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Faithfully flatness of extensions of a commutative ring

By

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All rings are assumed to be commutative and contain an identity element. Moreover, when we write "$R \subseteqq S$", we mean that R is a subring of a ring S and that the identity of R is the identity of S. Let $a_1, \dots, a_s$ be elements of a ring R. Then $(a_1, \dots, a_s)$ means the ideal of R generated by $a_1, \dots, a_s$. The symbol $\subset$ means proper inclusion.

Let R and S be rings with $R \subseteqq S$. If S is generated as a ring by a set of elements in S over R, then S may be best described by an exact sequence of R-homomorphisms

$$(*) \qquad 0 \longrightarrow I \longrightarrow R[X] \longrightarrow S \longrightarrow 0$$

where X is a set of variables over R and $R[X]$ is the polynomial ring in X over R. If $w = r_0 + r_1 X^{(1)} + r_2 X^{(2)} + \cdots + r_n X^{(n)}$ is an element of $R[X]$ where $r_0, r_1, \dots, r_n \in R$ and $X^{(i)}$ monomials with degree $\geqq 1$ such that $X^{(i)} \neq X^{(j)}$ if $i \neq j$, then we define $c(w)$ (and $c'(w)$) to be $(r_0, r_1, \dots, r_n)$ (and $(r_1, \dots, r_n)$).

We say that an R-module M is faithfully flat if M is flat over R and $PM \subset M$ for any maximal ideal P of R. Let the notations be as above. In this note we seek some conditions which are necessary and sufficient in order that S is, as an R-module, faithfully flat. They will be characterized in terms of the R-module I, which is also the ideal in $R[X]$.

In order to prove the Theorem 1 we need the following two Lemmas. Lemmas A and B may be found in [1, Chapter 1, § 3, Prop. 9] and [2, Prop. 2.1], respectively.

Lemma A. *Let R and S be rings with $R \subseteq S$. Then S is faithfully flat over R if and only if S/R is flat over R.*

Lemma B. *Let the notations be as in (*). Then S is flat over R if and only if $u \in c(u)I$ for each element u of I.*

Theorem 1. *Let the notations be as in (*). Then S is faithfully flat over R if and only if $u \in c'(u)I$ for each element u of I.*

Proof. We remark that $R+I$ is the internal direct sum of R and I in $R[X]$. Since the following diagram,

$$\begin{array}{ccccc} 0 \longrightarrow & R & \longrightarrow & S \\ & \wr\downarrow & & \wr\downarrow \\ 0 \longrightarrow & R+I/I & \longrightarrow & R[X]/I \end{array}$$

is commutative, we first observe that $S/R \simeq R[X]/R+I$ as an R-module.

Assume that S is faithfully flat over R. By Lemma B it is enough to prove that $c(u)=c'(u)$ for each element u of I. We consider the exact sequence of R-modules,

$$0 \longrightarrow R+I \longrightarrow R[X] \longrightarrow R[X]/R+I \longrightarrow 0.$$

Tensoring it with $R/c'(u)$ over R and since $R[X]/R+I$ is R-flat by Lemma A we get an exact sequence of R-modules,

$$0 \longrightarrow R+I \otimes R/c'(u) \longrightarrow R[X] \otimes R/c'(u).$$

If $u=r_0+r_1X^{(1)}+\cdots+r_nX^{(n)}$ is an element of I we have that $-r_0+u \in R+I$ and $(-r_0+u)\otimes(1+c'(u))=0$ in $R[X]\otimes R/c'(u)$, which imply $(-r_0+u)\otimes(1+c'(u))=0$ in $R+I\otimes R/c'(u)$. By the remark above we must have $-r_0\otimes(1+c'(u))=0$ in $R\otimes R/c'(u)$, because the tensor product commutes with the direct sum. Thus $r_0 \in c'(u)$ and hence $c(u)=c'(u)$.

For the converse, it suffices to prove that $PS \subset S$ if P is a proper ideal of R. Since $R[X]/I$ is isomorphic to S as an R-module we may use $R[X]/I$ instead of S and suppose that $P(R[X]/I)=R[X]/I$, that is,

$P[X]+I=R[X]$. Then there exist $p_0, p_1,\dots, p_m \in P$ such that $1+p_0+p_1X^{(1)}+\cdots+p_mX^{(m)}$ belongs to I. Since P is proper, $1+p_0$ does not belong to P. It gives a contradiction to the primitive converse hypothesis. **q.e.d.**

Corollary. *Let the notations be as in* (*). *If* $I\subseteqq XR[X]$ *where* $XR[X]$ *is the ideal of* $R[X]$ *generated by* X, *then* S *is faithfully flat over* R *if and only if* S *is flat over* R.

Remark. Let the notations be as in (*). If S is a faithfully flat ring extension of R, then $c'(u)=c(u)$ for any u of I by Theorem 1. Therefore if there is an element u of I such that $c'(u)\subset c(u)$, then S is not faithfully flat over R. Let S be a ring such that $R\subseteqq S\subseteqq K$, where K is the total quotient ring of R. Then it follows by the above that S is faithfully flat over R only when $S=R$. But it is not convenient to prove that S is faithfully flat over R. Later we have a complete result (Theorem 2) in case that I is a principal ideal of $R[X]$.

In order to prove Theorem 2, we need the following ring-theoretic proposition.

Proposition. *Let* $u=r_0+r_1x+r_2x^2+\cdots+r_nx^n$ *be an element of* $R[x]$, *the polynomial ring in one variable over a ring* R. *We assume that* $c'(u)=(r_1,\dots, r_n)$ *is a direct summand of* R. *Then* $c'(uw)=c'(u)c(w)+c(u)c'(w)$ *for any element* w *of* $R[x]$.

Proof. Let $w=a_0+a_1x+a_2x^2+\cdots+a_mx^m$ and we prove this by induction *on* m. In case $m=0$ its proof is trivial. Let $m\geqq 1$ and $n\geqq 1$ (if $n=0$, obvious). We first show the following assertions $P(i)$, $0\leqq i\leqq n$,

$$P(i):\qquad c'(uw)\supseteqq(r_{n-i}, r_{n-i+1},\dots, r_n)a_m.$$

$P(0)$ is clearly true. Suppose that $i<n$ and $P(i)$ is true. Then it follows that $c'(uw)\supseteqq r_jc'(uw)=c'(u(r_jw))$ and $c'(uw)\supseteqq c'(u(r_jw-r_ja_mx^m))$ for $j=n-i, n-i+1,\dots, n$.

On the other hand, $r_jw-r_ja_mx^m=r_ja_0+r_ja_1x+\cdots+r_ja_{m-1}x^{m-1}$, therefore by induction on m, we obtain that $c'(uw)\supseteqq c'(u(r_jw-r_ja_mx^m))\supseteqq c'(u)c(r_jw-r_ja_mx^m)=r_j(a_0, a_1,\dots, a_{m-1})$, for, $c'(u)$ is a direct summand of R and $j\geqq 1$. Consequently we have that $c'(uw)$ includes r_ja_k for $n-i\leqq j\leqq n$ and $0\leqq k\leqq m$. Investigating the coefficient of degree $n+m-i-1$ of the polynomial uw, we immediately can see that $r_{n-i-1}a_m\in c'(uw)$. Thus $P(i+1)$ is true and hence $P(n)$ is true by induction on i. The above argument gives $c'(uw)=c'(u)c(w)+c(u)c'(w)$. **q.e.d.**

Corollary. *The above Proposition is also valid for* $R[X]$, *the polynomial ring in a set of variables* X *over a ring* R.

Proof. As to the proof we may assume that X is a finite set. Let $X=\{X_1,\dots, X_s\}$ and let $u=r_0+r_1X^{(1)}+\cdots+r_nX^{(n)}$ and $w=a_0+a_1X^{(1)}+\cdots+a_mX^{(m)}$ where $X^{(i)}$ monomials in $X_1,\dots, X_s$ with degree$\geqq 1$ such that $X^{(i)}\neq X^{(j)}$ if $i\neq j$. Of course we are moreover assuming that $c'(u)$ is a direct summand of R. If $M(X)=\pi_{i=1}^{s}X_i^{e(i)}$ then let us call the integer $\sum_{i=1}^{s}e(i)d(i)$ the weight of the monomial $M(X)$ with respect to $d(1),\dots, d(s)$. Obviously by a suitable choice of $d(1),\dots, d(s)\ (\geqq 1)$ we can see to it that no two of the monomials $X^{(1)},\dots, X^{(p)}$ ($p=\mathrm{Max}(m, n)$) have the same weight.

Put $X_i=x^{d(i)}$, $i=1, 2,\dots, s$. Then u and w become to $U=r_0+r_1x^{t(1)}+\cdots+r_nx^{t(n)}$ and $W=a_0+a_1x^{t(1)}+\cdots+a_mx^{t(m)}$, where $t(i)=$ weight $X^{(i)}\geqq 1$ and $t(i)\neq t(j)$ if $i\neq j$. Then it easily follows that $c'(uw)\supseteqq c'(UW)$. By the previous Proposition we have that $c'(UW)=c'(U)c(W)+c(U)c'(W)=c'(u)(w)+c(u)c'(w)$. Thus $c'(uw)=c'(u)c(w)+c(u)c'(w)$. **q.e.d.**

Remark 1. By the Corollary to Proposition it follows that $c'(uw)=c(uw)$ for every element w of $R[X]$ if $c'(u)=c(u)$ and $c'(u)$ is a direct summand of R.

Remark 2. If $c(u)$ is a direct summand of R then $c(uw)=c(u)c(w)$ for any element w of $R[X]$ (use induction on deg. w, $w\in R[x]$ and generalize it to $R[X]$). In general Dedekind [4, Satz 7] proved that $c(u)^{m+1}c(w)=c(u)^m c(uw)$ and $c(u)c(w)^{n+1}=c(uw)c(w)^n$ for arbitrary elements u, w of $R[x]$, where $n=$deg. u and $m=$deg. w. Remark 2 is easily follows from Satz 7 of Dedekind. The technique of the proof in Satz 7 also gives the proof of Remark 1 above.

Professor Nagata proved the following result. We give here a simple proof.

Application. *Let the notations be as in* $(*)$. *If* $I=(u)$ *is a principal ideal of* $R[X]$, *then* S *is flat over* R *if and only if* $c(u)$ *is a direct summand of* R.

Proof. The only if part is well known and its proof is easy being $c(u)$ finitely generated. For the if part, it suffices to prove that $uw \in c(uw)I$ for each element w of $R[X]$ by Lemma B. By Remark 2 it follows that $c(uw)I=c(u)c(w)I=c(w)I \ni uw$. **q.e.d.**

By virtue of Theorem 1, Remark 1 and Application we have

Theorem 2. *Let the notations be as in* $(*)$. *If* $I=(u)$ *is a principal ideal of* $R[X]$, *then* S *is faithfully flat over* R *if and only if* $c'(u)=c(u)$ *and* $c'(u)$ *is a direct summand of* R.

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