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A formulation of the Riemann-Roch theorem in terms of differentials with singularities at the ideal boundary

By

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In spite of a plenty of fruitful investigations on the Riemann-Roch theorem and Abel's theorem for open Riemann surfaces, important problems seem to be still remained. As for the Riemann-Roch theorem we discussed some of them in [18], and in the present paper we shall treat it in more general setting. Concerning other results based on [18], see [10] and [19] for example.

The Riemann-Roch theorem on open Riemann surfaces given by Kusunoki [7], [9] is described in terms of "canonical semiexact differentials" introduced by himself in [7]. This concept and his method stimulated the works of Mizumoto [11] and Yoshida [20]. One of the most important devices in these papers and [18] is the evaluation of curvelinear integrals of type $\int f\omega$ with a function f and a differential ω, namely it is shown that the imaginary part of that integral along the boundary of regular subregion tends to zero as the subregion exhausts the surface. This is the first key. The second key is a skillful use of the bilinear relations for Abelian differentials of the second and third kinds, which is essentially a combination of the residue theorem and the Green's formula for multi-valued functions (cf. [6], [7] and [9]). In [18] we made use of these keys effectively with the aid of a new notion, "dual boundary behaviors".

In this paper, we shall regard the boundary integral $\int f\omega$ mentioned above as a kind of residues at (some subset of) the ideal boundary, if it would be well-defined. Routine cases, residues at isolated interior points, are trivially included. For those points can be considered as the ideal boundary of the punctured surface (which is obtained by deleting those points from the original surface). We often encounter such an idea in the proof of the existence theorems ([2], [5] and [17]). In the following it will be seen that such observation would play an essential role not only in the proof of existence theorems but also in the formulation of our theorems.

Our formulation brings us some byproducts. Namely, a) we can more naturally reduce the theorem (on open surfaces) to the classical cases (compare with [12]); b) functions (resp. differentials) in the final formulae need to be square integrable only on some compacta, and this is a superfluous requirement; c) more general singularities than poles may be granted; d) the concept of divisor is fairly generalized; e) semiexactness of differentials is, to some extent, weakened and so on. In order to give the proper examples with such new characters it seems necessary much more detailed studies on the behaviors of analytic functions in the neighborhood of large closed sets like simple closed arcs, although the author can not yet accomplish this.

1. Preliminaries. Let R be an arbitrary open Riemann surface and R^* its Kerékjártó-Stoïlow compactification. We denote by ∂R the Kerékjártó-Stoïlow ideal boundary of R, $\partial R = R^* - R$. Let g be the genus of R, which may be infinity. We fix, once for all, a canonical regular exhaustion $\{\Omega\}$ of R. We denote by $\Xi = \Xi(R) = \{A_j, B_j\}_{j=1}^{g}$ the canonical homology basis of R modulo the ideal boundary ∂R, which is associated with the above exhaustion. Let $\mathscr{E}(R)$ be the collection of all U such that $R-\bar{U}$ is a canonical regular region.

It is well known that all the square integrable real (resp. complex) differentials on R form a real (resp. complex) Hilbert space $\Gamma=\Gamma(R)$ (resp. $\tilde{\Lambda}=\tilde{\Lambda}(R)$) with respect to the usual inner product:

$$(\omega_1, \omega_2)=\iint_R (a_1\bar{a}_2+b_1\bar{b}_2)\ dxdy,$$

where $z=x+iy$ stands for a generic local parameter and $\omega_j=a_j(x, y)dx+b_j(x, y)dy$ is the representation of $\omega_j\in\Gamma(R)$ (resp. $\tilde{\Lambda}(R)$) in z, $j=1, 2$. If we introduce into $\tilde{\Lambda}(R)$ a new inner product defined by

$$<\omega_1, \omega_2>=\mathrm{Re}(\omega_1, \omega_2),$$

we can consider $\tilde{\Lambda}(R)$ as a real Hilbert space, which we denote by $\Lambda(R)$. Note that the norm in $\Lambda(R)$ is the same as in $\tilde{\Lambda}(R)$, but the orthogonality relations in $\Lambda(R)$ are finer than those in $\tilde{\Lambda}(R)$. It is easily seen that

$$\Lambda(R)=\Gamma(R)\oplus i\Gamma(R)\ \text{(direct sum)},\ i^2=-1.$$

In this paper we shall use the standard notations ω^* and $\bar{\omega}$ to represent the conjugate differential and the complex conjugate of ω respectively ($*$: the Hodge star operator).

Some important subspaces of $\Lambda=\Lambda(R)$ are: Λ^1; Λ^1_e, Λ^1_{e0}, Λ_e, Λ_{e0}; Λ^1_c, Λ^1_{c0}, Λ_c, Λ_{c0}; Λ_h, Λ_{hse} etc.. The strict definitions of these and their relations to the various subclasses of $\Gamma(R)$ (and interrelations between themselves also) are found in [18]. Above all the following are fundamental:

$$\Lambda_h=\Lambda_c\cap\Lambda_c^* \qquad \text{(Weyl's lemma)};$$

$$\Lambda=\Lambda_h\oplus\Lambda_{e0}\oplus\Lambda_{e0}^* \qquad \text{(de Rham's decomposition)};$$

$$\Lambda_c=\Lambda_h\oplus\Lambda_{e0} \qquad \text{(Dirichlet principle)}.$$

The fundamental tool in our theory is the following

Lemma 1. *Let Ω be a canonical regular region of R. Let φ_1, φ_2 be closed C^1-differentials on $\bar{\Omega}$ and further we assume that φ_1 be semiexact. Then*

$$(\varphi_1, \varphi_2)_\Omega = -\int_{\partial\Omega}\left(\int\varphi_1\right)\bar{\varphi}_2 + \sum_\Omega\left(\int_{A_j}\varphi_1\int_{B_j}\bar{\varphi}_2 - \int_{B_j}\varphi_1\int_{A_j}\bar{\varphi}_2\right).$$

Here $\int\varphi_1$ is a primitive function[1] *of φ_1 on the planar surface Ω_0 which is obtained by cutting Ω along all A_j- and B_j-cycles, and $\sum_\Omega$ denotes the sum for only those A_j and B_j belonging to $\Xi(\Omega) = \Xi(R) \cap \Omega$.*[2]

Proof. Omitted. See, for example, [1], [18] etc..

Consider a (not necessarily closed) subspace Λ_0 of Λ_{hse} for which the following conditions are fulfilled:

a) there exists a *closed* subspace Λ_1 of Λ_h such that $\Lambda_0 \supset \Lambda_1 + i\Lambda_1^{\perp *}$ (vector sum), where $\Lambda_1^\perp$ means the orthogonal complement of Λ_1 in Λ_h;

b) for all $\lambda \in \Lambda_0$, $<\lambda, i\lambda^*> = 0$;

c) there is a family $\mathscr{L}$ of real one-dimensional subspaces L_j of $\mathbf{C}$ such that

$$\int_{\substack{A_j \\ B_j}} \lambda \in L_j \qquad (j = 1, 2, \ldots, g)$$

for all $\lambda \in \Lambda_0$.

A space Λ_0 with these properties is denoted by $\Lambda_0 = \Lambda_0(\Lambda_1, \mathscr{L})$ and is called a *behavior space* associated with Λ_1 and $\mathscr{L}$.

Let $\Lambda_0' = \Lambda_0(\Lambda_1', \mathscr{L}')$ and $\Lambda_0'' = \Lambda_0(\Lambda_1'', \mathscr{L}'')$ be two behavior spaces associated with Λ_1', $\mathscr{L}' = \{L_j'\}$; Λ_1'', $\mathscr{L}'' = \{L_j''\}$ respectively. We say

1) This is determined up to an additive constant on account of the semiexactness of φ_1; and hence together with the closedness of φ_2 we know that the first term in the right is well-defined.

2) Strictly speaking, this expression is not legitimate; by definition, $\Xi(\Omega)$ stands for the collection of Aj, Bj which are contained in Ω. (They form the canonical homology basis of Ω mod. $\partial\Omega$.)

that Λ_0' and Λ_0'' are *dual behavior spaces* to each other with respect to a real one-dimensional subspace L_0 [3] of **C**, if and only if

1°) $(\lambda', \overline{\lambda''^*}) \equiv 0 \mod L_0$ [4] for all $\lambda' \in \Lambda_0'$ and all $\lambda'' \in \Lambda_0''$;

2°) $L_j' \circ L_j'' = L_0$ $(j=1,2,\ldots,g)$. Here $L_j' \circ L_j''$ is, by definition, the real vector space spanned by $\zeta_j'\zeta_j''$ with $\zeta_j' \in L_j'$ and $\zeta_j'' \in L_j''$.

2. $(P)\Lambda_0$-divisors, existence theorems.

Let $\Lambda_0 = \Lambda_0(\Lambda_1, \mathscr{L})$ be a behavior space, $\mathscr{L} = \{L_j\}_{j=1}^g$. An analytic differential φ, defined on a neighborhood of ∂R, is called to have *Λ_0-behavior* ([18]) if there are $\lambda_0 \in \Lambda_0$, $\lambda_{e0} \in \Lambda_{e0} \cap \Lambda^1$ and $U \in \mathscr{E}(R)$ such that

$$\varphi = \lambda_0 + \lambda_{e0} \qquad \text{on } U.$$

In connection with this definition, cf. Ahlfors' distinguished differentials ([3], [4], [5]), Kusunoki's canonical semiexact differentials ([7], [8], [9]) and Yoshida's Γ_x-behaviors ([20]). See also Mizumoto [11] and Royden [15].

Let P be a (regular) partition of the ideal boundary of R. Let $\mathscr{A}^P$ be the family of all analytic (P)semiexact [5] differential φ whose domain of definition is some $U \in \mathscr{E}(R)$ [6], and $\mathscr{A}^P_{\mathscr{L}}$ be the subfamily of $\mathscr{A}^P$ which consists of all $\varphi \in \mathscr{A}^P$ such that

$$\int_{\substack{A_j\\B_j}} \varphi \in L_j \qquad \text{for each } A_j,\ B_j \in \Xi(U),\ U = \operatorname{dom}\varphi.$$

For $\Lambda_0 = \Lambda_0(\Lambda_1, \mathscr{L})$ we set $\mathscr{A}^P_{\Lambda_0} = \{\varphi \in \mathscr{A}^P_{\mathscr{L}} \mid \varphi \text{ has } \Lambda_0\text{-behavior}\}$. $\mathscr{A}^P_{\mathscr{L}}$

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3) We shall restrict ourselves to the simplest case that $L_0 = \mathbf{R}$ in the present paper. (In this case the condition 1°) is equivalent to $<\lambda', i\overline{\lambda''^*}> = 0$.) The general cases will be analogously treated.

4) We use the notation $\alpha \equiv 0 \mod L_0$ to express that a complex number α belongs to L_0. For example, the condition c) is rewritten as follows: $\int_{\substack{A_j\\B_j}} \lambda \equiv 0 \mod L_j$ $(j=1, 2,\ldots, g)$.

5) For each cycle d which is dividing with respect to P and is contained in U, $\int_d \varphi = 0$.

6) As a matter of fact, it is sufficient to assume that the domain of definition of φ contains some $U \in \mathscr{E}(R)$.

and $\mathscr{A}^P_{\Lambda 0}$ are both real vector spaces. We call elements of the quotient vector space

$$V^P_{\Lambda 0} = \mathscr{A}^P_{\mathscr{L}} / \mathscr{A}^P_{\Lambda 0}$$

$(P)\Lambda_0$-singularities, and subspaces of $V^P_{\Lambda 0}$ are called *$(P)\Lambda_0$-divisors*. As for the connection with the classical terminologies, see §4, (III).

Let $V = V(P, \Lambda_0)\,(\subset V^P_{\Lambda 0})$ be a $(P)\Lambda_0$-divisor. A regular analytic differential λ on R is said to be a *multiple* of V if there exist $\sigma \in V$, $\lambda_0 \in \Lambda_0$ and $\lambda_{e0} \in \Lambda_{e0} \cap \Lambda^1$ such that

$$\lambda = \sigma + \lambda_0 + \lambda_{e0}$$

on some $U \in \mathscr{E}(R)$. Here and also in the sequel, we sometimes continue to denote by σ a representative of $\sigma \in V$ (cf. Propositions 1, 2; Lemma 2 and its corollary). We also say that λ has a $(P)\Lambda_0$-singularity σ. We set

$$\mathscr{D}(V) = \left\{\lambda \,\middle|\, \begin{array}{l}\lambda \text{ is a regular analytic differential} \\ \text{on } R \text{ and is a multiple of } V.\end{array}\right\}.$$

The following theorem was proved in [18]:

Theorem 1′. *If φ is a regular analytic differential on R which has Λ_0-behavior, then φ is identically zero provided that*

$$\int_{\substack{A_j \\ B_j}} \varphi \in L_j, \qquad j = 1, 2, \ldots, g.$$

Under our new terminologies, we have the corresponding

Theorem 1. *If $\varphi \in \mathscr{D}(V)$ is $(P)\Lambda_0$-singularity free, then φ is identically zero provided that*

$$\int_{\substack{A_j \\ B_j}} \varphi \in L_j, \qquad j = 1, 2, \ldots, g.$$

Remarks. The condition $\int_{\substack{A_j \\ B_j}} \varphi \in L_j$ is meaningful only for those A_j, B_j which lie in some compact subset of R, since it is automatically satisfied for other elements of $\Xi(R)$. Furthermore, as in [18], this condition can be weakened; a finite number of L_j's may be replaced by other L_j''s.

The partition P gives rise to the parts β_ν of ∂R ($\nu \in I$: the set of indices). Suppose that $\beta \in \{\beta_\nu\}_{\nu \in I}$. A $(P)\Lambda_0$-singularity σ is said to be zero outside β if we can find a representative ds ($\in \mathscr{A}^P_{\mathscr{L}}$) of σ such that $ds \equiv 0$ on a neighborhood of $\partial R - \beta$. We shall call such ds a *nice* representative of σ. A $(P)\Lambda_0$-divisor V is said to be zero outside β if all the elements of V are zero outside β in the above sense. (We would be able to define the "support" of a $(P)\Lambda_0$-divisor rigorously, if we wish. However, for our present aim it suffices to use the above terminology.) We denote by $V(P, \Lambda_0; \beta, m)$ a $(P)\Lambda_0$-divisor which is zero outside β and is of dimension m (as a real vector space), $0 \leq m \leq \infty$. We agree to $m \neq 0$ whenever $\beta \neq \phi$.

In the remaining part of this section, we shall show the existence of the elementary differentials. Let Λ_0 be a behavior space and $V = V(P, \Lambda_0; \beta, m)$ a $(P)\Lambda_0$-divisor. First, we claim the following theorem without proof (cf. [18], Th. 2).

Theorem 2. *For any complex numbers ξ_j, η_j such that $\xi_j \not\equiv 0$, $\eta_j \not\equiv 0$* mod L_j, *we can find holomorphic differentials $\phi_{\xi_j}(A_j)$, $\phi_{\eta_j}(B_j)$ with the following properties:*

i) $\phi_{\xi_j}(A_j)$, $\phi_{\eta_j}(B_j)$ *are multiples of* V,

$$\left.\begin{array}{ll} ii) & \int_{A_k} \phi_{\xi_j}(A_j) \equiv \xi_j \cdot (A_j \times A_k) = 0 \\[2ex] & \int_{B_k} \phi_{\xi_j}(A_j) \equiv \xi_j \cdot (A_j \times B_k) = \xi_j \delta_{jk} \\[2ex] ii)' & \int_{A_k} \phi_{\eta_j}(B_j) \equiv \eta_j \cdot (B_j \times A_k) = -\eta_j \delta_{kj} \\[2ex] & \int_{B_k} \phi_{\eta_j}(B_j) \equiv \eta_j \cdot (B_j \times B_k) = 0 \end{array}\right\} \text{mod } L_k.$$

Here δ_{jk} is the Kronecker delta. These differentials are uniquely determined. Not only $\phi_{\xi_j}(A_j)$ and $\phi_{\eta_j}(B_j)$ are square integrable but also they have Λ_0-behaviors, i.e., they are $(P)\Lambda_0$-singularity free.

As for $(P)\Lambda_0$-singularities, we have

Theorem 3. *Let $\sigma \in V = V(P, \Lambda_0; \beta, m)$. Then there is a regular analytic differential φ on R whose $(P)\Lambda_0$-singularity is exactly σ. If we normalize the periods of φ so that*

$$\int_{\substack{A_j \\ B_j}} \varphi \equiv 0 \bmod L_j, \qquad j = 1, 2, \ldots, g,$$

φ is uniquely determined.

Proof. (Although our proof is only a modification of standard ones, we shall reproduce it for completeness.)

By the very definition of σ, it is representable by an analytic differential defined near ∂R, which we denote by σ again. We may assume that the domain of definition of σ contains the closure of some $U \in \mathscr{E}(R)$. Because of the (P)semiexactness of σ, we can extend $\sigma|_U$ to $\hat{\sigma}$, a closed C^1-differetial on R, so that $\mathrm{Supp.}\,\hat{\sigma} \cap \overline{R-U}$ is compact (cf. [16], Lemma 2). Since σ is analytic on U, $\sigma - i\sigma^* = 0$ there. Therefore $\hat{\sigma} - i\hat{\sigma}^* \in \Lambda^1(R)$.

Now by use of the de Rham's decomposition and the orthogonal decomposition $\Lambda_h = \Lambda_1 \oplus \Lambda_1^\perp$, there are differentials $\lambda_1 \in \Lambda_1$, $\lambda_1^\perp \in \Lambda_1^\perp$; $\lambda'_{e0}, \lambda''_{e0} \in \Lambda_{e0}$, for which

$$\hat{\sigma} - i\hat{\sigma}^* = \lambda_1 + \lambda_1^\perp + \lambda'_{e0} + \lambda''^*_{e0}$$

holds. Arranging this, we obtain a closed and coclosed and hence harmonic differential

$$\omega = \hat{\sigma} - \lambda_1 - \lambda'_{e0} = i\hat{\sigma}^* + \lambda_1^\perp + \lambda''^*_{e0}.$$

From this expression, the smoothness of λ'_{e0} and λ''_{e0} is deduced. It is easily seen that $\varphi = \frac{1}{2}(\omega + i\omega^*)$ is a requested differential. The possibility of periods-normalization and uniqueness discussion are trivial, q. e. d.

Remark. It should be noted that $|||\varphi-\sigma|||_W<+\infty$ for some $W\in\mathscr{E}(R)$, although φ itself is possibly not square integrable (cf. [5], p. 299).

3. A duality theorem.

Hereafter we fix two (regular) partitions of ∂R; the one is the canonical partition Q and the other is P: $\partial R=\alpha\cup\beta\cup\gamma$ such that $\beta\cup\gamma\neq\phi$.

Let $\Lambda_0'=\Lambda_0(\Lambda_1', \mathscr{L}')$ and $\Lambda_0''=\Lambda_0(\Lambda_1'', \mathscr{L}'')$ be two behavior spaces which are dual to each other with respect to $\mathbf{R}$. Let $\mathscr{L}'=\{L_j'\}_{j=1}^{g}$ and $\mathscr{L}''=\{L_j''\}_{j=1}^{g}$. Suppose that a $(Q)\Lambda_0'$ divisor $V_Q=V(Q, \Lambda_0'; \beta, m)$ and a $(P)\Lambda_0''$-divisor $V_P=V(P, \Lambda_0''; \gamma, n)$ are given.

We begin with the definition of important families of functions and differentials. The first one is $\mathscr{D}(V_P)$ whose general definition appeared in the preceding section. The second is

$$\mathscr{M}(V_Q)=\left\{f=\int\varphi\,\middle|\,\varphi\in\mathscr{D}(V_Q)\,;\ \int_{\substack{A_j\\B_j}}\varphi\equiv 0 \bmod L_j',\ 1\leq j\leq g.\right\}.$$

Elements of $\mathscr{M}(V_Q)$ are, in general, multi-valued analytic functions. In case that $\gamma\neq\phi$, two elements f_1, f_2 of $\mathscr{M}(V_Q)$ are identified when their difference is a (complex) constant (cf. [8], for example.).

To obtain a well-defined bilinear mapping from $\mathscr{M}(V_Q)\times\mathscr{D}(V_P)$ into $\mathbf{R}$, we need some propositions. Before stating them we note that P induces the partition of each $\partial\Omega$, P_Ω: $\partial\Omega=\alpha_\Omega\cup\beta_\Omega\cup\gamma_\Omega$, where α_Ω, β_Ω and γ_Ω are dividing cycles homologous to α, β and γ respectively.

Proposition 1. *Let $\sigma=ds\in\mathscr{A}^0_{\mathscr{L}'}$, and $\omega\in\mathscr{D}(V_P)$. Then*

$$\lim_{\Omega\to R}\operatorname{Im}\int_{\beta_\Omega}s\omega$$

exists and is finite.

Proof. Let $\{G\}$ be a canonical regular exhaustion towards β which is induced by $\{\Omega\}$. That is, $G=\Omega\cup\bar{U}$ for some neighborhood U of $\alpha\cup\gamma$. Let Ω_1, $\Omega_2(\supset\Omega_1)$ be sufficiently large canonical regular regions. We set $\beta_1=\partial G_1$, $\beta_2=\partial G_2$ and $G=G_2-G_1$. Then the (relative) boundary of G is exactly $\beta_2-\beta_1$.

Applying Lemma 1 to $G_1\cap\Omega_2$ and Ω_2 (or directly to G in a modified form), we have

$$\int_{\beta_2-\beta_1} s\omega=-(\sigma,\ \bar{\omega}^*)_G+\sum_G\left(\int_{A_j}\sigma\int_{B_j}\omega-\int_{B_j}\sigma\int_{A_j}\omega\right).$$

Now $(\sigma,\ \bar{\omega}^*)_G=-i(\sigma,\ \bar{\omega})_G=0$, for σ and ω are both analytic on G. On the other hand, by the hypothesis that $\sigma\in\mathscr{A}^{0}_{\mathscr{L}'}$ and ω be a multiple of V_P, it is easily seen that

$$\int_{\substack{A_j\\ B_j}}\sigma\in L_j', \qquad \int_{\substack{A_j\\ B_j}}\omega\in L_j'' \qquad \text{for } A_j,\ B_j\in\Xi(G)$$

and hence (in virtue of the condition 2°) of dual behaviors)

$$\operatorname{Im}\sum_G\left(\int_{A_j}\sigma\int_{B_j}\omega-\int_{B_j}\sigma\int_{A_j}\omega\right)=0.$$

Therefore $\operatorname{Im}\int_{\beta_\Omega}s\omega$ is independent of the choice of Ω provided that Ω is sufficiently large. This completes the proof.

Similarly we can prove

Proposition 2. *If $f\in\mathscr{M}(V_Q)$ and $\tau\in\mathscr{A}^{P}_{\mathscr{L}''}$, then*

$$\lim_{\Omega\to R}\operatorname{Im}\int_{\tau_\Omega}f\tau$$

exists and is finite.

Further we have the following

Lemma 2. *Let $f\in\mathscr{M}(V_Q)$ and $\omega\in\mathscr{D}(V_P)$. Suppose that f has*

$(Q)\Lambda_0'$-singularity σ [7] *and ω has $(P)\Lambda_0''$-singularity τ, respectively. Then*

$$\lim_{\Omega\to R} \operatorname{Im}\int_{\partial\Omega} f\omega = \lim_{\Omega\to R} \operatorname{Im}\int_{\beta_\Omega} s_0\omega + \lim_{\Omega\to R} \operatorname{Im}\int_{\gamma_\Omega} f\tau_0,$$

for any nice representatives ds_0 of σ and τ_0 of τ.

Proof. We note, first of all, that there exist some $U\in\mathscr{E}(R)$; $\lambda_0'\in\Lambda_0'$, $\lambda_0''\in\Lambda_0''$; λ_{e0}', $\lambda_{e0}''\in\Lambda_{e0}\cap\Lambda^1$ such that

$$\left.\begin{aligned} df &= ds_0 + \lambda_0' + \lambda_{e0}' \\ \omega &= \tau_0 + \lambda_0'' + \lambda_{e0}'' \end{aligned}\right\} \text{ on } U$$

We take a canonical regular region Ω whose boundary $\partial\Omega$ is contained in U. Then, since ds_0 and τ_0 are zero outside β and γ respectively,

$$\int_{\partial\Omega} f\omega = \int_{\alpha_\Omega+\beta_\Omega+\gamma_\Omega} \left(\int (ds_0+\lambda_0'+\lambda_{e0}')\right)(\tau_0+\lambda_0''+\lambda_{e0}'')$$

$$= \int_{\alpha_\Omega+\beta_\Omega+\gamma_\Omega} \left(\int (\lambda_0'+\lambda_{e0}')\right)(\lambda_0''+\lambda_{e0}'') + \int_{\beta_\Omega} s_0(\lambda_0''+\lambda_{e0}'')$$

$$+ \int_{\gamma_\Omega} \left(\int (\lambda_0'+\lambda_{e0}')\right)\tau_0,$$

applying Lemma 1 to the first term,

$$= -(\lambda_0', \overline{\lambda_0''^*})_\Omega + \varepsilon_\Omega + \sum_\Omega \left(\int_{A_j}\lambda_0' \int_{B_j}\lambda_0'' - \int_{B_j}\lambda_0' \int_{A_j}\lambda_0''\right)$$

$$+ \int_{\beta_\Omega} s_0\omega + \int_{\gamma_\Omega} f\tau_0,$$

where $\varepsilon_\Omega = -[(\lambda_0', \overline{\lambda_{e0}''^*})_\Omega + (\lambda_{e0}', \overline{\lambda_0''^*})_\Omega + (\lambda_{e0}', \overline{\lambda_{e0}''^*})_\Omega]$ tends to zero as $\Omega\to R$.

On the other hand, we have

$$\lim_{\Omega\to R} \operatorname{Im}(\lambda_0', \overline{\lambda_0''^*})_\Omega = 0 \text{ and } \operatorname{Im}\sum_R \left(\int_{A_j}\lambda_0' \int_{B_j}\lambda_0'' - \int_{B_j}\lambda_0' \int_{A_j}\lambda_0''\right) = 0,$$

for Λ_0' and Λ_0'' are dual to each other *w. r. t.* **R.**

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7) This means that df has $(Q)\Lambda_0'$-singularity σ (although this terminology is not in accordance with the conventional one).

Hence

$$\lim_{\Omega\to R} \operatorname{Im}\int_{\partial\Omega} f\omega = \lim_{\Omega\to R} \operatorname{Im}\int_{\beta_\Omega} s_0\omega + \lim_{\Omega\to R} \operatorname{Im}\int_{\tau_\Omega} f\tau_0,$$

which is to be proved.

Corollary. *Let $f\in\mathcal{M}(V_Q)$ have $(Q)\Lambda_0'$-singularity σ and $\omega\in\mathcal{D}(V_P)$ have $(P)\Lambda_0''$-singularity τ, respectively. Then the quantities*

$$\underset{\beta}{\mathfrak{Res}}\; s\omega = -\lim_{\Omega\to R} \operatorname{Re}\frac{1}{2\pi i}\int_{\beta_\Omega} s_0\omega = -\frac{1}{2\pi}\lim_{\Omega\to R} \operatorname{Im}\int_{\beta_\Omega} s_0\omega$$

and

$$\underset{\tau}{\mathfrak{Res}}\; f\tau = -\lim_{\Omega\to R} \operatorname{Re}\frac{1}{2\pi i}\int_{\tau_\Omega} f\tau_0 = -\frac{1}{2\pi}\lim_{\Omega\to R} \operatorname{Im}\int_{\tau_\Omega} f\tau_0$$

are independent of the choice of nice representatives ds_0 of σ and τ_0 of τ.

Due to this corollary we may adopt

Definition. $\underset{\beta}{\mathfrak{Res}}\; s\omega$ (resp. $\underset{\tau}{\mathfrak{Res}}\; f\tau$) is called the (**R**-) *residuum*[8] of $s\omega$ (resp. $f\tau$), provided that $\sigma=ds$ (resp. τ) is the $(Q)\Lambda_0'$-(resp. $(P)\Lambda_0''$-) singularity of $f\in\mathcal{M}(V_Q)$ (resp. $\omega\in\mathcal{D}(V_P)$). Similarly, we can define, for those f and ω in Lemma 2, $\underset{\alpha}{\mathfrak{Res}}\; f\omega$, $\underset{\beta}{\mathfrak{Res}}\; f\omega$, $\underset{\tau}{\mathfrak{Res}}\; f\omega$ and $\underset{\partial R}{\mathfrak{Res}}\; f\omega$.

With these definitions we have

Lemma 2$'$. *For f and ω in Lemma 2,*

$$\underset{\alpha}{\mathfrak{Res}}\; f\omega + \underset{\beta}{\mathfrak{Res}}\; f\omega + \underset{\tau}{\mathfrak{Res}}\; f\omega = \underset{\beta}{\mathfrak{Res}}\; s\omega + \underset{\tau}{\mathfrak{Res}}\; f\tau.$$

The following lemma is substantially the residue theorem and

8) If we consider two behavior spaces which are dual to one another *w.r.t.* L_0, a general one-dimensional subspace of **C**, we have to define the L_0-residuum. Its strict definition will be self-explanatory.

the lemmata of this type have played essential role in forerunners' works (Kusunoki [6], [7]; Mizumoto [11]; Yoshida [20]. cf. also [18]). However, ours takes a variant form; this originates in our starting-point that all differentials under consideration must be regular on all over R.

Lemma 3. *If* $f\in\mathcal{M}(V_Q)$ *and* $\omega\in\mathcal{D}(V_P)$, *then we have*

$$-2\pi \underset{\partial R}{\mathfrak{Res}}\ f\omega=\mathrm{Im}\sum_{j=1}^{g}\left(\int_{A_j}df\int_{B_j}\omega-\int_{B_j}df\int_{A_j}\omega\right)\quad (a\ \textit{finite sum}).$$

Let $(f,\omega)\in\mathcal{M}(V_Q)\times\mathcal{D}(V_P)$ and $ds=\sigma$ be the $(Q)\Lambda_0'$-singularity of f. We set

$$h(f,\omega)=\underset{\beta}{\mathfrak{Res}}\ s\omega.$$

It is easily seen that h is a well-defined bilinear mapping from $\mathcal{M}(V_Q)\times\mathcal{D}(V_P)$ into $\mathbf{R}$ (cf. Corollary to Lemma 2).

We set

$$\mathcal{S}(V_Q||V_P)=\left\{f\in\mathcal{M}(V_Q)\ \middle|\ \begin{array}{l} f \text{ is single-valued on } R;\\ \underset{\tau}{\mathfrak{Res}}\ f\tau=0,\ {}^{\forall}\tau\in V_P.\end{array}\right\}$$

$$\mathcal{D}(V_P||V_Q)=\left\{\omega\in\mathcal{D}(V_P)\ \middle|\ \underset{\beta}{\mathfrak{Res}}\ s\omega=0,\quad {}^{\forall}ds\in V_Q.\right\}.$$

These are evidently real vector spaces.

Now we are ready to prove

Theorem 4. *Under the finiteness condition*

$$(F)\qquad \dim\left[\mathcal{M}(V_Q)/\mathcal{S}(V_Q||V_P)\right]<+\infty,$$

the duality relation

$$\mathcal{M}(V_Q)/\mathcal{S}(V_Q||V_P)\cong\mathcal{D}(V_P)/\mathcal{D}(V_P||V_Q)$$

holds.

Proof. (Only outline. cf. [7], [18] etc..). Combining Lemma 2′ and Lemma 3, h is rewritten:

$$h(f, \omega) = -\frac{1}{2\pi} \operatorname{Im} \sum_{j=1}^{g} \left(\int_{A_j} df \int_{B_j} \omega - \int_{B_j} df \int_{A_j} \omega\right) - \mathfrak{Res}_{\gamma}\, f\tau.$$

It is not difficult to see that the left-kernel of h is exactly $\mathscr{S}(V_Q||V_P)$ and that the right-kernel is $\mathscr{D}(V_P||V_Q)$. A purely algebraic lemma accomplishes the proof, q. e. d.

4. Reductions to the special cases.

In this section, we shall see that the abstractly formulated Theorem 4 above reduces to the already known results when the surfaces are specialized.

We specialize the theorem step by step.

(I) If m is finite, then not only (F) is trivially satisfied but also $\mathscr{M}$ and $\mathscr{S}$ themselves are finite dimensional. Hence we have the equality

$$\dim \mathscr{M}(V_Q) - \dim \mathscr{S}(V_Q||V_P) = \dim [\mathscr{D}(V_P)/\mathscr{D}(V_P||V_Q)].$$

But by the definition of $\mathscr{M}$,

$$\dim \mathscr{M}(V_Q) = \begin{cases} m, & if\ \gamma \neq \phi \\ m+2, & if\ \gamma = \phi \end{cases}$$

$$= m + 2 - 2\min(\#\gamma, 1),$$

and therefore the theorem has the following form:

Theorem 4-A. *If m is finite,*

$$\dim \mathscr{S}(V_Q||V_P) = m + 2 - 2\min(\#\gamma, 1) - \dim [\mathscr{D}(V_P)/\mathscr{D}(V_P||V_Q)].$$

This is an immediate generalization of Theorem 4 in [18]. However, since β and γ are not always finite sets and further n may be infinite, the claim seems to be more wealthy.

(II) In the case that m, n and g be finite, we can easily see that $\dim\mathscr{D}(V_P)=2g+n$, and so we have

Theorem 4 B. *As for surfaces of finite genus* g,

$$\dim\mathscr{S}(1/\Delta)-\dim\mathscr{D}(\Delta)=\mathrm{ind}\,\Delta-2g+2.$$

Here we set $\Delta=V_P||V_Q$, $1/\Delta=V_Q||V_P$ *symbolically, and* ind $\Delta=m-n-2\min(\sharp\gamma,1)$ *is the index of* Δ.

Remarks. (1) β and γ may yet be infinite sets. (2) Since m and n are not always even integers, so is $\mathrm{ind}\Delta$ (cf. (III)).

(III) We proceed to the more concrete cases. Let R_0 be any Riemann surface, compact or not. Let $B=\{p_1, p_2,\dots, p_r\}$ and $C=\{q_1, q_2,\dots, q_s\}$ be disjoint finite subsets of R_0. Then $R=R_0-B\cup C$ is always an *open* Riemann surface and the ideal boundary of R is $\partial R=\alpha\cup\beta\cup\gamma$, where we set $\alpha=\partial R_0$, $\beta=B$ and $\gamma=C$. We suppose that $B\cup C$ is not empty, while α may be an empty set. It is easily seen that α, β and γ are all closed sets in the Kerékjártó-Stoïlow compactification R^* of R. For non-empty B, we associate an ordered set of positive integers $(m_1, m_2, \dots, m_r)$ and set $m_0=\sum_i m_i$. In case that B be empty we agree to $m_0=0$. Similarly, let $(n_1, n_2, \dots, n_s)$ be another ordered set (of positive integers) which corresponds to C, $n_0=\sum_j n_j$.

Let $\sigma_i^{\mu_i}$, $\tilde{\sigma}_i^{\mu_i}$ $(1\le i\le r,\ 1\le\mu_i\le m_i)$ be differentials defined and meromorphic near ∂R such that

$$\sigma_i^{\mu_i}=\begin{cases}\dfrac{dz_i}{z_i^{\mu_i+1}} & \text{near } p_i,^{9)}\\[2ex] 0 & \text{near } \partial R-\{p_i\},\end{cases}\quad\text{and}\quad \tilde{\sigma}_i^{\mu_i}=\begin{cases}\sqrt{-1}\,\dfrac{dz_i}{z_i^{\mu_i+1}} & \text{near } p_i,\\[2ex] 0 & \text{near } \partial R-\{p_i\}.\end{cases}$$

9) z_i is a fixed local parameter near p_i which satisfies $z_i(p_i)=0$. Similarly, ζ_j will stand for a local parameter near q_j such that $\zeta_j(q_j)=0$.

We consider the vector space $V(\beta)$ which is spanned by (the equivalence classes of) $\sigma_i^{\mu_i}$'s and $\tilde{\sigma}_i^{\mu_i}$'s (modulo Λ_0'-behavior). Cf. section 2.

The other vector space with which we shall deal is $V(\gamma)$ spanned by (the equivalence classes of) $\tau_j^{\nu_j}$, $\tilde{\tau}_j^{\nu_j}$; φ_k, $\tilde{\varphi}_k$ $(1\leq j\leq s$, $2\leq\nu_j\leq n_j$; $2\leq k\leq s)$, where

$$\tau_j^{\nu_j}=\begin{cases}\dfrac{d\zeta_j}{\zeta_j^{\nu_j}} & \text{near } q_j,\\ 0 & \text{near } \partial R-\{q_j\},\end{cases}\qquad \tilde{\tau}_j^{\nu_j}=\begin{cases}\sqrt{-1}\,\dfrac{d\zeta_j}{\zeta_j^{\nu_j}} & \text{near } q_j,\\ 0 & \text{near } \partial R-\{q_j\};\end{cases}$$

$$\varphi_k=\begin{cases}\dfrac{d\zeta_1}{\zeta_1} & \text{near } q_1,\\ -\dfrac{d\zeta_k}{\zeta_k} & \text{near } q_k,\\ 0 & \text{near } \partial R-\{q_1, q_k\},\end{cases}\qquad \tilde{\varphi}_k=\begin{cases}\sqrt{-1}\,\dfrac{d\zeta_1}{\zeta_1} & \text{near } q_1,\\ -\sqrt{-1}\,\dfrac{d\zeta_k}{\zeta_k} & \text{near } q_k,\\ 0 & \text{near } \partial R-\{q_1, q_k\},\end{cases}$$

provided that $\gamma\neq\phi$. And if $\gamma=\phi$ we set $V(\gamma)=\{0\}$. Then $m\equiv\dim V(\beta)=2m_0$ and

$$n\equiv\dim V(\gamma)=\begin{cases}2n_0-2 & \text{if } \gamma\neq\phi \ (i.e., n_0\neq 0)\\ 0 & \text{if } \gamma=\phi \ (i.e., n_0=0)\end{cases}$$
$$=2n_0-2\min(\#\gamma, 1)=2n_0-2\min(n_0, 1).$$

Let P be the partition of the ideal boundary: $\partial R=\alpha\cup\beta\cup\gamma$ and Λ_0' and Λ_0'' *any* behavior spaces which are dual to one another (*w. r. t.* some L_0, *e. g.*, $\mathbf{R}$). [10] Then it is obvious that $V(\beta)$ is a $(Q)\Lambda_0'$-divisor and $V(\gamma)$ is a $(P)\Lambda_0''$-divisor. We write $V_Q=V(Q, \Lambda_0'; \beta, m)$ for $V(\beta)$ and $V_P=V(P, \Lambda_0''; \gamma, n)$ for $V(\gamma)$. If we set $\Delta=V_P||V_Q$ as before, $\mathrm{ind}\Delta=m-n-2\min(\#\gamma, 1)=2(m_0-n_0)$. Therefore Theorem 4 reduces to

10) Each $\lambda\in\Lambda_0'\oplus\Lambda_{e0}\cap\Lambda^1$ is written as $\lambda=df$, where f is single-valued near $p_i(1\leq i\leq r)$ and assumes a constant on each boundary component $\{p_i\}$ of β. As for any $\lambda\in\Lambda_0''\oplus\Lambda_{e0}\cap\Lambda^1$, we can discuss analogously.

Theorem 4-C.

(*) $$\dim \mathscr{S}(1/\varDelta) - \dim \mathscr{D}(\varDelta) = 2(m_0 - n_0) - 2g + 2.$$

This is the Riemann-Roch theorem on R_0 stated and proved in [18]. (In particular (*) contains the results in [6], [7], [9], [11] and [20].) In fact, β and γ are nothing other than the supports of integral divisors $\delta_p = p_1^{m_1} p_2^{m_2} \ldots p_r^{m_r}$ and $\delta_q = q_1^{n_1} q_2^{n_2} \ldots q_s^{n_s}$ (on $R_0!$) respectively. And ind $\varDelta = 2(m_0 - n_0) = 2\text{ord}\,\delta$, $\delta = \delta_p/\delta_q$. Note that there is a correspondence between V_P(resp. V_Q) and δ_p(resp. δ_q). Furthermore, $\alpha = \partial R_0$ is the proper(original) ideal boundary of R_0. Thus, $\mathscr{S}(1/\varDelta)$ and $\mathscr{D}(\varDelta)$ have much more concrete meanings: $\mathscr{S}(1/\varDelta) = \{f \mid$ (i) f is a single-valued meromorphic function on R_0, (ii) f has $\varLambda_0'$-behavior, (iii) f is a multiple of $1/\delta.\}$ and $\mathscr{D}(\varDelta) = \{\omega \mid$ (i) ω is a meromorphic differential on R_0, (ii) ω has $\varLambda_0''$-behavior, (iii) ω is a multiple of $\delta.\}$. If we consider $\varDelta$ as the pair $(\varLambda_0'', \delta)$ and $1/\varDelta$ as the pair $(\varLambda_0', 1/\delta)$, (*) is completely identical with the corollary to Theorem 4 in [18].

(III′) Our theorem (esp. formula (*)) includes the classical Riemann-Roch theorem very naturally. Indeed, the results on compact surfaces are obtained by considering the case $\alpha = \phi$.

It seems very important to find some other examples and applications showing the merits of our standpoint, although the author can not give them just now. For the further developments much more detailed knowledge will be needed and research on irregularities of differentials (or functions) near the linear boundaries will be strongly requested.

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