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On Finsler spaces with Randers' metric and special forms of important tensors

By

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In 1941 G. Randers introduced first a special Finsler metric

$$ds=(g_{ij}(x)dx^i dx^j)^{1/2}+b_i(x)dx^i,$$

in a viewpoint of general relativity [12]*. Since then many physicists have developed the general relativity based on this metric. (See References of [5]).

From the standpoint of Finsler geometry itself Randers' metric is very interesting, because its form is simple and properties of the Finsler space equipped with this metric must be described by the ones of the Riemannian space equipped with the metric $L(x, dx)=(g_{ij}(x)dx^i dx^j)^{1/2}$ together with the 1-form $\beta(x, dx)=b_i(x)dx^i$. For example the curvature tensors R_{hijk}, P_{hijk} and S_{hijk} of the Finsler space must be written in terms of Riemannian tensors, that is, the curvature tensor, b_i and its covariant derivatives with respect to the Riemannian connection. But we have few papers concerned with the Finsler space in viewpoint of Finsler geometry [4], [5], [10], [13]. This situation seems to come from the fact that we must hit at once against insuperable difficulty of exhausting calculations to obtain the concrete form of Cartan's $\Gamma^{*i}_{j\ k}$.

The purpose of the present paper is to write the torsion and curvature tensors of the *Randers space* (the Finsler space equipped

* Numbers in brackets refer to the references at the end of the paper.

with Randers' metric) in terms of Riemannian tensors. The idea to overcome exhausting calculations is only the equation (1.3), from which the algebraic equations satisfied by the differences of $\Gamma^{*}{}_{j}{}^{i}{}_{k}$ from Christoffel symbols of the Riemannian space are derived easily and then $\Gamma^{*}{}_{j}{}^{i}{}_{k}$ are completely found by supplying certain algebraic equations. The torsion and curvature tensors are found by making use of those algebraic equations, not of the concrete form of $\Gamma^{*}{}_{j}{}^{i}{}_{k}$.

From a geometrical standpoint it seems to be more important that a Randers space is C-reducible [10] and we show in the third section that the notion of C-reducibility proposes interesting special forms of torsion and curvature tensors of Finsler space. Therefore important problems arise to study Finsler space with torsion and curvature tensors of such special forms.

The terminology and notations are referred to the author's monograph [8], which are a little different from Cartan's ones.

§1. Common quantities of Finsler spaces F^n and $^*F^n$

Let M^n be an n-dimensional differentiable manifold and F^n be a Finsler space equipped with a fundamental function $L(x, y)(y^i=\dot{x}^i)$ on M^n. If a differential 1-form $\beta(x, dx)=b_i(x)dx^i$ is given on M^n, then we obtain another Finsler space $^*F^n$ on M^n whose fundamental function is defined by

$$(1.1) \qquad {}^*L(x, y)=L(x, y)+\beta(x, y).$$

Throughout the paper we assume that $^*L(x, y)$ satisfies the ordinary conditions as fundamental function. If $L(x, y)$ is Riemannian, then $^*F^n$ is called a *Randers space*. In the first two sections we shall be concerned with a generalization $^*F^n$ of a Randers space such that $L(x, y)$ is a general Finsler metric.

It follows from (1.1) that

$$(1.2) \qquad {}^*L_i=L_i+b_i.$$

Throughout the paper we shall use the notations

$$L_i=\dot{\partial}_iL, \qquad L_{ij}=\dot{\partial}_j\dot{\partial}_iL \quad \text{and etc.}$$

Following E. Cartan we shall denote by l^i the normalized supporting element y^i/L. Then (1.2) gives the relation between the normalized covariant supporting elements:

$$(1.2') \qquad {}^*l_i = l_i + b_i .$$

Next we obtain from (1.2)

$$(1.3) \qquad {}^*L_{ij} = L_{ij} .$$

If we denote by $g_{ij}(x, y)$ the fundamental tensor $\dot{\partial}_j\dot{\partial}_i L^2/2$ and put

$$(1.4) \qquad h_{ij} = g_{ij} - l_i l_j ,$$

then (1.3) is written in the form

$$(1.3') \qquad {}^*h_{ij}/{}^*L = h_{ij}/L .$$

In virtue of (1.4) the equation (1.3′) is rewritten as the relation between the fundamental tensors:

$$(1.3'') \qquad {}^*g_{ij} = \tau(g_{ij} - l_i l_j) + {}^*l_i {}^*l_j, \qquad (\tau = {}^*L/L) .$$

From (1.3″) the relation between the covariant components of the fundamental tensors will be easily derived as follows:

$$(1.5) \qquad {}^*g^{ij} = \tau^{-1} g^{ij} + \mu l^i l^j - \tau^{-2}(l^i b^j + l^j b^i) ,$$

where we put $\mu = (Lb^2 + \beta)/({}^*L\tau^2)$, $b^2 = b_i b^i$ and $b^i = g^{ij} b_j$.

The equation (1.3) is essential to discuss the Finsler space ${}^*F^n$, because it is equivalent to (1.1) and characterizes the fundamental function ${}^*L$ of ${}^*F^n$. Further (1.3) shows that all common quantities to F^n and ${}^*F^n$ consist of L_{ij} and their successive derivatives with respect to x^i and y^i.

In particular L_{ij}, L_{ijk}, L_{ijkh} and etc. are components of tensors common to F^n and ${}^*F^n$. We have already shown $L_{ij} = h_{ij}/L$. We next treat the common tensor

$$(1.6) \qquad {}^*L_{ijk} = L_{ijk} .$$

From the equation

$$(1.7)\qquad \dot{\partial}_k h_{ij} = 2C_{ijk} - L^{-1}(l_i h_{jk} + l_j h_{ik}),$$

we obtain

$$(1.8)\qquad L_{ijk} = 2L^{-1}C_{ijk} - L^{-2}(h_{ij}l_k + h_{jk}l_i + h_{ki}l_j),$$

where we put $C_{ijk} = \dot{\partial}_k g_{ij}/2$. Therefore (1.6) is rewritten as the relation between C_{ijk} and $*C_{ijk}$:

$$(1.6')\qquad *C_{ijk} = \tau C_{ijk} + (h_{ij}m_k + h_{jk}m_i + h_{ki}m_j)/(2L),$$

where we put

$$(1.9)\qquad m_i = b_i - (\beta/L)l_i .$$

It is noted that the vector m_i is orthogonal to the supporting element.

Finally we deal with another common tensor

$$(1.10)\qquad *L_{hijk} = L_{hijk} .$$

From (1.7), (1.8) and $\dot{\partial}_j l_i = h_{ij}/L$ we obtain easily

$$(1.11)\qquad L_{hijk} = 2\dot{\partial}_k C_{hij}/L - 2(l_h C_{ijk} + l_i C_{hjk} + l_j C_{hik} + l_k C_{hij})/L^2$$
$$- (h_{hi}h_{jk} + h_{hj}h_{ki} + h_{hk}h_{ij})/L^3 + 2(h_{hi}l_j l_k + h_{hj}l_k l_i$$
$$+ h_{hk}l_i l_j + h_{ij}l_k l_h + h_{jk}l_i l_h + h_{ki}l_j l_h)/L^3 .$$

Thus (1.10) will be rewritten as the relation between $\dot{\partial}_k *C_{hij}$ and $\dot{\partial}_k C_{hij}$, which will be used later on.

§2. Cartan connection of the space $*F^n$

Continuing the last section we shall consider the common quantities

$$(2.1)\qquad \partial_k *L_{ij} = \partial_k L_{ij},$$

which are not components of a tensor. We shall be concerned with Cartan's connection of F^n and $*F^n$. The connection parameters of the connection are denoted by $(F_j{}^i{}_k, N^i_k, C_j{}^i{}_k)$. That is, the h- and v-covariant derivatives $X_{i|j}$, $X_i|_j$ of a covariant vector field X_i are defined by

$$X_{i|j}=\partial_j X_i - N^r_j \dot{\partial}_r X_i - X_r F_i{}^r{}_j\,,$$

$$X_i|_j=\dot{\partial}_j X_i - X_r C_i{}^r{}_j\,,$$

where $N^r_j=F_0{}^r{}_j\ (=y^k F_k{}^r{}_j)$ and $C_i{}^r{}_j=g^{rk}C_{ikj}$. Thus we obtain

$$L_{ij|k}=\partial_k L_{ij}-L_{ijr}N^r_k-L_{rj}F_i{}^r{}_k-L_{ir}F_j{}^r{}_k\,.$$

In virtue of $L_{ij|k}=0$ we obtain

$$\partial_k L_{ij}=L_{ijr}N^r_k+L_{rj}F_i{}^r{}_k+L_{ir}F_j{}^r{}_k\,. \tag{2.2}$$

The equation (2.1) serves the purpose to find the relation between Cartan connections of F^n and $*F^n$. For this purpose we put

$$D_j{}^i{}_k={}^*F_j{}^i{}_k-F_j{}^i{}_k\,. \tag{2.3}$$

The difference D^i_{jk} is obviously a tensor of (1, 2)-type. In virtue of (2.2) the equation (2.1) is written in the tensorial form

$$L_{ijr}D_0{}^r{}_k+L_{rj}D_i{}^r{}_k+L_{ir}D_j{}^r{}_k=0\,. \tag{2.1'}$$

In order to find the difference $D_j{}^i{}_k$, we have to construct supplementary equations to (2.1′). From (1.2) we obtain

$$\partial_j {}^*L_i=\partial_j L_i+\partial_j b_i\,. \tag{2.4}$$

From $L_{i|j}=0$ the equation (2.4) is written in the form

$${}^*L_{ir}{}^*N^r_j+{}^*L_r{}^*F_i{}^r{}_j=L_{ir}N^r_j+L_rF_i{}^r{}_j+b_{i|j}+b_rF_i{}^r{}_j\,,$$

or, by means of (1.2), (1.3) and (2.3), in the tensorial form

$$L_{ir}D_0{}^r{}_j+(l_r+b_r)D_i{}^r{}_j=b_{i|j}\,. \tag{2.4'}$$

The difference tensor $D_j{}^i{}_k$ is now found from (2.1′) and (2.4′), namely,

Proposition 1. *The Cartan connection of* $*F^n$ *is completely determined by the equations* (2.1′) *and* (2.4′) *in terms of the one of* F^n.

To prove this, we shall note the following fact:

Lemma. *The system of algebraic equations*

$$(2.5)\qquad (1)\quad L_{ir}A^r = B_i, \qquad (2)\quad (l_r + b_r)A^r = B,$$

has a unique solution (A^r) *for given* B *and* B_i *such that* $B_i l^i = 0$.

It follows from (1.4) that (1) of (2.5) is written in the form

$$(2.6)\qquad g_{ir}A^r = LB_i + l_i(l_r A^r).$$

Contraction of (2.6) by b^i gives

$$b_r A^r = LB_\beta + (\beta/L)l_r A^r,$$

where and in the remainder of the paper we shall use the subscript β to denote the contraction by b^i. Then (2) of (2.5) is written as

$$(2.7)\qquad l_r A^r = \tau^{-1}(B - LB_\beta).$$

Therefore (2.6) is written as

$$(2.6')\qquad A^i = LB^i + \tau^{-1}(B - LB_\beta)l^i,$$

which is the concrete form of the solution of (2.5).

We are now in a position to show the proof of Proposition 1. It is obvious that (2.4′) is equivalent to the two equations

$$(2.8)\qquad L_{ir}D_0{}^r{}_j + L_{jr}D_0{}^r{}_i + 2(l_r + b_r)D_i{}^r{}_j = 2E_{ij},$$

$$(2.9)\qquad L_{ir}D_0{}^r{}_j - L_{jr}D_0{}^r{}_i = 2F_{ij},$$

where we put

$$(2.10)\qquad 2E_{ij} = b_{i|j} + b_{j|i}, \qquad 2F_{ij} = b_{i|j} - b_{j|i}.$$

On the other hand (2.1′) is clearly equivalent to

$$(2.11)\qquad 2L_{jr}D_i{}^r{}_k + L_{ijr}D_0{}^r{}_k + L_{jkr}D_0{}^r{}_i - L_{kir}D_0{}^r{}_j = 0.$$

Contraction of (2.8) by y^j gives

(2.12) $$L_{ir}D_0{}^r{}_0+2(l_r+b_r)D_0{}^r{}_i=2E_{i0}\,.$$

Similarly we obtain from (2.9) and (2.11) respectively

(2.13) $$L_{ir}D_0{}^r{}_0=2F_{i0}\,,$$

(2.14) $$L_{ir}D_0{}^r{}_j+L_{jr}D_0{}^r{}_i+L_{ijr}D_0{}^r{}_0=0\,.$$

Moreover contraction of (2.12) gives

(2.15) $$(l_r+b_r)D_0{}^r{}_0=E_{00}\,.$$

Now we shall first consider (2.13) and (2.15):

(I) $$L_{ir}D_0{}^r{}_0=2F_{i0}, \qquad (l_r+b_r)D_0{}^r{}_0=E_{00}\,.$$

We can apply Lemma to (I) to obtain

(2.16) $$l_rD_0{}^r{}_0=\tau^{-1}(E_{00}-2LF_{\beta 0})\,,$$

(2.17) $$D_0{}^i{}_0=2LF^i_0+\tau^{-1}(E_{00}-2LF_{\beta 0})l^i\,,$$

where we put $F^i_0=g^{ij}F_{j0}$.

Secondly we add (2.9) and (2.14) to obtain

(2.18) $$L_{ir}D_0{}^r{}_j=G_{ij}\,,$$

where we put

(2.19) $$G_{ij}=F_{ij}-L_{ijr}D_0{}^r{}_0/2\,.$$

The equation (2.12) is written in the form

(2.12′) $$(l_r+b_r)D_0{}^r{}_j=G_j\,,$$

where we put

(2.20) $$G_j=E_{j0}-L_{jr}D_0{}^r{}_0/2\,.$$

Substituting from (2.17) in (2.19) we obtain

$$(2.19') \qquad G_{ij}=F_{ij}-LL_{ijr}F^r_0+L_{ij}(E_{00}-2LF_{\beta 0})/(2^*L).$$

In virtue of (2.13) G_j are written as

$$(2.20') \qquad G_j=E_{j0}-F_{j0}.$$

Thus we have obtained the system of equations (2.18) and (2.12′):

$$(\mathrm{II}) \qquad L_{ir}D_0{}^r{}_j=G_{ij}, \qquad (l_r+b_r)D_0{}^r{}_j=G_j.$$

Applying Lemma to (II) we obtain

$$(2.21) \qquad l_rD_0{}^r{}_j=\tau^{-1}(G_j-LG_{\beta j}),$$

$$(2.22) \qquad D_0{}^i{}_j=LG^i_j+\tau^{-1}(G_j-LG_{\beta j})l^i,$$

where we put $G^i_j=g^{ir}G_{rj}$.

Finally we deal with (2.11) and (2.8):

$$(\mathrm{III}) \qquad L_{ir}D_j{}^r{}_k=H_{ijk}, \qquad (l_r+b_r)D_j{}^r{}_k=H_{jk},$$

where we put

$$(2.23) \qquad \begin{cases} H_{ijk}=2^{-1}(L_{jkr}D_0{}^r{}_i-L_{kir}D_0{}^r{}_j-L_{ijr}D_0{}^r{}_k), \\ H_{jk}=E_{jk}-2^{-1}(L_{jr}D_0{}^r{}_k+L_{kr}D_0{}^r{}_j). \end{cases}$$

In virtue of (2.22) H_{ijk} and H_{jk} are written in terms of known quantities. Then, applying Lemma to the system of equations (III) we can find the concrete representations of $D_j{}^i{}_k$ and the proof is completed.

§3. On C-reducible Finsler spaces

The remainder of the paper we shall restrict our consideration to the case where $L(x, y)$ is Riemannian, i.e., the space $^*F^n$ is a Randers space. Then (1.6′) and (1.3′) give

$$(3.1) \qquad {}^*C_{ijk}={}^*h_{ij}{}^*M_k+{}^*h_{jk}{}^*M_i+{}^*h_{ki}{}^*M_j,$$

where we put $^*M_i=m_i/(2^*L)$. In a previous paper [10] the author introduced a notion of C-reducibility of Finsler space:

Definition. *A non-Riemannian Finsler space of dimension $n \geqq 3$ is called C-reducible if the torsion tensor C_{ijk} is of the form*

$$\boxed{C_{ijk} = h_{ij}M_k + h_{jk}M_i + h_{ki}M_j}\,. \tag{3.2}$$

It is remarked that the equation (3.2) holds good for any 2-dimensional Finsler space and $M_i = C^j_{ij}/(n+1)$. It then follows from (3.1) and (1.9) that a Randers space $^*F^n$ $(n \geqq 3)$ is C-reducible, provided $\beta \neq 0$.

In this section we shall treat a general C-reducible Finsler space F^n and consider components of the $(v)hv$-torsion tensor P_{hij}, hv-curvature tensor P_{hijk}, v-curvature tensor S_{hijk} and another important tensor T_{hijk}.

First components P_{hij} are equal to $C_{hij|0}$. Hence (3.2) gives immediately

$$\boxed{P_{hij} = h_{hi}P_j + h_{ij}P_h + h_{jh}P_i}\,, \qquad (P_i = M_{i|0})\,. \tag{3.3}$$

Secondly components P_{hijk} are written in the two forms as follows:

$$\begin{aligned} P_{hijk} &= C_{ijk|h} - C_{hjk|i} + C_{hjr}P^r_{ik} - C_{ijr}P^r_{hk} \\ &= P_{ijk}|_h - P_{hjk}|_i + P_{ikr}C_h{}^r{}_j - P_{hkr}C_i{}^r{}_j\,. \end{aligned}$$

Therefore (3.2) or (3.3) gives the following form of P_{hijk}:

$$\boxed{P_{hijk} = h_{hj}P_{ik} - h_{ij}P_{hk} + h_{hk}P'_{ji} - h_{ik}P'_{jh} - h_{jk}P''_{hi}}\,, \tag{3.4}$$

where we put

$$\begin{cases} P_{ik} = (M_rP^r/2)h_{ik} - M_{k|i} + M_kP_i \\ \qquad = (M_rP^r/2)h_{ik} - (l_iP_k + l_kP_i)/L - P_k|_i + M_kP_i\,, \\ P'_{ik} = -(M_rP^r/2)h_{ik} - M_{i|k} - P_iM_k \\ \qquad = -(M_rP^r/2)h_{ik} - (l_iP_k + l_kP_i)/L - P_k|_i - P_kM_i\,, \\ P''_{ik} = P_{ik} - P_{ki} = -P'_{ik} + P'_{ki}\,. \end{cases} \tag{3.5}$$

Hence we have

$$P_k|_i+(l_kP_i+l_iP_k)/L=M_{k|i}. \tag{3.6}$$

Thirdly components S_{hijk} are given by

$$S_{hijk}=C_{hkr}C_i{}^r{}_j-C_{hjr}C_i{}^r{}_k.$$

Hence (3.2) gives immediately

$$\boxed{S_{hijk}=h_{hk}M_{ij}-h_{hj}M_{ik}+h_{ij}M_{hk}-h_{ik}M_{hj}}\,, \tag{3.7}$$

where we put $M_{ij}=(M_rM^r/2)h_{ij}+M_iM_j$.

Finally we consider the tensor

$$T_{hijk}=LC_{hij}|_k+C_{hij}l_k+C_{hik}l_j+C_{hjk}l_i+C_{ijk}l_h. \tag{3.8}$$

We were recently conscious of the importance of this tensor [7], [11]. It was proved [10] that there exists a scalar M such that

$$M_i|_j+(M_il_j+M_jl_i)/L=Mh_{ij}.$$

In virtue of this and $M_i|_j=M_j|_i$ we obtain easily

$$\boxed{T_{hijk}=LM(h_{hi}h_{jk}+h_{hj}h_{ki}+h_{hk}h_{ij})}\,. \tag{3.9}$$

These simple forms (3.3), (3.4), (3.7) and (3.9) of important tensors lead us to some interesting problems of Finsler geometry. In fact these tensors are all equal to zero if the space is Riemannian and $P_{hij}=P_{hijk}=0$ if the space is locally Minkowskian. Thus some important problems arose from this situation, for example, to consider a Finsler space with $S_{hijk}=0$ (Brickell's theorem [2]), or $P_{hij}=0$ (Landsberg space [5]), or $T_{hijk}=0$ [7]; those correspond only to the trivial problem of Riemannian geometry to consider a Riemannian space with vanishing curvature. So far as the author knows, interesting special forms ($\neq 0$) of these tensors don't be known yet except $L^2S_{hijk}=S(h_{hj}h_{ik}-h_{hk}h_{ij})$ noticed by the author [9]. We now obtain more interesting problems of Finsler geometry, for example, to consider a Finsler space with the

$(v)hv$-torsion tensor P_{hij} of the special form (3.3). It is noted that the author gave another C-reducible space in the paper [10]. It is remarkable that *the tensor* h_{ij} *plays an important role in those special forms.*

§4. The v-curvature tensor of a Randers space

We shall return to the consideration of a Randers space $^*F^n$. Then (1.6′) reduces to

$$(4.1)\qquad {}^*C_{ijk}=(h_{ij}m_k+h_{jk}m_i+h_{ki}m_j)/(2L).$$

In virtue of (1.5) we obtain easily

$$(4.2)\qquad \begin{aligned}{}^*C_j{}^i{}_k=&(h^i_jm_k+h^i_km_j+h_{jk}m^i)/(2^*L)\\ &-(L/2^*L^2)(2m_jm_k+(b^2-\beta^2/L^2)h_{jk})l^i,\end{aligned}$$

from which we obtain $^*C_j=(n+1)m_j/(2^*L)$, or, in virtue of (1.2′) and (1.9)

$$(4.3)\qquad \dot\partial_j\log(^*g)^{1/2}=(n+1)(b_j-(\beta/^*L)^*l_j)/(2L),$$

where $^*g=\det{}^*g_{ij}$. This equation verifies Deicke's theorem [3] such that $^*C_j=0$ is necessary and sufficient for $^*F^n$ to be Riemannian, because $\beta=0$ is immediately obtained from vanishing of the right-hand side of (4.3).

We see from (4.1) and (4.2)

$$(4.4)\qquad \begin{aligned}{}^*C_{ijr}{}^*C_h{}^r{}_k=&(m^2h_{ij}h_{hk}+2h_{ij}m_hm_k+2h_{hk}m_im_j+h_{ih}m_jm_k\\ &+h_{ik}m_jm_h+h_{jk}m_im_h+h_{jh}m_im_k)/(4L^*L),\end{aligned}$$

where we put $m^2=m^im_i$.

Now we shall consider the v-curvature tensor $^*S_{hijk}={}^*C_{ijr}{}^*C_h{}^r{}_k-{}^*C_{ikr}{}^*C_h{}^r{}_j$. It follows from (4.4) that

Proposition 2. *The v-curvature tensor $^*S_{hijk}$ of a Randers space $^*F^n$ is of the form*

$$(4.5)\qquad {}^*L^2{}^*S_{hijk}=h_{hk}m_{ij}-h_{hj}m_{ik}+h_{ij}m_{hk}-h_{ik}m_{hj},$$

where we put

$$(4.6)\qquad m_{ij}=(\tau/4)((m^2/2)h_{ij}+m_i m_j).$$

The form (4.5) of $^*S_{hijk}$ has been known from (3.7), but it gives $^*S_{hijk}$ in terms of Riemannian tensors.

In virtue of (1.5) the Ricci tensor $^*S_{ik}={}^*g^{hj}\,{}^*S_{hijk}$ is of the form

$$(4.7)\qquad {}^*L^2\,{}^*S_{ik}=-((n-1)m^2/4\tau){}^*h_{ik}-((n-3)/4)m_i m_k.$$

Hence we have

Theorem 1. *In a Randers space of dimension* $n\geqq 4$ *there exists a scalar* H *such that the matrix* $\|{}^*L^2\,{}^*S_{ik}+H\,{}^*h_{ik}\|$ *is of rank less than two.*

It is well known that $^*S_{hijk}$ of any two-dimensional Finsler space vanishes. As to a three-dimensional Finsler space it is shown [9] that $^*S_{hijk}$ is always of the form

$$(4.8)\qquad {}^*L^2\,{}^*S_{hijk}={}^*S({}^*h_{hj}\,{}^*h_{ik}-{}^*h_{hk}\,{}^*h_{ij}),$$

which implies ${}^*L^2\,{}^*S_{ik}={}^*S\,{}^*h_{ik}$. If a Finsler space $^*F^n$ of dimension more than three is such that $^*S_{hijk}$ is of the form (4.8), we shall call $^*F^n$ *S3-like*. We consider an S3-like Randers space. It then follows from (4.8), (4.5) and (1.3′) that

$$S(h_{hj}h_{ik}-h_{hk}h_{ij})=h_{ij}m_{hk}+h_{hk}m_{ij}-h_{ik}m_{hj}-h_{hj}m_{ik},$$

where we put $S={}^*S\tau^2$. Contraction by g^{hj} gives

$$S(n-2)h_{ik}=-(n-3)m_{ik}-(m_{hj}g^{hj})h_{ik}.$$

Moreover contraction by g^{ik} gives $S(n-1)=-2m_{hj}g^{hj}$, hence we obtain $(S/2)h_{ik}=-m_{ik}$, from which and the definition of m_{ij} it follows that $m_i=0$, i.e., $\beta=0$ from (1.9). Thus

Theorem 2. *A Randers space is S3-like if and only if* $\beta=0$, *that is, the space is Riemannian.*

The following theorem will be easily verified from (4.5):

Theorem 3. *The v-curvature tensor* $*S_{hijk}$ *of a Randers space vanishes if and only if* $\beta=0$, *that is, the space is Riemannian.*

It is noticed that Brickell's theorem [2] can not be directly applied to a Randers space, because the fundamental function $*L(x, y)$ is not symmetric.

§5. The tensor T_{hijk} of a Randers space

We shall treat the tensor T_{hijk} given by (3.8) of a Randers space $*F^n$. In virtue of (1.11), (4.1), (1.2′) and (1.9) the equation (1.10) for $*F^n$ is written in the form

$$(5.1)\qquad \dot{\partial}_k{}^*C_{hij} = -(h_{hi}n_{jk}+h_{hj}n_{ik}+h_{hk}n_{ij}+h_{ij}n_{hk}+h_{jk}n_{hi}+h_{ki}n_{hj})/(2L^2) + (\beta/2L^3)(h_{hi}h_{jk}+h_{hj}h_{ki}+h_{hk}h_{ij}),$$

where we put $n_{ij}=l_im_j+l_jm_i$. Therefore we obtain from (4.4) and (5.1)

$$(5.2)\qquad {}^*C_{hij}|_k = -(h_{hi}{}^*n_{jk}+h_{hj}{}^*n_{ki}+h_{hk}{}^*n_{ij}+h_{ij}{}^*n_{hk} + h_{jk}{}^*n_{hi}+h_{ik}{}^*n_{hj}) - (T/4L^3\,{}^*L)(h_{hi}h_{jk} + h_{hj}h_{ki}+h_{hk}h_{ij}),$$

where we put ${}^*n_{ij}={}^*l_im_j+{}^*l_jm_i$ and $T=L^2b^2+\beta^2+2L\beta$.

Consequently we obtain ${}^*T_{hijk}$ of a Randers space $*F^n$ as follows:

Proposition 3. *The tensor* ${}^*T_{hijk}$ *of a Randers space is written in the form*

$$(5.3)\qquad {}^*T_{hijk} = -(T/4L^3)(h_{hi}h_{jk}+h_{hj}h_{ki}+h_{hk}h_{ij}),$$

where $T=L^2b^2+\beta^2+2L\beta$.

We consider ${}^*T_{hijk}=0$. Because $T=0$ reduces easily $\beta=0$, we obtain

Theorem 4. *The tensor* ${}^*T_{hijk}$ *of a Randers space vanishes if and only if* $\beta=0$, *that is, the space is Riemannian.*

§6. The $(v)hv$-torsion tensor and hv-curvature tensor of a Randers space

In virtue of (1.8) the tensor G_{ij} of a Randers space $*F^n$ given by (2.19′) is written as

$$G_{ij} = F_{ij} + (l_i F_{j0} + l_j F_{i0})/L + G h_{ij}, \tag{6.1}$$

where we put $G = (E_{00} - 2LF_{\beta 0})/(2L*L)$. Differentiation of (2.13) and (2.15) by y^j leads us respectively to

$$L_{ir}\dot{\partial}_j D_0{}^r{}_0 = 2F_{ij} - L_{ijr} D_0{}^r{}_0,$$

$$(l_r + b_r)\dot{\partial}_j D_0{}^r{}_0 = 2E_{j0} - L_{jr} D_0{}^r{}_0.$$

The right-hand sides of the above equations are equal to $2G_{ij}$ and $2G_j$ respectively, hence Lemma and the system of equations (II) of the second section yield

$$\dot{\partial}_j D_0{}^r{}_0 = 2D_0{}^r{}_j. \tag{6.2}$$

We are next concerned with the difference $\dot{\partial}_k D_0{}^i{}_j - D_j{}^i{}_k$, which are nothing but the components of the $(v)hv$-torsion tensor $*P^i_{jk}$, because of the definition

$$*P^i_{jk} = \dot{\partial}_k *N^i_j - *F_j{}^i{}_k$$

and (2.3). Since $*l_i *P^i_{jk} = *l_i *C_j{}^i{}_{k|0} = 0$, we obtain from (1.3″)

$$*P_{hjk} = *L L_{hr}(\dot{\partial}_k D_0{}^r{}_j - D_j{}^r{}_k).$$

It follows from (2.18) that

$$L_{hr}\dot{\partial}_k D_0{}^r{}_j = \dot{\partial}_k G_{hj} - L_{hkr} D_0{}^r{}_j,$$

which and (2.11) yield

$$*P_{hjk} = *L(\dot{\partial}_k G_{hj} - L_{hkr} D_0{}^r{}_j - H_{hjk}),$$

or, we obtain in virtue of (6.2)

$$*P_{hjk} = -(*L/2)(L_{hjkr}D_0{}^r{}_0 + L_{jkr}D_0{}^r{}_h + L_{hkr}D_0{}^r{}_j + L_{hjr}D_0{}^r{}_k).$$

Consequently, by means of (1.11), (2.13), (2.16), (2.20′) and (6.1) we obtain

Proposition 4. *The* (*v*)*hv-torsion tensor* $*P_{hjk}$ *of a Randers space is written in the form*

$$*P_{hjk} = h_{hj}p_k + h_{jk}p_h + h_{kh}p_j, \tag{6.3}$$

where we put

$$2p_j = (*L/L^2)F_{j0} + E_{j0}/L - F_{\beta j} - pl_j - Gm_j, \tag{6.4}$$

and $G = (E_{00} - 2LF_{\beta 0})/(2L*L)$, $p = \tau(2G + F_{\beta 0}/*L)$.

We now find a condition for a Randers space $*F^n$ such that the torsion $*P_{hjk}$ vanishes, namely, $*F^n$ is a *Landsberg space*. It follows easily from (6.3) that $*P_{hjk} = 0$ is equivalent to $p_j = 0$. We shall first treat the weaker condition $p_\beta = p_i b^i = 0$. From (6.4) we obtain

$$\begin{aligned} 4*LL^3 p_\beta = {} & [\beta(6L^2F_{\beta 0} + 2L^2E_{\beta 0} - \beta E_{00}) - L^2b^2E_{00}] \\ & + 2L[(L^2 + \beta^2 + L^2b^2)F_{\beta 0} + L^2E_{\beta 0} - \beta E_{00}]. \end{aligned} \tag{6.5}$$

The term in the first (resp. second) brackets of the right-hand side of (6.5) is a polynomial of the fourth (resp. third) order with respect to y^i. Therefore $p_\beta = 0$ is equivalent to

$$\beta(6L^2F_{\beta 0} + 2L^2E_{\beta 0} - \beta E_{00}) - L^2b^2E_{00} = 0, \tag{6.6}$$

$$(L^2 + \beta^2 + L^2b^2)F_{\beta 0} + L^2E_{\beta 0} - \beta E_{00} = 0. \tag{6.7}$$

Assuming $\beta \neq 0$ (6.6) shows that β must be a factor of E_{00}, so that there exists a covariant vector $c_i(x)$ such that $2E_{ij} = b_ic_j + b_jc_i$. Then (6.6) reduces to $L^2(6F_{\beta 0} + \beta c_\beta) - \beta^2 c_0 = 0$, which yields $c_0 = 0$ ($c_i = 0$) and $E_{ij} = 0$. Now (6.6) reduces only to $F_{\beta 0} = 0$ ($F_{\beta j} = 0$). From $E_{ij} = 0$ and $F_{\beta j} = 0$ the equation (6.7) becomes trivial. It is noted that this condition is equivalent to $G = 0$.

We now return to the condition $p_j = 0$, which reduces to $F_{j0} = 0$ ($F_{ij} = 0$) only. As a consequence we have

Theorem 5. *A Randers space $^*F^n$ is a Landsberg space* ($^*P_{hjk}=0$) *if and only if the covariant vector field b_i is parallel with respect to the Riemannian connection of the Riemannian space F^n.*

This is a generalization of Theorem 1′ of the paper [5], in which we referred to isothermal coordinates. Further it is known [10] that the Randers space stated in Theorem 5 is Berwald's affinely-connected space.

If a Randers space $^*F^n$ is a Landsberg space (*Landsberg-Randers space*), then we obtain $D_j{}^i{}_k=0$ from Proposition 1, so that $^*F_j{}^i{}_k$ are nothing but Christoffel symbols constructed from $g_{ij}(x)$. Then the h-curvature tensor $^*R_h{}^i{}_{jk}$ of $^*F^n$ is of the form

$$^*R_h{}^i{}_{jk}=R_h{}^i{}_{jk}+{}^*C_h{}^i{}_rR_0{}^r{}_{jk},$$

where $R_h{}^i{}_{jk}$ are components of the Riemannian curvature tensor. The Ricci identities lead us to

$$b_{i|j|k}-b_{i|k|j}=-b_rR_i{}^r{}_{jk}=R_{\beta ijk}=0.$$

Paying attention to this we obtain in virtue of (1.3″) and (4.1)

$$^*R_{hijk}=\tau R_{hijk}+(m_hR_{0ijk}-m_iR_{0hjk})/(2L). \tag{6.8}$$

We shall consider a condition for a Landsberg-Randers space $^*F^n$ to be *of scalar curvature* in Berwald's sense [1]. Since the condition is $^*R_{0i0k}={}^*R{}^*L^2{}^*h_{ik}$, we obtain from (6.8) and (1.3′)

$$R_{0i0k}={}^*R{}^*L^2h_{ik}. \tag{6.9}$$

Because R_{0i0k} is a polynomial of the second order with respect to y^j and $^*L=L+\beta$, it is easily seen that $^*R$ is not a constant, provided $\beta\neq 0$. If we put $RL^2={}^*R{}^*L^2$ and differentiate (6.9) by y^j, then we obtain

$$R_{ji0k}+R_{0ijk}=L^2h_{ik}\dot{\partial}_jR+2RLh_{ik}l_j-RL(l_ih_{jk}+l_kh_{ji}).$$

Subtraction from this the equation obtained by interchanging indices j and k yields

$$3R_{0ijk}=h_{ik}T_j-h_{ij}T_k, \tag{6.10}$$

where we put $T_j=L^2\dot{\partial}_jR+3RLl_j$. Differentiating (6.10) by y^h and making use of $R_{hijk}+R_{ihjk}=0$ we obtain $R=$constant, so that F^n is of constant curvature R. Consequently we have

Theorem 6. *A Landsberg-Randers space is of scalar curvature $^*R$ if and only if F^n is of constant curvature R, where $^*R=RL^2/^*L^2$. The space is of constant curvature if and only if $\beta=0$ and F^n is of constant curvature.*

We return to the consideration of a general Randers space and find components of the hv-curvature tensor $^*P_{hijk}$. It is well known that these are derived from $^*P_{hjk}$ as

$$^*P_{hijk}=\dot{\partial}_h{}^*P_{ijk}-\dot{\partial}_i{}^*P_{hjk}+{}^*P_{hjr}{}^*C_i{}^r{}_k-{}^*P_{ijr}{}^*C_h{}^r{}_k.$$

Thus, from (6.3) and (4.2), we have

Proposition 6. *The hv-curvature tensor $^*P_{hijk}$ of a Randers space $^*F^n$ is written in the form*

$$^*P_{hijk}=h_{hj}\underset{(1)}{P}_{ik}-h_{ij}\underset{(1)}{P}_{hk}+h_{hk}\underset{(2)}{P}_{ji}-h_{ik}\underset{(2)}{P}_{jh}-h_{jk}\underset{(3)}{P}_{hi}, \tag{6.11}$$

where we put

$$\begin{cases} \underset{(1)}{P}_{ij}=p_\beta h_{ij}/(4^*L)-(l_ip_j+l_jp_i)/L-\dot{\partial}_ip_j+m_ip_j/(2^*L), \\ \underset{(2)}{P}_{ij}=-p_\beta h_{ij}/(4^*L)-(l_ip_j+l_jp_i)/L-\dot{\partial}_ip_j-p_im_j/(2^*L), \\ \underset{(3)}{P}_{ij}=\underset{(1)}{P}_{ij}-\underset{(1)}{P}_{ji}=-\underset{(2)}{P}_{ij}+\underset{(2)}{P}_{ji}. \end{cases} \tag{6.12}$$

§7. The $(v)h$-torsion tensor and h-curvature tensor of a Randers space.

We shall consider the $(v)h$-curvature tensor $^*R^i_{jk}$ defined by

$$^*R^i_{jk}=\partial_k{}^*N^i_j-\partial_j{}^*N^i_k-{}^*N^r_k\dot{\partial}_r{}^*N^i_j+{}^*N^r_j\dot{\partial}_r{}^*N^i_k,$$

which is the contracted tensor $*R_0{}^i{}_{jk}$ of the h-curvature tensor $*R_h{}^i{}_{jk}$. Paying attention to (2.3) and $P^i_{jk}=0$ we obtain

$$(7.1)\qquad *R^i{}_{jk}=R_0{}^i{}_{jk}+D_0{}^i{}_{j|k}-D_0{}^i{}_{k|j}-D_0{}^r{}_k\dot{\partial}_r D_0{}^i{}_j+D_0{}^r{}_j\dot{\partial}_r D_0{}^i{}_k,$$

where $R_0{}^i{}_{jk}$ is the contracted tensor of the curvature tensor $R_h{}^i{}_{jk}$ of F^n. In virtue of (1.3″) the covariant components $*R_{hjk}$ are written in the form

$$(7.2)\qquad *R_{hjk}=\tau R_{0hjk}+*LL_{hi}(D_0{}^i{}_{j|k}-D_0{}^i{}_{k|j}-D_0{}^r{}_k\dot{\partial}_r D_0{}^i{}_j + D_0{}^r{}_j\dot{\partial}_r D_0{}^i{}_k).$$

In order to discuss a condition for $*F^n$ to be of scalar curvature or of constant curvature in Berwald's sense [1] it is sufficient to find moreover contracted tensor $*R_{h0k}$. From (2.18) we obtain

$$L_{hi}(D_0{}^i{}_{j|k}-D_0{}^i{}_{k|j})y^j=G_{h0|k}-G_{hk|0},$$

or from (6.1)

$$=2F_{h0|k}-F_{hk|0}-(l_hF_{k0|0}+l_kF_{h0|0})/L-G_{|0}h_{hk}.$$

In virtue of Bianchi identities we obtain the symmetric form as follows:

$$(7.3)\qquad L_{hi}(D_0{}^i{}_{j|k}-D_0{}^i{}_{k|j})y^j=F_{h0|k}+F_{k0|h}-(l_hF_{k0|0}+l_kF_{h0|0})/L-G_{|0}h_{hk}.$$

Next it follows from (2.18) that

$$L_{hi}(D_0{}^r{}_k\dot{\partial}_r D_0{}^i{}_j-D_0{}^r{}_j\dot{\partial}_r D_0{}^i{}_k)y^j=D_0{}^r{}_k y^j\dot{\partial}_r G_{hj}-D_0{}^r{}_0\dot{\partial}_r G_{hk}.$$

In virtue of (2.22) and (6.1) we have

$$D_0{}^r{}_k y^j\dot{\partial}_r G_{hj}=-LG^2h_{hk}+F_{h0}F_{k0}/L-LF_{rh}F^r_k-F_{r0}F^r_0 l_hl_k/L -2G(l_hF_{k0}+l_kF_{h0})-(l_hF^r_k+l_kF^r_h)F_{r0}.$$

From (2.17) we obtain

$$D_0{}^r{}_0\dot{\partial}_r G_{hk}=2Lh_{hk}F^r_0\dot{\partial}_r G+4F_{h0}F_{k0}/L-2G(l_hF_{k0}+l_kF_{h0})$$
$$-2(l_hF^r_k+l_kF^r_h)F_{r0}\,.$$

Therefore we obtain

$$L_{hi}(D_0{}^r{}_k\dot{\partial}_r D_0{}^i{}_j-D_0{}^r{}_j\dot{\partial}_r D_0{}^i{}_k)y^j=-L(G^2+2F^r_0\dot{\partial}_r G)h_{hk}$$
$$-F_{r0}F^r_0 l_h l_k/L-LF_{rh}F^r_k-3F_{h0}F_{k0}/L+(l_hF_{rk}+l_kF_{rh})F^r_0\,. \tag{7.4}$$

Consequently substitution from (7.3) and (7.4) in (7.2) yields

Proposition 6. *The components $^*R_{i0j}$ of the contracted (v)h-torsion tensor or the contracted h-curvature tensor of a Randers space $^*F^n$ are written in the form*

$$^*R_{i0j}=\tau R_{0i0j}+{}^*LLG'h_{ij}+L^2K_{ij}-L(l_iK_{j0}+l_jK_{i0})+K_{00}l_il_j\,, \tag{7.5}$$

where R_{0i0j} is the contracted curvature tensor of F^n and we put

$$G'=G^2+2F^r_0\dot{\partial}_r G-G_{|0}/L\,, \tag{7.6}$$

$$K_{ij}=L(F_{i0|j}+F_{j0|i}+LF_{ri}F^r_j+3F_{i0}F_{j0}/L)/{}^*L^2\,. \tag{7.7}$$

Now, according to the definition of Berwald [1], the space $^*F^n$ is *of scalar curvature* $^*R$ if the equation $^*R_{i0j}={}^*R{}^*L^2{}^*h_{ij}$ holds good. If the scalar $^*R$ is constant, then $^*F^n$ is called to be *of constant curvature* $^*R$. In virtue of (1.3') the equation is written as $^*R_{i0j}={}^*R{}^*L^3h_{ij}/L$.

Lemma. *If the Riemannian space F^n is such that there exists a scalar R satisfying the equation $R_{0i0j}=RL^2h_{ij}$, then F^n is of constant curvature R.*

Proof. Differentiating the given equation by y^k and y^h and referring to the identities satisfied by R_{hijk} we obtain

$$3R_{hikj}=3R(h_{hk}h_{ij}-h_{hj}h_{ik})+Lh_{hi}(l_k\dot{\partial}_jR-l_j\dot{\partial}_kR)$$
$$-h_{hj}(Ll_i\dot{\partial}_kR+3Rl_il_k)+h_{hk}(Ll_i\dot{\partial}_jR+3Rl_il_j)$$

$$+h_{ij}(L^2\dot{\partial}_h\dot{\partial}_k R+2Ll_h\dot{\partial}_k R+Ll_k\dot{\partial}_h R+Rl_hl_k)$$

$$-h_{ik}(L^2\dot{\partial}_h\dot{\partial}_j R+2Ll_h\dot{\partial}_j R+Ll_j\dot{\partial}_h R+Rl_hl_j).$$

Thus the identity $R_{hikj}+R_{ihkj}=0$ is written in the form

$$2Lh_{hi}(l_k\dot{\partial}_j R-l_j\dot{\partial}_k R)+h_{ij}R_{hk}+h_{hj}R_{ik}-h_{ik}R_{hj}-h_{hk}R_{ij}=0,$$

where we put $R_{ij}=L^2\dot{\partial}_i\dot{\partial}_j R+L(l_i\dot{\partial}_j R+l_j\dot{\partial}_i R)-2Rl_il_j$. Contraction by g^{hi} gives immediately $l_k\dot{\partial}_j R-l_j\dot{\partial}_k R=0$, so that $\dot{\partial}_j R=0$. Therefore the proof is completed.

From Lemma we can show immediately

Theorem 7. *Assume that the covariant vector b_i satisfies the equation $K_{ij}=Kg_{ij}$. Then the Randers space $*F^n$ is of scalar curvature $*R$ if and only if the Riemannian space F^n is of constant curvature R, and $*R=(*LL^2R+*LL^2G'+L^3K)/*L^3$.*

It is rather complicated to discuss a condition for $*F^n$ to be of scalar curvature or even of constant curvature [6]. If $*F^n$ is of constant curvature $*R$, then (7.5) is written in the form

$$L^3A^{(4)}_{ij}+L^2A^{(5)}_{ij}=(B^{(5)}+LB^{(4)})(L^2g_{ij}-y_iy_j),$$

where $A^{(r)}_{ij}$ and $B^{(r)}$ are polynomials of the order r with respect to y^i defined by

(7.8) $$A^{(4)}_{ij}=L^2K^{(2)}_{ij}-y_iK^{(2)}_{j0}-y_jK^{(2)}_{i0}+F_{r0}F^r_0y_iy_j+(L^2+3\beta^2)R_{0i0j},$$

(7.9) $$A^{(5)}_{ij}=L^4K^{(1)}_{ij}-L^2y_iK^{(1)}_{j0}-L^2y_jK^{(1)}_{i0}+(3L^2\beta+\beta^3)R_{0i0j},$$

(7.10) $$K^{(2)}_{ij}=L^2F_{ri}F^r_j+3F_{i0}F_{j0},\qquad K^{(1)}_{ij}=F_{i0|j}+F_{j0|i},$$

(7.11) $$B^{(5)}=*R(5L^4\beta+10L^2\beta^3+\beta^5)-L^2G^{(3)}-\beta G^{(4)},$$

(7.12) $$B^{(4)}=*R(L^4+10L^2\beta^2+5\beta^4)-G^{(4)}-\beta G^{(3)},$$

(7.13) $$\begin{cases} G^{(3)}=-3E_{00}F_{\beta 0}+2L^2F_{r\beta}F^r_0-E_{00|0}/2+\beta(3E_{r0}F^r_0 \\ \qquad +F_{r0|0}b^r+F_{r0}F^r_0), \\ G^{(4)}=(3/4)(E_{00})^2+3L^2(F_{\beta 0})^2+L^2(3E_{r0}F^r_0+F_{r0|0}b^r \\ \qquad +F_{r0}F^r_0)+\beta(2L^2F_{r\beta}F^r_0-E_{00|0}/2). \end{cases}$$

It is observed that $K^{(r)}_{ij}$ and $G^{(r)}$ are also polynomials of the order r with respect to y^i. Therefore we obtain

$$(7.14)\qquad \begin{cases} L^2 A^{(4)}_{ij} = B^{(4)}(L^2 g_{ij} - y_i y_j), \\ L^2 A^{(5)}_{ij} = B^{(5)}(L^2 g_{ij} - y_i y_j). \end{cases}$$

Hence it is obvious that L^2 must be a factor of $B^{(4)}$ and $B^{(5)}$. This reduces to

$$(7.15)\qquad {}^*R\beta^4 - (3/4)(E_{00})^2 + \beta E_{00|0}/2 = L^2 D^{(2)},$$

$$(7.16)\qquad 4{}^*R\beta^3 - \beta(3E_{r0}F^r_0 + F_{r0|0}b^r + F_{r0}F^r_0) + E_{00|0}/2 + 3E_{00}F_{\beta 0} = L^2 D^{(1)},$$

where $D^{(r)}$ is a polynomial of the order r with respect to y^i. Then (7.14) reduces to

$$(7.17)\qquad \begin{cases} A^{(4)}_{ij} = B^{(2)}(L^2 g_{ij} - y_i y_j), \\ A^{(5)}_{ij} = B^{(3)}(L^2 g_{ij} - y_i y_j), \end{cases}$$

where $B^{(r)}$ is a polynomial of the order r with respect to y^i. Summarizing the above we obtain

Theorem 8. *A Randers space* $^*F^n$ *is of constant curvature* $^*R$ *if and only if* (7.15), (7.16) *and* (7.17) *are satisfied.*

The condition $K_{ij} = K g_{ij}$ of Theorem 7 is imposed on the skew-symmetric parts of $b_{i|j}$. We shall now consider the condition imposed on the symmetric parts of $b_{i|j}$; $E_{ij} = E g_{ij}$, that is, b_i is supposed to be conformally Killing. It then follows from (7.15) easily that

Theorem 9. *Assume that the covariant vector* b_i *is conformally Killing. If the Randers space* $^*F^n$ *is of constant curvature, the curvature vanishes.*

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