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An algebraic characterization of the affine plane

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1. Statements of results

C. P. Ramanujam [9] characterized the affine plane over the complex field as follows: Let X be a non-singular algebraic surface which is contractible and simply connected at infinity. Then X is isomorphic to the affine two space as an algebraic variety. The purpose of the present article is to prove the following algebraic characterizations of the affine plane.

Theorem 1. *Let k be an algebraically closed field of arbitrary characteristic, let A be a finitely generated k-domain of dimension two and let X be the affine surface defined by A. Then X is isomorphic to the affine plane over k if and only if the following conditions are satisfied:*

(i) *A is a unique factorization domain.*

(ii) *The set A^* of all invertible elements of A coincides with $k^* = k-(0)$.*

(iii) *There is a non-trivial action of the additive group scheme G_a on X defined over k.*

Theorem 2. *Let k be an algebraically closed field of characteristic zero, let A be a finitely generated, regular, rational k-domain of dimen-*

sion two and let X be the affine surface defined by A.

If the conditions (i) *and* (ii) *of Theorem* 1 *are satisfied, the condition* (iii) *is equivalent to the condition:*

(iii)′ *There is an algebraic system F of closed curves on X parametrized by a rational curve such that a general member of F is an affine rational curve with only one place at infinity and that two distinct general members of F have no intersection on X.*

Theorem 3. *Let k be an algebraically closed field of characteristic zero and let X be an affine non-singular surface defined by an affine k-domain A. Assume that the following conditions are satisfied:*

(1) *A is a unique factorization domain and $A^* = k^*$.*

(2) *There exist non-singular irreducible closed curves C_1 and C_2 on X such that $C_1 \cap C_2 = \{v\}$, and C_1 and C_2 intersect transversally at v.*

(3) *C_1 (resp. C_2) has only one place at infinity.*

(4) *Let a_2 be a prime element of A defining the curve C_2. Then $a_2 - \alpha$ is a prime element of A for all $\alpha \in k$.*

(5) *There is a non-singular complete surface V containing X such that the closure $\bar{C}_2$ of C_2 in V is non-singular and $(a_2)_0 = \bar{C}_2$.*

Then X is isomorphic to the affine plane $\mathbf{A}^2$.

2. Proof of Theorem 1.

Let k be a field, let A be a k-domain and let $X = \mathrm{Spec}(A)$. An action of the additive group scheme G_a on X defined over k can be described by means of a locally finite iterative higher derivation D on A. (For the definition and relevant results, see [3] or [4].)

Let A_0 be the invariant subring of A with respect to the given G_a-action. Then we have

Lemma 1. *Let k, A and A_0 be as above. Then A_0 is an inert subring of A. Namely, if $a = a_1 a_2$ with $a \in A_0$ and $a_1, a_2 \in A$, then both a_1 and a_2 belong to A_0. In particular, if A is a unique factorization domain and if A_0 is a noetherian ring, A_0 is a unique factorization* *domain.*

For the proof, see [7].

It seems difficult in general to show or deny that given a finitely generated k-domain A and a non-trivial G_a-action on Spec(A), the invariant subring A_0 is finitely generated over k. However we have

Lemma 2. *Let k be an algebraically closed field, let A be a finitely generated, unique factorization domain defined over k of dimension two and with $A^* = k^*$. Assume that there is a non-trivial G_a-action on* Spec(A) *defined over k. Then the invariant subring A_0 of A is a one-parameter polynomial ring over k.*

Proof. Let K and K_0 be the quotient fields of A and A_0 respectively. It is known [7] that there are an element a of A_0 and an element t of A such that $A[a^{-1}]=A_0[a^{-1}][t]$. Since $A[a^{-1}]$ is a unique factorization domain, $A_0[a^{-1}]$ is a unique factorization domain of dimension 1 and is finitely generated over k. Therefore $A_0[a^{-1}]$, (hence $A[a^{-1}]$), is rational over k. Namely $K_0=k(u)$ and $K=k(u, t)$.

We shall show that there is an element c of A_0 such that $K_0=k(c)$. Since $K_0=k(u)=Q(A_0)$ (where $Q(\ )$ means the quotient field), there are elements a and b of A_0 such that $u=a/b$. Consider a subring $A_1=k[a, b]$ of A_0, and let C be the normalization of A_1 in $Q(A_1)=K_0$. Then C is finitely generated over k. Since the assumption that $A^*=k^*$ implies that $C^*=k^*$, C is a one-parameter polynomial ring over k. Write $C=k[c]$ with $c\in A_0$. Then $K_0=k(c)$.

We shall show that $A_0=k[c]$. Otherwise, take any element a of $A_0-k[c]$ and consider a subring $A_2=k[c, a]$ of A_0. Let C' be the normalization of A_2 in K_0. Then C' is finitely generated over k and $C'^*=k^*$. Hence C' is a one-parameter polynomial ring over k. Moreover, since $Q(C)=Q(C')=K_0$, we should have $C=C'$. Then

$a \in k[c]$, and this is a contradiction. q. e. d.

The key to prove the "if" part of Theorem 1 is

Lemma 3. *Let k be an algebraically closed field of arbitrary characteristic and let A be a finitely generated k-domain of dimension two. Assume the following conditions:*

(i) *A is a unique factorization domain.*

(ii) *There is a non-trivial G_a action on* Spec(A) *defined over k.*

(iii) *The invariant subring A_0 of A with respect to the G_a-action is finitely generated over k.*

Then A is a one-parameter polynomial ring over A_0.

Proof. Our proof consists of several steps.

(1) Let $X = \mathrm{Spec}(A)$, let $Y = \mathrm{Spec}(A_0)$ and let $f: X \longrightarrow Y$ be the canonical morphism defined by the canonical injection $A_0 \hookrightarrow A$. Since A_0 is a finitely generated, unique factorization domain over k, Y is isomorphic to the affine line which might be deleted a finitely many points. Hence there is an element a of A_0 such that $A_0 = k[a, h(a)^{-1}]$, where $h(a) \neq 0$, $\in k[a]$.

(2) Let $D = \{D_0, D_1, \dots\}$ be the locally finite iterative higher derivation on A associated with the given G_a-action on Spec(A) and let $\varphi : A \longrightarrow A[u]$ (u being an indeterminate) be the k-algebra homomorphism defined by $\varphi(x) = \sum_{i \geq 0} D_i(x) u^i$ for every x of A. Define the length $l(x)$ of an element x of A by $l(x) = \deg_u \varphi(x)$. It is then easy to show that if $l(x) \neq 0$ and $l(x)$ is the shortest among the lengths of all elements of $A - A_0$, $D_1(x), \dots, D_{l(x)}(x)$ are G_a-invariant (cf. [7], Appendix). Choose an element t in $A - A_0$ so that (i) $l(t)$ is the shortest and that (ii) if we write $D_{l(t)}(t) = c a_1^{\alpha_1} \dots a_n^{\alpha_n}$ with an invertible element c and mutually distinct prime elements $a_1, \dots, a_n$, then $\sum_{1 \leq i \leq n} \alpha_i$ is minimal. Then for any α of k, $t - \alpha$ is a prime element of A. For, otherwise, $t - \alpha = t_1 t_2$ with $t_1, t_2 \in A$. Then either t_1 or t_2 has the same length as t, and

the other one is G_a-invariant. Assume that t_2 is G_a-invariant, and let $a_1 = D_{l(t_1)}(t_1)$. Then $D_{l(t)}(t) = a_1 t_2$, which is contrary to the choice of t since t_2 is not invertible.

(3) Let $B = A_0[t]$ and let $Z = \mathrm{Spec}(B)$. Then, by the canonical inclusions $A_0 \hookrightarrow B \overset{\phi}{\hookrightarrow} A$, Z is a Y-scheme (with the projection $g: Z \longrightarrow Y$), and we have a Y-morphism $\rho: X \longrightarrow Z$ such that $f = g \circ \rho$. ρ is birational since there is an element c of A_0 such that $A[c^{-1}] = A_0[c^{-1}][t]$ (cf. [7], Appendix or the proof of Lemma 2). G_a acts on Z via the restriction of the locally finite iterative higher derivation D on B, and ρ commutes with the G_a-actions on X and Z. On the other hand, each fibre of f is irreducible since $a - \alpha$ (which defines the fibre of f at the point $y: a = \alpha$) is a prime element in A for every element α of k with $h(\alpha) \neq 0$ (cf. Lemma 1). We shall show that f is surjective and that for every $y \in Y$, the restriction ρ_y of ρ onto $f^{-1}(y)$ is a generically surjective morphism from $f^{-1}(y)$ to $g^{-1}(y)$. For this purpose it suffices to show that for any $\alpha \in k$ such that $h(\alpha) \neq 0$, $\bar{\phi}: B/(a-\alpha)B \longrightarrow A/(a-\alpha)A$ is injective, where $\bar{\phi}$ is induced from ϕ. Since $B/(a-\alpha)B \cong k[t]$, assume that $\bar{\phi}(q(t)) = 0$ for some $q(t) \neq 0, \in k[t]$. Since $q(t) = \beta \prod_{1 \le i \le m} (t - \gamma_i)$ with β and γ_i's in k, $\prod_{1 \le i \le m} (t - \gamma_i) \in (a-\alpha)A$. Since $a - \alpha$ is a prime element of A, there are an integer i $(1 \le i \le m)$ and an element h' of A such that $t - \gamma_i = (a-\alpha)h'$. Since $D_{l(t)}(t) = (a-\alpha) D_{l(h')}(h')$ and $l(t) = l(h')$, this contradicts to the choice of t. Therefore $\bar{\phi}$ is injective, and it is easy to see that ρ is quasi-finite since each fibre of f (or g) has dimension 1.

(4) Since ρ is a birational quasi-finite morphism and since X and Z are normal, ρ is an open immersion by the Main Theorem of Zariski (cf. [1]). The image $\rho(X)$ is an affine open set. Since G_a acts on Z and ρ commutes with the G_a-actions on X and Z, it is easy to see that $\rho(X)$ has the complement of codimension two in Z. Then $\rho(X) = Z$. Hence $A = A_0[t]$.

Now the "if" part of Theorem 1 follows easily from Lemmas

2 and 3. The "only if" part is obvious. Thus, Theorem 1 is completely proved.

Remarks. (1) Lemma 3 is false if A is not a unique factorization domain, as is shown in the following example : Let k be an algebraically closed field of characteristic $\neq 2$. Let $A=k[t, X, Y]/(Y^2-tX-1)$. A is a rational, regular k-domain, but A is not a unique factorization domain. In fact, A is the affine ring of an affine surface of the form: $\mathbf{P}^1\times\mathbf{P}^1-$(an ample irreducible curve). Define a G_a-action on $\mathrm{Spec}(A)$ by a k-homomorphism $\varphi: A\longrightarrow A[u]$; $\varphi(t)=t$, $\varphi(X)=X+2Yu+tu^2$ and $\varphi(Y)=Y+tu$. Then the invariant subring of A is $k[t]$. Hence A is not a polynomial ring over $k[t]$.

(2) Let k be an algebraically closed field and let A be a finitely generated normal k-domain. Then A^* is isomorphic to a direct product of k^* and a torsion-free $\mathbf{Z}$-module of finite rank.

Proof. Let X be the affine variety defined by A and let V be a complete normal variety which contains X as a dense open set. Let Y be the complement of X in V. Then Y has pure codimension 1. Let $Y_1,\dots, Y_n$ be irreducible components of Y. If f is an invertible element of A, then $(f)=\sum_{1\leq i\leq n} m_iY_i$. Define a mapping $\nu: A^*\longrightarrow \bigoplus_{1\leq i\leq n}\mathbf{Z}$ by $\nu(f)=(m_1,\dots, m_n)$. Then ν is a homomorphism of abelian groups and Ker $\nu=k^*$. Therefore A^*/k^* is a $\mathbf{Z}$-submodule of $\bigoplus_{1\leq i\leq n}\mathbf{Z}$, hence A^*/k^* is a torsion-free $\mathbf{Z}$-module of finite rank. It is then obvious to see that A^* is a direct product of k^* and a free $\mathbf{Z}$-module A^*/k^* of finite rank.

3. Proof of Theorem 2

First of all, we shall treat the implication (iii)$'\Longrightarrow$(iii) of Theorem 2. Let k be an algebraically closed field of characteristic zero and let A be a finitely generated, regular, rational k-domain of dimension two. Assume that A is a unique factorization domain and that $A^* = k^*$. Let X be the affine surface defined by A. Then there is a non-singular projective surface V containing X as an open set.

We shall summarize rather elementary results in the following two Lemmas.

Lemma 4. *Let A, X and V be as above. If $V-X$ is irreducible, then V is isomorphic to the projective plane $\mathbf{P}^2$ and $V-X$ is isomorphic to a hyperplane.*

Proof. V dominates a relatively minimal rational projective surface V_0, which is isomorphic to $\mathbf{P}^2$ or F_n with $n \geq 0$ and $n \neq 1$, (cf. [8]). V is obtained from V_0 by repeating local quadratic transformations with non-singular centers ; $V = V_r \longrightarrow V_{r-1} \longrightarrow \dots \longrightarrow V_0$. Then Pic$(V)$ is a direct sum of Pic(V_0) and a free $\mathbf{Z}$-module of rank r. The facts that Pic$(V_0) \cong \mathbf{Z}$ (if $V_0 \cong \mathbf{P}^2$) or Pic$(V_0) \cong \mathbf{Z} \oplus \mathbf{Z}$ (if $V_0 \cong F_n$) and that Pic$(X) = (0)$ imply that $V = V_0 \cong \mathbf{P}^2$ if $V-X$ is irreducible. If $V = \mathbf{P}^2$ and $V-X$ is irreducible, it is easy to see that $V-X$ is a hyperplane. q. e. d.

Lemma 5. *Let A, X and V be as above. If X has an algebraic system F of closed curves which satisfies the condition* (iii)$'$ *of Theorem* 2, *there is a linear pencil L of divisors on V such that a general member of L is irreducible and of multiplicity* 1 *and that for a general member C of L, $C \cap X$ is a member of F.*

Proof. By the condition (iii)$'$ there is a rational curve T and an irreducible subvariety W of $X \times T$ such that if we denote by p and q the canonical projections of W onto X and T respectively, then for any point $t \in T$, $W_t = q^{-1}(t)$ is a member of F, identifying W_t with $p(W_t)$ by p. Replacing T by an affine open set $(\neq \phi)$ of T, we may assume that T is an affine open set of $\mathbf{A}^1$, i. e., $T =$ Spec$(k[u, g(u)^{-1}])$ with $g(u) \neq 0$ and $g(u) \in k[u]$. Let $R = k[u, g(u)^{-1}]$.

Then the affine algebra $k[W]$ of W is of the form $k[W]=A\underset{k}{\otimes}R/I$, where I is a prime ideal of $A\underset{k}{\otimes}R$. The condition (iii)′ implies that the canonical homomorphism $\rho : A\hookrightarrow A\underset{k}{\otimes}R\longrightarrow k[W]$ $(a\longmapsto a\otimes 1$ (mod I)) yields an isomorphism $\rho : k(X)\xrightarrow{\sim} k(W)$. Namely we have a commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{\ \ \rho\ \ } & k[W] \\ \cap\downarrow & & \cap\downarrow \\ k(X) & \xrightarrow[\rho]{\sim} & k(W) \end{array}$$

We shall identify A with a subalgebra $\rho(A)$ of $k[W]$ and $k(X)$ with $k(W)$ by ρ. Since A is a unique factorization domain and $k[W]$ is finitely generated over A, there exists a set of prime elements $(b_1,\dots, b_r)$ of A such that

$$A\hookrightarrow k[W]\hookrightarrow A[1/b_1,\dots, 1/b_r].$$

Let $\bar{u}=1\otimes u$ (mod I) and write $\bar{u}=a_1/a_0$, where a_0, $a_1\in A$, $(a_0, a_1)=1$ and $a_0=b_1^{e_1}\dots b_r^{e_r}$ with non-negative integers $e_1,\dots, e_r$. Then for any point $\alpha\in T(k)\subset k$, $(\bar{u}-\alpha)A[1/b_1,\dots, 1/b_r]=(a_1-\alpha_0 a)A[1/b_1,\dots, 1/b_r]$. This implies that the curve on X defined by $a_1-\alpha a_0$ has support in the union of $p(W_\alpha)$ and the curves defined by b_i $(i=1,\dots, r)$. Therefore, for any point $(\beta, \gamma)\in \mathbf{P}^1$ the divisor $(a_1\beta-a_0\gamma)$ on V can be written in the form; $(a_1\beta-a_0\gamma)=C_\alpha+D_0+D_1-D_2$, where the following conditions are satisfied:

(1) $\alpha=\gamma/\beta$.

(2) $C_\alpha, D_2>0$; D_0, $D_1\geq 0$; $\mathrm{Supp}(D_1)\cup\mathrm{Supp}(D_2)\subset V-X$; Supp (D_0) is contained in the union of the closures in V of the curves on X defined by $b_i=0$ for $i=1,\dots, r$; D_0, D_1 and D_2 are fixed divisors (independent of α).

(3) For a general point α of $\mathbf{P}^1$, C_α is irreducible and $C_\alpha\cap X=p(W_\alpha)$.

Then the divisors $\{C_\alpha\}_{\alpha \in P^1}$ form a linear pencil L. From the construction of L, a general member of L is irreducible and of multiplicity 1. q. e. d.

Now we shall prove the implication (iii)$'\Longrightarrow$(iii) of Theorem 2. By the second theorem of Bertini, a general member C of the linear pencil L constructed in Lemma 5 has no singular points outside base points of L. Therefore, $C \cap X$ is isomorphic to the affine line $\mathbf{A}^1$, and L has at most one base point which will be situated on $V-X$ if it exists. Let $f: V \longrightarrow \mathbf{P}^1$ be the rational mapping defined by L, which is regular outside a base point. If L has a base point P $(\in V-X)$, there exists a succession of locally quadratic transformations $T: V^* \longrightarrow V$ with centers P and its infinitely near base points of L such that the linear system L^* on V^*, which is the total transform of L by T deleted all fixed components, has no base points. Let $f^*: V^* \longrightarrow \mathbf{P}^1$ be the morphism defined by L^*. Then it is not hard to show that for a general member C^* of L^*, $C^* \cap X$ is a member of the algebraic system F on X fixed in the condition (iii)$'$ of Theorem 2 and that the restriction of f^* onto X $(\subset V^*)$ is identical with the restriction of f onto X.

Replacing V, L and f by V^*, L^* and f^* respectively, we may assume that L has no base points. Then a general member C of L is non-singular and rational. Hence C is isomorphic to $\mathbf{P}^1$. Since $C \cap X$ is isomorphic to $\mathbf{A}^1$, C cuts an irreducible component E of $V-X$ at only one point. Since the characteristic of k is zero, the restriction of f onto E yields a birational mapping $f|_E: E \longrightarrow \mathbf{P}^1$. This implies, in particular, that a general member C of L cuts E transversally at only one point. Then there is an affine open set $U(\neq \phi)$ of $\mathbf{P}^1$ such that $f^{-1}(U)$ is a trivial $\mathbf{P}^1$-bundle and that $E \cap f^{-1}(U)$ is a section of $f^{-1}(U)$ (cf. [2], Theorem 1. 8). Then $f^{-1}(U) \cap X = f^{-1}(U) - E \cap f^{-1}(U)$ is a trivial $\mathbf{A}^1$-bundle over U.

On the other hand, $X - f^{-1}(U) \cap X$ consists of a finitely many (mutually distinct) irreducible curves $G_1, \ldots, G_r$ which are defined by prime elements $a_1,\dots, a_r$ of A respectively. Then $f^{-1}(U)\cap X=\mathrm{Spec}(A[a^{-1}])$ where $a=a_1\dots a_r$. Let $U=\mathrm{Spec}(B)$. Then B is a subring of $A[a^{-1}]$, and there exists an element t of A such that $A[a^{-1}]=B[t]$ (= a polynomial ring over B). Since $A^*=k^*$ and A is a unique factorization domain, $(A[a^{-1}])^*/k^*=$ a free $\mathbf{Z}$-module of rank r generated by $a_1,\dots, a_r$. Since $A[a^{-1}]=B[t]$, we have $(A[a^{-1}])^*=B^*$. If we write B in the form: $B=k[u, g(u)^{-1}]$ with $u\in B$ and $g(u)=\prod_{1\le i\le s}(u-\alpha_i)\in k[u]$ ($\alpha_1,\dots, \alpha_s$ being mutually distinct elements of k), we have that $r=s$.

We shall show that $f(X)$ is an affine open set of $\mathbf{P}^1$. Assume the contrary: $f(X)=\mathbf{P}^1$. Here we may assume that $V-X$ has more than two irreducible components. In fact, if $V-X$ is irreducible, Lemma 4 says that V is isomorphic to $\mathbf{P}^2$ and $V-X$ is isomorphic to a hyperplane. Therefore X is isomorphic to $\mathbf{A}^2$, and we have nothing to prove. Now since L has no base point and a general member of L cuts $V-X$ transversally at only one point of the irreducible component E of $V-X$, the irreducible components of $V-X$ other than E correspond to a finite number of points $Q_1,\dots, Q_m$ of $\mathbf{P}^1$ by f, i.e., $f(V-X\cup E)=\{Q_1,\dots, Q_m\}$. Then the assumption that $f(X)=\mathbf{P}^1$ implies that for every $i(1\le i\le m)$, $f^{-1}(Q_i)\cap X$ is not empty and consists of a finite number of irreducible curves of X which belong to $\{G_1,\dots, G_r\}$. We may assume that $\bigcup_{1\le i\le m}(f^{-1}(Q_i)\cap X)=G_1\cup\dots\cup G_{r'}$, with $r'\le r$. Let $f(G_{r'+1}\cup\dots\cup G_r)=\{Q_{m+1},\dots, Q_{s'}\}$. Then $s'=s+1$ since U is obtained from $\mathbf{P}^1$ deleting the points $u=\alpha_1,\dots, u=\alpha_s$ and the point of infinity $u=\infty$, and $s'\le r$ since all irreducible curves of $X-f^{-1}(U)\cap X$ are sent onto the points $Q_1,\dots, Q_{s'}$ by f. However, this is absurd since $r=s$. Therefore $f(X)$ is an affine open set of $\mathbf{P}^1$.

Let $f(X)=\mathrm{Spec}(A_0)$. Then A_0 is a subring of A. Moreover, there is an element a_0 of A_0 such that $U=\mathrm{Spec}(A_0[a_0^{-1}])$, $f^{-1}(U)\cap X=\mathrm{Spec}(A[a_0^{-1}])$ and that $A[a_0^{-1}]=A[a_0^{-1}][t]=$ a polynomial ring over $A_0[a_0^{-1}]$ with $t\in A$. Now define a locally finite iterative higher derivation $D=\{D_0=id., D_1,\dots\}$ by setting $D_i=(1/i!)D_1^i$,

$D_1(b)=0$ for any element b of A_0 and $D_1(t)=a_0^\alpha$ with sufficiently large integer α, (cf. [7], Theorem 2. 9 and its proof, or Appendix). Therefore there is a non-trivial G_a-action on X. We have thus proved the implication (iii)$'\Longrightarrow$(iii) in Theorem 2.

Conversely, assume the condition (iii). Let $\sigma: G_a\times X\longrightarrow X$ be the given G_a-action on X. Let $\Phi=(\sigma, p_2): G_a\times X\longrightarrow X\times X$, p_2 being the projection of $G_a\times X$ to X. Let $\Gamma=\Phi(G_a\times X)$ and let $\bar{\Gamma}$ be the closure of Γ in $X\times X$. We know by ([3], Theorems 2. 1 and 2. 3) that there exists a G_a-stable open set $U(\neq\phi)$ of X such that there exists a quotient variety Y (in the sense of [3]) of U by the induced action of G_a. Then since the projection $p: U\longrightarrow Y$ is faithfully flat and U is rational, Y is isomorphic to the affine line deleted a finitely many points, (if $Y=\mathbf{P}^1$, replace U by $U-p^{-1}$(a point)). Then U is a G_a-homogeneous space over Y (cf. [5]). Therefore U/Y has a section T' (cf. Théorème 4. 13, *ibid.*). Let T be the closure of T' in X. Then T meets (transversally) with a general G_a-orbit at only one point. Let $\tilde{F}=(X\times T)\cap\bar{\Gamma}$. Then $\tilde{F}$ gives rise to a required algebraic system F on X satisfying the condition (iii), shrinking T to a smaller open set of T if necessary. This completes the proof of Theorem 2.

4. Proof of Theorem3

We shall start with a less restrictive situation and add the conditions of Theorem 3 step by step.

Let k be an algebraically closed field of characteristic zero and let X be an affine non-singular surface defined by an affine k-domain A such that A is a unique factorization domain and $A^*=k^*$. Assume that there exists a maximal ideal $\mathfrak{m}$ of A which is generated by two elements: $\mathfrak{m}=a_1A+a_2A$ with $a_1, a_2\in A$. Let C_1 and C_2 be curves defined by a_1 and a_2 respectively. We may assume without loss of generality that C_1 and C_2 are irreducible. Let v be the point of X corresponding to $\mathfrak{m}$. Then $C_1\cap C_2=\{v\}$, C_1 and C_2

intersect transversally at v, and v is a non-singular point on C_1 and C_2.

Lemma 6. *Under the above situation assume moreover that C_1 is non-singular and has only one place at infinity. Then C_1 is rational. For any element α of k, denote by C_2^α the curve on X defined by $a_2-\alpha$. Then for almost all α of k, C_2^α is irreducible, $C_1\cap C_2^\alpha=\{v_\alpha\}$ and C_1 and C_2^α intersect transversally at v_α.*

Proof. Put $d=a_2$(modulo a_1A). Then d is a regular function on C_1. Let $\bar{C}_1$ be a non-singular irreducible complete curve containing C_1 and let $P_\infty=\bar{C}_1-C_1$. Denote by w the normalized discrete valuation corresponding to P_∞. Then $(d)=v+w(d)P_\infty$. Hence $w(d)=-1$. For any element α of k, $w(d-\alpha)=w(d(1-\alpha d^{-1}))=w(d)=-1$. Hence $(d-\alpha)=v_\alpha-P_\infty$, where $C_1\cap C_2^\alpha=\{v_\alpha\}$. C_1 and C_2^α intersect transversally at v_α. Since $(d)=v-P_\infty$, C_1 must be rational.

q. e. d.

Lemma 7. *Let A be an affine k-domain and let a be an element of $A-k$. Assume the following conditions:*

(1) *A is a unique factorization domain.*

(2) *For any $\alpha\in k$, $a-\alpha$ is a prime element of A.*

(3) *$A^*=k^*$.*

Let $S=k[a]-0$ and let $A'=S^{-1}A$. Then we have:

(i) *A' is a unique factorization domain.*

(ii) *$A'^*=K^*$ where $K=k(a)$.*

(iii) *The quotient field $Q(A')$ of A' is a regeular extension of K. Therefore A' defines an affine variety defined over K with dimension one less than the dimension of the variety defined by A over k.*

Proof. The assertion (i) is well-known. If $A'^*\neq K^*$, there exist elements x and y of $A-k[a]$ such that $xy=\varphi(a)\neq 0$, $\in k[a]$.

Since A is a unique factorization domain and $a-\alpha$ is a prime element of A for all α of k, x and $y \in k[a]$. This is a contradiction, and the assertion (ii) is proved. As for the assertion (iii), we have only to show that K is algebraically closed in $Q(A')$ since $\mathrm{char}(k)=0$. Assume that f/g is algebraic over K, f and g being elements of A such that $(f, g)=1$. Then there exist $\varphi_0, \ldots, \varphi_n$ of $k[a]$ such that the greatest common divisor of $\varphi_0, \ldots, \varphi_n$ is 1 and that

$$\varphi_0(f/g)^n+\varphi_1(f/g)^{n-1}+\ldots+\varphi_n=0.$$

Then it is easy to see that f and g divide φ_n and φ_0 respectively. Hence f and $g \in k[a]$. Thus $f/g \in K$. q. e. d.

Lemma 8. *Besides the assumptions of Lemma* 6, *assume the following additional conditions :*

(1) *C_2 has only one place at infinity.*

(2) *There exists a non-singular complete surface V containing X, on which the closure $\bar{C}_2$ of C_2 is non-singular and $(a_2)_0=\bar{C}_2$.*

(3) *For any element α of k, $a_2-\alpha$ is a prime element of A. Then for almost all element α of k, C_2^α is rational and has only one place at infinity.*

Proof. Our proof consists of several steps.

(I) For a general element $\alpha \in k$, the principal divisor $(a_2-\alpha)$ on V is of the form ; $(a_2-\alpha)=\bar{C}_2^\alpha+D-$ (the polar divisor), where $D \geq 0$ is contained in $V-X$ and independent of α. Specializing α to 0, we have : $(a_2)=\bar{C}_2-$ (the polar divisor) by the last condition of the assumption (2). Hence $D=0$. It is then easy to show that there exists a linear pencil L of divisors on V such that $\bar{C}_2$ is a member of L and the closure $\bar{C}_2^\alpha$ of C_2^α is a member of L for almost all α of k. If L has a base point (which is the unique base point), by repeating the blowings-up with center at the base point and its appropriate infinitely near points, we have a non-singular complete surface $\tilde{V}$ containing X and a linear pencil $\tilde{L}$ of divisors on $\tilde{V}$, which is obtained from the total transform of L deleting the fixed components, such that:

(i) $\tilde{L}$ has no base points.

(ii) The closure $\tilde{C}_2$ of C_2 and the closure $\tilde{C}_2^{\alpha}$ of C_2^{α} (for almost all α of k) in $\tilde{V}$ are members of $\tilde{L}$.

(iii) The closure $\tilde{C}_1$ of C_1 does not pass through the point $\tilde{C}_2 - C_2$.

Let $p: \tilde{V} \longrightarrow P$ be the morphism defined by $\tilde{L}$, and let $y_0 = p(\tilde{C}_2)$. Then there exists an open neighbourhood Y of y_0 in $\mathbf{P}^1$ such that $\tilde{C}_1$ intersects transversally with each fibre $p^{-1}(y)$ for all $y \in Y$. Then by [2, p. 3], $\tilde{C}_1 \cap p^{-1}(Y)$ is p-ample, and $p: W = p^{-1}(Y) \longrightarrow Y$ is flat. Restricting Y to a smaller open neighbourhood of y_0 if necessary, we may assume that $p: W \longrightarrow Y$ is smooth. The curve $\tilde{C}_1 \cap W$ gives rise to a section s of p.

(II) Since $p: W \longrightarrow Y$ is a smooth projective morphism whose fibres are geometrically integral curves, the Picard scheme $\mathrm{Pic}_{W/Y}$ is representable and $\mathrm{Pic}^0_{W/Y}$ is a smooth group scheme over Y. Moreover, for any Y-scheme T, $\mathrm{Pic}(W \underset{Y}{\times} T) = \mathrm{Pic}_{W/Y}(T) \times \mathrm{Pic}(T)$ (a direct product) since p has a section s. Therefore $\mathrm{Pic}_{W/Y} \underset{Y}{\times} T = \mathrm{Pic}_{W \underset{Y}{\times} T/T}$ for any Y-scheme T. In particular $(\mathrm{Pic}_{W/Y})_{y_0} \cong \mathrm{Pic}_{C_2/k} \cong \mathbf{Z}$. Since $\mathrm{Pic}^0_{W/Y}$ is smooth and connected, $\mathrm{Pic}^0_{W/Y} = 0$. Let K be the function field of Y and let $W_K = W \underset{Y}{\times} \mathrm{Spec}(K)$. Then $\mathrm{Pic}^0_{W_K/K} = 0$. This implies that the arithmetic genus of W_K is zero.

(III) W_K is in fact the non-singular complete model of the affine curve C defined by $A' = S^{-1}A$ over $K = k(a_2)$, where $S = K[a_2] - 0$ (cf. Lemma 2). C has a K-rational point P which is provided by the sectional curve $\tilde{C}_1 \cap W$. Since the arithmetic genus of W_K is zero and W_K has a K-rational point P, W_K is K-isomorphic to $\mathbf{P}^1$.

(IV) Since C ($\subset W_K$) is defined over K, $W_K - C$ consists of a finite number of K-rational prime cycles. Introduce a homogenneous coordinate (x_0, x_1) in $\mathbf{P}^1_K$ such that $P = (1, 0)$ and let $x = x_0/x_1$. Then there exist irreducible polynomials $f_1, \ldots, f_n$ of $K[x]$

such that the affine ring of $C-P$ is $K[x, f_1^{-1},\dots, f_n^{-1}]$. Then $(K[x, f_1^{-1},\dots, f_n^{-1}])^* \cong K^* \times \mathbf{Z}^n$. However since the affine ring A' of C is a unique factorization domain and $A'^* = K^*$, we must have $n=1$. This means that $W_K - C$ consists of only one K-rational prime cycle. On the other hand, P is linearly equivalent to the K-rational prime cycle $W_K - C$ with an appropriate multiplicity. This implies that $W_K - C$ consists of only one K-rational point. Hence C is K-isomorphic to the affine line $\mathbf{A}^1$. This implies that for almost all α of k, the curve C_2^α defined by $a_2 - \alpha$ is isomorphic to $\mathbf{A}^1$ and that C_2^α has therefore only one place at infinity. This completes the proof of Lemma 8.

Lemma 8 says that X is rational and has a rational pencil of curves $\{C_2^\alpha ; \alpha \in k\}$ satisfying the condition (iii)′ of Theorem 2. Thus we have proved our Theorem 3, applying Theorem 2. Finally we shall remark that if X is isomorphic to the affine plane all conditions of our Theorem 3 are satisfied.

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