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Convergence theorems of Abelian differentials with applications to conformal mappings I

By

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§1. Introduction.

1A. Koebe was the first who proved the existence of conformal mapping of an arbitrary planar surface onto a horizontal slit region on the extended plane $\bar{\mathbf{C}}$. Concerning an open Riemann surface $\mathbf{R}$ of positive genus g, Kusunoki [6] showed the existence of a meromorphic function f such that

(i) Re (f) has, roughly speaking, the constant value on each boundary component. Precisely speaking, it can be said that Re (f) has Γ_{hm} behavior in the sense of Yoshida [18],

(ii) the divisor of f is a multiple of $(P_1P_2 \ldots P_{g+1})^{-1}$, where $P_1, P_2, \ldots P_{g+1}$ are suitable points of $\mathbf{R}$,

(iii) residue of f at P_1 is equal to 1 (or i),

(iv) $f(\mathbf{R})$, the image of $\mathbf{R}$ under f, is of at most $g+1$ sheets over the extended plane $\bar{\mathbf{C}}$.

Further, Mori [9] showed that there exists a meromorphic function f on $\mathbf{R}$ such that

(i) Re (f) has Γ_{hm} behavior,

(ii) the divisor of f is $(P)^{-g-1}$, where P is a properly given point on $\mathbf{R}$,

(iii) $f(\mathbf{R})$ is of $g+1$ sheets over $\bar{\mathbf{C}}$.

For the case of finite surface, Shiba [13] proved the next theorem.

Theorem. Let $\mathbf{R}$ be the interior of a compact bordered Riemann surface of genus g and its border consists of K contours β_1, $\beta_2, \dots \beta_K$. Then there exists a meromorphic function on $\mathbf{R}$ which satisfies the following conditions:

(i) f maps $\mathbf{R}$ onto a slit region on $\bar{\mathbf{C}}$ such that each contour β_k corresponds to a slit with direction ℓ_k, $k=1, 2 \dots K$, where each ℓ_k is an arbitrarily given straight line,

(ii) the divisor of f is a multiple of $(P_1P_2 \dots P_{g+1})^{-1}$, where $P_1, P_2, \dots P_{g+1}$ are arbitrarily given points on $\mathbf{R}$,

(iii) $f(\mathbf{R})$ is of at most $g+1$ sheets over $\bar{\mathbf{C}}$.

But the condition on the boundary in the last theorem is very restrictive, and our intention in this paper is to remove this restriction and generalize these theorems stated above. In §2, as the preparations for our purpose, we shall consider some subspaces (over the real number field) of real differentials on an open Riemann surface $\mathbf{R}$. In §3, we shall consider the convergence theorems of Abelian differentials with certain boundary behaviors (Cf *Lemma 3.3*) and show, in case of $\mathbf{R}$ with finite genus, the existence of meromorphic functions with these boundary behaviors (Cf *Lemma 3.5* and *Lemma 3.7*). In §4, we prove the convergence theorems of Abelian differentials with special boundary behaviors and the existence theorems of Shiba's behavior spaces (Cf *Theorem 1* and *Theorem 2*). In §5, we shall generalize the above slitmapping theorems by use of the results in §2-§4 (Cf *Theorem 3, 4, 5*).

§2. Orthogonal decompositions for differentials.

2A. Let $\mathbf{R}$ be an arbitrary Riemann surface. The totality of square integrable complex (resp. real) differentials on $\mathbf{R}$ forms a real Hilbert space $\Lambda=\Lambda(\mathbf{R})$ (resp. $\Gamma=\Gamma(\mathbf{R})$) over the real number field with the inner product defined by

$$\langle \lambda_1, \lambda_2 \rangle = \mathrm{Re}\iint_{\mathbf{R}} (a_1\bar{a}_2 + b_1\bar{b}_2)\ dxdy,$$

where $\lambda_j = a_j(z)dx + b_j(z)\ dy$ for a local parameter $z = x + iy$. It should be noticed that the meanings of the letters Λ and Γ are different from those in Ahlfors and Sario [3]. With only these exceptions, we inherit the terminologies and notations of Ahlfors and Sario [3], if not mentioned further. For example, we define Λ_e (resp. Λ_{eo}) to be the closure of Λ_e^1 (resp. Λ_{eo}^1) in Λ where Λ_e^1 (resp. Λ_{eo}^1) is the linear space formed by all exact complex C^1 differentials (resp. exact complex C^1 differentials with compact supports), Λ_c (resp. Λ_{co}) to be the orthogonal complement of Λ_{eo}^* (resp. Λ_e^*) with respect to $\langle\ \rangle$, and Λ_h to be the space $\Lambda_c^{1*} \cap \Lambda_c^1$. With these notations, the following relations hold:

$$\Lambda = \Lambda_c \dot{+} \Lambda_{eo}^* = \Lambda_h \dot{+} \Lambda_{eo} \dot{+} \Lambda_{eo}^* = \Lambda_c \cap \Lambda_c^* \dot{+} \Lambda_{eo} \dot{+} \Lambda_{eo}^*,$$

$$(\Lambda_1 = \Gamma_1 \dot{+} i\Gamma_2)^{\perp} = \Gamma_1^{\perp} \dot{+} i\Gamma_2^{\perp}, \quad (\zeta\Lambda_1)^{\perp} = \zeta\Lambda_1^{\perp},$$

where $\Gamma_j^{\perp}$ (resp. $\Lambda_1^{\perp}$) means the orthogonal complemnt of Γ_j (resp. Λ_1) in Γ (resp. Λ), and ζ a complex number (Cf Shiba [13]).

2B. Next we consider the real subspaces (subspaces over the real number field) of real differentials.

Let $\bar{\Omega}$ be a compact bordered Riemann surface. We divide $\partial\bar{\Omega}$ into two sets α, β of contours and consider the following linear subspaces of real differentials:

$$\Gamma_{eo}^1(\beta, \bar{\Omega}) = \{df : df \in \Gamma_e^1(\bar{\Omega}) \text{ such that } f = 0 \text{ on } \beta\},$$

$$\Gamma_{co}^1(\alpha, \bar{\Omega}) = \{\omega : \omega \in \Gamma_c^1(\bar{\Omega}) \text{ such that } \omega = 0 \text{ along } \alpha\}.$$

Lemma 2.1. *$\Gamma_{co}^1(\alpha, \bar{\Omega})^*$ is the orthogonal complement of $\Gamma_{eo}^1(\beta, \bar{\Omega})$ in $\Gamma^1(\bar{\Omega})$.*

Proof. Omitted (analogous as in Ahlfors and Sario [3] p 275).

Next on an open Riemann surface **R** we consider the same linear subspaces as above stated. Let $\mathbf{R}^*$ be the Kerékjártó-Stoïlow's compactification of $\mathbf{R}$ and $P(\Delta)=\alpha\cup\beta$ a regular partition of $\Delta=\mathbf{R}^*-\mathbf{R}$. We call G (resp. G') an end towards α (resp. β) if $\bar{D}^c=G\cup G'$, $\bar{G}\ni\alpha$, $\bar{G}'\ni\beta$ and $\bar{G}\cap\bar{G}'=\phi$ where D is a regular region and $\bar{G}$ the closure in $\mathbf{R}^*$. Now we consider the following linear subspaces:

$\Gamma^1_{eo}(\beta, \mathbf{R})=\{df: (a)\ df\in\Gamma^1_e(\mathbf{R}),\ (b)$ there exists an end U_β towards β which is disjoint with the support of $f\}$,

$\Gamma^1_{co}(\alpha, \mathbf{R})=\{\omega: (a)\ \omega\in\Gamma^1_c(\mathbf{R}),\ (b)$ there exists an end U_α towards α which is disjoint with the support of $\omega\}$,

$\Gamma_{eo}(\beta, \mathbf{R})=$ the closure of $\Gamma^1_{eo}(\beta, \mathbf{R})$ in Γ,

$\Gamma_{co}(\alpha, \mathbf{R})=$ the orthogonal complement of $\Gamma_{eo}(\beta, \mathbf{R})^*$ in Γ.

Note 2. 1. *We identify all consatnt functions with zero.*

Lemma 2. 2. (i) $\Gamma^1_{co}(\alpha, \mathbf{R})\subset\overline{\Gamma^1_{co}(\alpha, \mathbf{R})}\subset\Gamma_{co}(\alpha, \mathbf{R})\subset\Gamma_c$,

$$\Gamma_{eo}(\Delta, \mathbf{R})=\Gamma_{eo}\subset\Gamma_{eo}(\beta, \mathbf{R})\subset\Gamma_e,$$

where bar stands for the closure in Γ.

(ii) *With the notations* $\Gamma_{ho}(\alpha, \mathbf{R})=\Gamma_h\cap\Gamma_{co}(\alpha, \mathbf{R})$ *and* $\Gamma_{heo}(\alpha, \mathbf{R})=\Gamma_h\cap\Gamma_{eo}(\alpha, \mathbf{R})$, *we have*

$$\Gamma=\Gamma_{eo}(\beta, \mathbf{R})\dot{+}\Gamma_{ho}(\alpha, \mathbf{R})^*\dot{+}\Gamma^*_{eo},$$

$$\Gamma_h=\Gamma_{heo}(\beta, \mathbf{R})\dot{+}\Gamma_{ho}(\alpha, \mathbf{R})^* \quad \text{(Weill. G [15])}.$$

Proof. Since (i) is evident, we prove only the first relation of (ii). Because $\Gamma_{eo}(\beta, \mathbf{R})\subset\Gamma_e\perp\Gamma^*_{eo}$, we have

$$\Gamma_{ho}(\alpha, \mathbf{R})^{*\perp}=[\Gamma_{eo}(\beta, \mathbf{R})^\perp\cap(\Gamma_{eo}\dot{+}\Gamma^*_{eo})^\perp]^\perp$$

$$=\overline{\Gamma_{eo}(\beta, R)+\Gamma_{eo}+\Gamma^*_{eo}}=\Gamma_{eo}(\beta, \mathbf{R})\dot{+}\Gamma^*_{eo}.$$

Through this paper we shall use the following notations and terminologies:

(i) $E_\alpha(\mathbf{R})=\{\gamma:\gamma$ is a local rectifiable curve in R, starting from any point of K and tending to α where K denotes a compact region that separates α and $\beta\}$.

(ii) $\bar{O}$ denotes the closure (in $\mathbf{R}^*$) of an open set O on $\mathbf{R}$.

(iii) $\{R_n\}$ being a regular canonical exhaustion of $\mathbf{R}$, we write $R'_n = R_n \cup D_n \cup \partial D_n$ where D_n is an end towards β and consists of the components of $\bar{R}^c_n$, then we say $\{R'_n\}$ is a normal exhaustion of $\mathbf{R}$ towards α.

(iv) For any normal exhaustion $\{R'_n\}$ towards α and a differentials $\lambda \in \Lambda_c$, if the limit

$$\lim_{n\to\infty}\int_{\partial R'_n} \lambda$$

exists, then we write this value as $\int_\alpha \lambda$.

(v) A property will be said to be hold for almost all curves of a family if the extremal length of the subfamily of exceptional curves is infinite (Cf Ohtsuka [11]).

(vi) If $g_0 \in C(\mathbf{R})$ is equal to a constant value c on an end towards α, we say $g_0 = c$ near α.

Lemma 2. 3. (i) *Let G, G' be two ends towards α such that $G' \supset G \cup \partial G$, then we have for $\omega \in \Gamma_{e0}(\alpha, G')$*

$\omega|_G$ (=restriction of ω to G) $\in \Gamma_{e0}(\alpha, G)$.

(ii) *For $\omega \in \Gamma_{e0}(\alpha, \mathbf{R}) \cap \Gamma^1$ and $f \in C^1(\mathbf{R})$, f being a Dirichlet function on an end towards α, we have*

$$\int_\alpha f\omega = 0.$$

(iii) *$du \in \Gamma_{e0}(\alpha, \mathbf{R}) \cap \Gamma^1$ if and only if $du \in \Gamma^1_e$ and for almost all curves γ of $E_\alpha(\mathbf{R})$ $\lim_{\gamma \ni P \to \alpha} u(P) = 0$.*

(iv) *For $du \in \Gamma_{e0}(\alpha, \mathbf{R}) \cap \Gamma^1$ and $\omega \in \Gamma^1_c$ we have $\int_\alpha u\ \omega = 0$.*

Proof. (i) Denoting $\{\omega|_G : \omega \in \Gamma_{e0}(\alpha, G')\}$ by $\Gamma_{e0}(\alpha, G')|_G$, we have

$$\Gamma_{e0}(\alpha, G')|_G = \overline{\Gamma^1_{e0}(\alpha, G')}|_G \subset \Gamma^1_{e0}(\alpha, G')|_G \subset \Gamma_{e0}(\alpha, G).$$

(ii) is evident from the definition of $\Gamma_{eo}(\alpha, \mathbf{R})$.

(iii) For $du \in \Gamma_{eo}(\alpha, \mathbf{R}) \cap \Gamma^1$ there exists a sequence $\{df_n\}$ such that $df_n \in \Gamma^1_{eo}(\alpha, \mathbf{R})$ and $\|df_n - df\| \to 0$. Therefore by Theorem 2.4 in Fuji-i-e [5] we get for almost all curves γ of $E_\alpha(\mathbf{R})$

$$\lim_{\gamma \ni P \to \alpha} u(P) = 0.$$

Conversely, by Lemma 1 in Yamaguchi [16] we have $d(g_0 u) \in \Gamma_{eo}$ where $g_0 \in C^\infty(\mathbf{R})$, $g_0 = 1$ near α and $g_0 = 0$ near β. Therefore, for $\omega \in \Gamma_{co}(\beta, \mathbf{R}) \cap \Gamma^1$ we have from (i) (ii) of this Lemma and Lemma 2.2

$$\langle du^*, \omega \rangle = \int_\alpha u\omega = \langle d(g_0 u)^*, \omega \rangle = 0,$$

$$du \in \Gamma_{eo}(\alpha, \mathbf{R}) \cap \Gamma^1.$$

(iv) For $du \in \Gamma_{eo}(\alpha, \mathbf{R}) \cap \Gamma^1$ we have $d(g_0 u) \in \Gamma_{eo} \cap \Gamma^1$, and so by the Proposition 6 in Yamaguchi [17] we have

$$\int_{\alpha \cup \beta} (g_0 u)\, \omega = 0 = \int_\alpha u\omega.$$

Lemma 2.4 (Weill, G. [15]). (i) *Let Ω be the interior of a bordered Riemann surface $\bar{\Omega}$ whose border α consists of a finite number of contours, then we have*

$\Gamma_{ho}(\alpha, \Omega) = \{\omega : \omega \in \Gamma_h(\bar{\Omega})$ *such that* $\omega = 0$ *along* $\alpha\}$.

(ii) *Let* $\mathbf{R}$ *be an open Riemann surface and* $\{R'_n\}$ *a normal exhaustion towards* α, *then we get*

$$\Gamma_{ho}(\alpha, \mathbf{R}) = \{\omega : \text{there exists a sequence } \{\omega_n\} \text{ with } \omega_n \in \Gamma_{ho}(\partial R'_n, R'_n) \text{ such that } \|\omega - \omega_n\|_{R'_n} \longrightarrow 0\}.$$

Proof. (i) We can get easily this result by the same way as in Ahlfors and Sario [3] p 288— p 291.

(ii) For $\omega \in \Gamma_{ho}(\alpha, \mathbf{R})$, $\omega|_{R'_n}$ has a decomposition of the form

$$\omega|_{R'_n}=\omega_n+\omega'_n,$$

where $\omega_n\in\Gamma_{ho}(\partial R'_n,\ R'_n)$ and $\omega'_n\in\Gamma_{heo}(\beta,\ R'_n)^*$. Making use of Lemma 2.3 (i), we get a differential $\tilde{\omega}\in\Gamma_{ho}(\alpha,\ \mathbf{R})$ such that

$$||\tilde{\omega}-\omega_n||_{R'_n}\longrightarrow 0 \text{ (analogous as in [3])}.$$

But from Lemma 2.3 (iii) and Theorem 2.4 in Fuji-i-e [5] we can get for almost all curves γ of $E_\beta(\mathbf{R})$

$$\lim_{\gamma\ni P\to\beta} f(P)=0 \text{ where } df=(\tilde{\omega}-\omega)^*.$$

Therefore we have

$$\omega-\tilde{\omega}\in\Gamma_{heo}(\beta,\ \mathbf{R})^*\cap\Gamma_{ho}(\alpha,\ \mathbf{R})=0, \text{ i. e., } \omega=\tilde{\omega}.$$

Conversely, for $\sigma\in\Gamma_{heo}(\beta,\ \mathbf{R})$ we get from Lemma 2.3 (i)

$$\langle\omega,\ \sigma^*\rangle=\lim_{n\to\infty}\langle\omega_n,\ \sigma^*\rangle_{R'_n}=0,$$

and so $\omega\in\Gamma_{ho}(\alpha,\ R)$.

2C. Lemma 2.5. *Let $\{\Omega_n\}$ be an exhaustion of* $\mathbf{R}$ *by regularly imbeded regions, and*

$$\Lambda_h(\mathbf{R})=\Lambda_o(\mathbf{R})\dotplus\Lambda_o(\mathbf{R})^\perp \ (\textit{resp. } \Gamma_h(\mathbf{R})=\Gamma_o(\mathbf{R})\dotplus\Gamma_o(\mathbf{R})^\perp)$$

$$\Lambda_h(\Omega_n)=\Lambda_o(\Omega_n)\dotplus\Lambda_o(\Omega_n)^\perp \ (\textit{resp. } \Gamma_h(\Omega_n)=\Gamma_o(\Omega_n)\dotplus\Gamma_o(\Omega_n)^\perp)$$

be orthogonal decompositions of $\Lambda_h(\mathbf{R})$ or $\Lambda_h(\Omega_n)$ (resp. $\Gamma_h(\mathbf{R})$ or $\Gamma_h(\Omega_n)$). Furthermore, if for each n and each $\lambda'\in\Lambda_o(\mathbf{R})^\perp$ (resp. $\lambda'\in\Gamma_o(\mathbf{R})^\perp$), we have $\lambda'|_{\Omega_n}\in\Lambda_o(\Omega_n)^\perp$ (resp. $\lambda'|_{\Omega_n}\in\Gamma_o(\Omega_n)^\perp$), then each limit of locally uniformly convergent subsequence of $\{\lambda_n\}$ with $\lambda_n\in\Lambda_o(\Omega_n)$ (resp. $\Gamma_o(\Omega_n)$) belongs to $\Lambda_o(\mathbf{R})$ (resp. $\Gamma_o(\mathbf{R})$), provided $\sup_n||\lambda_n||_{\Omega_n}<\infty$

Proof. We prove only the case $\Lambda_o(\mathbf{R})$. It is evident that the limit λ of locally uniformly convergent subsequence $\{\lambda_{n_k}\}$ on $\mathbf{R}$ belongs to $\Lambda_h(\mathbf{R})$. For $\varepsilon>0$ and λ' with $\lambda'\in\Lambda_o(\mathbf{R})^\perp$ there exists a regular canonical region G such that $||\lambda'||_{\mathbf{R}-G}<\varepsilon$. Therefore we have

$$|\langle\lambda,\ \lambda'\rangle|\leqslant|\lim_{k\to\infty}\langle\lambda_{n_k},\ \lambda'\rangle_G|+\varepsilon||\lambda||$$

$$\leqslant|\lim_{k\to\infty}\langle\lambda_{n_k},\ \lambda'\rangle_{\Omega_{n_k}-G}|+\varepsilon||\lambda||<2||\lambda||\varepsilon.$$

Consequently $\lambda\perp\Lambda_o(\mathbf{R})^\perp$ and so $\lambda\in\Lambda_o(\mathbf{R})$.

Corollary 2.1. (i) *Let* $\{R_n\}$ *be a regular canonical exhaustion and* $\Lambda_o=\Gamma_{hm}+i\overset{\bullet}{\Gamma}_{hse}$, *then the result of Lemma 2.5 is true.*

(ii) *Let* $\{R'_n\}$ *be a normal exhaustion towards* β *and* $\Gamma_o=\Gamma_{ho}(\beta,*)$, *then the result of Lemma 2.5 is true.*

§3. The convergence theorems of Abelian differentials and the existence of certain meromorphic functions.

3A. Suppose $\mathbf{R}$ is an open Riemann surface of genus g which may be infinity, and $\{R_n\}$ is a regular canonical exhaustion of $\mathbf{R}$, then we can choose a canonical homology basis $\{A_j, B_j\}_{j=1}^{g}$ of $\mathbf{R}$ modulo dividing curves such that $\{A_j, B_j\}\cap D_n^k$ is also a canonical homology basis of D_n^k modulo ∂D_n^k for each n and k, where D_n^k denotes a component of $R_{n+1}-\bar{R}_n$ (Cf Ahlfors and Sario [3]). Furthermore, suppose $\mathscr{L}=\{L_j\}_{j=1}^{g}$ is a family of straight lines on the complex plane each of which passes through the origin. We consider a space $\Lambda_o=\Lambda_o(\mathscr{L})=\Lambda_o(\mathscr{L}, \mathbf{R})$ which satisfies the following conditions:

(a) $\Lambda_o(\mathscr{L}, \mathbf{R})\subset\Lambda_{hse}(\mathbf{R})=\Gamma_{hse}(R)\overset{\bullet}{+}i\Gamma_{hse}(\mathbf{R})$,

(b) $\Lambda_o(\mathscr{L}, \mathbf{R})=i\Lambda_o(\mathscr{L}, \mathbf{R})^{*\perp}$,

(c) $\int_{A_j}\lambda\in L_j$, $\int_{B_j}\lambda\in L_j$ for each $\lambda\in\Lambda_o(\mathscr{L}, \mathbf{R})$ and $j=1, 2..g$.

Such a space Λ_o will be called the "restricted behavior space on $\mathbf{R}$ associated with $\mathscr{L}$". Further we consider an another space $\tilde{\Lambda}_o=\Lambda_o(\mathscr{L}, \hat{\Lambda})=\Lambda_o(\mathscr{L}, \hat{\Lambda}, \mathbf{R})$ which satisfies the following conditions:

(a) $\tilde{\Lambda}_o$ is a linear subspace (not necessarily closed) of Λ_{hse},

(b) there exists a closed subspace $\hat{\Lambda}$ of Λ_h such that

$$\tilde{\Lambda}_o\supset\hat{\Lambda}+i\hat{\Lambda}^{*\perp},$$

(c) $\langle\lambda, i\lambda^*\rangle=0$ for any $\lambda\in\tilde{\Lambda}_o$,

(d) $\int_{A_j}\lambda\in L_j$, $\int_{B_j}\lambda\in L_j$ for each $\lambda\in\tilde{\Lambda}_o$ and $j=1, 2...g$.

Such a space $\tilde{\Lambda}_o$ will be called simply the "behavior space (or Shiba's behavior space) on $\mathbf{R}$ associated with $\mathscr{L}$ and $\hat{\Lambda}$" (Cf. Shiba [13]). For a restricted behavior space Λ_o, $\bar{\Lambda}_o=\{\lambda:\bar{\lambda}\in\Lambda_o$ where $\bar{\lambda}$

denotes the complex conjugate of λ} is also a restricted behavior space which will be called hereafter "the dual behavior space of Λ_0".

Note 3.1 Hereafter we denote a behavior space (resp. restricted behavior space) by the notation $\tilde{\Lambda}_0$ (resp. Λ_0).

Let $\tilde{\Lambda}_0$ be a behavior space. A meromorphic differential λ on R is called to have $\tilde{\Lambda}_0$ *behavior* (*on* U) if there exist $\lambda_0 \in \tilde{\Lambda}_0$ and $\lambda'_{e0} \in \Lambda_{e0} \cap \Lambda^1$ such that

$$\lambda = \lambda_0 + \lambda_{e0} \text{ on } U,$$

where U denotes an end towards Δ. A single valued meromorphic function f on R is called to have $\tilde{\Lambda}_0$ behavior (on U) if the differential df has $\tilde{\Lambda}_0$ behavior (on U). Then the following theorems hold (Cf. Shiba [13]).

Existence Theorem 1. *Let* α_j, β_j *be given complex numbers such that* $\alpha_j \overline{\in} L_j$ *and* $\beta_j \overline{\in} L_j$. *Then there exists uniquely a regular differential* $\phi_{\alpha_j}(A_j) = \phi_{\alpha_j}(A_j, \tilde{\Lambda}_0, \mathbf{R}) \in \Lambda_a(\mathbf{R})$ (*resp.* $\phi_{\beta_j}(B_j) = \phi_{\beta_j}(B_j, \tilde{\Lambda}_0, \mathbf{R}) \in \Lambda_a(\mathbf{R})$) *such that*

(a) $\phi_{\alpha_j}(A_j)$ (resp. $\phi_{\beta_j}(B_j)$) *has* $\tilde{\Lambda}_0$ *behavior,*

(b) $\int_{A_k} \phi_{\alpha_j}(A_j) \in L_k$, $\int_{B_k} \phi_{\alpha_j}(A_j) - \delta_{kj}\alpha_j \in L_k$ *for* $k = 1, 2..g$.

(resp. $\int_{A_k} \phi_{\beta_j}(B_j) + \delta_{kj}\ \beta_j \in L_k$, $\int_{B_k} \phi_{\beta_j}(B_j) \in L_k$ *for* $k = 1, 2, \ldots g$),

where δ_{kj} *denotes the Kronecker delta.*

Existence Theorem 2. *Let* P *be an arbitrary point on* **R**. *For an integer* $n \geqslant 2$ *there exists a differential* $\phi_{P^n} = \phi(P^n, \tilde{\Lambda}_0, \mathbf{R})$ (resp. $\psi_{P^n} = \psi(P^n, \tilde{\Lambda}_0, \mathbf{R})$) *uniquely such that*

(a) ϕ_{P^n} (*resp.* ψ_{P^n}) *has* $\tilde{\Lambda}_0$ *behavior on* $\bar{D}^c$ *where* D *is a parametric disk at* P,

(b) ϕ_{P^n} (*resp.* ψ_{P^n}) *is regular except* P, *where* ϕ_{P^n} (*resp.* ψ_{P^n}) *has the singularity* $\frac{dz}{z^n}$ (*resp.* $\frac{idz}{z^n}$).

Note 3.2. For simlicity we denote ϕ_{P^n} and ψ_{P^n} by $\phi_\theta=\phi(\theta, \tilde{\Lambda}_o, \mathbf{R})$.

3B. With these preparations we define the convergence of the sequence of the restricted behavior spaces.

Definition 1. Suppose $\{\Lambda_o(\mathscr{L}_n, R_n)\}_{n=1}^{\infty}$ is a sequence of the restricted behavior spaces and $\hat{\Lambda}(\mathbf{R})$ a subspace of $\Lambda_h(\mathbf{R})$ where $\mathscr{L}_n$ denotes a subset of $\mathscr{L}$ such that $\mathscr{L}_n=\{L_j\}_{j=1}^{g_n}$, g_n being the genus of R_n. Further, let $\{\Lambda_o(\mathscr{L}_n, R_n)\}_{n=1}^{\infty}$ and $\hat{\Lambda}(\mathbf{R})$ satisfy the following conditions:

(a) for each $\lambda\in\hat{\Lambda}(\mathbf{R})$, there exists $\{\lambda_n\}$ with $\lambda_n\in\Lambda_o(\mathscr{L}_n, R_n)$ such that $\|\lambda-\lambda_n\|_{R_n}\longrightarrow 0$,

(b) if $\{\lambda_n\}$ with $\lambda_n\in\Lambda_o(\mathscr{L}_n, R_n)$ is a sequence such that $\sup\|\lambda_n\|_{R_n}<\infty$, then each limit of locally uniformly convergent subsequence of $\{\lambda_n\}$ belongs to $\hat{\Lambda}(\mathbf{R})$.

Then we say $\{\Lambda_o(\mathscr{L}_n, R_n)\}_{n=1}^{\infty}$ *is convergent to* $\hat{\Lambda}(\mathbf{R})$ *and denote by*

$$\Lambda_o(\mathscr{L}_n, R_n)\Longrightarrow\hat{\Lambda}(\mathbf{R}).$$

Lemma 3.1. *If* $\Lambda_o(\mathscr{L}_n, R_n)\Longrightarrow\hat{\Lambda}(\mathbf{R})$, $\hat{\Lambda}(\mathbf{R})$ *is a restricted behavior space on* $\mathbf{R}$ *associated with* $\mathscr{L}$.

Proof. For simplicity we write $\Lambda_n=\Lambda_o(\mathscr{L}_n, R_n)$ and $\Lambda=\hat{\Lambda}(\mathbf{R})$. Since Λ is a subspace of $\Lambda_{hse}(\mathbf{R})$ and $\|\lambda-\lambda_n\|_{R_n}\longrightarrow 0$ for each λ with $\lambda\in\Lambda$, λ_n being the projection of $\lambda|_{R_n}$ on Λ_n, we can get easily $\Lambda\subset i\Lambda^{*\perp}$. Next we prove $\Lambda\supset i\Lambda^{*\perp}$. Suppose $\lambda\perp i\Lambda^*$ and

$$\lambda=\lambda_o+\lambda_o' \text{ where } \lambda_o\in\Lambda \text{ and } \lambda_o'\in\Lambda^{\perp},$$

then from the fact $\Lambda\subset i\Lambda^{*\perp}$ we have

$$\lambda_o'=(\lambda-\lambda_o)\perp i\Lambda^*.$$

But, the restriction of λ_o' to R_n has a decomposition of the form

$$\lambda_o'|_{R_n}=\lambda_n+\lambda_n' \text{ where } \lambda_n\in\Lambda_n \text{ and } \lambda_n'\in\Lambda_n^{\perp}=i\Lambda_n^*,$$

and so we have $||\lambda_n||_{R_n} \leqslant ||\lambda||$, and $||\lambda'_n||_{R_n} \leqslant ||\lambda||$. Therefore there exists $\{n_k\}$ such that $\{\lambda_{n_k}\}$ (resp. $\{\lambda'_{n_k}\}$) is locally uniformly convergent to $\sigma_0 \in \Lambda$ (resp. $\sigma'_0 \in i\Lambda^*$) and so we get

$$\sigma_0 + \sigma'_0 = \lambda'_0 \perp i\Lambda^*.$$

Consequently from $\sigma_0 \in \Lambda \perp i\Lambda^*$ we have successively

$$\sigma'_0 = 0, \ \Lambda^{\perp} \ni \lambda'_0 = \sigma_0 \in \Lambda, \ \lambda = \lambda_0 \in \Lambda.$$

Therefore we have $\Lambda \supset i\Lambda^{*\perp}$. Thus Λ is a restricted behavior space on $\mathbf{R}$ associated with $\mathscr{L}$.

Lemma 3. 2. *Let* W *be a fixed end towards* Δ *and* $\{R_n\}$ *a regular canonical exhaustion. We consider the sequence* $\{df_n\}$ *with* $df_n \in \Gamma_{e0}(R_n) \cap \Gamma_{he}(R_n \cap W)$ *such that* $\sup_n ||df_n||_{R_n} < \infty$, *then there exists a subsequence* $\{df_{n_k}\}$ *such that*

(i) $\{df_{n_k}\}$ *is locally uniformly convergent to* df *on* W′, W′ *being an end towards* Δ *and* $W' \cup \partial W' \subset W$,

(ii) $df|_{W'} = dg|_{W'}$ *where* $dg \in \Gamma_{e0}(\mathbf{R})$, *and* $df|_{W'}$ *is harmonic.*

Proof. By the conditions there exists $\{df_{n_k}\}$ which is locally uniformly convergent on W to an exact harmonic differential df. Now we take $g \in C^2(\mathbf{R})$ such that $g=1$ on W', $g=0$ on W^c and $1 \geqslant g \geqslant 0$ where W' is an fixed end towards Δ such that $W' \cup \partial W' \subset W$, then $\{dG_{n_k}\}$ with $dG_n = d(gf_n)$ is locally uniformly convergent to $dG = dgf$. Further, for each $\omega \in \Gamma_h$ and $\varepsilon > 0$, there exists a regular region Ω such that $||\omega||_{\mathbf{R}-\Omega} < \varepsilon$ and so we have

$$|\langle \omega, dG \rangle| = |\langle \omega, dG \rangle_{\mathbf{R}-\Omega} + \lim_{k\to\infty} \langle \omega, dG_{n_k} \rangle_{\Omega}|$$
$$< \varepsilon ||dG|| + |\lim_{k\to\infty} \langle \omega, dG_{n_k} \rangle_{R_{n_k}-\Omega}| < \varepsilon \text{ constant.}$$

Consequently, $dG \in \Gamma_{e0}(\mathbf{R})$.

Lemma 3. 3. *When* $\Lambda_n = \Lambda_0(\mathscr{L}_n, R_n) \Longrightarrow \Lambda_0(\mathscr{L}, \mathbf{R}) = \Lambda_0$, *there exists* $\{n_k\}$ *such that*

(i) *for each* j, $\phi_{\alpha_j}(A_j, \Lambda_{n_k}, R_{n_k}) \longrightarrow \phi_{\alpha_j}(A_j, \Lambda_o, \mathbf{R})$,

$\phi_{\beta_j}(B_j, \Lambda_{n_k}, R_{n_k}) \longrightarrow \phi_{\beta_j}(B_j, \Lambda_o, \mathbf{R})$,

(ii) *for a* θ, $\phi(\theta, \Lambda_{n_k}, R_{n_k}) \longrightarrow \phi(\theta, \Lambda_o, \mathbf{R})$,
where the convergences are locally uniform on $\mathbf{R}$.

Proof. We prove only the case $\phi(\mathrm{P}^m, \Lambda_o, \mathbf{R})$, because other cases are proved analogously. At first we write

$$\theta = \frac{dz}{z^m}, \quad m \geqslant 2 \text{ in } V,$$

where V is a parametric disk at P and z a parameter of V such that $z(\mathrm{P})=0$. Then we can extend θ to a closed $C^1(\mathbf{R}\text{-P})$ differential $\hat{\theta}$ such that

$$\hat{\theta}\text{-}i\hat{\theta}^* = 0 \text{ on } V' \cup \bar{D}^c \text{ and the support of } \hat{\theta} \subset D,$$

where D and V' denote parametric disks such that $V \supset \bar{D} \supset \mathrm{D} \supset \bar{V}'$. Therefore $\hat{\theta}\text{-}i\hat{\theta}^*$ has the decomposition of the form

$$[1] \quad \begin{cases} \hat{\theta}\text{-}i\hat{\theta}^* = \lambda_c + df_o^*, \\ \hat{\theta}\text{-}i\hat{\theta}^* = \lambda_{nc} + df_{on}^*, \end{cases}$$

where $\lambda_c \in \Lambda_c(\mathbf{R})$, $\lambda_{nc} \in \Lambda_c(R_n)$, $df_o \in \Lambda_{eo}(\mathbf{R}) \cap \Lambda^1$ and $df_{on} \in \Lambda_{eo}(R_n) \cap \Lambda^1$. From [1] df_o (resp. df_{on}) is harmonic on $\bar{D}^c$ (resp. $\bar{D}^c \cap R_n$), and so is λ_c (resp. λ_{nc}). Next by the orthogonal decompositions we obtain

$$[2] \quad \begin{cases} \lambda_c = \lambda + \lambda^\perp + dg_o, \\ \lambda_{nc} = \lambda_n + \lambda_n^\perp + dg_{on}, \end{cases}$$

where $\lambda \in \Lambda_o(\mathbf{R})$, $\lambda^\perp \in \Lambda_o(\mathbf{R})^\perp$, $dg_o \in \Lambda_{eo}(\mathbf{R}) \cap \Lambda^1 \cap \Lambda_h(\bar{D}^c)$, $\lambda_n \in \Lambda_o(\mathscr{L}_n, R_n)$, $\lambda_n^\perp \in \Lambda_o(\mathscr{L}_n, R_n)^\perp = i\Lambda_o(\mathscr{L}_n, R_n)^*$ and $dg_{on} \in \Lambda_{eo}(R_n) \cap \Lambda^1 \cap \Lambda_h(\bar{D}^c \cap R_n)$. We set

$$\tau_n = \hat{\theta} - \lambda_n - dg_{on} = \lambda_n^\perp + df_{on}^* + i\hat{\theta}^*,$$

$$\tau = \hat{\theta} - \lambda - dg_o = \lambda^\perp + df_o^* + i\hat{\theta}^*.$$

Then we can get (Cf Shiba [13])

$$\phi(\mathrm{P}^m,\ \Lambda_o,\ \mathbf{R}) = \frac{1}{2}(\tau + i\tau^*),$$

$$\phi(\mathrm{P}^m,\ \Lambda_n,\ R_n) = \frac{1}{2}(\tau_n + i\tau_n^*).$$

But from [1] and [2] we have

$$\|\phi(\mathrm{P}^m,\ \Lambda_n,\ R_n) - \phi(\mathrm{P}^m,\ \Lambda_o,\ \mathbf{R})\| < 4\|\hat{\theta} - i\hat{\theta}^*\| < \text{constant}.$$

Consequently, there exists $\{n_k\}$ such that (Cf Lemma 3. 2)

$$\phi(\mathrm{P}^m,\ \Lambda_{n_k},\ R_{n_k}) - \phi(\mathrm{P}^m,\ \Lambda_o,\ R) \longrightarrow \Psi,$$

where Ψ is a regular differential. On the other hand, from the assumption $\Lambda_n \Longrightarrow \Lambda_o$, Ψ is a regular differential with Λ_o behavior on $\bar{D}^c$ (Cf Lemma 3. 1), and so by the uniqueness theorem (Cf Shiba [13]) Ψ reduces to 0.

Corollary 3. 1 (Kusunoki [6]). *Let $\{R_n\}$ be a regular canonical exhaustion of* $\mathbf{R}$, $\Lambda_o = \Gamma_{hm}(\mathbf{R}) \dot{+} i\Gamma_{hse}(\mathbf{R})$ *and* $\Lambda_n = \Gamma_{hm}(R_n) \dot{+} i\Gamma_{hse}(R_n)$, *then by the Corollary* 2. 1 (i) *we have* $\Lambda_n \Longrightarrow \Lambda_o$, *therefore there exists a subsequence* $\{n_k\} = \{k\}$ *such that*

for any j $\quad \phi_1(A_j,\ \Lambda_k,\ R_k) \longrightarrow \phi(A_j,\ \Lambda_o,\ \mathbf{R}),$

$\quad \phi_1(B_j,\ \Lambda_k,\ R_k) \longrightarrow \phi(B_j,\ \Lambda_o,\ \mathbf{R}),$

for a θ, $\quad \phi_\theta(\Lambda_k,\ R_k) \longrightarrow \phi_\theta(\Lambda_o,\ \mathbf{R}).$

3C. Suppose $\mathbf{R}$ is of finite genus g and δ is a finite divisor, then with g and δ we associate the following linear subset over the real number field:

$S\left(\Lambda_o,\ \frac{1}{\delta}\right) = \Big\{f : f$ is a single valued meromorphic function with Λ_o behavior on $\mathbf{R}$ and the divisor of f is a multiple of $\frac{1}{\delta}\Big\}$,

$D(\Lambda_o,\ \delta) = \{\alpha : \alpha$ is a meromorphic differential with Λ_o behavior and the divisor of α is a multiple of $\delta\}$.

Then Shiba [13] showed the following theorems:

(i) $\{\phi_{\alpha j}(A_j,\ \Lambda_o,\ \mathbf{R}),\ \phi_{\beta j}(B_j,\ \Lambda_o,\ \mathbf{R})\}_{j=1}^{g}$ forms a basis of $D(\Lambda_o,\ 1)$,

(ii) Riemann-Roch's Theorem:

$$\dim S\left(\Lambda_0, \frac{1}{\delta}\right)=2\ (ord\ \delta-g+1)+\dim D(\Lambda'_0, \delta),$$

where Λ'_0 denotes the dual behavior space of Λ_0.

Definition 2. Let $\mathbf{R}$ be an open Riemann surface of finite genus g and Λ_0 a behavior space on $\mathbf{R}$, then we call a point P a *Weierstrass point for* Λ_0 if there exists a non constant meromorphic function with Λ_0 behavior which has the only singularity of order at most g at P. We denote *the set consisting of all Weierstrass points for* Λ_0 *by* $W(\Lambda_0, \mathbf{R})$.

Lemma 3.4. *Suppose* $\mathbf{R}$ *is an open Riemann surface with finite genus* $g: g>1$, *then the complement of* $W(\Lambda_0, \mathbf{R})$ *is open dense on* $\mathbf{R}$.

Proof. Suppose Λ'_0 is the dual behavior space of Λ_0, and

$$\begin{aligned}\phi_{\alpha_j}(A_j, \Lambda'_0, \mathbf{R}) &= f_j\, dz,\\ \phi_{\beta_j}(B_j, \Lambda'_0, \mathbf{R}) &= f_{j+g}\, dz,\end{aligned} \qquad j=1, 2, \ldots\ldots, g,$$

where $z=x+iy$ is a local parameter at P such that $z(\mathrm{P})=0$. We consider the real analytic function with respect to x and y such that

$$V_{2k}(z)=|R^0_{2k}\ I^0_{2k}\ R^1_{2k}\ I^1_{2k} \ldots\ldots R^{k-1}_{2k}\ I^{k-1}_{2k}|,\ k=1, 2..g,$$

where R^r_h and I^r_h denote the transposed matrices of

$$(\mathrm{Re}\ f^r_1, \mathrm{Re}\ f^r_2, \ldots\ldots \mathrm{Re}\ f^r_h), \text{ and } (\mathrm{Im}\ f^r_1, \mathrm{Im}\ f^r_2, \ldots \mathrm{Im}\ f^r_h)$$

respectively, and $f^r_j=\dfrac{d^r f_j}{dz^r}$, $f^0_j=f_j$. For $\mathrm{P}\in W(\Lambda_0, \mathbf{R})$, we have $V_{2g}(\mathrm{P})=0$ and the converse is also true, that is to say, $W(\Lambda_0, \mathbf{R})$ is closed on $\mathbf{R}$. If $\mathrm{E}=\{$the closure in $\mathbf{R}$ of $W(\Lambda_0, \mathbf{R})^c\}$ is not equal to $\mathbf{R}$, we put $\mathbf{R}-\mathrm{E}=\mathrm{U}$. From the fact $V_2(z)\not\equiv 0$ on U(Cf. Kusunoki [6] p 255) and $\{\phi_{\alpha_j}(A_j, \Lambda'_0, \mathbf{R}), \phi_{\beta_j}(B_j, \Lambda'_0, \mathbf{R})\}_{j=1}^{g}$ is a basis of $D(\Lambda'_0, 1)$ we can prove this theorem by the same way as in

Mori [8].

Corollary 3. 2. *If we set* $\Lambda_0=\Gamma_{hm}(\mathbf{R})+i\dot{\Gamma}_{hse}(\mathbf{R})$, *we get the Theorem* 1 *in Mori* [8].

Lemma 3. 5. *Suppose* **R** *is an open Riemann surface of genus* g, $1<g<\infty$, *and* $\{R_n\}$ *is a regular canonical exhaustion of* **R**. *If we assume*

$$\Lambda_n=\Lambda_0(\mathscr{L}_n, R_n)\Longrightarrow\Lambda_0(\mathscr{L}, \mathbf{R})=\Lambda_0,$$

then we have

(i) *when* $\mathrm{P}\in W(\Lambda_0, \mathbf{R})^c$, *then* $\mathrm{P}\in W(\Lambda_{n_k}, R_{n_k})^c$ *for* $k>k_0$ *where* k_0 *is sufficiently large,*

(ii) *for the function* f *which has* Λ_0 *behavior and has only a pole of order* $g+1$ *at* P, *there exists a sequence* $\{f_{n_k}\}=\{f_k\}$ *such that*

(a) f_k *is a meromorphic function on* R_k *which has* Λ_k *behavior on* $\bar{D}^c\cap R_k$. *Here* D *denotes a fixed parametric disk at* P,

(b) $f_k-f\longrightarrow 0$ *locally uniformly on* **R**.

Proof. Let the dual behavior space of Λ_0 (resp. Λ_n) be Λ_0' (resp. Λ_n'), and the basis of $D(\Lambda_0', 1)$ (resp. $D(\Lambda_n', 1)$) be $\{\phi_{\alpha j}', \phi_{\beta j}'\}_{j=1}^{g}$ (resp. $\{\phi_{\alpha j}^{n\prime}, \phi_{\beta j}^{n\prime}\}_{j=1}^{g}$). Further, denoting the determinant of order 2g which is constructed from $\{\phi_{\alpha j}', \phi_{\beta j}'\}$ (resp. $\{\phi_{\alpha j}^{n\prime}, \phi_{\beta j}^{n\prime}\}$) in the proof of Lemma 3. 4 by $V_{2g}(z)$ (resp. $V_{2g}^{n}(z)$), we have the subsequence $\{n_k\}=\{h\}$ such that $V_{2g}^{h}(z)\longrightarrow V_{2g}(z)$. But for $\mathrm{P}\in W(\Lambda_0, \mathbf{R})^c$, $V_{2g}(z)\neq 0$, therefore we have $V_{2g}^{h}(z)\neq 0$ by Lemma 3. 3 and so $\mathrm{P}\in W(\Lambda_h, R_h)^c$. Concerning (ii), suppose the singularity of df at P is

$$\sum_{r=2}^{g+2}\frac{a_r+ib_r}{z^r}dz,$$

where all a_r, b_r are real constant, then by the uniqueness theorem (Shiba [13]) df is equal to a differential σ such that

$$\sum_{r=2}^{g+2} a_r\ \phi(\mathrm{P}^r, \Lambda_0, R)+b_r\psi(\mathrm{P}^r, \Lambda_0, R)=\sigma$$

if and only if $\int_{A_k}\sigma=0$, $\int_{B_k}\sigma=0$, $k=1,2,\ldots g$.

On the other hand, the matrix T of the coefficients of the following system of the linear (with respect to a_r, b_r $r=2,3..g+2$) equations

$$0=\sum_{r=2}^{g+2} a_r\int_{\substack{A_k\\B_k}}\phi(\mathrm{P}^r, \Lambda_0, \mathbf{R})+b_r\int_{\substack{A_k\\B_k}}\psi(\mathrm{P}^r, \Lambda_0, \mathbf{R}),\ k=1,2..g$$

has rank 2g (because dim $S(\Lambda_0, P^{-g-1})=4$ and dim $S(\Lambda_0, P^{-g})=2$), that is to say, T has a minor determinant $|T'|$ of order 2g such that $|T'|\neq 0$. Consequently the matrix T_n of the coefficients of the following system of the linear equations

$$0=\sum_{r=2}^{g+2} a_r\int_{\substack{A_k\\B_k}}\phi(\mathrm{P}^r, \Lambda_n, R_n)+b_r\int_{\substack{A_k\\B_k}}\psi(\mathrm{P}^r, \Lambda_n, R_n),\ k=1,\ldots g.$$

has rank 2g, i. e., T_n has the minor determinant $|T'_n|\neq 0$ which corresponds to $|T'|$ in T. Therefore, taking proper constants, from Lemma 3. 3 there exists $\{f_n\}$ which satisfyes the conditions (a) and (b).

Lemma 3. 6. *Let* **R** *be an open Riemann surface of finite genus* $g>0$, *then it is possible to find g distinct points* $\mathrm{P}_1, \mathrm{P}_2, \ldots\ldots \mathrm{P}_g$ *of* **R** *such that*

$$\dim D(\Lambda_0, \delta)=0,$$

where $\delta=\mathrm{P}_1\mathrm{P}_2\ldots\mathrm{P}_g$. (Cf. Kusunoki [6] p 255)

Lemma 3. 7. *Let* **R** *be an open Riemann surface of finite genus* $g>0$, $\{R_n\}$ *a regular canonical exhaustion of* **R**, $\Lambda_n\Longrightarrow\Lambda_0$ *and* P_1 *a given point of* **R**. *Then for suitable choice of g points* $\mathrm{P}_2, \mathrm{P}_3, \ldots \mathrm{P}_{g+1}$ *there exists a meromorphic function f on* **R** *with* Λ_0 *behavior such that*

(i) *the divisor of f is a multiple of* $(\mathrm{P}_1\mathrm{P}_2\ldots\mathrm{P}_{g+1})^{-1}$ *and the residue of f at* P_1 *is equal to* 1,

(ii) *there exists a sequence* $\{f_{n_l}\}=\{f_l\}$ *such that*

(a) f_l *is a meromorphic function on* R_l *with* Λ_l *behavior,*

(b) $f_\ell - f \longrightarrow 0$ *locally uniformly on* **R**.

Proof. From Lemma 3.6 and its proof, we can take g distinct points $P_2, P_3, \ldots P_{g+1}$ such that

$P_1 \neq P_j$, $j=2, 3 \ldots g+1$ and $\dim S(\Lambda_o, (P_2P_3P_4 \ldots P_{g+1})^{-1}) = 2$.

Therefore, the determinant of the coefficients of the following system of the linear (with respect to x_j, y_j $j=2, 3 \ldots g+1$) equations

$$[1] \quad 0 = \sum_{j=1}^{g+1} x_j \int_{B_k}^{A_k} \phi(P_j^2, \Lambda_o, \mathbf{R}) + y_j \int_{B_k}^{A_k} \psi(P_j^2, \Lambda_o, \mathbf{R}), \ k=1, 2..g$$

is different from zero. Consequently we have the required (in (i)) function f such that

$$df = \sum_{j=1}^{g+1} x_j \phi(P_j^2, \Lambda_o, \mathbf{R}) + y_j \psi(P_j^2, \Lambda_o, \mathbf{R})$$

where x_j, y_j $j=2, 3 \ldots g+1$ are the solution of [1] after choices of $x_1 = -1$, $y_1 = 0$. Next by Lemma 3.3 there exists $\{n_\ell\} = \{\ell\}$ such that each determinant of the following linear (with respect to $\{x_j, y_j\}_{j=2}^{g+1}$) equations

$$[2] \quad 0 = \sum_{j=1}^{g+1} x_j \int_{B_k}^{A_k} \phi(P_j^2, \Lambda_\ell, R_\ell) + y_j \int_{B_k}^{A_k} \psi(P_j^2, \Lambda_\ell, R_\ell), \ k=1, ..g$$

is different from zero. Accordingly, taking proper constant, for each ℓ we can get f_ℓ from [2] after choices of $x_1 = -1$; $y_1 = 0$ and moreover we see that $\{f_\ell\}$ satisfies the condition (ii).

§4. Special behavior spaces and the convergence theorems.

4A. Suppose **R** is an open Riemann surface of genus g which may be infinity, $\{R_n\}$ a regular canonical exhaustion, $P(\Delta) = \bigcup_{k=1}^{K} \beta_k$ a finite regular partition of Δ and $\Omega = R_{n_o}$ with sufficiently large n_o. Further we write $\beta_k^n = \beta_k(\partial R_n)$, $\beta_k(\partial R_n)$ being induced by β_k on ∂R_n, and $\Omega^c = \bigcup_{k=1}^{K} W_k$ where W_k denotes the end towards β_k. Besides $\mathscr{L} = \{L_j\}_{j=1}^{g}$ and $\mathscr{L}_n = \{L_j\}_{j=1}^{g_n}$ (g_n : the genus of R_n) we consider the families of lines on the complex plane **C** and the families of differentials on **R** such that:

$\ell = \{l_k,\ k=1,2,\dots K,$ where each l_k is a line passing through the origin of $\mathbf{C}\}$,

${}_{\Omega}\mathscr{L} = \{L'_j,\ j=1,2,\dots g$ where $L'_j=L_j$ for j such that $A_j,\ B_j\subset\Omega$ and $L'_j=il_k=\{\sqrt{-1}\ t: t\in l_k\}$ for j and k such that $A_j,\ B_j\subset W_k\}$,

$\Lambda_q(\mathbf{R},\ \mathscr{L},\ \ell)=\{\lambda$: (a) $\lambda\in\Lambda_{hse}(\mathbf{R})$, (b) Im $\bar{z}_k\lambda\in\Gamma_{ho}(\beta_k,\ R)$, $k=1,2,\dots K$, (c) $\int_{A_j\atop B_j}\lambda\in L_j,\ j=1,2,\dots g$ where z_k is a complex number such that $z_k\in l_k$ and $|z_k|=1\}$,

$\Lambda_q(\mathbf{R}_n,\ \mathscr{L}_n,\ \ell)=\{\lambda$: (a) $\lambda\in\Lambda_{hse}(\mathbf{R}_n)$, (b) Im $\bar{z}_k\lambda\in\Gamma_{ho}(\beta''_k,\ R_n)$, $k=1,2,\dots K$, (c) $\int_{A_j\atop B_j}\lambda\in L_j,\ j=1,2,\dots g_n$ where $z_k\in l_k,\ |z_k|=1\}$,

$\Lambda'_q(\mathbf{R},\ \mathscr{L},\ \ell)=\{\lambda$: there exists a sequence $\{\lambda_n\}$ with $\lambda_n\in\Lambda_q(R_n,\ \mathscr{L}_n,\ \ell)$ such that $\|\lambda_n-\lambda\|_{R_n}\longrightarrow 0\}$.

Note 4.1. For the finite Riemann surface **R**, Λ_q $(\mathbf{R},\ \mathscr{L},\ \ell)$ is a restricted behavior space (Cf. Shiba [13]). On the contrary when **R** is open, it is a question whether Λ_q $(\mathbf{R},\ \mathscr{L},\ \ell)$ is a behavior space or not.

Note 4.2 For $\lambda\in\Lambda_q(\mathbf{R},\ {}_{\Omega}\mathscr{L},\ \ell)$, Re $\bar{z}_k\lambda|_{W_k}$ is exact on W_k, $k=1,2,\dots K$.

Theorem 1. *Under above notations we have*

(i) $\Lambda_q(\mathbf{R},\ {}_{\Omega}\mathscr{L},\ \ell)$ *is a restricted behavior space on* **R**,

(ii) $\Lambda_{qn}=\Lambda_q(\bar{R}_n,\ {}_{\Omega}\mathscr{L}_n,\ \ell)\Longrightarrow\Lambda_q(R,\ {}_{\Omega}\mathscr{L},\ \ell)$,

therefore from Lemma 3.3 we get for each j and a singularity θ

$$\phi_{\alpha_j}(A_j,\ \Lambda_{qn},\ R_n)\longrightarrow\phi_{\alpha_j}(A_j,\ \Lambda_q,\ \mathbf{R}),$$

$$\phi_{\beta_j}(B_j,\ \Lambda_{qn},\ R_n)\longrightarrow\phi_{\beta_j}(B_j,\ \Lambda_q,\ \mathbf{R}),$$

$$\phi(\theta,\ \Lambda_{qn},\ R_n)\longrightarrow\phi(\theta,\ \Lambda_q,\ \mathbf{R}),$$

Before proceeding the proof, we state the following special cases.

Corollary 4.1. *For the identity partition and* $\mathscr{L} = \ell = \{$*the imaginary axis*$\}$, $\Lambda_q(\mathbf{R}, \mathscr{L}, \ell)$ *coincides with* $i\,\Gamma_{he}$ *behavior space in Yoshida's sense (Cf. Yoshida [18]).*

Corollary 4.2. *If* $\mathbf{R}$ *is of finite genus, we have for any pair of* $\mathscr{L}$ *and* ℓ

$$\text{(i)}\quad \Lambda'_q(\mathbf{R}, \mathscr{L}, \ell) = \Lambda_q(\mathbf{R}, \mathscr{L}, \ell),$$

$$\text{(ii)}\quad \Lambda_q(R_n, \mathscr{L}_n, \ell) \Longrightarrow \Lambda_q(\mathbf{R}, \mathscr{L}, \ell).$$

4B. Proof of Theorem 1.

Proof of the Theorem 1 consists of the following four steps, namely Lemma 4.1 - Lemma 4.4.

Lemma 4.1. $\Lambda'_q(\mathbf{R}, \mathscr{L}, \ell) \subset \Lambda_q(\mathbf{R}, \mathscr{L}, \ell)$ *for any pair of* $\mathscr{L}$ *and* ℓ.

Proof. We have only to prove $\mathrm{Im}\ \bar{z}_k\lambda \in \Gamma_{ho}(\beta_k, \mathbf{R})$, $k=1, 2..K$ for each $\lambda \in \Lambda'_q(\mathbf{R}, \mathscr{L}, \ell)$. Let $\{R'_n\}$ be normal exhaustion towards β_k which is constructed by $\{R_n\}$, and D a doubly connected region such that $\bar{D}^c = G \cup G'$ where G (resp. G') is an end towards β_k (resp. $\beta' = \Delta - \beta_k$) such that $\bar{G} \cap \bar{G}' = \phi$. Further we take $g_o \in C^2(\mathbf{R})$ such that $g_o = 1$ on G and $g_o = 0$ on G'. For the sequence $\{\lambda_n\}$ with $\lambda_n \in \Lambda_q(R_n, \mathscr{L}_n, \ell)$ defining $\lambda \in \Lambda'_q(\mathbf{R}, \mathscr{L}, \ell)$ we can construct another sequence $\{\lambda'_n\}$ such that

$$\lambda'_n = \mathrm{Im}\ \bar{z}_k\tilde{\lambda}_n \text{ on } R'_n \cup \partial R'_n,$$

$$= 0 \text{ on } \mathbf{R} - R'_n \cup \partial R'_n,$$

where $\tilde{\lambda}_n = \lambda_n$ on $R_n \cap G$, $\tilde{\lambda}_n = \lambda$ on G' and $\tilde{\lambda}_n = dg_o f_n + d(f - fg_o)$ on D, df (resp df_n) being equal to λ (resp λ_n) on D. Evidently we have $\lambda'_n \in \Gamma_{co}(\beta_k, \mathbf{R})$ (Cf. Constantinescu. *C* und Cornea. *A* [4] Hilfssatz 7.2) and $\|\lambda'_n - \mathrm{Im}\ \bar{z}_k\lambda\| \to 0$. Consequently we have

Im $\bar{z}_k\lambda\in\Gamma_h\cap\Gamma_{co}(\beta_k, \mathbf{R})=\Gamma_{ho}(\beta_k, \mathbf{R})$. q. e. d.

Lemma 4. 2. *For each $\mathscr{L}$, we have*

$$i\Lambda_q(\mathbf{R}, \mathscr{L}, \nearrow)^{*\perp}\subset\Lambda_q'(\mathbf{R}, \mathscr{L}, \nearrow)$$
$$\subset i\Lambda_q'(\mathbf{R}, \mathscr{L}, \nearrow)^{*\perp}\subset\Lambda_q(\mathbf{R}, \mathscr{L}, \nearrow).$$

Proof. At first we show $\lambda\in\Lambda_{hse}(\mathbf{R})$ if $\lambda\perp\Lambda_q'(\mathbf{R}, \mathscr{L}, \nearrow)^*$. A dividing curve $\gamma\subset W_k$ induces a partition $\mathrm{P}_\gamma(\Delta)$ of Δ such that

$$\mathrm{P}_\gamma(\Delta)=\beta_k'\cup\beta_k'', \text{ where } \beta_k\supset\beta_k' \text{ and } \beta_k''\supset\Delta-\beta_k.$$

Now we take the function f such that $f=H_g^R$ where $g\in C^2(\mathbf{R}^*)$, $g=$real constant $\neq0$ near β_k' and $g=0$ near β_k'', then we know

$$df'=iz_kdf\in\Lambda_q'(\mathbf{R}, \mathscr{L}, \nearrow),$$

because $df_n'=iz_k\, dH_g^{R_n}\in\Lambda_q(R_n, \mathscr{L}_n, \nearrow)$ and

$$\|df_n'-df'\|_{R_n}\longrightarrow 0.$$

Therefore we have for λ with $\lambda\perp\Lambda_q'(\mathbf{R}, \mathscr{L}, \nearrow)^*$

$$0=\langle df', \lambda^*\rangle=\lim_{n\to\infty}\langle df_n', \lambda^*\rangle_{R_n}=\mathrm{Re}\{iz_k \text{ real const} \int_\gamma\bar{\lambda}\}.$$

By the same way we can get

$$0=\mathrm{Re}\ z_k\int_\gamma\bar{\lambda}, \text{ and so } \lambda\in\Lambda_{hse}(\mathbf{R}).$$

Next we show $\int_{A_j}\lambda\in iL_j$, $\int_{B_j}\lambda\in iL_j$ for $\lambda\perp\Lambda_q'(\mathbf{R}, \mathscr{L}, \nearrow)$ where iL_j means $\{\sqrt{-1}\ t: t\in L_j\}$. Let $\sigma(A_j)$ (resp. $\sigma_n(A_j)$) be the reproducing differential in $\Gamma_c(\mathbf{R})$ (resp. $\Gamma_c(R_n)$) associated with A_j and $\zeta_j\in L_j$, then we have by Accola [2]

$$\|\zeta_j\sigma_n(A_j)-\zeta_j\sigma(A_j)\|_{R_n}\leqslant|\zeta_j|\,\{\text{Extremal length of } \{A_j\}_n$$
$$-\text{Extremal length of } \{A_j\}\},$$

where $\{A_j\}_n$ (resp. $\{A_j\}$) stands for the family of curves which are homologous to A_j on R_n (resp. $\mathbf{R}$). From the continuity lemma (Cf. Suita [14]), we get $\|\zeta_j\sigma_n(A_j)-\zeta_j\sigma(A_j)\|_{R_n}\longrightarrow 0$. Evidently we have $\zeta_j\sigma_n(A_j)\in\Lambda_q(R_n,\ \mathscr{L}_n,\ \not\ )$ and so $\zeta_j\sigma(A_j)\in\Lambda_q'(\mathbf{R},\ \mathscr{L},\ \not\ )$. Therefore we get

$$0=\langle\lambda^*,\ \zeta_j\sigma(A_j)\rangle=-\mathrm{Re}\ \bar{\zeta}_j\int_{A_j}\lambda,\ \int_{A_j}\lambda\in iL_j.$$

Analogously we get $\int_{B_j}\lambda\in iL_j$ for $\lambda\perp\Lambda_q'(\mathbf{R},\ \mathscr{L},\ \not\ )^*$. At last we show $\mathrm{Re}\ \bar{z}_k\lambda\in\Gamma_{ho}(\beta_k,\ \mathbf{R})$ if $\lambda\perp\Lambda_q'(\mathbf{R},\ \mathscr{L},\ \not\ )^*$. Taking $df\in\Gamma_{heo}(\beta',\ \mathbf{R})$ where $\beta'=\Delta-\beta_k$, then $z_k\ df\in\Lambda_q'(\mathbf{R},\ \mathscr{L},\ \not\ )$ and so

$$0=\langle z_kdf,\ \lambda^*\rangle=\mathrm{Re}\int_\Delta z_kf\lambda=\mathrm{Re}\int_{\beta_k}z_kf\lambda=\langle df^*,\ \mathrm{Re}\ \bar{z}_k\lambda\rangle.$$

$$i\Lambda_q'(\mathbf{R},\ \mathscr{L},\ \not\ )^{*\perp}\subset\Lambda_q(\mathbf{R},\ \mathscr{L},\ \not\ ).$$

Furthermore since $\Lambda_q(R_n,\ \mathscr{L}_n,\ \not\ )$ is a restricted behavior space, we have

$$i\Lambda_q'(\mathbf{R},\ \mathscr{L},\ \not\ )^{*\perp}\supset\Lambda_q'(\mathbf{R},\ \mathscr{L},\ \not\ ).\ \text{q. e. d.}$$

Lemma 4.3. *$\Lambda_q(\mathbf{R},\ {}_{\Omega}\mathscr{L},\ \not\ )$ is a restricted behavior space.*

Proof. For any pair of $\lambda,\ \lambda'\in\Lambda_q(\mathbf{R},\ {}_{\Omega}\mathscr{L},\ \not\ )$ we have (Cf. Shiba [13] Lemma 6)

$$\langle\lambda,\ i\lambda'^*\rangle=\lim_{n\to\infty}\ \mathrm{Re}\left[\sum_{j=1}^{g_n}\left\{\int_{A_j}\lambda\int_{B_j}\overline{i\lambda'}-\int_{A_j}\overline{i\lambda'}\int_{B_j}\lambda\right\}-\sum_{k=1}^{K}\int_{\beta_k^n}(f_k\ \overline{i\lambda'})\right],$$

where we write $\lambda=df_k$ near β_k^n. But since

$$\mathrm{Re}\ \sum_{j=1}^{g_n}\left\{\int_{A_j}\lambda\int_{B_j}\overline{i\lambda'}-\int_{A_j}\overline{i\lambda'}\int_{B_j}\lambda\right\}=0,$$

$$\mathrm{Re}\ \bar{z}_k\lambda=dF_k\ \text{on}\ W_k\ \text{and}\ \mathrm{Re}\ \bar{z}_k\lambda'=dF_k'\ \text{on}\ W_k,$$

where F_k, F_k' denote the Dirichlet functions on W_k (Cf. Note 4.2), we have from Lemma 2.3 successively

$$\mathrm{Re}\int_{\beta_k^n}f_k\overline{i\lambda'}=\int_{\beta_k^n}\{\mathrm{Re}\ \bar{z}_kf_k\ \mathrm{Re}\ \bar{z}_ki\lambda'+\mathrm{Im}\ \bar{z}_kf_k\ \mathrm{Im}\ \bar{z}_ki\lambda'\}$$

$$=-\int_{\beta_k^n}(F_k\ \mathrm{Im}\ \bar{z}_k\lambda'-F_k'\ \mathrm{Im}\ \bar{z}_k\lambda)\longrightarrow 0,$$

$$\langle\lambda,\ i\lambda'^*\rangle=0,\ \Lambda_q(\mathbf{R},\ {}_{\Omega}\mathscr{L},\ \not\ )\subset i\Lambda_q(\mathbf{R},\ {}_{\Omega}\mathscr{L},\ \not\ )^{*\perp}.$$

Consequently from Lemma 4.2 we have the result of Lemma 4.3.

Lemma 4.4. $\Lambda_q(R_n, {}_{\Omega}\mathscr{L}_n, \diagup) \Longrightarrow \Lambda_q(\mathbf{R}, {}_{\Omega}\mathscr{L}, \diagup)$.

Proof. For the sequence $\{\lambda_n\}$ with $\lambda_n \in \Lambda_q(R_n, {}_{\Omega}\mathscr{L}_n, \diagup)$ such tnat $\sup_n \|\lambda_n\|_{R_n} < \infty$, there exists a subsequence $\{\lambda_{n_\nu}\}$ which are locally uniformly convergent to a differential $\lambda \in \Lambda_{hse}(\mathbf{R})$ on $\mathbf{R}$. We have only to prove the above differential λ satisfies the condidtion $\mathrm{Im}\ \bar{z}_k \lambda \in \Gamma_{ho}(\beta_k, \mathbf{R})$, $k=1, 2, \ldots K$. At first we take a doubly connected region D such that $\bar{D}^c = G \cup G'$ where G (resp. G') is an end towards β_k (resp. $\Delta - \beta_k = \beta'$) such that $\bar{G} \cap \bar{G} = \phi$ and consider the differential ω_n such that

$$\omega_n = \lambda_n \text{ on } R_n \cap G,\ \omega_n = d(g_o f_n) + d[f(1-g_o)] \text{ on } D, \text{ and}$$
$$\omega_n = \lambda \text{ on } G',$$

where $\lambda_n = df_n$ on D, $\lambda = df$ on D and $g_o \in C^2(\mathbf{R})$ such that $g_o = 1$ on G and $g_o = 0$ on G'. Then we get easily the following:

(i) $\omega_n \in \Lambda_c(R_n' = R_n \cup G')$ and $\mathrm{Im}\ \bar{z}_k \omega_n \in \Gamma_{co}(\partial R_n', R_n')$,

(ii) $\tau_{n_\nu} = \mathrm{Im}\ \bar{z}_k \omega_{n_\nu} \longrightarrow \mathrm{Im}\ \bar{z}_k \lambda = \tau$ locally uniformly on $\mathbf{R}$.

For $\sigma \in \Gamma_{eo}(\beta', \mathbf{R}) \cap \Gamma^1$ and $\varepsilon > 0$, there exists a regular canonical region Ω such that $\|\sigma\|_{\mathbf{R}-\Omega} < \varepsilon$ and so

$$|\langle \tau, \sigma^* \rangle| < \varepsilon\|\tau\| + \lim_{\nu\to\infty} |\langle \tau_{n_\nu}, \sigma^* \rangle_\Omega|$$
$$< \varepsilon\|\tau\| + \lim_{\nu\to\infty} |\langle \tau_{n_\nu}, \sigma^* \rangle_{R'_{n_\nu} - \Omega}| < 4\varepsilon\|\tau\|$$

Consequently $\langle \tau, \sigma^* \rangle = 0$, $\tau \in \Gamma_{co}(\beta_k, \mathbf{R})$ and so have

$$\mathrm{Im}\ \bar{z}_k \lambda \in \Gamma_{co}(\beta_k, \mathbf{R}).$$

Thus the proof of Theorem 1 is comlete.

4C. Making use of the Theorem 1, for each finite regular partition $\mathrm{P}(\Delta)$ we can show the existence of a behavior space associated with any given pair of $\mathscr{L}$ and $\diagup$. Suppose $\{R_n\}$ is a canonical exhaustion. We set

$${}_n\Lambda=\Lambda_0(\mathbf{R},\ {}_{R_n}\mathscr{L},\ \swarrow),$$

$$\mathfrak{A}(\mathrm{P}(\Delta),\mathscr{L},\ \swarrow)=\mathfrak{A}=\bigcap_{n=1}^{\infty}\overline{\bigcup_{k=n}^{\infty}{}_k\Lambda},\ \text{and}\ \mathfrak{B}=\overline{\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}{}_k\Lambda}.$$

Lemma 4. 5. $i\mathfrak{A}^{*\perp}=\mathfrak{B}\subset\mathfrak{A}$.

Proof. From the theorem 1 we can get successively

$$\overline{\bigcup_{k=n}^{\infty}{}_k\Lambda^*}=(\bigcap_{k=n}^{\infty}{}_k\Lambda^{*\perp})^{\perp}=(\bigcap_{k=n}^{\infty}i\ {}_k\Lambda)^{\perp},$$

$$\mathfrak{A}^{*\perp}=(\bigcap_{n=1}^{\infty}\overline{\bigcup_{k=n}^{\infty}{}_k\Lambda^*})^{\perp}=(\bigcap_{n=1}^{\infty}(\bigcap_{k=n}^{\infty}i\ {}_k\Lambda)^{\perp})^{\perp}=\overline{\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}i\ {}_k\Lambda}=i\ \mathfrak{B}.$$

Theorem 2. *Let* $\mathrm{P}(\Delta)$ *be a finite regular partition of* Δ. *Then we have the following:*

(i) $\mathfrak{A}=\mathfrak{A}(\mathrm{P}(\Delta),\ \mathscr{L},\ \swarrow)$ *is a behavior space associated with* $\mathscr{L}$,

(ii) *the necessary and sufficient condition that* $\mathfrak{A}$ *should be a restricted behavior space is that* $\mathfrak{A}=\mathfrak{B}$.

Proof. (i) For $\varepsilon>0$ and $\lambda\in\mathfrak{A}$, there exists λ_1 such that

$$\lambda_1\in{}_{n_1}\Lambda\ \text{and}\ \|\lambda-\lambda_1\|<\varepsilon.$$

Further for a fixed integer m with $m>n_1$ there exists λ_2 such that

$$\lambda_2\in{}_{n_2}\Lambda,\ n_2>m\ \text{and}\ \|\lambda-\lambda_2\|<\frac{\varepsilon}{2}$$

Thus we can get the sequence $\{\lambda_p\}$ such that

(a) $\lambda_p\in{}_{n_p}\Lambda$ with $n_p\longrightarrow\infty$ as $p\longrightarrow\infty$.

(b) $\|\lambda-\lambda_p\|<\frac{\varepsilon}{2^p}$.

Therefore we have for any $\lambda\in\mathfrak{A}$

$$\langle\lambda,\ i\lambda^*\rangle=\lim_{p\to\infty}\langle\lambda_p,\ i\lambda_p^*\rangle=0,\ \text{and}$$

$$\mathfrak{A}=\mathfrak{A}+i\mathfrak{A}^{*\perp}\ \text{(Cf. Lemma 4. 5)},$$

that is to say, $\mathfrak{A}$ is a behavior space associated with $\mathscr{L}$. The proof of (ii) is omitted (Cf. Lemma 4. 5).

4D. Furthermore for some finite non regular partitions we can show the existence of a behavior space associated with any given pair of $\mathscr{L}$ and $\diagup$. For example, let $\tilde{\mathrm{P}}(\Delta)=\alpha\cup\beta\cup\gamma$ be a partition of Δ where α, $\alpha\cup\beta$ are closed, then there exists a sequence of regular partitions $\{\mathrm{P}_n(\Delta)\}$ with $\mathrm{P}_n(\Delta)=\alpha_n\cup\beta_n\cup\gamma_n$ such that $\alpha_n\cup\beta_n\downarrow\alpha\cup\beta$ and $\alpha_n\downarrow\alpha$. We set

$$\mathfrak{A}_n=\mathfrak{A}(\mathrm{P}_n(\Delta),\ \mathscr{L},\ \diagup),$$

$$\tilde{\mathfrak{A}}=\tilde{\mathfrak{A}}(\tilde{\mathrm{P}}(\Delta),\ \mathscr{L},\ \diagup)=\bigcap_{n=1}^{\infty}\overline{\bigcup_{k=n}^{\infty}\mathfrak{A}_n}.$$

Corollary 4.3. *$\tilde{\mathfrak{A}}(\tilde{\mathrm{P}}(\Delta),\ \mathscr{L},\ \diagup)$ is a behavior space associated with $\mathscr{L}$, where $\tilde{\mathrm{P}}(\Delta)=\alpha\cup\beta\cup\gamma$ is a partition such that α and $\alpha\cup\beta$* are closed.

Proof. From the Theorem 2 we can get successively the following:

$$\overline{\bigcup_{k=n}^{\infty}\mathfrak{A}_n^*}=(\bigcap_{k=n}^{\infty}\mathfrak{A}_n^{*\perp})^{\perp}\supset(\bigcap_{k=n}^{\infty}i\mathfrak{A}_n)^{\perp},$$

$$\tilde{\mathfrak{A}}^{*\perp}=(\bigcap_{n=1}^{\infty}\overline{\bigcup_{k=n}^{\infty}\mathfrak{A}_k^*})^{\perp}\subset(\bigcap_{n=1}^{\infty}(\bigcap_{k=n}^{\infty}i\mathfrak{A}_k)^{\perp})^{\perp}=\overline{\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}i\mathfrak{A}_k}\subset i\tilde{\mathfrak{A}}.$$

Therefore $\tilde{\mathfrak{A}}$ is a behavior space associated with $\mathscr{L}$.

§5. Applications to conformal mappings.

5A. At first we show the relation between the functions with $\Lambda_q(\mathbf{R},\ \mathscr{L},\ \diagup)$ behavior and the canonical exact Abelian differentials in Kusunoki's sense. For an open Riemann surface $\mathbf{R}$ of finite genus, we set $\Lambda_{\bullet}=\Lambda_q(\mathbf{R},\ \mathscr{L},\ \diagup)$ where $\mathscr{L}=\diagup=$ {the imarginary axis}, so that the corresponding partition is the identity partition.

Lemma 5.1. *Let* **R** *be of finite genus, then the meromorphic function f has $\Lambda_{\bullet}$ behavior if and only if df is canonical exact in Kusunoki's sense.*

Proof. Let df be canonical exact, then by Yoshida [18] df has

Λ_e behavior.[1] Conversely, let df be an exact differential with Λ_e behavior on W, W being an end towards Δ, then we can write

$$df=\phi+\lambda_{eo} \text{ on } W,$$

where $\phi\in\Lambda_e$ and $\lambda_{eo}\in\Lambda_{eo}$. If we set

$$\sigma=\sum_{k=1}^{g}\{b_k\sigma(A_k)-a_k\sigma(B_k)\},\ \phi'=\phi-\sigma,$$

where $\sigma(A_k)$ (resp. $\sigma(B_k)$) is the reproducing differentials associated with A_k (resp. B_k), $b_k=\int_{B_k}\phi$ and $a_k=\int_{A_k}\phi$, then we have

Im $i\phi'\in\Gamma_{he}\cap\Gamma_{ho}=\Gamma_{hm}$[1] (as **R** is of finite genus).

On the other hand Re $\sigma|_W=du$ satisfies the condition $u=(I)L_{1W}u$ on W, $(I)L_{1W}$ being Sario's principal operator on ∂W corresponding to the identity partition (Cf. Rodin and Sario [12] p 100). Moreover u satisfies the condition

$$(I)\ L_{1W}\ u=constant+f_{eo}\ on\ W,$$

where $df_{eo}\in\Gamma_{eo}$ (Cf. Nakai and Sario [10] Theorem 1). Therefore we can write

$$df=\phi+\lambda_{eo}=\omega_{hm}+i\omega_{hse}+\lambda'_{eo} \text{ on } W,$$

where $\lambda'_{eo}\in\Lambda_{eo}$, $\omega_{hm}\in\Gamma_{hm}$ and $\omega_{hse}\in\Gamma_{hse}$, hence df is canonical exact differential.

5B. Now we shall generalize the classical theorems stated in §1 by making use of the results in §2~§4.

Theorem 3. *Suppose* **R** *is an open Riemann surface with finite*

1) If **R** is of finite genus, then $\Gamma_{hm}=\Gamma_{ho}\cap\Gamma_{he}$. In fact, for $\omega\in\Gamma_{ho}$ we have $\omega|_{R_n}=\omega_n+\omega'_n$ where $\omega_n\in\Gamma_{ho}(R_n)$ and $\omega'_n\in\Gamma_{he}(R_n)^*$ so that $||\omega_n-\omega||_{R_n}\longrightarrow 0$. Therefore for any $\tau\in\Gamma_{hse}$ we have

$$<\omega,\tau^*>=\lim_{n\to\infty}\Big[\sum_{j=1}^{g}\Big\{\int_{A_j}\omega_n\int_{B_j}\tau-\int_{A_j}\tau\int_{B_j}\omega_n\Big\}-\int_{\partial R_n}f_n\tau\Big],$$

wherel $df_n=\omega_n$ near ∂R_n so that f_n=constant on each component of ∂R_n. Consequenty we get $\lim_{n\to\infty}\int_{\partial R_n}f_n\tau=0$, and so the special bilinear relation between Γ_{ho} and Γ_{hse} is valid, therefore $\Gamma_{hm}=\Gamma_{he}\cap\Gamma_{ho}$ (Cf. Accola [1]).

genus g, $\mathrm{P}(\Delta)=\bigcup_{k=1}^{K}\beta_k$ *a finite regular partition of Kerékjártó-Stoilow's boundary and* P *a point of* $\mathbf{R}$ *which is non Weierstrass point for* $\Lambda_q=\Lambda_q(\mathbf{R}, \mathscr{L}, \measuredangle)$, *then there exists a meromorphic function* f *on* $\mathbf{R}$ *which satisfies the following conditions :*

(i) f *has* $\Lambda_q(\mathbf{R}, \mathscr{L}, \measuredangle)$ *behavior,*

(ii) f *has only a pole of order* $g+1$ *at* P,

(iii) $f(\mathbf{R})$, *the image of* $\mathbf{R}$ *under* f, *is of at most* $g+1$ *sheets over the Riemann sphere.*

Proof. Let $\{R_n\}$ be a regular canonical exhaustion, then by Lemma 3.5 and Corollary 4.2 there exists sequence $\{f_{n_\nu}\}$ and a function f which satisfies the following conditions:

(a) f_{n_ν} is a meromorphic function on R_{n_ν} with $\Lambda_{q n_\nu}=\Lambda_q(R_{n_\nu}, \mathscr{L}_{n_\nu}, \measuredangle)$ behavior on $V^c\cap R_{n_\nu}$ where V is a fixed parametric disk at P,

(b) f has only a pole of order $g+1$ at P,

(c) $f_{n_\nu}-f \longrightarrow 0$, locally uniformly on $\mathbf{R}$,

(d) $f_{n_\nu}(R_{n_\nu})$ is at most $g+1$ sheeted over the Riemann sphere (Cf. Shiba [13]).

In the following, for simplicity we write $\{n_\nu\}=\{\nu\}$. Now we fix a complex number α arbitrarily, then by the argument principle we have

$$n(f, \alpha, R_\nu)-(g+1)=\frac{1}{2\pi i}\int_{\partial R_\nu}\frac{df}{f-\alpha} \text{ for } \nu > n_0'.$$

But since

$$\frac{1}{2\pi i}\int_{\partial R_\nu}\frac{df_\mu}{f_\mu-\alpha}=\text{an integer for } \mu > n_0',$$

$$\frac{1}{2\pi i}\int_{\partial R_\nu}\frac{df_\mu}{f_\mu-\alpha}\longrightarrow\frac{1}{2\pi i}\int_{\partial R_\nu}\frac{df}{f-\alpha} \text{ as } \mu\to\infty,$$

we can get

$$n(f, \alpha, R_\nu)-(g+1)=\frac{1}{2\pi i}\int_{\partial R_\nu}\frac{df_\mu}{f_\mu-\alpha} \text{ for larg } \mu.$$

On the other hand, since f_μ is of $\Lambda_{q\mu}$ behavior, we have

$$\int_{\partial R_\mu} d\ \arg\ (f_\mu-\alpha)=\sum_{k=1}^{K}\int_{\beta_k^\mu} d\ \arg(f_\mu-\alpha)=0.$$

Consequently we have $n(f,\ \alpha,\ R_\mu)\leqslant g+1$ for $\mu\geqslant n_0'$.

Corollary 5. 1. *If we set* $\mathscr{L}=\ell=\{$*the imaginary axis*$\}$, *Theorem 3 coincides partially with the Theorem 3 in Mori* [9].

Theorem 4. *Suppose* **R** *is of finite genus and* $P_1,\ P_2,\dots P_{g+1}$ *are arbitrarily* $g+1$ *points of* **R**, *then there exists a meromorphic function* f *with* $\Lambda_q(\mathbf{R},\ \mathscr{L},\ \ell)$ *behavior such that*

(i) *the divisor of* f *is a multiple of* $(P_1P_2\dots\ P_{g+1})^{-1}$,

(ii) $f(\mathbf{R})$ *is of at most* $g+1$ *sheets over the Riemann sphere.*

Proof. Omitted.

Theorem 5. *Let* **R** *be an open Riemann surface with finite genus, and* P_1 *an arbitrarily given point of* **R**, *then for suitable choice of* g *points* $P_2,\ P_3,\dots\ P_{g+1}$ *of* **R** *there exists a meromorphic function* f *which satisfies the following conditions :*

(i) f *has* $\Lambda_q(\mathbf{R},\ \mathscr{L},\ \ell)$ *behavior and the residue of* f *at* P_1 *is equal to* 1,

(ii) *the divisor of* f *is a multiple of* $(P_1P_2\dots P_{g+1})^{-1}$,

(iii) $f(\mathbf{R})$ *is of at most* $g+1$ *sheets over the Riemann sphere.*

Proof. From Lemma 3. 7 and Corollary 4. 2, we can prove this theorem by the same way as in Theorem 3.

Corollary 5. 2. *If we set* $\mathscr{L}=\ell=\{$*the imaginary axis*$\}$, *then the Theorem 5 reduces to the Theorem 14 in Kusunoki* [6].

Corollary 5. 3 *Assume that the regular partition* $P(\Delta)=\beta_1\cup\beta_2$ *and* W_1(resp W_2) *is an end towards* β_1(resp β_2). *If we set*

$$\Lambda_q=\{\lambda:\lambda\in\Lambda_{h_0},\ \int_{A_j}\lambda\in L_j,\ \int_{B_j}\lambda\in L_j,\ i=1,2,\dots g,$$

$$\mathrm{Im}\ \bar{z}_k\lambda\in\Gamma_{ho}(\beta_k,\ \mathbf{R})\ \textit{where}\ z_k\in l_k,\ k=1,2,$$
$$l_1=\textit{real axis and } l_2=\textit{imaginary axis}\},$$

then Theorem 5 coincides partially with the Theorem 5.3 in Mizumoto [7].

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