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Projective modules over polynomial rings over division rings

By

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Introduction. It was proved in [3] that if D is any (non-commutative) division ring, then there exist non-free projective ideals in $D[X, Y]$. The aim of this paper is to study the set of isomorphism classes of finitely generated projective modules over $D[X, Y]$, where D is a division algebra which is finite-dimensional over its centre. In §1. we prove a proposition on projective modules over matrix rings and deduce (Cor. 1. 3) that if D is a finite-dimensional central division algebra of dimension n^2 over K and L a splitting field for D, then for any finitely generated projective module P over $D[X, Y]$, $L\otimes_K P$ is free over $M_n(L)[X, Y]$. If we choose a splitting field L for D which is a finite Galois extension of K with Galois group G and an isomorphism $L\otimes_K D[X, Y]\xrightarrow{\sim} M_n(L)[X, Y]$, we get a cocycle $f: G\longrightarrow \mathrm{Aut}_{L[X,Y]-alg} M_n(L)[X, Y]$. For any integer $m\geqslant 1$, let $Z^1(m)$ denote the set of maps $T: G\longrightarrow \mathrm{Aut}_{L[X,Y]} M_n(L)[X, Y]^m$, where T satisfies a suitable cocycle condition and $T(\sigma)$ is $f(\sigma)$-semilinear for every $\sigma\in G$. We prove (Th. 2. 1) in §2, that for $m\geqslant 1$, the set of isomorphism classes of finitelygenerated projective modules of rank m (where rank is defined in a suitable manner) is in bijection with a quotient set $H^1(m)$ of $Z^1(m)$ modulo an equivalence relation. In § 3, we show (Cor. 3. 2) that the

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isomorphism $\mathbf{Z}\xrightarrow{\sim}K^0(D)\xrightarrow{\sim}K^0(D[X, Y])$ implies that the cohomologies $H^1(m)$ are 'stably trivial'. In § 4, we give an explicit description, in terms of matrices, of projective modules over $\mathbf{H}[X,Y]$, $\mathbf{H}$ being the algebra of real quaternions. In the final section, we prove (Prop. 5. 1) that two projective modules of rank 1 over $\mathbf{H}[X, Y]$ are isomorphic if and only if certain systems of equations over $\mathbf{R}[X, Y]$ have solutions. In particular (Cor. 5. 2), a projective module of rank 1 is free if and only if a certain diophantine equation is solvable over $\mathbf{R}[X, Y]$. Using this, we incidentally prove that there exist infinitely many non-isomorphic projective modules of rank 1 over $\mathbf{H}[X, Y]$.

All rings considered here are assumed to have unit elements and all modules are unitary. By a module, we generally mean a finitely generated left module.

§1. Projective modules over matrix rings.

Proposition 1. 1. *Let A be a ring such that every finitely generated projective module is free. Then every finitely generated projective module over $M_n(A)$ is isomorphic to $\bigoplus_{m\ \text{copies}} A^n$, A^n being the standard left $M_n(A)$-module.*

Proof. We have [1, p. 69] an equivalence of categories $\mathbf{mod}\ A \longrightarrow \mathbf{mod}\ M_n(A)$ given by $M \longmapsto A^n \otimes_A M$, A^n being considered as a right A- and left $M_n(A)$-module. Since this equivalence preserves projective modules, it follows that every projective module over $M_n(A)$ is of the form $A^n \otimes_A P$, where P is a projective A-module. By our assumption on A, P is free. This proves the proposition.

Corollary 1. 2. *Let L be any field. Then any projective module over $M_n(L)[X, Y] = M_n(L[X, Y])$ is isomorphic to a direct sum of copies of $L[X, Y]^n$.*

Proof. Immediate from the above proposition, using Seshadri's theorem on projective modules over $L[X, Y]$.

Corollary 1. 3. *Let D be a finite dimensional central division algebra over K and let L be a splitting field for D so that $L\otimes_K D \simeq M_n(L)$. If P is any projective module over $D[X, Y]$, then $L\otimes_K P$ is free over $M_n(L[X, Y])$.*

Proof. By Grothendieck's theorem [1, p. 643], we know that the inclusion $D \hookrightarrow D[X, Y]$ induces an isomorphism $K^0(D) \xrightarrow{\sim} K^0(D[X, Y])$, the inverse mapping being induced by the supplementation $D[X, Y] \longrightarrow D$ defined by $X \longmapsto 0$, $Y \longmapsto 0$. Thus if P is any projective module over $D[X, Y]$ and $m = \dim_D(D \otimes_{D[X,Y]} P)$, the image of $P - D[X, Y]^m$ in $K^0(D)$ is zero so that $P - D[X, Y]^m = 0$ in $K^0(D[X,Y])$. This implies that $P \oplus D[X, Y]^r \simeq D[X, Y]^{m+r}$ for some integer r. Tensoring with L, we get an isomorphism $L\otimes_K P \oplus M_n(L[X, Y])^r \simeq M_n(L[X, Y])^{m+r}$ of $M_n(L[X, Y])$-modules. By Corollary 1. 1, we have $L\otimes_K P$ is isomorphic to $(L[X, Y]^n)^s$. Comparison of ranks over $L[X, Y]$ yields $sn + rn^2 = (m+r)n^2$, i. e. $s = m \cdot n$ so that $L\otimes_K P \simeq \bigoplus_m \bigoplus_n L[X, Y]^n \simeq \bigoplus_m M_n(L[X, Y])$.

If P is a projective module over $D[X, Y]$, the dimension of $D \otimes_{D[X,Y]} P$ as a vector space over D is called the *rank of* P over $D[X, Y]$. If m is the rank of P over $D[X, Y]$, then $L\otimes_K P \simeq (M_n(L[X, Y]))^m$.

Remark. In [2, p. 18], Bass states that the construction of non-free projective modules over $D[X, Y]$ given in [3, p. 504] holds for any ring A in which there exist units a, b such that $ab - ba$ is a unit and over which free modules have well-defined ranks. This statement, however, is not true unless A is a domain. Let us take for instance $A = M_n(K)$, K being a field. Let a, $b \in M_n(K)$ be invertible such that $ab - ba$ is also invertible. Let P be defined by the exact sequence

$$0 \longrightarrow P \longrightarrow M_n(K[X, Y])^2 \xrightarrow{\phi} M_n(K[X, Y]) \longrightarrow 0,$$

where $\phi(1, 0) = X+a$, $\phi(0, 1) = Y+b$. By Proposition 1.1, P is isomorphic to $\bigoplus_m K[X, Y]^n$ and a comparison of ranks over $K[X, Y]$ in the above exact sequence shows that $m=n$. Thus $P \simeq \bigoplus_n K[X, Y]^n \simeq M_n(K[X, Y])$, i. e. P is free.

For example, in the case where $A = M_2(\mathbf{Z}_2)$, $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $ab-ba$ being 1, the projective module constructed as above is isomorphic through the first projection to the ideal generated by $\begin{pmatrix} 1+X+Y+XY & 1+Y \\ 1+X & X+Y+XY \end{pmatrix}$ and $\begin{pmatrix} 1+Y^2 & 0 \\ 0 & 1+Y^2 \end{pmatrix}$. This ideal in fact is principal, generated by $\begin{pmatrix} 1+Y^2 & 0 \\ (1+X)(X+Y+XY) & 1 \end{pmatrix}$, in view of the equations

$$\begin{pmatrix} 1+X+Y+XY & 1+Y \\ 1+X & X+Y+XY \end{pmatrix} = \begin{pmatrix} 1+X^2 & 1+Y \\ 1+X+X^2+X^3 & X+Y+XY \end{pmatrix} \times \begin{pmatrix} 1+Y^2 & 0 \\ (1+X)(X+Y+XY) & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1+Y^2 & 0 \\ 0 & 1+Y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (1+X)(X+Y+XY) & 1+Y^2 \end{pmatrix} \times \begin{pmatrix} 1+Y^2 & 0 \\ (1+X)(X+Y+XY) & 1 \end{pmatrix}$$

§2. A classification of projective modules over $D[X, Y]$.

In this section, we shall write $A = D[X, Y]$, D a central division algebra over K. If P is a projective A-module of rank m (in the sense defined in § 1), we know that $P \oplus A^r \simeq A^{m+r}$ for some r. If $m \geqslant 3$, since dim max $K[X, Y] = 2$, it follows by the cancellation theorem of Bass [1, p. 184] that $P \simeq A^m$, i. e. P is free. Thus any projective module over $D[X, Y]$ is either free or is of rank $\leqslant 2$.

Let L be a finite Galois extension of K which is a splitting field for D. Let $G = G(L/K)$ and $\phi : L \otimes_K D \xrightarrow{\sim} M_n(L)$ be an L-algebra isomorphism. This gives rise to an $L[X, Y]$-algebra isomorphism $L\underset{K}{\otimes}A \xrightarrow{\phi} M_n(L[X, Y])$. The group G operates on L and hence on $M_n(L[X, Y])$ entrywise. For each $\sigma\in G$, the map $f(\sigma)$: $M_n(L[X, Y])\longrightarrow M_n(L[X, Y])$ defined by $f(\sigma)=\sigma^{-1}\circ\phi\circ\sigma\otimes 1\circ\phi^{-1}$ is easily checked to be an $L[X, Y]$-algebra automorphism of $M_n(L[X, Y])$. It is also easily seen that

$$f(\sigma\tau)=\tau^{-1}\circ f(\sigma)\circ\tau\circ f(\tau)\ ;\ \sigma,\ \tau\in G.$$

Changing the isomorphism ϕ is equivalent to altering $f(\sigma)$ in its cohomology class. In what follows, however, we shall fix a ϕ and therefore an $f(\sigma)$ once and for all.

Let P be a projective module over A of rank m and $\psi: L\underset{K}{\otimes}P \simeq M_n(L[X, Y])^m$ a ϕ-semilinear isomorphism. The Galois group G operates on $M_n(L[X, Y])^m$ in an obvious manner. For any $\sigma\in G$, define

$$T_P(\sigma)=\sigma^{-1}\circ\psi\circ\sigma\otimes 1\circ\psi^{-1}$$

Clearly, $T_P(\sigma)$ is an additive map and for $\lambda\in M_n(L[X, Y])$, $x\in M_n(L[X, Y])^m$, we have

$$\begin{aligned}T_P(\sigma)(\lambda x)&=\sigma^{-1}\circ\psi\circ\sigma\otimes 1(\phi^{-1}(\lambda)\cdot\psi^{-1}(x))\ (\text{since } \psi \text{ is } \phi\text{-semi-}\\&\quad\text{linear})\\&=\sigma^{-1}\circ\psi(\sigma\otimes 1(\phi^{-1}(\lambda)\cdot\sigma\otimes 1(\psi^{-1}(x)))\\&=\sigma^{-1}(\phi\circ\sigma\otimes 1\circ\phi^{-1}(\lambda)\cdot\psi\circ\sigma\otimes 1\circ\psi^{-1}(x))\\&=f(\sigma)(\lambda)\cdot T_P(\sigma)(x).\end{aligned}$$

In particular, $T_P(\sigma)$ is $L[X, Y]$-linear. For σ, $\tau\in G$,

$$\begin{aligned}T_P(\sigma\tau)&=(\sigma\tau)^{-1}\circ\psi\circ\sigma\tau\otimes 1\circ\psi^{-1}\\&=\tau^{-1}\circ T_P(\sigma)\circ\tau\circ T_P(\tau).\end{aligned}$$

Thus for each $\sigma\in G$, $T_P(\sigma)$ is an $f(\sigma)$-semilinear automorphism of $M_n(L[X, Y])^m$. If $\psi': L\underset{K}{\otimes}P \xrightarrow{\simeq} M_n(L[X, Y])^m$ is another ϕ-semilinear isomorphism and $T'_P(\sigma)=\sigma^{-1}\circ\psi'\circ\sigma\otimes 1\circ\psi'^{-1}$, then

$$\sigma \circ T'_P(\sigma) = (\psi' \circ \psi^{-1}) \circ \sigma \circ T_P(\sigma) \circ (\psi' \circ \psi^{-1})^{-1}$$
$$= \theta \circ \sigma \circ T_P(\sigma) \circ \theta^{-1},$$

where $\theta = \psi' \circ \psi^{-1}$ is an $M_n(L[X, Y])$-linear automorphism of $M_n(L[X, Y])^m$. In particular, changing P in the isomorphism class of P amounts to changing $T_P(\sigma)$ in the above manner.

For any integer $m \geqslant 1$, let $Z^1(m)$ denote the set of all maps $T: G \longrightarrow \mathrm{Aut}_{L[X,Y]} M_n(L[X, Y])^m$ such that: (1) for every $\sigma \in G$, $T(\sigma)$ is $f(\sigma)$-semilinear and (2) for $\sigma, \tau \in G$, $T(\sigma\tau) = \tau^{-1} \circ T(\sigma) \circ \tau \circ T(\tau)$. We define a relation on $Z^1(m)$ by setting $T \sim T'$ if and only if there exists an $M_n(L[X, Y])$-linear automorphism θ of $M_n(L[X, Y])^m$ such that $T'(\sigma) = (\sigma^{-1} \circ \theta \circ \sigma) \circ T(\sigma) \circ \theta^{-1}$. This is an equivalence relation and we denote by $H^1(m)$ the quotient set $Z^1(m)/\sim$.

Theorem 2.1. *Let $\mathscr{P}(m)$ denote the set of isomorphism classes of projective modules of rank m over A. Let i be the map $\mathscr{P}(m) \xrightarrow{i} H^1(m)$ given by $i([P]) = [T_P]$ where $[T_P]$ denotes the class of T_P in $H^1(m)$. Then i is a bijection.*

Proof. We first check the injectivity of the map i. Let P, P' be projective modules of rank m over A such that $[T_P] = [T_{P'}]$. Let θ be an $M_n(L[X, Y])$-linear automorphism of $M_n(L[X, Y])^m$ such that $\sigma \circ T_{P'}(\sigma) = \theta \circ \sigma \circ T_P(\sigma) \circ \theta^{-1}$. Let $\psi: L \otimes_K P \xrightarrow{\sim} M_n(L[X, Y])^m$, $\psi': L \otimes_K P' \xrightarrow{\sim} M_n(L[X, Y])^m$ be ϕ-semilinear isomorphisms such that $\psi \circ \sigma \otimes 1 \circ \psi^{-1} = \sigma \circ T_P(\sigma)$, $\psi' \circ \sigma \otimes 1 \circ \psi'^{-1} = \sigma \circ T_{P'}(\sigma)$. The map $\psi'^{-1} \circ \theta \circ \psi: L \otimes_K P \longrightarrow L \otimes_K P'$ is $L \otimes_K D[X, Y]$-linear and $\psi'^{-1} \circ \theta \circ \psi \circ \sigma \otimes 1 = \sigma \otimes 1 \circ \psi'^{-1} \circ \theta \circ \psi$. Thus $\psi'^{-1} \circ \theta \circ \psi$ induces a $D[X, Y]$-isomorphism of P onto P', which proves the injectivity of i.

We now prove the surjectivity of i. Let $[T]$ be an element of $H^1(m)$ and $T \in Z^1(m)$ be a representative. Define

$$P = \{x \in M_n(L[X, Y])^m \mid \sigma \circ T(\sigma)(x) = x, \ ^\forall \sigma \in G\}.$$

We will show that $[P]\in\mathscr{P}(m)$ and that $i([P])=[T]$. Set $B=M_n(L[X, Y])$. We regard A as a subring of B through the isomorphism $\phi: L\underset{K}{\otimes}A \xrightarrow{\sim} B$ and consider B in what follows as a right A-module. The isomorphism ϕ induces a ring isomorphism

$$(\mathrm{End}_K L)\underset{K}{\otimes}A \longrightarrow \mathrm{End}_A(L\underset{K}{\otimes}A) \xrightarrow[\sim]{\bar{\phi}} \mathrm{End}_A B,$$

where $\bar{\phi}$ is defined by $\bar{\phi}(g)=\phi\circ g\circ\phi^{-1}$, for $g\in\mathrm{End}_A L\underset{K}{\otimes}A$. By the normal basis theorem, $\mathrm{End}_K L$ is generated as a K-algebra by G and L so that $(\mathrm{End}_K L)\underset{K}{\otimes}A$ is generated by $L\underset{K}{\otimes}A$ and $\sigma\otimes 1$, $\sigma\in G$. Thus $\mathrm{End}_A B$ is generated by $\bar{\phi}(\sigma\otimes 1)=\sigma\circ f(\sigma)$, $\sigma\in G$ and B, where B is regarded as a subring of $\mathrm{End}_A B$ through left multiplication. The relations between these generators are given by

$$\sigma\circ f(\sigma)\cdot\tau\circ f(\tau)=\sigma\tau\circ f(\sigma\tau)$$

$$\sigma\circ f(\sigma)\cdot b=\sigma\circ f(\sigma)(b)\cdot\sigma\circ f(\sigma);\ \sigma,\ \tau\in G,\ b\in B.$$

These relations are obtained from the relations between G and L in $\mathrm{End}_K L$ and are therefore the only relations.

On the other hand, the elements $\sigma\circ T(\sigma)$, $\sigma\in G$ belong to $\mathrm{End}_A B^m$ and satisfy the relations

$$\sigma\circ T(\sigma)\circ\tau\circ T(\tau)=\sigma\tau\circ T(\sigma\tau)$$

and $$\sigma\circ T(\sigma)\circ b=\sigma\circ f(\sigma)(b)\circ\sigma\circ T(\sigma),$$

where $B\hookrightarrow \mathrm{End}_A B^m$ as left multiplications. We therefore have a K-algebra homomorphism

$$\alpha: \mathrm{End}_A B \longrightarrow \mathrm{End}_A B^m,$$

defined by $$\alpha(\sigma\circ f(\sigma))=\sigma\circ T(\sigma),$$

$$\alpha(b)=b.$$

Let ${}_\alpha B^m$ denote B^m regarded as a left $\mathrm{End}_A B$-module through α. By Morita equivalence applied to the pair $(A, \mathrm{End}_A B)$ [1, p. 69], we have an $\mathrm{End}_A B$-isomorphism

$$B\underset{A}{\otimes}\mathrm{Hom}_{\mathrm{End}_A B}(B, {}_\alpha B^m) \xrightarrow{\sim} {}_\alpha B^m$$

induced by $(b, g) \longmapsto b \cdot g(1)$, $b \in B$, $g \in \mathrm{Hom}_{\mathrm{End}_A B}(B, {}_\alpha B^m)$. We assert that under this isomorphism, $\mathrm{Hom}_{\mathrm{End}_A B}(B, {}_\alpha B^m)$ maps onto the module P defined above. For, if $g \in \mathrm{Hom}_{\mathrm{End}_A B}(B, {}_\alpha B^m)$, then $\sigma \circ T(\sigma) \cdot g(1) = \sigma \circ f(\sigma) \cdot g(1)$ (by the definition of the $\mathrm{End}_A B$-module structure on ${}_\alpha B^m$) $= g(\sigma \circ f(\sigma)(1))$ (since g is $\mathrm{End}_A B$-linear) $= g(1)$, i. e. $g(1) \in P$ for every $g \in \mathrm{Hom}_{\mathrm{End}_A B}(B, {}_\alpha B^m)$. Conversely, let $x \in P$. Define $g: B \longrightarrow {}_\alpha B^m$ by $g(b) = bx$ for every $b \in B$. We prove that g is $\mathrm{End}_A B$-linear. Since $\mathrm{End}_A B$ is generated by B and $\sigma \circ f(\sigma)$, $\sigma \in G$, and since g is clearly B-linear, it is enough to check that $g((\sigma \circ f(\sigma) \cdot b) = \sigma \circ f(\sigma) \cdot g(b)$. Now $g(\sigma \circ f(\sigma) \cdot b) = g(\sigma \circ f(\sigma)(b)) = \sigma \circ f(\sigma)(b) \cdot x$ and $\sigma \circ f(\sigma) \cdot g(b) = \sigma \circ f(\sigma) \cdot bx = \sigma \circ T(\sigma) \cdot bx$ (by the definition of module structure on ${}_\alpha B^m$) $= \sigma \circ f(\sigma)(b) \cdot \sigma \circ T(\sigma)(x) = \sigma \circ f(\sigma)(b) \cdot x$, x being in P. Thus g is $\mathrm{End}_A B$-linear and $g(1) = x$.

The module ${}_\alpha B^m$ over $\mathrm{End}_A B$ is A-free and hence is $\mathrm{End}_A B$-projective. (For example, through a choice of a basis of B over A, we can identify $\mathrm{End}_A B$ with $M_k(A)$. If $\eta: M_k(A)^r \longrightarrow {}_\alpha B^m$ is a B-linear epimorphism and $t: {}_\alpha B^m \longrightarrow M_k(A)^r$ is an A-linear section to η, $\sum_{1 \leqslant i \leqslant k} e_{i1} t e_{1i}$, e_{ij} being the standard basis of $M_k(A)$ over A, is easily checked to be an $M_k(A)$-linear section of η). Hence, $\mathrm{Hom}_{\mathrm{End}_A B}(B, {}_\alpha B^m)$ and therefore P, is A-projective. The inclusion $P \hookrightarrow B^n$ gives rise to an isomorphism $\psi: L \otimes_K P \simeq B^m$. We prove that $\sigma^{-1} \circ \psi \circ \sigma \otimes 1 \circ \psi^{-1} = T(\sigma)$. In fact, for $l \in L$, $x \in P$, $\psi \circ \sigma \otimes 1 (l \otimes x) = \psi(\sigma(l) \otimes x) = \sigma(l) \cdot x$ and $\sigma \circ T(\sigma) \circ \psi(l \otimes x) = \sigma \circ T(\sigma)(lx) = \sigma(l) \sigma \circ T(\sigma)(x)$ (since $T(\sigma)$ is L-linear) $= \sigma(l) \cdot x$. This proves the surjectivity of i and the proof of the theorem is complete.

§3. Stabilisation.

With the notation of the previous section, for integers $m, m' \geqslant 1$, we have a map $\alpha: \mathscr{P}(m) \times \mathscr{P}(m') \longrightarrow \mathscr{P}(m+m')$ given by $([P], [P']) \longmapsto [P \oplus P']$. We also have a map $\beta: Z^1(m) \times Z^1(m') \longrightarrow Z^1(m+m')$, given by $(T, T') \longmapsto T \oplus T'$, where $T \oplus T': G \longrightarrow \mathrm{Aut}_{L[X,Y]} M_n(L[X, Y])^{m+m'}$ is defined by $(T \oplus T')(\sigma) = T(\sigma) \oplus T'(\sigma)$.

Let $T_1 \sim T_2$, $T'_1 \sim T'_2$, T_1, $T_2 \in Z^1(m)$, T'_1, $T'_2 \in Z^1(m')$, and let θ_1, θ_2 be automorphisms of $M_n(L[X, Y])^m$, $M_n(L[X, Y])^{m'}$ respectively, such that $\sigma \circ T_2(\sigma) = \theta_1 \circ \sigma \circ T_1(\sigma) \circ \theta_1^{-1}$, $\sigma \circ T'_2(\sigma) = \theta_2 \circ \sigma \circ T'_1(\sigma) \circ \theta_2^{-1}$, for every $\sigma \in G$. Let $\theta : M_n(L[X, Y])^{m+m'} \longrightarrow M_n(L[X, Y])^{m+m'}$ be the automorphism $\theta_1 \oplus \theta_2$. Then, clearly, $\sigma \circ (T_2 \oplus T'_2)(\sigma) = \theta \circ \sigma \circ (T_1 \oplus T'_1)(\sigma) \circ \theta^{-1}$. Thus the map β induces a map of the quotient sets $H^1(m) \times H^1(m') \xrightarrow{\beta} H^1(m+m')$. We write $\beta([T_1], [T_2]) = [T_1] \oplus [T_2]$.

Proposition 3.1. *The diagram*

$$\begin{array}{ccc} \mathscr{P}(m) \times \mathscr{P}(m') & \xrightarrow{\alpha} & \mathscr{P}(m+m') \\ \downarrow{\scriptstyle i \times i} & & \downarrow{\scriptstyle i} \\ H^1(m) \times H^1(m') & \xrightarrow{\beta} & H^1(m+m') \end{array}$$

is commutative; *i.e.* $i([P \oplus P']) = i([P]) \oplus i([P'])$.

Proof. Let T, T' be representatives in $Z^1(m)$, $Z^1(m')$ respectively of $i([P])$ and $i([P'])$. Let $\phi : L \otimes_K P \xrightarrow{\sim} M_n(L[X, Y])^m$, $\phi' : L \otimes_K P' \xrightarrow{\sim} M_n(L[X, Y])^{m'}$ be ϕ-semilinear isomorphisms such that $\sigma^{-1} \circ \phi \circ (\sigma \otimes 1) \circ \phi^{-1} = T(\sigma)$, $\sigma^{-1} \circ \phi' \circ (\sigma \otimes 1) \circ \phi'^{-1} = T'(\sigma)$.

We have a string of isomorphisms

$$L \otimes_K (P \oplus P') \overset{\lambda}{\simeq} L \otimes_K P \oplus L \otimes_K P' \xrightarrow{\phi \oplus \phi'} M_n(L[X, Y])^m \oplus M_n(L[X, Y])^{m'}$$
$$= M_n(L[X, Y])^{m+m'},$$

where λ is canonical. Let $\phi'' = (\phi \oplus \phi') \circ \lambda$. It is easily checked that $\sigma^{-1} \circ \phi'' \circ \sigma \otimes 1 \circ \phi''^{-1} = (\sigma^{-1} \circ \phi \circ \sigma \otimes 1 \circ \phi^{-1}$, $\sigma^{-1} \circ \phi' \circ \sigma \otimes 1 \circ \phi'^{-1}) = T(\sigma) \oplus T'(\sigma)$. This proves the proposition.

Corollary 3.2. *For any* $T \in Z^1(m)$, $T \oplus f^2 \sim f^{m+2}$, *where for any integer* k, $f^k \in Z^1(k)$ *is difined by* $f^k(\sigma) = f(\sigma)^k : M_n(L[X, Y])^k \longrightarrow M_n(L[X, Y])^k$.

Proof. For the free module $D[X, Y]^k = F$ of rank k, taking

$\psi=\phi^k$, we see that $T_F(\sigma)=f(\sigma)^k$. On the other hand, for any projective module P over $D[X, Y]$, of rank $m\geqslant 1$ we know that $P\oplus D[X, Y]^2\simeq D[X, Y]^{m+2}$. Hence $T_P\oplus f^2\sim f^{m+2}$.

§4. A description of projective modules over H[X, Y].

Let $\mathbf{H}$ denote the division algebra of quaternions over the field K of either real numbers or rational numbers. Then $L=K(i)$ is a splitting field of $\mathbf{H}$. The Galois group $G=G(L/K)$ is generated by the automorphism σ which takes i to $-i$. An L-algebra isomorphism $\phi: L\otimes_K \mathbf{H} \xrightarrow{\sim} M_2(L)$ is given by

$$\phi(\lambda\otimes(a+ib+jc+kd))=\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}a+ib & c+id\\ -c+id & a-ib\end{pmatrix},$$

so that ϕ^{-1} is given by

$$\phi^{-1}\begin{pmatrix}\lambda_1+i\lambda_2, & \mu_1+i\mu_2\\ \nu_1+i\nu_2, & \delta_1+i\delta_2\end{pmatrix}=1\otimes\left(\frac{\lambda_1+\delta_1}{2}+i\frac{\lambda_2-\delta_2}{2}+j\frac{\mu_1-\nu_1}{2}+k\frac{\mu_2+\nu_2}{2}\right)$$

$$+i\otimes\left(\frac{\lambda_2+\delta_2}{2}+i\frac{\delta_1-\lambda_1}{2}+j\frac{\mu_2-\nu_2}{2}+k\frac{-\mu_1-\nu_1}{2}\right).$$

The map ϕ induces an $L[X, Y]$-algebrai somorphism $\phi: L\otimes_K \mathbf{H}[X, Y] \xrightarrow{\sim} M_2(L[X, Y])$. Then $f(\sigma)=\sigma^{-1}\circ\phi\circ\sigma\otimes 1\circ\phi^{-1}=\text{Int } a$, where Int a denotes the inner automorphism of $M_2(L[X, Y])$ given by $a=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$.

Let P be a projective module of rank two over $\mathbf{H}[X, Y]$. Let $\psi: L\otimes_K P \xrightarrow{\sim} M_2(L[X, Y])^2$ be a ϕ-semilinear isomorphism and let $T(\sigma)=\sigma^{-1}\circ\psi\circ\sigma\otimes 1\circ\psi^{-1}$. The map $T(\sigma): M_2(L[X, Y])^2\longrightarrow M_2(L[X, Y])^2$ is uniquely determined by its values on the standard basis (e_1, e_2) of $M_2(L[X, Y])^2$, since it is semilinear. Let $T(\sigma)(e_i)=A_{i1}e_1+A_{i2}e_2$, $i=1, 2$; $A_{ij}\in M_2(L[X, Y])$; Then

$$T(\sigma)(\lambda_1 e_1+\lambda_2 e_2)=a\lambda_1 a^{-1}\cdot(A_{11}e_1+A_{12}e_2)+a\lambda_2 a^{-1}\cdot(A_{21}e_1+A_{22}e_2),$$

$$\lambda_1, \lambda_2\in M_2(L[X, Y]).$$

The cocycle condition for $T(\sigma)$ gives $Id=T(\sigma^2)=\sigma^{-1}\circ T(\sigma)\circ\sigma\circ T(\sigma)$, so that

$$\begin{aligned} e_1=\sigma\circ Id(e_1) &= T(\sigma)\circ\sigma\circ T(\sigma)(e_1) \\ &= T(\sigma)(\bar{A}_{11}e_1+\bar{A}_{12}e_2) \\ &= a\bar{A}_{11}a^{-1}\cdot(A_{11}e_1+A_{12}e_2)+a\bar{A}_{12}a^{-1}\cdot(A_{21}e_1+A_{22}e_2), \end{aligned}$$

where bar denotes the effect of σ. Also

$$\begin{aligned} e_2=\sigma\circ Id(e_2) &= T(\sigma)\circ\sigma\circ T(\sigma)(e_2) \\ &= a\bar{A}_{21}a^{-1}(A_{11}e_1+A_{12}e_2)+a\bar{A}_{22}a^{-1}(A_{21}e_1+A_{22}e_2), \end{aligned}$$

We thus have

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}=\begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}. \tag{1}$$

Conversely, every block matrix $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $A_{ij}\in M_2(L[X, Y])$ satisfying the above condition gives rise to a cocycle T. The cocycle f^2 which corresponds to the free module of rank 2 over $\mathbf{H}[X, Y]$ gives the matrix $\begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}$. Two such block matrices $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ give rise to equivalent T if and only if there exists an invertible 2×2 matrix (C_{ij}) with entries in $M_2(L[X, Y])$ such that

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}=\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}^{-1}\begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} \bar{C}_{11} & \bar{C}_{12} \\ \bar{C}_{21} & \bar{C}_{22} \end{pmatrix}. \tag{2}$$

Thus the set of isomorphism classes of projective modules of rank 2 over $\mathbf{H}[X, Y]$ is in bijection with the set of classes of matrices $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $A_{ij}\in M_2(L[X, Y])$ satisfying (1) with the equivalence relation given by (2).

Remark. The result on stabilisation proved in § 3 applies in this particular case to show that given any block matrix $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $A_{ij} \in M_2(L[X, Y])$ satisfying (1), the matrix $\begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 1_2 \end{pmatrix}$, $1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, is equivalent to $\begin{pmatrix} 1_2 & 0 & 0 \\ 0 & 1_2 & 0 \\ 0 & 0 & 1_2 \end{pmatrix}$.

We now consider projective modules of rank 1 over $\mathbf{H}[X, Y]$. Let P be a projective module of rank 1 over $\mathbf{H}[X, Y]$ with an isomorphism $\psi : L \underset{K}{\otimes} P \xrightarrow{\sim} M_2(L[X, Y])$, which is ϕ-semilinear. Then, $T(\sigma) = \sigma^{-1} \circ \psi \circ \sigma \otimes 1 \circ \psi^{-1} : M_2(L[X, Y]) \longrightarrow M_2(L[X, Y])$ is uniquely determined by $T(\sigma)(1)$. Let $T(\sigma)(1) = A$. Then $T(\sigma)(\lambda) = f(\sigma)(\lambda)\ T(\sigma)(1) = a\lambda a^{-1}A$ for every $\lambda \in M_2(L[X, Y])$. $Id = T(\sigma^2) = \sigma^{-1} \circ T(\sigma) \circ \sigma \circ T(\sigma)$ gives the following condition on A:

$$a\bar{A}a^{-1}A = Id. \tag{3}$$

Conversely, any $A \in M_2(L[X, Y])$ satisfying the above condition represents some T. The cocycle f which corresponds to the class of free modules of rank 1 over $\mathbf{H}[X, Y]$ gives the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Two matrices A, B satisfying the above condition represent equivalent T if and only if there exists an invertible matrix $u \in M_2(L[X, Y]$ with

$$B = au^{-1}a^{-1}A\bar{u}. \tag{4}$$

Thus, the set of isomorphism classes of projective modules of rank 1 over $\mathbf{H}[X, Y]$ is in bijection with the set of equivalence classes of matrices $A \in M_2(L[X, Y])$ satisfying (3), the equivalence being defined by (4).

Remark. As in the rank 2 case, given any matrix $A \in M_2(L[X, Y])$ satisfying (3), the matrix $\begin{pmatrix} A & 0 & 0 \\ 0 & 1_2 & 0 \\ 0 & 0 & 1_2 \end{pmatrix}$ is equivalent to

$$\begin{pmatrix} 1_2 & 0 & 0 \\ 0 & 1_2 & 0 \\ 0 & 0 & 1_2 \end{pmatrix}.$$

§5. Some explicit computations for rank 1 projective modules over the polynomial ring over the real quaternions.

Let $A \in M_2(\mathbf{C}[X, Y])$ such that $a\bar{A}a^{-1} = A^{-1}$. Then $\det A \cdot \det \bar{A} = 1$, i. e. $\det A = e^{i\theta}$, θ real. The matrix $B = au^{-1}a^{-1}A\bar{u}$, where $u = \begin{pmatrix} e^{i\theta}/4 & 0 \\ 0 & e^{i\theta}/4 \end{pmatrix}$ is equivalent to A and has determinant 1. Thus to find out whether two matrices are equivalent in the sense defined in §4, we may and do assume that the matrices have determinant 1.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det A = 1$. Then clearly A satisfies (3) if and only if $a = \bar{a}$, $d = \bar{d}$, $b = \bar{c}$, (bar denoting complex conjugation); i. e., the matrix A has the form $\begin{pmatrix} a_1 & b_1 + ib_2 \\ b_1 - ib_2 & d_1 \end{pmatrix}$ with $a_1, b_1, b_2, d_1 \in \mathbf{R}[X, Y]$. Conversely, any matrix A of the form $\begin{pmatrix} a_1 & b_1 + ib_2 \\ b_1 - ib_2 & d_1 \end{pmatrix}$ with $\det A = 1$ satisfies (3). If $B = \begin{pmatrix} e_1 & f_1 + if_2 \\ f_1 - if_2 & h_1 \end{pmatrix}$ is another such matrix which is equivalent to A, we know that there exists an invertible matrix $u = \begin{pmatrix} \lambda_1 + i\lambda_2 & \mu_1 + i\mu_2 \\ \nu_1 + i\nu_2 & \delta_1 + i\delta_2 \end{pmatrix} \in M_2(\mathbf{C}[X, Y])$ such that condition (4) is satisfied. Since $\det A = \det B = 1$, $\det u$ is real. By replacing u by $u' = \frac{1}{\sqrt{|\det u|}} u$, we assume that $\det u = \pm 1$. We have

$$\begin{aligned} \det u &= (\lambda_1 + i\lambda_2)(\delta_1 + i\delta_2) - (\mu_1 + i\mu_2)(\nu_1 + i\nu_2) \\ &= \lambda_1\delta_1 - \lambda_2\delta_2 - \mu_1\nu_1 + \mu_2\nu_2, \qquad (5) \end{aligned}$$

since $\det u$ is real. Equating the real and imaginary parts of matrix entries in equation (4), we get the following sets of equations :

$$\left.\begin{aligned} d_1\nu_1 &= -e_1\mu_1 + (f_1-b_1)\lambda_1 + (f_2+b_2)\lambda_2 \\ d_1\nu_2 &= e_1\mu_2 + (f_2-b_2)\lambda_1 - (f_1+b_1)\lambda_2 \\ d_1\delta_1 &= h_1\lambda_1 - (f_1+b_1)\mu_1 + (f_2+b_2)\mu_2 \\ d_1\delta_2 &= -h_1\lambda_2 + (f_2-b_2)\mu_1 + (f_1-b_1)\mu_2 \end{aligned}\right\} \tag{6}$$

and

$$\left.\begin{aligned} a_1\lambda_1 &= e_1\delta_1 - (f_1+b_1)\nu_1 - (f_2+b_2)\nu_2 \\ a_1\lambda_2 &= -e_1\delta_2 - (f_2-b_2)\nu_1 + (f_1-b_1)\nu_2 \\ a_1\mu_1 &= -h_1\nu_1 + (f_1-b_1)\delta_1 - (f_2+b_2)\delta_2 \\ a_1\mu_2 &= h_1\nu_2 - (f_2-b_2)\delta_1 - (f_1+b_2)\delta_1 \end{aligned}\right\}. \tag{6_*}$$

Since $\det A = \det B = 1$, we have $a_1d_1 = 1 + b_1^2 + b_2^2$ and $e_1h_1 = 1 + f_1^2 + f_2^2$. In particular, d_1 and h_1 are non-zero. Using these facts, one can verify that (6_*) is a consequence of (6). We shall show, for instance, how one obtains the first equation of (6_*) from (6).

$$\begin{aligned} d_1\{e_1\delta_1 - (f_1+b_1)\nu_1 - (f_2+b_2)\nu_2\} &= e_1d_1\delta_1 - (f_1+b_1)d_1\nu_1 - (f_2+b_2)d_1\nu_2 \\ &= e_1\{h_1\lambda_1 - (f_1+b_1)\mu_1 + (f_2+b_2)\mu_2\} \\ &\quad - (f_1+b_1)\{-e_1\mu_1 + (f_1-b_1)\lambda_1 + (f_2+b_2)\lambda_2\} \\ &\quad - (f_2+b_2)\{e_1\mu_2 + (f_2-b_2)\lambda_1 - (f_1+b_1)\lambda_2\}, \end{aligned}$$

using (6).

Thus, we have

$$d_1\{e_1\delta_1 - (f_1+b_1)\nu_1 - (f_2+b_2)\nu_2\} = (e_1h_1 - f_1^2 + b_1^2 - f_2^2 + b_2^2)\lambda_1 = a_1d_1\lambda_1.$$

The first equation of (6_*) now follows by cancelling d_1 in the above.

Substituting for δ_1, δ_2, ν_1, ν_2 from (6) in the equation (5) and using the fact that $e_1h_1 - f_1^2 - f_2^2 = 1$, we get

$$h_1d_1 \det u = (h_1\lambda_1 - f_1\mu_1 + f_2\mu_2)^2 + (h_1\lambda_2 - f_2\mu_1 - f_1\mu_2)^2 + \mu_1^2 + \mu_2^2. \tag{7}$$

Proposition 5.1. *A necessary and sufficient condition that the matrices* $A = \begin{pmatrix} a_1 & b_1+ib_2 \\ b_1-ib_2 & d_1 \end{pmatrix}$, $B = \begin{pmatrix} e_1 & f_1+if_2 \\ f_1-if_2 & h_1 \end{pmatrix}$ *with* $\det A = \det B = 1$ *are equivalent is that there exist* $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbf{R}[X, Y]$ *such that the four expressions on the right hand side of* (6) *belong to the ideal generated by* d_1 *and such that*

$$\varepsilon h_1d_1 = (h_1\lambda_1 - f_1\mu_1 + f_2\mu_2)^2 + (h_1\lambda_2 - f_2\mu_1 - f_1\mu_2)^2 + \mu_1^2 + \mu_2^2,$$

where $\varepsilon=\pm 1$.

(Note that at the most one of $\pm h_1 d_1$ *can be written as a sum of four squares in* $\mathbf{R}[X, Y]$*!).*

Proof. The remarks preceding the proposition prove the necessity of the conditions. Suppose conversely that the conditions of the proposition are satisfied. Then, we can solve for ν_1, ν_2, δ_1, δ_2 from (6) and the matrix $u=\begin{pmatrix}\lambda_1+i\lambda_2 & \mu_1+i\mu_2\\ \nu_1+i\nu_2 & \delta_1+i\delta_2\end{pmatrix}$ clearly satisfies $aua^{-1}B=A\bar{u}$. Further, by the very choice of u, equation (7) is satisfied. Hence $h_1d_1 \det u=h_1d_1\varepsilon$ which implies that $\det u=\varepsilon$ and u is invertible.

Corollary 5. 2. *The matrix* $B=\begin{pmatrix}e_1 & f_1+if_2\\ f_1-if_2 & h_1\end{pmatrix}$ *with* $\det B=1$ *is equivalent to the identity matrix if and only if the diophantine equation*

$$\varepsilon h_1=(h_1\lambda_1-f_1\mu_1+f_2\mu_2)^2+(h_1\lambda_2-f_2\mu_1-f_1\mu_2)^2+\mu_1^2+\mu_2^2$$

is solvable for λ_1, λ_2, μ_1, $\mu_2\in\mathbf{R}[X, Y]$.

The exact sequence

$$0\longrightarrow P\longrightarrow \mathbf{H}[X, Y]^2\xrightarrow{\eta}\mathbf{H}[X, Y]\longrightarrow 0,$$

where $\eta(1, 0)=f+i$, $\eta(0, 1)=g+j$, f, $g\in\mathbf{R}[X, Y]$, gives rise to a projective module P of rank 1 over $\mathbf{H}[X, Y]$. We compute the matrix A corresponding to P. It is easily seen that P is generated by $(1-\frac{1}{2}(f+i)k(g+j),\ \frac{1}{2}(f+i)k\ (f+i))$ and $(-\frac{1}{2}(g+j)k(g+j),\ 1+\frac{1}{2}(g+j)k(f+i))$. The first porjection of $\mathbf{H}[X, Y]^2$ onto $\mathbf{H}[X, Y]$ maps P isomorphically onto a left ideal $\mathfrak{a}$ of $\mathbf{H}[X, Y]$ generated by $1-\frac{1}{2}(f+i)k\ (g+j)$ and $-\frac{1}{2}(g+j)k\ (g+j)$. Under the isomorphism $\phi:\mathbf{C}\otimes_{\mathbf{R}}\mathbf{H}[X, Y]\xrightarrow{\sim} M_2(\mathbf{C}[X, Y])$, the image of $\mathbf{C}\otimes_{\mathbf{R}}\mathfrak{a}$ in $M_2(\mathbf{C}[X, Y])$ is generated by $\begin{pmatrix}1+if & g(1-if)\\ -g(1+if) & 1-if\end{pmatrix}$ and $\begin{pmatrix}0 & 1+g^2\\ 1+g^2 & 0\end{pmatrix}$.

The equations

$$\begin{pmatrix} 1+if & g(1-if) \\ -g(1+if) & 1-if \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -g & \frac{1}{2}i(1-if) \end{pmatrix} \begin{pmatrix} 1+if & fg \\ -ig(1+if) & 1-ifg^2 \end{pmatrix} \times \begin{pmatrix} 1-ifg^2 & g(1+ifg^2) \\ ig(1+if) & -ig^2(1+if)-2i \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1+g^2 \\ 1+g^2 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}ig(1+fg) & \frac{1}{2}i(1+if) \\ \frac{1}{2}ig(1-fg) & \frac{1}{2}(2+ig-fg^2) \end{pmatrix} \times \begin{pmatrix} 1+if & fg-1 \\ -ig(1+if)+1 & -ifg^2 \end{pmatrix} \begin{pmatrix} 1-ifg^2 & g(1+ifg^2) \\ ig(1+if) & -ig^2(1+if)-2i \end{pmatrix}$$

show that $\mathbf{C}\otimes_{\mathbf{R}}\mathfrak{a}$ is contained in the principal left ideal of $M_2(\mathbf{C}[X, Y])$ generated by $c = \begin{pmatrix} 1-ifg^2 & g(1+ifg^2) \\ ig(1+if) & -ig^2(1+if)-2i \end{pmatrix}$. On the other hand,

$$c\begin{pmatrix} f+i & 0 \\ 0 & f-i \end{pmatrix} = \begin{pmatrix} 2fg & fg^2-f-i+if^2g^2 \\ -2(1+if) & -g(1+if)^2 \end{pmatrix} \begin{pmatrix} g & 1 \\ -1 & g \end{pmatrix}$$

shows that $c \in$ image of $\mathbf{C}\otimes_{\mathbf{R}}\mathfrak{a}$. Thus the image of $\mathbf{C}\otimes_{\mathbf{R}}\mathfrak{a}$ in $M_2(\mathbf{C}[X, Y])$ is the principal left ideal generated by c. Hence the isomorphism $\phi : \mathbf{C}\otimes_{\mathbf{R}}\mathfrak{a} \xrightarrow{\sim} M_2(\mathbf{C}[X, Y])$ is given by the composite map $\mathbf{C}\otimes_{\mathbf{R}}\mathfrak{a} \xrightarrow{\phi} M_2(\mathbf{C}[X, Y]) \cdot c \xrightarrow{j} M_2(\mathbf{C}[X, Y])$, where $j(c)=1$. We shall now compute $A' = T(\sigma)\ (1) = \sigma^{-1} \circ \phi \circ \sigma \otimes 1 \circ \phi^{-1}(1) = \sigma^{-1} \circ j \circ \phi \circ \sigma \circ \otimes 1 \circ \phi^{-1}(c)$. Noting that for any $\begin{pmatrix} \lambda & \mu \\ \nu & \delta \end{pmatrix} \in M_2(\mathbf{C}[X, Y])$, $\phi \circ \sigma \otimes 1 \circ \phi^{-1}\begin{pmatrix} \lambda & \mu \\ \nu & \delta \end{pmatrix} = \begin{pmatrix} \bar{\delta} & -\bar{\nu} \\ -\bar{\mu} & \bar{\lambda} \end{pmatrix}$, we have,

$$A' = T(\sigma)\,(1) = \sigma \circ j \begin{pmatrix} ig^2(1-if)+2i, & ig(1-if) \\ -g(1-ifg^2) & 1+ifg^2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2}i(4+g^2(1+f^2)) & -\frac{1}{2}i(fg(g^2+1)+ig(1+f^2g^2)) \\ -\frac{1}{2}i(fg(g^2+1)-ig(1+f^2g^2)) & -\frac{1}{2}i(1+f^2g^4) \end{pmatrix}.$$

Taking out the scalar $-\frac{1}{2}i$, we get a matrix

$$A = \begin{pmatrix} 4+g^2(1+f^2) & fg(g^2+1)+ig(1+f^2g^2) \\ fg(g^2+1)-ig(1+f^2g^2) & 1+f^2g^4 \end{pmatrix},$$

which represents the projective module P.

Proposition 5. 3. *The projective modules P_1 and P_2 constructed as above with $f=X$, $g=Y^n$ and $f=X$, $g=Y^m$ are not isomorphic if $n\neq m$, $n\geqslant 0$, $m\geqslant 0$.*

Proof. Let $m<n$. The matrices corresponding to P_1 and P_2 are respectively

$$A=\tfrac{1}{2}\begin{pmatrix}4+Y^{2m}(1+X^2) & XY^m(1+Y^{2m})+iY^m(1+X^2Y^{2m})\\ XY^m(Y^{2m}+1)-iY^m(1+X^2Y^{2m}) & 1+X^2Y^{4m}\end{pmatrix}$$

and

$$B=\tfrac{1}{2}\begin{pmatrix}4+Y^{2n}(1+X^2) & XY^n(1+Y^{2n})+iY^n(1+X^2Y^{2n})\\ XY^n(1+Y^{2n})-iY^n(1+X^2Y^{2n}) & 1+X^2Y^{4n}\end{pmatrix}.$$

We shall show that there cannot exist λ_1, λ_2, μ_1, μ_2, $\in \mathbf{R}[X, Y]$ satisfying

$$\varepsilon(1+X^2Y^{4m})(1+X^2Y^{4n})=(h_1\lambda_1-f_1\mu_1+f_2\mu_2)^2 + (h_1\lambda_2-f_2\mu_1-f_1\mu_2)^2+\mu_1^2+\mu_2^2, \tag{8}$$

where $h_1=1+X^2Y^{4n}$, $d_1=1+X^2Y^{4m}$, $f_1=XY^n(1+Y^{2n})$, $f_2=Y^n(1+X^2Y^{2n})$. Suppose there exist λ_1, μ_1, λ_2, μ_2 satisfying (8) ; then $\varepsilon=+1$ and each of $h_1\lambda_1-f_1\mu_1+f_2\mu_2$, $h_1\lambda_2-f_2\mu_1-f_1\mu_2$, μ_1, μ_2 is of X-degree$\leqslant 2$ and Y-degree $\leqslant 2m+2n$. Let $\mu_1=b_0+\theta_1X+\theta_3X^2$, $\mu_2=c_0+\theta_2X+\theta_4X^2$, $h_1\lambda_1-f_1\mu_1+f_2\mu_2=d_0+\phi_1X+\phi_3X^2$, $h_1\lambda_2-f_2\mu_1-f_1\mu_2=e_0+\phi_2X+\phi_4X^2$, where b_0, c_0, d_0, e_0, ϕ_i, θ_j are all polynomials in Y of degree $\leqslant 2(m+n)$; comparing the degrees of X and Y in the last two equations, we see that X-degree of $\lambda_i\leqslant 2$ and Y-degree of $\lambda_i\leqslant n+2m$, $i=1, 2$. Let $\lambda_1=g_0+g_1X+g_2X^2$, $\lambda_2=g_0^1+g_1^1X+g_2^1X^2$, where g_i, g_i^1 are polynomials in Y of degree$\leqslant n+2m$. From (8) we have

$$(1+X^2Y^{4m})(1+X^2Y^{4n})=(d_0+\phi_1X+\phi_3X^2)^2+(e_0+\phi_2X+\phi_4X^2)^2+ (b_0+\theta_1X+\theta_3X^2)^2+(c_0+\theta_2X+\theta_4X^2)^2.$$

Treating both sides as polynomials in X and comparing the constant terms and the coefficients of X^4, we get, d_0, e_0, b_0, $c_0 \in \mathbf{R}$ and $\phi_3 = d_2 Y^{2(n+m)}$, $\phi_4 = e_2 Y^{2(n+m)}$, $\theta_3 = b_2 Y^{2(n+m)}$, $\theta_4 = c_2 Y^{2(n+m)}$, d_2, e_2, b_2, $c_2 \in \mathbf{R}$. Comparing the coefficients of X^2, we get

$$\phi_1^2 + \phi_2^2 + \theta_1^2 + \theta_2^2 = Y^{4m} + Y^{4n} - 2Y^{2(n+m)}(d_0 d_2 + e_0 e_2 + b_0 b_2 + c_0 c_2).$$

This implies that degree ϕ_i, degree θ_i are $\leqslant 2n$ and Y^{2m} divides each of ϕ_i and θ_i $i=1,2$. Let $\phi_i = Y^{2m}\phi_i^1$, $\theta_i = Y^{2m}\theta_i^1$, $i=1,2$. Substituting these expressions in $h_1\lambda_1 - f_1\mu_1 + f_2\mu_2$ and $h_1\lambda_2 - f_2\mu_1 - f_1\mu_2$, we have

$$(1+X^2Y^{4n})(g_0 + g_1 X + g_2 X^2) - XY^n(1+Y^{2n})(b_0 + \theta_1 X + b_2 Y^{2(n+m)} X^2)$$
$$+ Y^n(1+Y^{2n}X^2)(c_0 + \theta_2 X + c_2 Y^{2(n+m)} X^2) = d_0 + \phi_1 X + d_2 Y^{2(n+m)} X^2 \quad (9)$$
$$(1+X^2Y^{4n})(g_0^1 + g_1^1 X + g_2^1 X^2) - Y^n(1+Y^{2n}X^2)(b_0 + \theta_1 X + b_2 Y^{2(n+m)} X^2)$$
$$- XY^n(1+Y^{2n})(c_0 + \theta_2 X + c_2 Y^{2(n+m)} X^2) = e_0 + \phi_2 X + e_2 Y^{2(n+m)} X^2 \quad (10)$$

Let $g_1 = \sum_{i=0}^{n+2m} l_i Y^i$, $g_1^1 = \sum_{i=0}^{n+2m} k_i Y^i$. We regard (9) as a polynomial equation in X and compare the coefficients of various powers of X. Equating the terms independent of X, we get $g_0 = d_0 - c_0 Y^n$. Comparing the coefficients of X^4, $g_2 = -c_2 Y^{n+2m}$; comparing the coefficients of X^3, we have

$$Y^n \sum_{i=0}^{2m+n} l_i Y^i - b_2 Y^{2m}(1+Y^{2n}) + \theta_2 = 0. \quad (11)$$

Since deg $\theta_2 \leqslant 2n$, $l_{2m+n} = b_2$, $l_i = 0$ for $i = n+1, \dots . 2m+n-1$. Comparing the coefficients of X, we have

$$\left(\sum_{i=0}^{n} l_i Y^i + b_2 Y^{2m+n}\right) - b_0 Y^n(1+Y^{2n}) + \theta_2 Y^n = \phi_1.$$

Since Y^{2m} divides ϕ_1, comparing the terms of degree $\leqslant m$, we get $l_i = 0$ for $i = 0, 1, \dots . m$. Thus,

$$\sum_{i=m+1}^{n} l_i Y^i + b_2 Y^{2m+n} - b_0 Y^n(1+Y^{2n}) + \theta_2 Y^n = \phi_1. \quad (12)$$

Substituting for θ_2 from (11), we get

$$\phi_1 = \sum_{i=m+1}^{n} l_i Y^i + 2b_2 Y^{2m+n} - b_0 Y^n - b_0 Y^{3n} - Y^{2n}\left(\sum_{i=m+1}^{n} l_i Y^i\right).$$

Since Y^{2m} divides ϕ_1, Y^{2m} divides $\sum_{i=m+1}^{n} l_i Y^i - b_0 Y^n$, i. e. $\sum_{i=m+1}^{n} l_i Y^i = \sum_{i=2m}^{n} l_i Y^i$. Hence,

$$\phi_1 = \sum_{i=2m}^{n} l_i Y^i + 2b_2 Y^{2m+n} - b_0 Y^n - b_0 Y^{3n} - Y^{2n}\Big(\sum_{i=2m}^{n} l_i Y^i\Big).$$

Since $\deg \phi_1 \leqslant 2n$, equating to zero terms of degree>max $(2n, 2m+n)$, we get $-b_0 Y^{3n} - Y^{2n}\Big(\sum_{i=2m}^{n} l_i Y^i\Big) = 0$, i. e. $\sum_{i=2m}^{n} l_i Y^i = -b_0 Y^n$. Thus we have $\phi_1 = -2b_0 Y^n + 2b_2 Y^{2m+n}$, $\theta_2 = b_2 Y^{2m} + b_0 Y^{2n}$, $g_1 = -b_0 Y^n + b_2 Y^{n+2m}$.

A similar comparison of coefficients in (10) gives

$$\phi_2 = -2c_0 Y^n + 2c_2 Y^{2m+n},\quad \theta_1 = -c_0 Y^{2n} - c_2 Y^{2m},\quad g_1^1 = -c_0 Y^n + c_2 Y^{n+2m}.$$

Substituting these values of θ_i, ϕ_i, g_1 and g_1^1 in (9) and (10) and equating the coefficients of X^2, we get

$$d_0 Y^{2n} + 2c_0 Y^n + 2c_2 Y^{n+2m} = d_2 Y^{2m},$$
$$e_0 Y^{2n} - 2b_0 Y^n - 2b_2 Y^{n+2m} = e_2 Y^{2m}. \tag{13}$$

We now consider two cases.

Case (1). $m=0$. From (13), we get, $c_2 = -c_0$, $b_2 = -b_0$, $d_2 = d_0 = e_2 = e_0 = 0$. Substituting these values in (8) and equating the coefficients of X^2, we have $b_2^2 + c_2^2 = 1$ and $12(b_2^2 + c_2^2) = 0$, a contradiction which proves the proposition in this case.

Case (2). $m \neq 0$. If, either $n < 2m$ or $n > 2m$, from (13) we get $d_0 = d_2 = c_0 = c_2 = e_0 = e_2 = b_0 = b_2 = 0$. Substituting in (8), we get $0 = (1 + X^2 Y^{4n})(1 + X^2 Y^{4m})$, a contradiction again. If $n = 2m$, from (13), we have $d_0 = -2c_2$, $d_2 = 2c_0$, $e_0 = 2b_2$, $e_2 = -2b_0$. Substituting these values in (8) and equating the constant terms, coefficients of X^4, of X^2 and of X, we get $b_0^2 + c_0^2 + 4b_2^2 + 4c_2^2 = 1$, $b_2^2 + c_2^2 + 4b_0^2 + 4c_0^2 = 1$, $b_0 c_2 - b_2 c_0 = 0$ and $c_0 c_2 + b_0 b_2 = 0$, i. e. we have $b_0^2 + c_0^2 = b_2^2 + c_2^2 = 1/5$ and $c_0 c_2 + b_0 b_2 = 0 = b_0 c_2 - b_2 c_0$. This is impossible. This proves the proposition.

Corollary 5.4. *The projective module P corresponding to* $f = X$, $g = Y^n$, $n \geqslant 1$ *is not free.*

Proof. By the above proposition, P is not isomorphic to the projective module P_0 obtained by taking $f=X$, $g=Y^0=1$. Since P_0 'comes from' the projective module over $\mathbf{H}[X]$ (defined as the kernel of the map $\mathbf{H}[X]^2 \xrightarrow{\eta} \mathbf{H}[X]$, $\eta(e_1)=X+i$, $\eta(e_2)=1+j$), it is free. This proves the corollary.

Remark. The results of this section are valid for the quaternion algebra over the field of rationals, with suitable modifications.

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