector. Then we have

$$g_k(u_h)=u_h \quad \text{for} \quad h=2k-1,\ 2k,$$

$$g_k(u_h)=0 \quad \text{for} \quad h\neq 2k-1,\ 2k,$$

$$f_{ik}(u_{2i})=u_{2k},\ f_{ik}(u_{2k})=2i,$$

$$f_{ik}(u_{2i-1})=u_{2k-1},\ f_{ik}(u_{2k-1})=u_{2i-1},$$

$$f_{ik}(u_h)=u_h \quad \text{for} \quad h\neq 2i-1,\ 2i,\ 2k-1,\ 2k.$$

Since D is non-symmetric, there exists $u(\neq 0)\in\mathfrak{r}^{-1}$. Changing by $f_{ik}g_k(u)$ if necessary, we may assume that $u_h=0$ for $h>2$. Then for any $v\in W$ such that $v_h=0$ for $h>2$, we have $[[I, v], v]=c[[I, u], u]$ $(c\in\boldsymbol{R})$. Therefore $v\in\mathfrak{r}^{-1}$. By considering the transformation f_{ik}, we have $W=\mathfrak{r}^{-1}$. q.e.d.

Next we shall prove

Lemma 7.10. *If* $r(1)=r(s)=2m-1$. *Then*

(1) $\mathfrak{g}^1 \cong M(1, s, \boldsymbol{C})$ and $\mathfrak{g}^2 \cong \boldsymbol{R}^1$.

(2) $[\mathfrak{r}^{-1}, \mathfrak{r}^{-1}] = \left\{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in H(m, \boldsymbol{K});\ x \in H(m-1, \boldsymbol{K})\right\}$.

(3) $S = \{(z, w) \in \boldsymbol{C}^1 \times M(1, s, \boldsymbol{C});\ \mathrm{Im}\, z - \frac{1}{2}uu^* > 0\}$.

Proof. Let $u, v \in \mathfrak{g}^{-1}$. Then the bracket rule is given by

$$(7.1) \qquad [u, v] = \sqrt{-1}(vu^* + J\bar{u}^t v^t J - uv^* - J\bar{v}^t u^t J).$$

We set

$$\mathfrak{t}^0 = \left\{(A, C) \in \mathfrak{gl}(m, \boldsymbol{K}) \times \mathfrak{gl}(s, \boldsymbol{C});\ A = \begin{pmatrix} 0 & * & * \\ 0 & \alpha & 0 \\ 0 & 0 & \bar{\alpha} \end{pmatrix},\ \begin{matrix} \alpha \in \boldsymbol{C}^1 \\ C + C^* = 0 \end{matrix}\right\}.$$

For any $(A, C) \in \mathfrak{t}^0$, define an element $\psi^0(A, C)$ of $\mathfrak{gl}(R \times W)$ by

$$(7.2) \qquad \begin{cases} \psi^0(A, C)X = AX + XA^*, & X \in H(m, \boldsymbol{K}) \\ \psi^0(A, C)u = Au + uC, & u \in W. \end{cases}$$

By direct calculations, we can see that $\psi^0(A, C)$ belongs to $\mathfrak{g}^0$. Next we put

$$\mathfrak{t}^1 = \{X \in M(2m, s, \boldsymbol{C});\ X_{kl} = 0 \text{ for all } k \neq 2m-1\}$$
$$(\cong M(1, s, \boldsymbol{C})).$$

Let $X \in \mathfrak{t}^1$, $Y \in W$. By direct calculations, we easily see that the pair (A, C) belongs to $\mathfrak{t}^0$, where $A = \sqrt{-1}(YX^* + J\bar{Y}^t XJ)$, $C = \sqrt{-1}(X^*Y + Y^*X)$. And for any $Z \in H(m, \boldsymbol{K})$, ZX belongs to W. Therefore we can define an element $\psi^1(X)$ of $\mathrm{Hom}(\mathfrak{g}^{-2}, \mathfrak{g}^{-1}) + \mathrm{Hom}(\mathfrak{g}^{-1}, \mathfrak{g}^0)$ by

$$(7.3) \qquad \begin{cases} \psi^1(X)Z = ZX & \text{for } Z \in \mathfrak{g}^{-2}, \\ \psi^1(X)Y = \psi^0(A, C) & \text{for } Y \in \mathfrak{g}^{-1}, \end{cases}$$

where $A = \sqrt{-1}(YX^* + J\bar{Y}^t XJ)$, $C = \sqrt{-1}(Y^*X + X^*Y)$. Clearly ψ^1 is injective. We shall show that $\psi^1(X)$ belongs to $\mathfrak{g}^1$. By Theorem 4.1, it is sufficient to check the following equalities:

(a) $\psi^1(X)([u, v]) = [\psi^1(X)u, v] + [u, \psi^1(X)v] \quad (u, v \in \mathfrak{g}^{-1})$,

(b) $[\psi^1(X)Z, u]+[Z, \psi^1(X)u]=0 \quad (u\in\mathfrak{g}^{-1}, Z\in\mathfrak{g}^{-2})$.

From (7.1), (7.2) and (7.3), we have

$$\psi^1(X)([u, v])=[u, v]X=\sqrt{-1}(vu^*-uv^*)X,$$

because ${}^t v {}^t JX={}^t u {}^t JX=0$ as is easily observed. And

$$\begin{aligned}&[\psi^1(X)u, v]+[u, \psi^1(X)v]\\ &\quad=\sqrt{-1}(vu^*X-uv^*X)+\sqrt{-1}(J\bar{u}{}^tXJv-J\bar{v}{}^tXJu)\\ &\quad=\sqrt{-1}(vu^*X-uv^*X),\end{aligned}$$

where we use the facts that ${}^tXJv={}^t({}^tv{}^tJX)=0$ and ${}^tXJu={}^t({}^tu{}^tJX)=0$. Thus we get (a). And

$$\begin{aligned}[\psi^1(X)Z, u]&=\sqrt{-1}(u(ZX)^*+J\bar{Z}\bar{X}{}^tu{}^tJ-ZXu^*+J\bar{u}{}^t(ZX){}^tJ)\\ &=\sqrt{-1}(uX^*Z+ZJ\bar{X}{}^tu{}^tJ-ZXu^*-J\bar{u}{}^tX{}^tJZ),\end{aligned}$$

because $J\bar{Z}=ZJ$ and $Z^*=Z$. On the other hand

$$\begin{aligned}[Z, \psi^1(X)u]&=-\sqrt{-1}(uX^*+J\bar{u}{}^tXJ)Z+\sqrt{-1}Z(Xu^*+J\bar{X}{}^tuJ)\\ &=-\sqrt{-1}(uX^*Z-ZJ\bar{X}{}^tuJ-ZXu^*+J\bar{u}{}^tXJZ).\end{aligned}$$

Therefore we have (b) because ${}^tJ=-J$. Consequently we have $\dim\mathfrak{g}^1\geqq\dim M(1, s, \boldsymbol{C})$.

We put

$$W'=\{u\in\mathfrak{g}^{-1};\ u_{kl}=0 \quad \text{for} \quad k>2m-2\}$$

$$W''=\{u\in\mathfrak{g}^{-1};\ u_{kl}=0 \quad \text{for} \quad k\leqq 2m-2\}.$$

Then $\mathfrak{g}^{-1}=W'+W''$ (direct sum). We assert that $\mathfrak{r}^{-1}=W'$ or $\mathfrak{r}^{-1}=W''$. In fact, there exist $g_k(1\leqq k\leqq m)$ of $\mathfrak{g}^0$ and $f_{ik}(1\leqq i, k\leqq m-1)$ of $GL(D)$ as in Proof of Lemma 7.8. Let $u\in W'(u\neq 0)$. Then W' is generated by the elements $Ad f_{ik}\ ad\, g_k u(1\leqq i, k\leqq m-1)$. Therefore if there exists $u(\neq 0)\in W'\cap\mathfrak{r}^{-1}$, then $W'\subset\mathfrak{r}^{-1}$. Furthermore if $u(\neq 0)\in W''\cap\mathfrak{r}^{-1}$, then $W''\subset\mathfrak{r}^{-1}$ because the space $[W'', W'']$ is generated by the element $[[I, u], u]$ in $\mathfrak{r}^{-2}$. On the other hand, there exists $u(\neq 0)$ in $\mathfrak{r}^{-1}$, since D is non-symmetric. If $g_m(u)=0$, then $u\in W'$ and hence $W'\subset\mathfrak{r}^{-1}$. And if $g_m(u)\neq 0$, then $g_m(u)\in\mathfrak{r}^{-1}\cap W''$ and hence $W''\subset\mathfrak{r}^{-1}$. Therefore the fact that $\mathfrak{g}^1\neq 0$ implies our assertion. We now suppose that $\mathfrak{r}^{-1}=W''$. Then $[[W'', W''], \psi^1(\mathfrak{t}^1)] \subset [[\mathfrak{r}^{-1}, \mathfrak{r}^{-1}], \mathfrak{g}^1]=0$. Clearly from (7.3), we have $[[W'', W''], \psi^1(\mathfrak{t}^1)] \neq 0$. This is a contradiction. Thus we can conclude that $\mathfrak{r}^{-1}=W'$. Hence $\dim \mathfrak{g}^1=\dim \mathfrak{g}^{-1}-\dim \mathfrak{r}^{-1}=\dim W''$ and $\dim \mathfrak{g}^2=1$ by Lemma 6.2 and from the fact that $\mathfrak{r}_0^{-2}=[\mathfrak{r}^{-1}, \mathfrak{r}^{-1}]$. It is not difficult to see that the graded subalgebra $[W'', W'']+W''+[W'', \mathfrak{g}^1]+\mathfrak{g}^1+\mathfrak{g}^2$ has the properties 1) and 2) in Theorem 2.1. q.e.d.

Lemma 7.11. *If* $s=2$, $r(1)=2m-1$ *and* $r(2)=2m$. *Then* $\mathfrak{g}^1=0$ *and* $\mathfrak{g}^2=0$.

Proof. Suppose that $\mathfrak{g}^1 \neq 0$. We set $W_1=\{\begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \in M(2m, 2, \boldsymbol{C})$; $u \in M(2m-1, 1, \boldsymbol{C})\}$ and $W_2=\{(0\ u) \in M(2m, 2, \boldsymbol{C})$; $u \in M(2m, 1, \boldsymbol{C})\}$. Then $W=W_1+W_2$ (direct sum) and $F(W_1, W_2)=0$. Then the domain D_1 is one considered in Lemma 7.10. Put $R_0=\left\{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in H(m, \boldsymbol{K})\right.$; $\left.x \in H(m-1, \boldsymbol{K})\right\}$. Then from (2) of Lemma 7.10, the subspace R_0 of $\mathfrak{g}^{-2}$ is invariant by $ad\,\rho_1^{(0)}(\mathfrak{g}^0)$. Therefore any element of $\alpha(\mathfrak{g}^0)$ is of the form:

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{matrix} 2m-2 \\ 2 \end{matrix} \qquad \begin{matrix} 2m-2 & 2 \end{matrix} \tag{7.4}$$

where α is the mapping of $\mathfrak{g}^0$ to $\mathfrak{gl}(m, \boldsymbol{K})$ as in § 6. We put

$$W'=\left\{\begin{pmatrix} u \\ 0 \end{pmatrix} \in M(2m, 2, \boldsymbol{C});\ u \in M(2m-2, 2, \boldsymbol{C})\right\}$$

$$W''=\left\{\begin{pmatrix} 0 & 0 \\ u_1 & u_2 \\ 0 & u_3 \end{pmatrix} \in M(2m, 2, \boldsymbol{C});\ u_1, u_2, u_3 \in \boldsymbol{C}\right\}.$$

Then $W=W'+W''$ (direct sum) and by the arguments as in Proof of Lemma 7.10, we have $\mathfrak{r}^{-1}=W'$ or $\mathfrak{r}^{-1}=W''$. Suppose that $\mathfrak{r}^{-1}=W''$. Then we have (cf. Proof of Proposition 6.4)

$$\alpha(\mathfrak{r}_s^0)=\left\{\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \in \mathfrak{gl}(m, \boldsymbol{K});\ a \in M(2, 2m-2, \boldsymbol{C})\right\}.$$

This contradicts (7.4). Therefore $\mathfrak{r}^{-1}=W'$. And hence

$$[\mathfrak{r}^{-1}, \mathfrak{r}^{-1}]=R_0$$

$$\mathfrak{g}^2=\left\{\begin{pmatrix}0 & 0\\ 0 & A\end{pmatrix}\in H(m, \boldsymbol{K});\ A=\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix},\ a\in\boldsymbol{R}\right\}$$

(cf. Proof of Proposition 6. 4).

Thus by using (6. 1), we can write $\alpha(E_s)=\begin{pmatrix}0 & B\\ 0 & C\end{pmatrix}$. Let $X\in\mathfrak{g}^2$ ($\subset H(m, \boldsymbol{K})$). Then $[E_s, X]=-(\alpha(E_s)^*X+X\alpha(E_s))=2X$. Hence we have $C=-e_2+\begin{pmatrix}\sqrt{-1}a & b\\ -\bar{b} & -\sqrt{-1}a\end{pmatrix}$ $(a=\boldsymbol{R}, b\in\boldsymbol{C})$. We put $P=\begin{pmatrix}e_{2m-2} & -BC^{-1}\\ 0 & e_2\end{pmatrix}$. Then $P\alpha(E_s)P^{-1}=\begin{pmatrix}0 & 0\\ 0 & C\end{pmatrix}$. It is easy to see that the matrix P belongs to $GL(m, \boldsymbol{K})$. Hence we can define the element $\tilde{P}$ of $GL(D)$ by

$$\tilde{P}(X)=PXP^* \quad \text{for} \quad X\in H(m, \boldsymbol{K})$$

$$\tilde{P}(u)=Pu \quad \text{for} \quad u\in W.$$

Since $\alpha(Ad\,\tilde{P}E_s)=\begin{pmatrix}0 & 0\\ 0 & C\end{pmatrix}$, we may assume that $\alpha(E_s)=\begin{pmatrix}0 & 0\\ 0 & C\end{pmatrix}$. Then we get $a=b=0$ as in Proof of Lemma 6. 2. As a result $\alpha(E_s)=\begin{pmatrix}0 & 0\\ 0 & -e_2\end{pmatrix}$ and hence

$$\mathfrak{s}^{-2}=\left\{\begin{pmatrix}0 & 0\\ 0 & X\end{pmatrix}\in H(m, \boldsymbol{K});\ X=\begin{pmatrix}x & 0\\ 0 & x\end{pmatrix},\ x\in\boldsymbol{R}\right\}.$$

Let $u\in W''$ ($\subset\mathfrak{g}^{-1}$). Then $[[I, u], u]\in\mathfrak{s}^{-2}$. Hence we have $u\in\mathfrak{s}^{-1}$. Thus we get $\mathfrak{s}^{-1}=W''$, because $W'\subset\mathfrak{r}^{-1}$. Let X_1 (resp. X_2) be the element of $\mathfrak{s}^{-1}$ such that $u_1=1$, $u_2=u_3=0$ (resp. $u_2=1$, $u_1=u_3=0$). By using the fact that the associated symmetric domain is given by $\{(z, u_1, u_2, u_3)\in\boldsymbol{C}^4;\ \operatorname{Im} z-\sum_{i=1}^{3}|u_i|^2>0\}$, we can easily observe that there exist $A_1, \cdots, A_n$ of $\mathfrak{s}^0$ such that $Ad(\exp A_1\circ\cdots\circ\exp A_n)X_1=X_2$. We set $\mathfrak{q}_i=\{X\in\mathfrak{r}^{-1}; F(X, X_i)=0\}$ $(i=1, 2)$. We then have $Ad f\mathfrak{q}_1=\mathfrak{q}_2$, where $f=\exp A_1\circ\cdots\circ\exp A_n$. Clearly

$$(7.5)\qquad \begin{cases}\mathfrak{q}_1=\left\{\begin{pmatrix}0 & u\\ 0 & 0\end{pmatrix}\in M(2m, 2, \boldsymbol{C});\ u\in M(2m-2, 1, \boldsymbol{C})\right\}\\[2ex] \mathfrak{q}_2=\left\{\begin{pmatrix}u & 0\\ 0 & 0\end{pmatrix}\in M(2m, 2, \boldsymbol{C});\ u\in M(2m-2, 1, \boldsymbol{C})\right\}.\end{cases}$$

Next we set $\mathfrak{p}_i=\{X\in\mathfrak{g}^{-1}; F(X, \mathfrak{q}_i)=0\}$ $(i=1, 2)$. Then from (7. 5) we have $\dim_{\boldsymbol{C}}\mathfrak{p}_1=2m-1$ and $\dim_{\boldsymbol{C}}\mathfrak{p}_2=2m$. On the other hand $\mathfrak{p}_2=Ad f\,\mathfrak{p}_1$ and hence $\dim_{\boldsymbol{C}}\mathfrak{p}_1=\dim_{\boldsymbol{C}}\mathfrak{p}_2$. This contradiction arises from the first assumption that $\mathfrak{g}^1 \neq 0$. Therefore $\mathfrak{g}^1=0$. Since D is non-degenerate we have $\mathfrak{g}^2=0$ by Corollary 1.4. q.e.d.

We are now in a position to prove the following

Theorem 7.12.[12] *Let D be the Siegel domain of the second kind corresponding to the cone $H^+(m, \boldsymbol{K})$ $(m \geqq 2)$ and the function $r(t)$ on the interval $[1, s]$. And let S be the associated symmetric domain.*

(1) *If $r(s)<2m-1$ and $n=[(r(s)+1)/2]$. Then $\mathfrak{g}^1=0$, $\mathfrak{g}^2 \cong H(m-n, \boldsymbol{K})$ and S is of the first kind corresponding to the cone $H^+(m-n, \boldsymbol{K})$.*

(2) *If $r(s)=2m-1$. Let s' be the integer $(s'<s)$ such that $r(s')<2m-1$ and $r(s'+1)=2m-1$. (In the case $r(1)=2m-1$, we put $s'=0$.) Then $\mathfrak{g}^1 \cong M(1, s-s', \boldsymbol{C})$, $\mathfrak{g}^2 \cong \boldsymbol{R}^1$ and $S=\{(z, w) \in \boldsymbol{C}^1 \times M(1, s-s', \boldsymbol{C})$; $\operatorname{Im} z-ww^*>0\}$.*

(3) *If $r(s-1)=2m$ $(s \geqq 2)$. Then $\mathfrak{g}^1=0$, $\mathfrak{g}^2=0$ and $S=(0)$.*

(4) *If $r(s)=2m$ and $r(s-1)<2m$. (In the case $s=1$, we put $r(0)=0$.) Let $n=[(r(s-1)+1)/2]$. Then $g^1 \cong M(2m-2n, 1, \boldsymbol{C})$, $\mathfrak{g}^2 \cong H(m-n, \boldsymbol{K})$ and S is the domain corresponding to the cone $H^+(m-n, \boldsymbol{K})$ and the function $r(t)$ such that $s=1$, $r(1)=2(m-n)$.*

Proof. (1) In this case, $\mathfrak{g}^1=0$ by Proposition 6.4. Other assertions can be proved similarly as Theorem 7.6.

(2) We set $W_1=\{u \in W;\ u_{kt}=0$ for $t>s'\}$ and $W_2=\{u \in W;\ u_{kt}=0$ for $t \leqq s'\}$. Then $W=W_1+W_2$ (direct sum) and $F(W_1, W_2)=0$. The domain D_1 (resp. D_2) is one considered in Lemma 7.9 (resp. in Lemma 7.10). Let $\mathfrak{s}=\sum_{\lambda=-2}^{2} \mathfrak{s}^\lambda$ be the semi-simple graded subalgebra of $\mathfrak{g}(D_2)$ as in Theorem 2.1. By Proposition 7.3, Lemma 7.9 and Lemma 7.10, we have $\dim \mathfrak{g}^1=\dim \mathfrak{g}^{-1}-\dim \mathfrak{r}^{-1} \leqq \dim \mathfrak{s}^{-1}$. On the other hand by Proposition 7.4 and Lemma 7.10, the Lie algebra $\mathfrak{s}$ is imbedded as a garded subalgebra of $\mathfrak{g}(D)$. Therefore we have $\mathfrak{g}^1=\mathfrak{s}^1$. Since D is non-degenerate, we know that $\mathfrak{g}^2=[\mathfrak{g}^1, \mathfrak{g}^1]$ from Corollary 1.4. As a result $\mathfrak{g}^2=\mathfrak{s}^2$, because $\mathfrak{s}^2=[\mathfrak{s}^1, \mathfrak{s}^1]$. Now it is clear that the subalgebra $\mathfrak{s}$ of $\mathfrak{g}(D)$ has properties 1) and 2) in Theorem 2.1.

[12] Tsuji [10] calculated $\mathfrak{g}^1$ and $\mathfrak{g}^2$ in special cases of this theorem.

(3) We set $W_1=\{u\in W;\ u_{kt}=0 \text{ for } t\geqq s-1\}$ and $W_2=\{u\in W;\ u_{kt}=0 \text{ for } k<s-1\}$. Then $W=W_1+W_2$ (direct sum) and $F(W_1, W_2)=0$. Then the domain D_2 is one considered in Lemma 7.9. Therefore $W_2\subset \mathfrak{r}^{-1}$ by Proposition 7.3. Since $[W, W]=[W_2, W_2]=\mathfrak{g}^{-2}$, we have $W=\mathfrak{r}^{-1}$ and $\mathfrak{g}^{-2}=\mathfrak{r}^{-2}$. Hence $\mathfrak{g}^1=0$ and $\mathfrak{g}^2=0$.

(4) If $n=m$, then $r(s-1)=2m-1$. We set $W_1=\{u\in W;\ u_{kt}=0 \text{ for } t\geqq s-1\}$ and $W_2=\{u\in W;\ u_{kt}=0 \text{ for } t<s-1\}$. Then the domain D_2 is one considered in Lemma 7.11. Therefore we have $\mathfrak{g}^1=0$ and $\mathfrak{g}^2=0$ by the same reason as in (3).

We now consider the case where $n<m$, i.e., $r(s-1)<2m-1$. We set $W_1=\{u\in W;\ u_{ks}=0\}$ and $W_2=\{u\in W;\ u_{kt}=0 \text{ for } t<s\}$. Then $W=W_1+W_2$ (direct sum) and $F(W_1, W_2)=0$. We have $W_1\subset\mathfrak{r}^{-1}$, because the domain D_1 is degenerate. Put $W'=\{u\in W;\ u_{kt}=0 \text{ for } k>2n\}$ and $W''=\{u\in W;\ u_{kt}=0 \text{ for } k\leqq 2n\}$. Since $[W', W']=[W_1, W_1]$, we have $W'\subset\mathfrak{r}^{-1}$. Therefore $\dim \mathfrak{g}^1\leqq \dim W''$. On the other hand, the domain D_2 is symmetric. By using the well known expressions of $\mathfrak{g}(D_2)$, as in Proof of (3) of Theorem 7.8, we can show that there exists a semi-simple graded subalgebra $\mathfrak{s}=\sum_{\lambda=-2}^{2}\mathfrak{s}^\lambda$ of $\mathfrak{g}(D_2)$ such that $\mathfrak{s}^{-1}\cong W''$, $[\mathfrak{s}^{-1}, \mathfrak{s}^{-1}]=\mathfrak{s}^{-2}$, $[\mathfrak{s}, [W_1, W_1]]=0$ and the adjoint representation of $\mathfrak{s}^0$ on $\mathfrak{s}^{-2}+\mathfrak{s}^{-1}$ is faithful. Now our assertions can be verified similarly as (3) of Theorem 7.8. q.e.d.

Remark 4. Let D be a Siegel domain of the second kind and let $\mathfrak{g}(D)=\sum_{\lambda=-2}^{2}\mathfrak{g}^\lambda$ be the graded Lie algebra of Aut(D). Take a point $v\in V$. Then the domain D is homogeneous if and only if $\mathfrak{g}^{-2}=[\mathfrak{g}^0, v]$. Therefore the homogeneity of D implies that $\mathfrak{g}^0$ is fairly large. By Theorem 4.1, $\mathfrak{g}^1$ is the first prolongation of $(\mathfrak{g}^{-2}+\mathfrak{g}^{-1}, \mathfrak{g}^0)$. Thus the following question arises: Is there an irreducible inhomogeneous Siegel domain with $\mathfrak{g}^1\neq 0$? As an answer, we give the following example.

Let $R=H(m, \boldsymbol{C})$ $(m\geqq 3)$, $V=H^+(m, \boldsymbol{C})$. And let $W=\boldsymbol{C}^1\times\boldsymbol{C}^1\times H(m, s, \boldsymbol{C})$ $(s\geqq 1)$. Define a V-hermitian form F on W by

$$F(u, w)=\begin{pmatrix} u_1\bar{w}_1 & 0 & 0\\ 0 & u_2\bar{w}_2 & 0\\ 0 & 0 & 0\end{pmatrix}+u_3w_3{}^*,$$

where $u=(u_1, u_2, u_3)$, $w=(w_1, w_2, w_3)$.

It is not difficult to see that the domain D associated with V and F is irreducible and inhomogeneous. By the same methods as in Proof of Theorem 7.8, we get $\mathfrak{g}^1 \cong M(m-2, s, \boldsymbol{C})$ and $\mathfrak{g}^2 \cong H(m-2, \boldsymbol{C})$.

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