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On Tanaka's imbeddings of Siegel domains

By

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Introduction

Let D be a Siegel domain of the second kind in $\mathbf{C}^N$. In the case where D is homogeneous, Tanaka [7] showed that there exists an imbedding h of $\mathbf{C}^N$ onto an open subset of a certain complex homogeneous space G_c/B such that every holomorphic transformation of D can be extended to a holomorphic transformation of G_c/B. One of the purposes of this paper is to obtain the same results as Tanaka's without the assumption of homogeneity of D, which is discussed in §2 and §3.

By using the imbedding h, we shall prove in §4 that every holomorphic transformation of D which leaves the Silov boundary of D invariant is an affine automorphism of D. This fact is stated in Pyatetski-Shapiro [5] in the case where D is of the first kind.

Finally, in §5, we shall see that D is a symmetric homogeneous domain if and only if the space G_c/B is compact.

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§1. The automorphisms of Siegel domains

1.1. Let R (resp. W) be a real (resp. complex) vector space of finite dimension. Denote by R_c the complexification of R. For every vector $z \in R_c$, we denote by $\mathrm{Re}\, z$ the real part of z and by $\mathrm{Im}\, z$ the imaginary part of z. Let D be the Siegel domain of the second kind in $R_c\times W$ associated with a convex cone V in R and a V-hermitian form F on W ([5]). Denote by $Aff(D)$ (resp. by $GL(D)$) the closed subgroup of the complex affine transformation group $Aff(R_c\times W)$ (resp. of the general linear group $GL(R_c\times W)$) of $R_c\times W$ which consists of all elements of $Aff(R_c\times W)$ (resp. of $GL(R_c\times W)$) leaving D invariant. Then $GL(D)$ is a closed subgroup of $Aff(D)$. An element f of $GL(R_c\times W)$ belongs to $GL(D)$ if and only if f has the form: $f(z, w)=(Az, Bw)$, where $A\in GL(R)$, $B\in GL(W)$, $AV=V$ and $AF(w, w')=F(Bw, Bw')$ for all $w, w'\in W$ ([5]). We denote by ρ (resp. by σ) the correspondence: $f\to\rho(f)=A$ (resp. $f\to\sigma(f)=B$). Then the mapping ρ (resp. σ) is a homomorphism of $GL(D)$ into $GL(R)$ (resp. into $GL(W)$). Let $\mathfrak{g}^a$ be the Lie algebra of $Aff(D)$. For every $a\in R$ (resp. for every $c\in W$) we denote by $s(a)$ (resp. by $s(c)$) the element of $\mathfrak{g}^a$ induced by the following one parameter group (with parameter t):

$$(z, w)\longrightarrow(z+ta, w)$$

$$(\text{resp. } (z, w)\longrightarrow(z+2\sqrt{-1}F(w, tc)+\sqrt{-1}F(tc, tc), w+tc)).$$

Then the correspondence: $a+c\to s(a)+s(c)$ gives an injective linear mapping s of $R+W$ into $\mathfrak{g}^a$ and the following equalities are easily verified:

$$\text{(1.1)}\qquad 1)\quad [s(a), s(b+c)]=0 \qquad (a, b\in R, c\in W).$$

$$2)\quad [s(c), s(c')]=4s(\operatorname{Im}F(c, c')) \qquad (c, c'\in W).$$

We denote by $\mathfrak{g}^0$ the subalgebra of $\mathfrak{g}^a$ corresponding to the subgroup $GL(D)$ of $Aff(D)$. Then the following equality holds:

$$\text{(1.2)}\qquad [g, s(a+c)]=s(\rho_*(g)a+\sigma_*(g)c) \qquad (g\in\mathfrak{g}^0, a\in R, c\in W),$$

where ρ_* (resp. σ_*) is the homomorphism of $\mathfrak{g}^0$ to $\mathfrak{gl}(R)$ (resp. to $\mathfrak{gl}(W)$) induced by ρ (resp. σ). Let E (resp. I) be the element of $\mathfrak{g}^0$ induced by the following one parameter group in $GL(D)$:

$$(z, w)\longrightarrow(e^{-2t}z, e^{-t}w)$$

$$(\text{resp. } (z, w) \longrightarrow (z, e^{\sqrt{-1}t}w)).$$

Clearly E and I are in the center of $\mathfrak{g}^0$ and the following equalities hold:

$$(1.3) \qquad [E, s(a)+s(c)] = -2s(a)-s(c)$$

$$[I, s(a)+s(c)] = s(\sqrt{-1}c) \qquad (a \in R,\ c \in W).$$

1.2. We denote by $Aut(D)$ the automorphism group of D, i.e., the group of all holomorphic transformations of D. Let $\mathfrak{g}$ be the Lie algebra of $Aut(D)$. Then $\mathfrak{g}^0 = \{X \in \mathfrak{g};\ [E, X]=0\}$. Moreover the following theorem is known:

Theorem 1.1 (Kaup-Matsushima-Ochiai [2]).

(1) $\mathfrak{g} = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2$ *as a graded Lie algebra where* $\mathfrak{g}^\lambda = \{X \in \mathfrak{g};\ [E, X] = \lambda X\}$.

(2) $\mathfrak{g}^a = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0$, $\mathfrak{g}^{-2} = \{s(a);\ a \in R\}$ *and* $\mathfrak{g}^{-1} = \{s(c);\ c \in W\}$.

Remark 1. In the earlier paper [4], the author showed that $\mathfrak{g}^1$ and $\mathfrak{g}^2$ are determined algebraically from $\mathfrak{g}^a$.

§2. Tanaka's imbeddings

2.1. Let $\mathfrak{g} = \sum_{\lambda=-2}^{2} \mathfrak{g}^\lambda$ be the graded Lie algebra given in Theorem 1.1 and let $\mathfrak{g}_c$ be the complexification of $\mathfrak{g}$. We denote by G_c the adjoint group of $\mathfrak{g}_c$. Since $\mathfrak{g}$ is centerless ([2]), we identify the Lie algebra of G_c with $\mathfrak{g}_c$. Define linear endomorphisms P and $\bar{P}$ of $\mathfrak{g}_c^{-1}$ by

$$(2.1) \qquad P(X) = \frac{1}{2}(X - \sqrt{-1}[I, X]) \text{ for } X \in \mathfrak{g}_c^{-1},$$

$$\bar{P}(X) = \frac{1}{2}(X + \sqrt{-1}[I, X]) \text{ for } X \in \mathfrak{g}_c^{-1}.$$

It is easy to see the following equalities hold:

$$(2.2) \qquad P([I, X]) = \sqrt{-1}P(X),$$

$$\bar{P}([I, X]) = -\sqrt{-1}\bar{P}(X).$$

Therefore $P(\mathfrak{g}_c^{-1})=P(\mathfrak{g}^{-1})$ and $\bar{P}(\mathfrak{g}_c^{-1})=\bar{P}(\mathfrak{g}^{-1})$. Both $P(\mathfrak{g}^{-1})$ and $\bar{P}(\mathfrak{g}^{-1})$ are complex subspaces of $\mathfrak{g}_c^{-1}$ and $\mathfrak{g}_c^{-1}=P(\mathfrak{g}^{-1})+\bar{P}(\mathfrak{g}^{-1})$. We put

$$(2.3)\qquad \begin{aligned} \mathfrak{n}&=\mathfrak{g}_c^{-2}+P(\mathfrak{g}^{-1}) \\ \mathfrak{b}&=\bar{P}(\mathfrak{g}^{-1})+\mathfrak{g}_c^0+\mathfrak{g}_c^1+\mathfrak{g}_c^2. \end{aligned}$$

Lemma 2.1 (cf. [7]).
(1) $\mathfrak{g}_c=\mathfrak{n}+\mathfrak{b}$ *(direct sum).*
(2) $\mathfrak{n}$ *is an abelian subalgebra of* $\mathfrak{g}_c$.
(3) $\mathfrak{b}$ *is a subalgebra of* $\mathfrak{g}_c$.

Proof. (1) is clear. Proof of (2) is the same as in [7]. We can also verify $\bar{P}(\mathfrak{g}^{-1})$ is abelian. Since I is in the center of $\mathfrak{g}_c^0$, $\bar{P}(\mathfrak{g}^{-1})$ is invariant by $ad\,\mathfrak{g}_c^0$. Thus we obtain (3). q.e.d.

Define a closed subgroup B of G_c by $B=\{a\in G_c;\ a\mathfrak{b}=\mathfrak{b}\}$. Since $\mathfrak{b}$ is a complex subalgebra, B is a complex Lie subgroup. It is not difficult to see that the Lie algebra of B coincides with $\mathfrak{b}$ (cf. [7]). We denote by π the projection of G_c onto the homogeneous space G_c/B. Let us define a mapping h' of $\mathfrak{n}$ to G_c/B by

$$h'(X)=\pi\exp X \qquad (X\in\mathfrak{n}).$$

Then h' is a holomorphic imbedding of the complex vector space $\mathfrak{n}$ onto an open set of the complex homogeneous space G_c/B ([7]).

2.2. Let h'' be the mapping of $R_c\times W$ onto $\mathfrak{n}$ defined by

$$h''(z,w)=s(z)+P(s(w)) \qquad (z\in R_c,\ w\in W),$$

where s is the mapping of $R+W$ onto $\mathfrak{g}^{-2}+\mathfrak{g}^{-1}$ given in §1 and is extended to a mapping of R_c+W onto $\mathfrak{g}_c^{-2}+\mathfrak{g}^{-1}$ in a natural fashion. Since $P(s(\sqrt{-1}w))=P([I,s(w)])=\sqrt{-1}P(s(w))$ by (1.3) and (2.2), h'' is a biholomorphic mapping (linear isomorphism) of $R_c\times W$ onto $\mathfrak{n}$. Then the mapping $h=h'\circ h''$ is a holomorphic imbedding of $R_c\times W$ onto an open set of G_c/B. The imbedding h was first introduced by Tanaka [7] in the case where D is homogeneous. We call it Tanaka's imbedding.

We denote by G (resp. by G_a) the identity component of $Aut(D)$ (resp. of $Aff(D)$). For every $a \in G$, there exists a unique element $\tau(a)$ of G_c such that $Ad\,a X = \tau(a)X$ $(X \in \mathfrak{g})$. Since G is centerless ([2]), the mapping τ is an injective homomorphism of G into G_c. Let $v \in V$ and let K be the isotropy subgroup of G at $(\sqrt{-1}v, 0)$. We denote by K^0 the identity component of K and by $\mathfrak{k}$ the Lie algebra of K.

Lemma 2.2 (cf. [7]).

(1) $\tau(k)h(\sqrt{-1}v, 0) = h(\sqrt{-1}v, 0) \qquad (k \in K^0)$

(2) $\tau(t)h(\sqrt{-1}v, 0) = h(t(\sqrt{-1}v, 0)) \qquad (t \in G_a)$.

Proof. (1) Let X be an element of $\mathfrak{k}$. We write $X = X^{-2} + X^{-1} + X^0 + X^1 + X^2$, $X^\lambda \in \mathfrak{g}^\lambda$ (cf. Theorem 1.1). Then from [2] we know $X^{-2} = \frac{1}{2}(ad\, s(v))^2 X^2$, $X^{-1} = ad\, I\; ad\, s(v) X^1$ and $ad\, s(v) X^0 = 0$. It follows

$$\begin{aligned} &Ad(\exp(-\sqrt{-1}\, s(v)))X \\ &= X^{-2} + X^{-1} + X^0 + X^1 + X^2 - \sqrt{-1}([s(v), X^1] + [s(v), X^2]) \\ &\quad - \frac{1}{2}(ad\; s(v))^2 X^2 \\ &\equiv X^{-1} - \sqrt{-1}[s(v), X^1] \equiv 0 \qquad (\mathrm{mod}\, \mathfrak{b}), \end{aligned}$$

because $P(X^{-1}) = P(\sqrt{-1}[s(v), X^1])$ holds. Then

$$\begin{aligned} &\tau(\exp X)h(\sqrt{-1}v, 0) \\ &= \pi \exp X \cdot \exp(\sqrt{-1}s(v)) \\ &= \pi \exp(\sqrt{-1}s(v)) \cdot \exp[Ad(\exp(-\sqrt{-1}s(v)))X] \\ &= \pi \exp(\sqrt{-1}s(v)). \end{aligned}$$

Therefore we get Assertion (1). Proof of (2) is the same as [7]. q.e.d.

If we put $K_a = K \cap G_a$, then we have

Lemma 2.3. $G/K = G_a/K_a$.

Proof. It is known ([2]) that $\dim G-\dim K=\dim G_a-\dim K_a$. Therefore G_a/K_a is an open set of G/K. Being a submanifold of D, G/K has a Riemannian metric invariant by G and hence by G_a. As a result, the open orbit G_a/K_a of G_a coincides with G/K. q.e.d.

Next we verify the following

Lemma 2.4 (cf. [7]). *For every $f\in G$ and for every $p\in D$, $h(fp)=\tau(f)h(p)$.*

Proof. For every $p\in D$, there exist $t\in G_a$ and $v\in V$ such that $t(\sqrt{-1}v, 0)=p$. Then by Lemma 2.2 $h(p)=\tau(t)h(\sqrt{-1}v, 0)$. We can choose a neighbourhood $\mathscr{U}$ of e(=the unit element) in G having the property that every element of $\mathscr{U}\cap K$ can be expressed as $\exp X$, $X\in\mathfrak{k}$. There exists a neighbourhood $\mathscr{U}_1$ of e in G such that $t^{-1}\mathscr{U}_1^{-1}\mathscr{U}_1 t\subset\mathscr{U}$. We put $\mathscr{U}_a=\mathscr{U}_1\cap G_a$, which is an open set of G_a. Then the subset $\mathscr{U}_2$ of G defined by $\mathscr{U}_2=\{g\in\mathscr{U}_1;\, gp\in\mathscr{U}_a p\}$ is open in G by Lemma 2.3. For every $f\in\mathscr{U}_2$, there exists $g\in\mathscr{U}_a$ such that $fp=gp$. Then $t^{-1}g^{-1}ft\in\mathscr{U}\cap K$ and by Lemma 2.2,

$$\tau(t^{-1}g^{-1}ft)h(\sqrt{-1}v, 0)=h(\sqrt{-1}v, 0).$$

It follows

$$\begin{aligned}\tau(f)h(p)&=\tau(gp)h(\sqrt{-1}v, 0)\\&=h(gt(\sqrt{-1}v, 0))=h(fp),\end{aligned}$$

because $gt\in G_a$. Thus the mapping: $f\to\tau(f)h(p)$ of G to G_c/B is real analytic and coincides on $\mathscr{U}_2$ with the mapping: $f\to h(fp)$. Therefore we conclude $h(fp)=\tau(f)h(p)$ for all $f\in G$. q.e.d.

In what follows we identify the space $R_c\times W$ with an open submanifold of G_c/B by the imbedding h. We also identify the group G with a closed subgroup of G_c by the injective homomorphism τ.

2.3. We denote by o the origin of $R_c\times W$. Then the space $R_c\times W$ is the orbit of the group $\exp(\mathfrak{g}_c^{-2}+P(\mathfrak{g}^{-1}))$ through o. Let T be the union of all singular orbits of $\exp(\mathfrak{g}_c^{-2}+P(\mathfrak{g}^{-1}))$. Then T is a proper analytic set of G_c/B which is locally defined by a single equation and $G_c/B-T=R_c\times W$ ([7]). For $X\in\mathfrak{g}_c$, denote by $\tilde{X}$ the holomorphic vector field on G_c/B generated by X. It is easy to see that the correspondence: $X\to\tilde{X}$ of $\mathfrak{g}_c$ into the space of all holomorphic vector fields on G_c/B is injective. We sometimes identify X with $\tilde{X}$.[1)]

Lemma 2.5. *Let $p\in G_c/B$ and f be a holomorphic function defined on a connected neighbourhood U of p. Assume that there exists a neighbourhood $\mathcal{U}$ of e in G such that f is constant on $U\cap\mathcal{U}p$. Then f is constant on the whole U.*

Proof. We put $U_1=U\cap\mathcal{U}p$. Then $Xf=0$ on U_1 for all $X\in\mathfrak{g}$. Since f is holomorphic, $Xf=0$ on U_1 for all $X\in\mathfrak{g}_c$.[2)] Therefore the first derivative of f is zero on U_1. The same argument shows that all derivatives of f vanish on U_1 and hence f is constant on U. q.e.d.

Let S be the real submanifold of $R_c\times W$ defined by

$$S=\{(z,w)\in R_c\times W;\ \mathrm{Im}\,z-F(w,w)=0\},$$

which is a subset of the boundary of D and is called the Silov boundary of D. It is easy to see that each element of $\mathit{Aff}(D)$ leaves S invariant and that the group $\exp(\mathfrak{g}^{-2}+\mathfrak{g}^{-1})$ acts simply transitively on S.

Lemma 2.6 ([7]). *Let $a\in G$ and $p\in S$. If $ap\in R_c\times W$, then $ap\in S$.*

Let M be the orbit of G through o, i.e., $M=G/G\cap B$. Since $\dim M=\dim\mathfrak{g}^{-2}+\dim\mathfrak{g}^{-1}$, S is an open submanifold of M. Moreover we obtain

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1) We consider $\mathfrak{g}_c$ as the Lie algebra of left invariant vector fields on G_c. Thus $\widetilde{[X,Y]}=-[\tilde{X},\tilde{Y}]$.

2) Considering $\mathfrak{g}_c=\mathfrak{g}+J(\mathfrak{g})$, where J is the complex structure of G_c/B, we have only to show that $Xf=0$ implies $(JX)f=0$ $(X\in\mathfrak{g})$. Let $z_1,\dots,z_n$ be a local coordinate system $(z_j=x_j+\sqrt{-1}\,y_j)$. For any vector $X=\sum_j a^j\frac{\partial}{\partial x_j}+\sum_j b^j\frac{\partial}{\partial y_j}$ $(a^j, b^j\in\mathbf{R})$, $JX=\sum_j a^j\frac{\partial}{\partial y_j}-\sum_j b^j\frac{\partial}{\partial x_j}$. Then $Xf=0$ $(f=u+\sqrt{-1}v)$ implies $\sum_j a^j\frac{\partial v}{\partial x_j}+\sum_j b^j\frac{\partial u}{\partial y_j}=0$ and $\sum_j a^j\frac{\partial v}{\partial x_j}+\sum_j b^j\frac{\partial v}{\partial y_j}=0$. Since f is holomorphic, $\frac{\partial u}{\partial x_j}=\frac{\partial v}{\partial y_j}$ and $\frac{\partial u}{\partial y_j}=-\frac{\partial v}{\partial x_j}$. Thus by a direct calculation we have $(JX)f=0$.

Proposition 2.7. *M is a closed submanifold of G_c/B and $M=\bar{S}$, where $\bar{S}$ is the closure of S in G_c/B.*

Proof. We shall show that for any point p of G_c/B there exists an R-valued real analytic function f defined on a neighbourhood U of p such that $M\cap U=\{q\in U; f(q)=0\}$. If $p\in R_c\times W$. Then we can take $\mathrm{Im}\, z-F(w, w)$ as f, because $M\cap R_c\times W=S$ by Lemma 2.6. Next we consider the case where $p\in T$. We assert that there exists an $a\in G$ such that $ap\in R_c\times W$. In fact, suppose $Gp\subset T$. Let $f'=0$ be a local equation of T at p. Then by Lemma 2.5 $f'=0$ on a neighbourhood of p. This contradicts the fact that T is a proper analytic set, proving our assertion. We choose a neighbourhood U of p such that $aU\subset R_c\times W$. Then the function $(\mathrm{Im}\, z-F(w, w))\circ a$ defined on U has the desired property by Lemma 2.6. As a consequence M is closed. By Lemma 2.6 $M-S\subset T$. Let $f'=0$ be a local equation of T. Then the restriction of the equation $f'=0$ to M defines the subset $M-S$. Clearly there is no open set U' of M such that f' vanishes on U' by Lemma 2.5. It follows immediately that S is an open dense subset of M.

q.e.d.

§3. Equivalence of Siegel domains

3.1. Let D be the Siegel domain of the second kind in $R_c\times W$ associated with a convex cone V in R and a V-hermitian form F on W. We use the notations given in §1 and §2.

Let D' be another Siegel domain of the second kind. For an object A such as a space, a group, etc., with respect to the domain D, we denote by A' the corresponding object with respect to the domain D'. We now assume that the two domains D and D' are holomorphically equivalent, i.e., there exists a biholomorphic mapping φ of D onto D'. The mapping φ induces an isomorphism Φ of G_c onto G'_c in a natural manner. Clearly

$$\Phi(a)=\varphi a\varphi^{-1} \qquad (a\in G).$$

For each $q\in G_c/B$, we put

$$\mathfrak{h}_q=\{X \in \mathfrak{g}_c;\ \tilde{X}_q=0\}.$$

Let $p \in D$ and $p'=\varphi(p) \in D'$. We assert that the following equality holds:

$$\mathfrak{h}'_{p'}=\Phi_* \mathfrak{h}_p .$$

Indeed, $(\Phi_* X)_{p'}=\varphi_* X_p$ for $X \in g$. (We can regard $\varphi_* X$ as a holomorphic vector field on G'_c/B'.) Since the mapping Φ_* and φ_* are complex linear, our assertion is clear. We can choose $a \in G_c$ (resp. $a' \in G'_c$) such that $ao=p$ (resp. $a'o'=p'$). We put $\hat{\Phi}(c)=a'^{-1}\Phi(aca^{-1})a'$ for each $c \in G_c$. Then the mapping $\hat{\Phi}$ is an isomorphism of G_c onto G'_c. Since $Ada\,\mathfrak{b}=Ada\,\mathfrak{h}_o=\mathfrak{h}_p$ and $Ada'\mathfrak{b}'=Ada'\mathfrak{h}'_{o'}=\mathfrak{h}'_{p'}$, we get $\hat{\Phi}_*\mathfrak{b}=Ada'^{-1}\Phi_* Ada\mathfrak{b}=\mathfrak{b}'$. Therefore $\hat{\Phi}(B)$ is a closed subgroup of G'_c with Lie algebra $\mathfrak{b}'$, and hence $\hat{\Phi}(B) \subset B'$. By considering the inverse of φ we conclude $\hat{\Phi}(B)=B'$. As a result, there exists a biholomorphic mapping $\hat{\varphi}$ of G_c/B onto G'_c/B' such that $\hat{\varphi}\circ\pi=\pi'\circ\hat{\Phi}$. For any $x=cp$ $(c \in G)$, we get $\varphi(x)=\Phi(c)\varphi(ao)=a'\hat{\Phi}(a^{-1}ca)o'=\pi'a'\hat{\Phi}(a^{-1})\hat{\Phi}(ca)=a'\hat{\Phi}(a^{-1})\hat{\varphi}(x)$. Thus the biholomorphic mapping $\tilde{\varphi}=a'\circ\hat{\Phi}(a^{-1})\circ\hat{\varphi}$ of G_c/B onto G'_c/B' coincides with φ on the orbit of G through p. By using Lemma 2.5, we can easily verify that $\tilde{\varphi}$ coincides with φ on the whole D. Thus we have proved

Proposition 3.1 (cf. [7]). *Every biholomorphic mapping of D onto D' can be extended to a biholomorphic mapping of G_c/B onto G'_c/B'.*

Let φ be a biholomorphic mapping of D onto D'. Denote by the same letter φ the extended biholomorphic mapping of G_c/B onto G'_c/B'. It is easy to see that $\varphi(cx)=\Phi(c)\varphi(x)$ for all $c \in G_c$ and $x \in G_c/B$. We now verify the following theorem. The proof is almost similar to the one in [7].

Theorem 3.2 (cf. [2], [7]). *Let D (resp. D') be the Siegel domain of the second kind associated with a convex cone V (resp. V') in R (resp. in R') and a V- (resp. V'-) hermitian form F (resp. F') on W (resp. on W'). Assume that there exists a biholomorphic mapping φ of D onto D'. Then*

(1) $\dim R = \dim R'$ *and* $\dim W = \dim W'$.

(2) φ *can be written as* $\varphi = c'\psi c$, *where* $c \in G_a$, $c' \in G'$ *and* ψ *is a complex linear isomorphism of* $R_c \times W$ *to* $R'_c \times W'$ *satisfying the followings*: $\psi(R) = R'$, $\psi(W) = W'$, $\psi(V) = V'$ *and* $\psi(F(u, w)) = F'(\psi(u), \psi(w))$ *for all* $u, w \in W$.

Proof. We can easily verify that there exists a point $p \in S$ such that $\varphi(p) \in S'$ ([7]). Choose $c \in G_a$ and $c' \in G'_a$ such that $co = p$ and $c'o' = \varphi(p)$. By considering $c'^{-1} \circ \varphi \circ c$ instead of φ, we may assume $\varphi(o) = o'$. Since $\varphi(ao) = \Phi(a)\varphi(o) = \Phi(a)o'$ for any $a \in G_c$, we have $\varphi(M) = M'$. Furthermore $\Phi(B) = B'$, because $\Phi(b)o' = \Phi(b)\varphi(o) = \varphi(bo) = o'$ for every $b \in B$. Therefore we get $\varphi \circ \pi = \pi' \circ \Phi$. Clearly $\mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2$ (resp. $\mathfrak{g}'^0 + \mathfrak{g}'^1 + \mathfrak{g}'^2$) is the Lie algebra of $G \cap B$ (resp. of $G' \cap B'$). It follows

1) $\Phi_*(\mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2) = \mathfrak{g}'^0 + \mathfrak{g}'^1 + \mathfrak{g}'^2$.

Since $\pi_*(\mathfrak{g}^{-1})_e = T_o(M) \cap JT_o(M)$, where J is the complex structure of G_c/B, and $\pi'_*(\mathfrak{g}'^{-1})_{e'} = T_{o'}(M') \cap J'T_{o'}(M')$, we have

2) $\Phi_*\mathfrak{g}^{-1} \equiv \mathfrak{g}'^{-1} (\mathrm{mod}\ \mathfrak{g}'^0 + \mathfrak{g}'^1 + \mathfrak{g}'^2)$.

For any $s(c) \in \mathfrak{g}^{-1}$ $(c \in W)$, $\varphi_*\pi_*[I, s(c)]_e = \varphi_*\pi_*s(\sqrt{-1}c)_e = J'\varphi_*\pi_*s(c)_e = J'\pi'_*(\Phi_*s(c))_{e'} = \pi'_*([I', \Phi_*s(c)])_{e'}$. On the other hand, $\varphi_*\pi_*[I, s(c)]_e = \pi'_*(\Phi_*[I, s(c)])_{e'}$. Thus we get

3) $\Phi_*[I, X] \equiv [I', \Phi_*X] \quad (\mathrm{mod}\ \mathfrak{g}'^0 + \mathfrak{g}'^1 + \mathfrak{g}'^2)$

for all $X \in \mathfrak{g}^{-1}$.

From 1) and 2), we obtain $\Phi_*E \equiv E' (\mathrm{mod}\ \mathfrak{g}'^1 + \mathfrak{g}'^2)$ ([7]). Hence we can write $\Phi_*E \equiv E' + X^1 (\mathrm{mod}\ \mathfrak{g}'^2)$, $X^1 \in \mathfrak{g}'^1$. Then $Ad(\exp X^1)\ \Phi_*E = E'$ $(\mathrm{mod}\ \mathfrak{g}'^2)$. Therefore there exists $X^2 \in \mathfrak{g}'^2$ such that $Ad(\exp X^1)\Phi_*E = E' + 2X^2$. It follows $Ad(\exp X^2)\ Ad(\exp X^1)\Phi_*E = E'$. We put $\psi = \exp X^2 \circ \exp X^1 \circ \varphi$. And denote by Ψ the induced isomorphism of G_c onto G'_c. Then it is clear $\Psi_*E = E'$ and hence $\Psi_*\mathfrak{g}^\lambda = \mathfrak{g}'^\lambda$ $(-2 \leqq \lambda \leqq 2)$. Therefore $\dim R = \dim R'$ and $\dim W = \dim W'$. By considering the equality: $\Psi_*\mathfrak{g}^{-1} = \mathfrak{g}'^{-1}$, we get from 3)

$$\Psi_*[I, X]=[I', \Psi_* X] \quad \text{for} \quad X \in \mathfrak{g}^{-1}.$$

As a result, $\Psi_* P(X)=P'(\Psi_* X)$ for $X \in \mathfrak{g}^{-1}$. Recalling the definition of Tanaka's imbeddings, we get

$$\begin{aligned}\psi(z, w) &= \psi(\exp[s(z)+P(s(w))]o) \\ &= \Psi(\exp[s(z)+P(s(w))])o' \\ &= \exp[\Psi_* s(z)+P(\Psi_* s(w))]o'.\end{aligned}$$

The above equality shows that there exist linear isomorphisms A_1 of R onto R' and A_2 of W onto W' such that $\psi(z, w)=(A_1 z, A_2 w)$. Clearly $A_1 V=V'$. And by a simple calculation, we get $A_1 F(u, w)=F'(A_2 u, A_2 w)$. q.e.d.

Applying Theorem 3.2 to the special case where $D'=D$, we get

Corollary 3.3. $Aut(D)=G \cdot GL(D)$.

Proof. Let $\varphi \in Aut(D)$. Then by Theorem 3.2, we can write $\varphi = c'\psi c$, where $c, c' \in G$ and $\psi \in GL(D)$. It follows $\varphi = c'\psi c \psi^{-1}\psi$ and clearly $\psi c \psi^{-1} \in G$. q.e.d.

§4. The Silov boundary S and the group $\mathcal{Aff}(D)$

4.1. Let D be the Siegel domain of the second associated with a convex cone V in R and a V-hermitian form F on W. And let D_1 be the associated Siegel domain of the first kind, i.e., $D_1=\{x \in R_c;$ $\operatorname{Im} z \in V\}$. We denote by $\mathfrak{g}$ (resp. by $\mathfrak{t}$) the Lie algebra of $Aut(D)$ (resp. $Aut(D_1)$). Then by Theorem 1.1, $\mathfrak{g}=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^{0}+\mathfrak{g}^{1}+\mathfrak{g}^{2}$ and $\mathfrak{t}=\mathfrak{t}^{-2}+\mathfrak{t}^{0}+\mathfrak{t}^{2}$. Since $\mathfrak{g}^{-2}+\mathfrak{g}^{0}+\mathfrak{g}^{2}$ is the subalgebra corresponding to the subgroup $\{f \in Aut(D); f$ leaves $D_1 \times (0)$ invariant$\}$ ([2]), there exists a homomorphism $\alpha: \mathfrak{g}^{-2}+\mathfrak{g}^{0}+\mathfrak{g}^{2} \to \mathfrak{t}^{-2}+\mathfrak{t}^{0}+\mathfrak{t}^{2}$ (as graded Lie algebras).

Lemma 4.1. *The homomorphism α is injective on* $\mathfrak{g}^{-2}+\mathfrak{g}^{2}$.

Proof. Clearly α is an isomorphism on $\mathfrak{g}^{-2}$. Let $X \in \mathfrak{g}^{2}$ be such that $\alpha(X)=0$. Then for any $Y\in\mathfrak{g}^{-2}$, $\alpha([Y,[Y,X]])=0$. Since $[Y,[Y,X]]\in\mathfrak{g}^{-2}$, we get $[Y,[Y,X]]=0$. This implies $X=0$ ([4], [8]), q.e.d.

Lemma 4.2. *Let $X\in\mathfrak{g}^1$. If $[[I,X],X]=0$. Then $X=0$.*

Proof. By a direct calculation we have

$$0=[Y,[Y,[[I,X],X]]]=2[[I,[Y,X]],[Y,X]] \quad \text{for } Y\in\mathfrak{g}^{-2}.$$

Since $[Y,X]\in\mathfrak{g}^{-1}$, we have $[Y,X]=0$ for all $Y\in\mathfrak{g}^{-2}$.[3)] Therefore we get $X=0$ ([4], [8]). q.e.d.

4.2. Let H be the subgroup of $Aut(D)$ which consists of all elements f of $Aut(D)$ leaving the Silov boundary S invariant, where f should be regarded as a holomorphic transformation of G_c/B. Clearly H contains $Aff(D)$. Since each element of $Aut(D)$ leaves M invariant (cf. Proof of Theorem 3.2) and since S is open dense subset of M, H is a closed subgroup of $Aut(D)$.

Lemma 4.3. *The Lie algebra of H coincides with the Lie algebra of $Aff(D)$.*

Proof. Denote by $\mathfrak{h}$ the Lie algebra of H. Since $\mathfrak{h}$ contains the element E, $\mathfrak{h}$ is a graded subalgebra of $\mathfrak{g}$, i.e.,

$$\mathfrak{h}=\mathfrak{h}^{-2}+\mathfrak{h}^{-1}+\mathfrak{h}^{0}+\mathfrak{h}^{1}+\mathfrak{h}^{2},\ \mathfrak{h}^{\lambda}=\mathfrak{h}\cap\mathfrak{g}^{\lambda}.$$

Let $X\in\mathfrak{h}^2$. Then $\exp\alpha(X)$ leaves the Silov boundary of D_1 invariant. It is known in [5] that a holomorphic transformation of a Siegel domain of the first kind which leaves the Silov boundary invariant is an affine transformation. Therefore we get $\alpha(X)=0$ and hence $X=0$ by Lemma 4.1. It follows $\mathfrak{h}^2=0$ and hence $\mathfrak{h}^1=0$ by Lemma 4.2. q.e.d.

3) Identifying $\mathfrak{g}_c^{-2}+\mathfrak{g}^{-1}$ with $R_c\times W$ by the map s, we have from (1.1) and (1.3)

$$[[I,Z],Z]=4F(Z,Z) \quad \text{for } Z\in\mathfrak{g}^{-1}.$$

Theorem 4.4. *The group H coincides with $Aff(D)$.*

Proof. Let $f\in H$. Then $fo\in S$ and hence there exists $c\in G_a$ such that $cfo=o$. We put $\varphi=cf$ and denote by Φ the induced isomorphism of G_c by φ. Then $\Phi_*(\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2)=\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2$. We recall that $\Phi(a)=\varphi a\varphi^{-1}$. Since $\varphi\in H$, we get $\Phi_*(\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^0)=Ad\varphi(\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^0)=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^0$ by Lemma 4.3. It follows $\Phi_*\mathfrak{g}^0=\mathfrak{g}^0$. By considering the equation $\Phi_*E\equiv E\ (\mathrm{mod}\ \mathfrak{g}^1+\mathfrak{g}^2)$, we conclude $\Phi_*E=E$ and hence $\Phi_*\mathfrak{g}^\lambda=\mathfrak{g}^\lambda$. Now from the Proof of Theorem 3.2, it is clear that φ is an element of $GL(D)$. q.e.d.

Corollary 4.5. *Let $f\in Aut(D)$. Assume that f and f^{-1} are continuous in $\bar{D}$, where $\bar{D}$ is the closure of D in $R_c\times W$. Then $f\in Aff(D)$.*

Proof. By Lemma 2.6 and by Corollary 3.3, we know $f\in H$. q.e.d.

§5. The compactness of C_c/B

5.1. Let D be the Siegel domain of the second kind as in §1. Assume that D is a symmetric homogeneous domain. Then it is well known that the Lie algebra $\mathfrak{g}$ is semi-simple. Therefore $\mathfrak{g}_c$ is semi-simple. We also define linear endomorphisms P and $\bar{P}$ of $\mathfrak{g}_c^1$ in the same way as in (2.1), i.e.,

$$P(X)=\frac{1}{2}(X-\sqrt{-1}[I, X]),$$

$$\bar{P}(X)=\frac{1}{2}(X+\sqrt{-1}[I, X]) \qquad \text{for } X\in\mathfrak{g}_c^1 .$$

We put $\mathfrak{s}^{-1}=\mathfrak{g}_c^{-2}+P(\mathfrak{g}^{-1})$, $\mathfrak{s}^0=\bar{P}(\mathfrak{g}^{-1})+\mathfrak{g}_c^0+P(\mathfrak{g}^1)$ and $\mathfrak{s}^1=\bar{P}(\mathfrak{g}^1)+\mathfrak{g}_c^2$. It is easy to see that $\mathfrak{g}_c=\mathfrak{s}^{-1}+\mathfrak{s}^0+\mathfrak{s}^1$ is a graded Lie algebra. According to [6] there exists an involutive automorphism θ of $\mathfrak{g}_c$ (as a real Lie algebra) associated with a certain Cartan decomposition of $\mathfrak{g}_c$ such that $\theta(\mathfrak{s}^{-1})=\mathfrak{s}^1$ and $\theta(\mathfrak{s}^0)=\mathfrak{s}^0$. Let G_θ be the Lie subgroup of G_c corresponding to the subalgebra $\{X\in\mathfrak{g}_c;\ \theta(X)=X\}$. Clearly the orbit of

G_θ through the origin o is open and compact in G_c/B and hence coincides with G_c/B. As a result G_c/B is compact.

The purpose of this section is to prove the following

Theorem 5.1. *Let D be a Siegel domain of the second and let G_c/B and $G/G \cap B$ be homogeneous spaces constructed in* §2. *Then the following conditions are equivalent*:

(1) *D is symmetric.*

(2) *G_c/B is compact.*

(3) *$G/G \cap B$ is compact.*

5.2. We put $\tilde{\mathfrak{b}} = \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2$. Let $m = \dim \mathfrak{g}$ and $n = \dim \tilde{\mathfrak{b}}$. Denote by $Q(m, n; \mathbf{R})$ the grassmann manifold of all n-dimensional subspaces of the m-dimensional real vector space $\mathfrak{g}$. Since the subgroup $G \cap B$ of G leaves $\tilde{\mathfrak{b}}$ invariant, we can define a mapping η of $Go = G/G \cap B$ to $Q(m, n; \mathbf{R})$ by

$$\eta(ao) = Ada\tilde{\mathfrak{b}} \qquad (a \in G).$$

Lemma 5.2. *Let $v \in V$ and let $\mathfrak{k}_v^0 = \{X \in \mathfrak{g}^0;\ [s(v), X] = 0\}$. Then we have*

$$\lim_{t\to\infty} \eta(\exp ts(v)o) = \mathfrak{k}_v^0 + [s(v), \tilde{\mathfrak{b}}] \qquad (\text{in } Q(m, n; \mathbf{R})).$$

Proof. We put $n_1 = \dim \mathfrak{g}^1$ and $n_2 = \dim \mathfrak{g}^0 + \mathfrak{g}^2$. Let $X^1 \in \mathfrak{g}^1$. Then

$$Ad(\exp ts(v))X^1 = X^1 + t[s(v), X^1].$$

Since the mapping: $X^1 \to [s(v), X^1]$ of $\mathfrak{g}^1$ to $\mathfrak{g}^{-1}$ is injective (cf. [8]), we have

$$\lim_{t\to\infty} Ad(\exp ts(v))\mathfrak{g}^1 = [s(v), \mathfrak{g}^1] \qquad (\text{in } Q(m, n_1; \mathbf{R})).^{4)}$$

4) Let $X_1, \ldots, X_{n_1}$ be a base of $\mathfrak{g}^1$. Put $Y_j = [s(v), X_j]$. Then $Y_1, \ldots, Y_{n_1}$ are linear-independent. Thus the subspace $Ad(\exp ts(v))\, \mathfrak{g}^1$ is generated by $\frac{1}{t}X_1 + Y_1, \ldots, \frac{1}{t}X_{n_1} + Y_{n_1}$.

Let $X^2 \in \mathfrak{g}^2$ and $X^0 \in \mathfrak{g}^0$. Then

$$Ad\,(\exp t\, s(v))X^2 = X^2 + t[s(v),\ X^2] + \frac{t^2}{2}[s(v),\ [s(v),\ X^2]]$$

$$Ad\,(\exp t\, s(v))X^0 = X^0 + t[s(v),\ X^0].$$

If we put $Y^0 = \frac{t}{2}[s(v),\ X^2]$, then

$$Ad(\exp ts(v))X^2 = X^2 + \frac{t}{2}[s(v),\ X^2] + Ad(\exp ts(v))Y^0.$$

Therefore

$$\begin{aligned} &Ad(\exp ts(v))(\mathfrak{g}^2 + \mathfrak{g}^0) \\ &= \left\{X^2 + \frac{t}{2}[s(v),\ X^2];\ X^2 \in \mathfrak{g}^2\right\} \\ &\quad + \{X^0 + t[s(v),\ X^0];\ X^0 \in \mathfrak{g}^0\}. \end{aligned}$$

Since the mapping: $X^2 \to [s(v),\ [s(v),\ X^2]]$ of $\mathfrak{g}^2$ to $\mathfrak{g}^{-2}$ is injective (cf. [8]), we know $[s(v),\ \mathfrak{g}^2] \cap \mathfrak{k}_v^0 = (0)$ and hence

$$\begin{aligned} &\lim_{t\to\infty} Ad(\exp ts(v))(\mathfrak{g}^2 + \mathfrak{g}^0) \\ &= [s(v),\ \mathfrak{g}^2 + \mathfrak{g}^0] + \mathfrak{k}_v^0 \qquad (\text{in } Q(m,\ n_2;\ \mathbf{R})). \end{aligned}$$

q.e.d.

We can now prove Theorem 5.1. Suppose that G_c/B is compact. Then by Proposition 2.7, $G/G \cap B$ is compact. Thus we have only to verify that (3) implies (1). Suppose that $G/G \cap B$ is compact. Then $\eta(G/G \cap B)$ is compact. Therefore by Lemma 5.2, there exists $a \in G$ such that $Ad\, a\, \tilde{\mathfrak{b}} = [s(v),\ \tilde{\mathfrak{b}}] + \mathfrak{k}_v^0$. Clearly $[E,\ Ad\, a\, \tilde{\mathfrak{b}}] \subset Ad\, a\, \tilde{\mathfrak{b}}$. Then $[Ad\, a^{-1} E,\ \tilde{\mathfrak{b}}] \subset \tilde{\mathfrak{b}}$. If we write $Ad\, a^{-1} E \equiv X^{-2} + X^{-1} \pmod{\tilde{\mathfrak{b}}}$, $X^{-2} \in \mathfrak{g}^{-2}$ and $X^{-1} \in \mathfrak{g}^{-1}$. Then $[Ad\, a^{-1} E,\ E] \equiv 2X^{-2} + X^{-1} \equiv 0 \pmod{\tilde{\mathfrak{b}}}$. Therefore $X^{-2} = X^{-1} = 0$ and hence $Ada^{-1}E \in \tilde{\mathfrak{b}}$. As a result $E \in Ad\, a\, \tilde{\mathfrak{b}}$. Thus we can write $E = Y + Z$ ($Y \in \mathfrak{k}_v^0$, $Z \in [s(v),\ \tilde{\mathfrak{b}}]$). We denote by $\mathrm{Tr}\, adX$ ($X \in \mathfrak{g}$)

the trace of the linear endomorphism adX of $\mathfrak{g}$. Since $\mathfrak{k}_v^0$ is compact, $\mathrm{Tr}\, adY=0$. Clearly $\mathrm{Tr}\, adZ=0$. And hence $\mathrm{Tr}\, adE=0$. This implies $\dim \mathfrak{g}^{-2}=\dim \mathfrak{g}^2$ and $\dim \mathfrak{g}^{-1}=\dim \mathfrak{g}^1$. Therefore $\mathfrak{g}^{-2}=(ad\, s(v))^2\mathfrak{g}^2$ and hence the domain D is homogeneous (cf. [8]). On the other hand, from [2] we know $\mathfrak{g}$ is semi-simple. Then by Borel [1] or Koszul [3] we can conclude that D is symmetric.

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