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Mixed problem for hyperbolic equation of second order

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§0. Introduction.

Mixed initial-boundary value problems for hyperbolic equation of second order have been treated by many authors; O. A. Ladyzenskaya [15], L. Hörmander [9], J. L. Lions [17], D. F. D. Duff [6], K. Yosida [24], S. Mizohata [20], M. Ikawa [11], [12], [13], R. Sakamoto [22], [23], R. Agemi [1] and others. However as is described below, an important and critical problem, so-called Neumann problem for wave equation with non-homogeneous boundary data, is not solved in framework of L^2-theory. Considering this problem we face on the following question 'Which class of first order boundary conditions should be posed for obtaining existence and uniqueness theory of the solution in L^2-space of mixed problem for hyperbolic equation of second order?

The purpose of this paper is to answer the above question, establishing the necessary and sufficient condition for obtaining existence and uniqueness theory in Sobolev slace, even if arbitrary lower order terms are added to the regularly hyperbolic operator of second order and to the boundary operator of first order. This theory just corresponds to 'Coercive' theory for elliptic boundary value problems. Our boundary conditions will be described in the relaxed form of so-called 'Uniformly Lopatinskii condition' treated at first by S. Agmon [3] and completed by R. Sakamoto [23], (in case where hyperbolic system, by H. O. Kreiss [14]) Now we point out some problems which remain open. Let $R^n_+=$

$\{(x, y_1, \cdots, y_{n-1}),\ x>0,\ y=(y_1, \ldots, y_{n-1})\in R^{n-1}\}$.

[1]. In $R^n_+\times(0. \infty)$, Neuman problem for wave equation is

$$(0.1)\begin{cases} \square u=-D_t^2u+(\sum_{j=1}^{n-1} D_{y_j}^2+D_x^2)u=f(t, x, y) & \text{in } R^n_+\times(0, \infty), \\ D_x u\big|_{x=0}=g(t, y) & \text{on } R^{n-1}\times(0, \infty) \\ D_t^j u\big|_{t=0}=u_j(x, y),\ (j=0, 1) & \text{on } R^n_+ \end{cases}$$

$$D_t=\frac{1}{i}\frac{\partial}{\partial t},\ etc.$$

If $g\equiv 0$, the solution u exists uniquely and holds

$$(0.2)\quad \sum_{j=0}^{1}\|D_x^j u(t)\|_{1-j}\leq Ce^{\beta t}(\sum_{j=0}^{1}\|u_j\|_{1-j}+\int_0^t\|f(s)\|_0 ds)$$

where $\|u\|_k^2=\sum_{\alpha\leq k}\|D_{x,y}^\alpha u\|_0^2,\ \|u\|_0^2=\iint|u(x, y)|^2\ dxdy_1\ldots dy_{n-1}$.

But if $g\not\equiv 0$, the following questions arise:

(Q_1) 'Whatever energy inequality should be held?' and

(Q_2) 'Does the existence theory follow in the frame-work of L^2 ?' and 'How about adding arbitrary lower order terms?'.

[2].

$$(0.3)\begin{cases} \square u=f(t, x, y) \\ Bu\big|_{x=0}\equiv(D_x+\sum_{j=1}^{n-1} b_j(t, y)D_{y_j}-c(t, y)D_t)u\big|_{x=0}=g(t, y) \\ D_t^j u\big|_{t=0}=u_j(x, y),\quad (j=0, 1). \end{cases}$$

How about the questions (Q_1), (Q_2) concerning problem (0.3)? Whatever condition must b_j and c satisfy?

[3]. We mainly consider the mixed problem:

$$(P)\begin{cases} P(t, x, y, D_t, D_x, D_y)u=f(t, x, t),\quad x>0,\ t>0, \\ Bu\big|_{x=0}=(D_x+\sum_{j=1}^{n-1} b_j(t, y)D_{y_j}-c(t, y)D_t)n\big|_{x=0}=g(t, y),\ x=0,\ t>0, \\ D_t^j u\big|_{t=0}=u_j(x, y),\ (j=0, 1),\quad x>0,\ t=0. \end{cases}$$

Here we assume that $x=0$ is non-characteristic for P.

$$(0.4)\quad P=-a(t,x,y)D_t^2+2\Big(\sum_{i=1}^{n-1}a_i(t,x,y)D_{y_i}+a_n(t,x,y)D_x\Big)D_t$$

$$+\Big(\sum_{i,j=1}^{n-1}a_{ij}D_{y_i}D_{y_j}+2\sum_{j=1}^{n-1}a_{nj}D_{y_j}D_x+D_x^2\Big),\ a_{ij}=a_{ij}(t,x,y),\ \text{etc.}$$

is regularly hyperbolic with respect to t.

When has the mixed problem $(P)=\{P, B\}$ the strongly hyperbolicity in L^2-sence defined below?

def. We say that the mixed problem $\{P, B\}$ has the strongly hyperbolicity if and only if not only $\{P, B\}$ but also the mixed problem $\{P+P', B+B'\}$ has a unique solution in L^2-space (or Sovolev spaces) and energy inequality holds in the L^2-sence, where P' and B' are arbitraly first order and zero order operators respectively.

[4]. If $g\not\equiv 0$, whatever energy inequality should correspond to L^2-well-posedness? In case $g=0$, R. Agemi [1] gave energy inequality. (Q in [1] is slightly different from our $\tilde{Q}$ in page 474.)

[5]. Whether $\{P, B\}$ has the same speed of propagation as Canchy problem, or not? How do we construct the solution of the mixed problem $\{P, B\}$ in the general domain $\Omega\times(0. \infty)$? Here Ω is the exterior or interior of a smooth and compact hypersurface $\partial\Omega$ in R^n.

This paper answers affirmatively all above questions. Since Dirichlet problem is well-known, considering our results we can regard that the mixed problem for second order hyperbolic equations with real boundary coefficients is solved so completely as coercive boundary value problem for elliptic operators. In §1, we state the result of our paper. In §4, Sharp form of Gärding's inequality is essentially used on the boundary plane. That is proved by Hörmander and in vector-valued case by Lax-Nirenberg. For the estimate of boundary integral, localiczations shown in §3 of the mixed problem (P) is indispensable. As in R. Sakamoto [23], the estimate of boundary norm is obtained from the energy inequality for the dual problem $\{P^*, B'\}$ in §7. All our arguments, especially, one for obtaining necessary condition in §5 depend on the lemmas given in §2. In §8, we discuss on the finiteness of propagation speed and go forward to the theorem on the mixed problem in general cylindrical domain $\Omega\times(0, \infty)$. Strongly hyperbolicity helps us to make the theory in general domain.

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§1. Statement of results

In this section we state our results in Theorem 1~6 and their corollaries, which answer to the questions in §0.

We need to introduce some notations and to consider slightly the energy inequalities.

Assume that all the coefficients of P and B are real and sufficiently smooth and constant outside of compact set K in $\overline{R_+^n}\times R^1$. From the hyperbolicity of P given in (0.4) the root τ of characteristic equation

$$P(t, x, y, \tau, \xi, \eta)=0$$

is real and distinct for $(t, x, y)\in R^1\times R_+^n$, $\xi^2+|\eta|^2=1$.

This means, denoting $a=a(t, x, y)$, etc.,

$$a+a_2^n>0, \quad a\neq 0, \tag{1.1}$$

$$d(\eta)\equiv(a+a_n^2)\Big\{\Big(\sum_{i=1}^{n-1}a_i\eta_i\Big)^2+a\sum_{i,j=1}^{n-1}a_{ij}\eta_i\eta_j\Big\}-a_n\sum_{i=1}^{n-1}a_i\eta_i \tag{1.2}$$

$$+a\sum_{j=1}^{n-1}a_{nj}\eta_j)^2>0, \text{ for } (x, y, t)\in\overline{R_+^n}\times R^1,\ |\eta|=\Big(\sum_{j=1}^{n-1}\eta_2^j\Big)^{1/2}\neq 0,$$

It is natural that we assume

$$a>0 \tag{1.3}$$

that means 'the number of boundary conditions is one'. In fact, in the theory of mixed problem for hyperbolic equation with constant coefficients the number of boundary condition is given by the number of the roots with positive imaginary part of

$$P(\tau, \xi, \eta)=0,\ \tau=\sigma-i\gamma,\ \sigma, \eta \text{ real},\ \gamma>0.$$

In our case since two root of $P(\tau, \xi, 0)=0$ is

$$\xi=(-a_n\pm\sqrt{a_n^2+a}\,)\tau$$

the number of boundary conditions must be $1\pm\text{sgn}\, a_n$, if $a<0$.

(H_1) *We assume* (1.3) *and* (1.2).

Now let us introduce the necessary and sufficient condition for that $\{P, B\}$ becomes strongly hyperbolic in L^2-sence. First, condsider the case where the initial data are zero.

$$(P_0)\quad\begin{cases} Pu=f(t, x, y) & x>0,\ t>0 \\ Bu\big|_{x=0}=g(t, y) & x=0,\ t>0 \\ D_t^j u\big|_{t=0}=0 & (j=0, 1)\ t=0,\ x>0 \end{cases}$$

If all the coefficients are constant and $f\equiv 0$, by Fourier-Laplace transformation we have

$$(1.4)\quad P(\tau, D_x, \eta)\hat{u}(\tau, x, \eta)=0,\quad \hat{u}(\tau, x, \eta)=\mathscr{F}(e^{-\gamma t}u(t, x, y))$$

$$(1.5)\quad B(\tau, D_x, \eta)\hat{u}(\tau, x, \eta)\big|_{x=0}=\hat{g}(\tau, \eta).$$

The solution of (1.4) converging to zero when x tends to ∞, is given by $c(\tau, \eta)e^{i\xi_+(\tau, \eta)x}$, where $\xi_+(\tau, \eta)$ is a root of $P(\tau, \xi, \eta)=0$, with positive imaginary part when $\gamma>0$. Substitute it into (1.5) then we see that $c(\tau, \eta)$ must satisfy

$$(1.5)'\qquad B(\tau, \xi_+(\tau, \eta), \eta)c(\tau, \eta)=g(\tau, \eta).$$

For $\gamma>0$, $B(\tau, \xi_+(\tau, \eta), \eta)\neq 0$ follows. So-called 'Uniformly Lopatinskii condition' is

$$(1.7)\quad |B(\tau, \xi_+(\tau, \eta), \eta)|\geqslant c>0 \text{ for } |\eta|^2+|\tau|^2=1,\ \gamma>0.$$

But (1.7) is not satisfied in case of Neumann problem [1] in §0. $B(\tau, \xi_+(\tau, \eta), \eta)=\xi_+(\tau, \eta)=(\tau^2-|\eta|^2)_+^{1/2}$, $(\tau=\sigma-i\gamma,\ \gamma>0)$ satisfies

$$(1.8)\quad |\xi_+(\tau, \eta)|\geqslant c(\sigma, \eta)\gamma^{\frac{1}{2}},\ c(\sigma, \eta)>0$$
$$\text{for } \{(\eta, \sigma, \gamma),\ |\eta|^2+|\tau|^2=1,\ \gamma>0\}.$$

Our condition for $\{P, B\}$ is

(H_2) $|L(y, t, \tau, \eta)|=|B(y, t, \tau, \xi_+(y, t, \tau, \eta), \eta)|\geqq c(y, t, \sigma, \eta)\gamma^{\frac{1}{2}}$,
$c(y, t, \sigma)>0$ for $(y, t)\in R^{n-1}\times R^1$, $|\eta|^2+|\tau|^2=1$, $\gamma>0$,

where $\xi_+(y, t, \tau, \eta)$ is a root with positive imaginary part of $P(t, 0, y, \tau, \xi, \eta)=0$.

The following two cases are possible if (H_2) is not satisfied.

(Case I) There exists $(y_0, t_0, \sigma_0, \eta_0, \gamma_0)$, $\sigma_0^2+|\eta|_0^2+\gamma_0^2=1$, $\gamma_0>0$.

$$L(y_0, t_0, \sigma_0-i\gamma_0, \eta_0)=0.$$

(Case II) Or, there exists $(y_0, t_0, \eta_0, \sigma_0)$ $(\sigma_0^2+|\eta_0|^2=1)$, such that

$$c(y_0, t_0, \sigma_0, \eta)\gamma>L(y_0, t_0, \sigma_0-i\gamma, \eta_0), \ {}^{\exists}c(y_0, t_0, \sigma_0, \eta_0)>0.$$

We state our theorems after considering some relations among various types of energy inequalities. First let us integrate (0.2) with $u_j\equiv 0$, $(j=0, 1)$ from 0 to t, applying Schwarz inequality, then we have

$$(1.9)\quad \int_0^t \sum_{j=1}^{1} \|D_t^j u(s)\|_{1-j}^2\, ds \leq \frac{t^2}{2} e^{2\beta t} \int_0^t \|f(s)\|_0^2 ds.$$

Muliply $e^{-2\gamma t}$ $(\gamma>\beta>0)$ and integrate from 0 to ∞, then

$$(1.9)'\quad \frac{1}{2\gamma}|u|_{1,\gamma,+}^2 \leq \int_0^\infty \left(\frac{s^2}{\gamma-\beta}+2\frac{s}{(\gamma-\beta)^2}+2\frac{1}{(\gamma-\beta)^3}\right)e^{-2(\gamma-\beta)s}\|f(s)\|_0^2 ds.$$

where

$$(1.10)\quad |u|_{m,\gamma,+}^2 \equiv \sum_{i+j+k+|\alpha|=m} \int_0^\infty |e^{-\gamma t}\gamma^i D_t^j D_x^k D_y^\alpha u|^2\, dxdydt.$$

Taking accout of

$$\{s(\gamma-\beta)\}^j e^{-(\gamma-\beta)s} \leq j!$$

We can see that there exist positive constant γ_0, and such that

$$(1.11)\quad \gamma|u|_{1,\gamma,+}^2 \leq c\frac{1}{\gamma}|f|_{0,\gamma/2,+}^2 \quad \text{for} \quad \gamma>\gamma_0.$$

Now we change a point of view and consider the following boundary value problem closely connected to (P_0) as is shown later.

$$(P_0') \quad \begin{cases} P(t, x, y, D_t, D_x, D_y)u=f(t, x, y) & x>0, \ -\infty<t<\infty \\ B(t, y, D_t, D_x, D_y)u\big|_{x=0}=g(t, y), & x=0, \ -\infty<t<\infty. \end{cases}$$

For description of energy inequality for (P_0'), let us introduce Hilbert spaces $\mathcal{H}_{m,\gamma}(\mathrm{R}_+^n\times \mathrm{I})$, $\mathcal{H}_{m,\gamma}(\mathrm{R}^{n-1}\times \mathrm{I})$ defined by the completion of $C_0^\infty(\overline{\mathrm{R}_+^n\times \mathrm{I}})$ and $C_0^\infty(\mathrm{R}^n\times\overline{\mathrm{I}})$ with the following norms respectively, $\mathrm{I}=(a, b)$, $\gamma>0$.

$$(1.12) \quad |u|^2_{\mathcal{H}_{m,\gamma}(R^n_+\times I)}\equiv|u|^2_{m,\gamma,I} = \sum_{i+j+k+|\alpha|=m}\int_a^b dt\int_0^\infty dx\int_{R^{n-1}}|e^{-\gamma t}\gamma^i D_t^j D_x^k D_y^\alpha u(t, x, y)|^2 dy$$

$$(1.13) \quad \langle v\rangle^2_{\mathcal{H}_{m,\gamma}(R^{n-1}\times I)}\equiv\langle v\rangle^2_{m,\gamma,I} = \sum_{i+j+|\alpha|=m}\int_a^b dt\int_{R^{n-1}}|e^{-\gamma t}\gamma^i D_t^j D_y^\alpha v(t, y)|^2 dy.$$

We denote in short $|u|_{m,\gamma,(-\infty,\infty)}=|u|_{m,\gamma}$, $|u|_{m,\gamma,(0,\infty)}=|u|_{m,\gamma,+}$, $|u|_{m,\gamma,(-\infty,0)}=|u|_{m,\gamma,-}$.

Denote

$$\Lambda_\gamma=\overline{\mathcal{F}}(|\eta|^2+\sigma^2+\gamma^2)^{\frac{1}{2}}\mathcal{F}_{(y,t)}$$

$$\Lambda_{y,\gamma}=\overline{\mathcal{F}}(|\eta|^2+\gamma^2)^{\frac{1}{2}}\mathcal{F}_{(y)},$$

$$\mathcal{F}_{(y,t)}u(t, x, y)=\int_{R^n}e^{-2\pi i(y\eta+t\sigma)}u(t, x, y)dydt.$$

In our theory $\Lambda_{y,\gamma}$ plays the most important role, relating to the localigation of u in §4.

Theorem 1. *There exist positive constants* γ_k and C_k $(k=0, 1, \ldots)$ *such that for every* $u\in\mathcal{H}_{k+2,\gamma}(R_+^n\times R^1)$

$$(1.14) \quad \gamma|u|^2_{1+k,\gamma}+\gamma\sum_{j=0}^{1}\langle\Lambda_{y,\gamma}^{-\frac{1}{2}}D_x^j u\rangle^2_{1+k-j,\gamma} \leq C_k\left\{\frac{1}{\gamma}|Pu|^2_{k,\gamma}+\frac{1}{\gamma}\langle\Lambda_{y,\gamma}^{\frac{1}{2}}Bu\rangle^2_{k,\gamma}\right\} \quad \text{for } \gamma>\gamma_k.$$

Corollary *If the solution u of (P_0) belongs to $\mathcal{H}_{2,\gamma,+}$, then*

$$\gamma|u|^2_{1,\gamma,+}+\gamma\sum_{j=0}^{1}<\Lambda^{-\frac{1}{2}}_{y,\gamma}D^j_x u>^2_{1-j,\gamma,+} \leq C_0\frac{1}{\gamma}\left\{|f|^2_{0,\gamma,+}+<\Lambda^{\frac{1}{2}}_{y,\gamma}g>^2_{0,\gamma,+}\right\} \tag{1.15}$$

Theorem 2. *For $f\in\mathcal{H}_{k,\gamma}$ $(R^n_+\times R^1)$, $\Lambda^{\frac{1}{2}}_{y,\gamma}g\in\mathcal{H}_{k,\gamma}$ $(R^{n-1}\times R^1)$, $(\gamma>\gamma_k)$, there exists a unique solution $u\in\mathcal{H}_{1+k,\gamma}$ $(R^n_+\times R^1)$ of (P'_0). If $\underset{t}{supp}\,[f]$, $\underset{t}{supp}\,[g]\subset[T,\infty)$ then $\underset{t}{supp}\,[u]\subset[T,\infty)$.*

Corollary *If $f\in\mathcal{H}_{1,\gamma}$ $(R^n_+\times R^1_+)$, $\Lambda^{\frac{1}{2}}_{y,\gamma}g\in\mathcal{H}_{1,\gamma}$ $(R^{n-1}\times R^1_+)$, $(\gamma>\gamma_1)$, and $f(0,x,y)\equiv g(0,y)\equiv 0$, then there exists a solution $u\in\mathcal{H}_{2,\gamma}$ $(R^n_+\times R^1)$ of*

$$\begin{cases} Pu=f(t,x,y) \\ Bu\,|_{x=0}=g(t,y) \\ D^j_t u\,|_{t=0}=0 \qquad (j=0,1) \end{cases} \tag{1.16}$$

and satisfy the energy inequality

$$\int_0^t\sum_{j=0}^{1}\|(D^j_t u)(s)\|^2_{1-j}\,ds \leq Ce^{\beta t}\left\{t^2\int_0^t\|f(s)\|^2_0 ds+t^2\int_0^t\ll(\Lambda^{\frac{1}{2}}_{y,\beta}g)(s)\gg^2_0 ds \right. \left. +t\int_0^t\ll g(s)\gg^2_0 ds,\right. \qquad (\beta>\gamma_0), \tag{1.17}$$

where $\ll g(s)\gg^2=\int|g(s,y)|^2\,dy$, c: constant.

(1.17) follows from Theorem 1 and corollary by putting $\gamma=\frac{1}{t}$ for small t. Conversely, in the same way as (1.11), we have from (1.17)

$$\gamma|u|^2_{1,\gamma,+}\leq c\frac{1}{\gamma}\left\{|f|^2_{0,\gamma/2,+}+<\Lambda^{\frac{1}{2}}_{y,\gamma}g>^2_{0,\gamma/2,+}\right\}, \qquad (\gamma>\gamma_0). \tag{1.18}$$

Next denote

$$(1.19)\qquad [u(t)]^2_{m,\gamma}=\sum_{\substack{i+j+k+|\alpha|=m\\ j=0,1}}\iint|e^{-\gamma t}\gamma^i D_x^k D_y^\alpha(D_t^j u)(t,x,y)|^2 dxdy$$

then we have our main energy inequality for (P).

Theorem 3. *There exist* c_k *and* γ_k $(k=0,1,2,\dots)$ *such that for* u *in* $\mathcal{H}_{2+k,\gamma}$ $(R^n_+\times(0,t))$, $(0<t\leq\infty)$, *and for* $\gamma>\gamma_k$, we have

$$(1.20)\qquad \gamma|u|_{1+k,\gamma,(0,t)}+\gamma\sum_{j=0}^{1}<\Lambda_{y,\gamma}^{-\frac{1}{2}}D_x^j u>^2_{1+k-j,\gamma,(0,t)}$$
$$+[u(t)]^2_{1+k,\gamma}\leq C_k\left\{\frac{1}{\gamma}|Pu|^2_{k,\gamma,(0,t)}+\frac{1}{\gamma}<\Lambda_{y,\gamma}^{\frac{1}{2}}Bu>^2_{k,\gamma,(0,t)}\right.$$
$$\left.+[u(0)]^2_{1+k,\gamma}\right\}.$$

Corollaries of Theorem 3.

1) There exist β_k and c_k such that we have, for $0<t<\infty$,

$$(1.20)'\quad \sum_{j=0}^{1}\|(D_t^j u)(t)\|^2_{1+k-j}+\sum_{j=0}^{1}<(t\Lambda_{y,1}+1)^{-\frac{1}{2}}D_x^j u>^2_{1+k-j,0,(0,t)}$$
$$\leq c_k e^{\beta_k t}\left\{t|Pu|^2_{k,0,(0,t)}+<(t\Lambda_{y,1}+1)^{\frac{1}{2}}Bu>^2_{k,0(0,t)}\right.$$
$$\left.+\sum_{j=0}^{1}\|(D_t^j u)(0)\|^2_{1+k-j}\right\}.\qquad k=0,1,2,\dots.$$

2) If the solution $u(t,x,y)$ of (P) belongs to $\mathcal{E}_t^0(H^2(R^n_+))\cap\mathcal{E}_t^1(H^1(R^n_+))$[1)] and $\Lambda_{y,1}^{-\frac{1}{2}}D_x^j u(0,y,t)$ belong to $H_{2-j}(R^{n-1}\times(0,t_0))$ $(j=0,1)$, then the following compatibility condition must be satisfied

$$(C_1)\qquad B_1(f,u_1,u_2)\equiv\sum_{j=0}^{1}B_{1,j}(0,y,D_x,D_y)u_j(0,y)=g(0,y),$$

where $\quad B(y,t,D_x,D_y,D_t)=\sum_{j=0}^{1}B_{1,j}(y,x,D_x,D_y)D_t^j.$

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1) $f(t)\in\mathcal{E}_t^k(H)$ means that $f(t)$ is k times continuously differentiable in t with values in H, $(k=0,1,2,\cdots)$.

3) Conversely if $f\in\mathcal{H}_{1-j}(R^n_+\times(0,\infty))$, $\Lambda^{\frac{1}{2}}_{y,\gamma}g\in\mathcal{H}_{1,\gamma}(R^{n-1}\times(0,\infty))$ and $u_j\in H^{2-j}(R^n_+)$, $(j=0, 1)$ satisfy (C_1), then the solution of (P) exists and satisfies (1.20) and (1.23) with $k=1$.
4) Moreover assume $f\in\mathcal{H}_{k,\gamma}$ $(R^n_+\times(0,\infty))$, $\Lambda^{\frac{1}{2}}_{y,\gamma}g\in\mathcal{H}_{k,\gamma}(R^{n-1}\times(0,\infty))$, $u_j\in H^{k+1-j}(R^n_+)$, $(j=0, 1)$ and that f, g and u_j $(j=0, 1)$ satisfy the following compatibility conditions (C_k) of order k. Then the solution $u(t, x, y)$ of (P) belongs to $\mathcal{E}^0_t(H^{1+k}(R^n_+))$ $\mathcal{E}^1_t(H^k(R^n_+))$ and satisfied (1.20) and (1.23).

$$(C_k)\quad B_m(f, u_1, u_2)=\sum_{j=0}^{m}\{B_{m,j}(0, y, D_x, D_y)u_j\}(0, y)=(D_t^{m-1}g)(0, y),\qquad (m=1, \ldots, k),$$

where $B_{m,j}$ is defined by

$$D_t^{m-1}B(t, y, D_t, D_x, D_y)u=\sum_{0\leq j\leq m}B_{m,j}(t, y, D_x, D_y)D_t^j u$$

and u_{2+i} is successively defined by

$$u_{2+i}(x, y)=a(0, x, y)^{-1}\{(D_t^i f)(0, x, y)-Q_i(0, x, y, D_x^j, D_y^\alpha, u_k)\}\qquad (i=0, 1, 2, \ldots),\ \text{where}$$

$$\begin{aligned}Q_i=Q_i(t, x, y:\ \ & D_x^j D_y^\alpha(D_t^k u),\ k\leq i+1, k+j+|\alpha|\leq i+2)\\ \equiv\ & D_t^i Pu-a(t, x, y)D_t^{2+i}u.\end{aligned}$$

Remark 1 Even if arbitrary lower order terms be added to P and B, Therorem 1, 2 and 3, and their corollaries are also true by making C_k and γ_k $(k=0, 1, 2, \ldots)$ larger if necessary.

Theorem 4 *Assume* (H_1). *If we have*

$$(1.14)'\qquad \gamma|u|^2_{1,\gamma,+}\leq C\left\{\frac{1}{\gamma}|Pu|^2_{0,\gamma,+}+\frac{1}{\gamma}\langle\Lambda^{\frac{1}{2}}_{y,\gamma}Bu\rangle^2_{0,\gamma,+}\right\},\ \gamma>\gamma_0$$

for the solution of (P_0) *belonnging to* $\mathcal{H}_{2,\gamma,+}$, *then* P, B *must satisfy the condition* (H_2). *It is true if we replace* (1.14)′ *by* (1.18).

Theorem 5 *Under* (H_1) *and* (H_2) *the solution of* (P) *has a finite* *speed of propagation and the speed is same to that of the solution of Cauchy problem. In other word the cone which describe the dependence domain is the same one as in case of Cauchy problem.*

Theorem 6 *We can extend Therorem* 1~5 *to the mixed problem in general cylindrical domain* $\Omega\times(0, \infty)$,

Dedailed statement of Theorem 6 is shown in §8. It plays a important role that the condition (H_2) is an intrinsic one. Strongly hyperbolicity and finiteness of propagation speed make it possible that all the results in quarter-space: $R^n_+\times(0, \infty)$ generate the corresponding ones in general cylindrical domain $\Omega\times(0, \infty)$.

Remark 2 We can replace $\sum_{j=1}^{n-1} b_j D_{y_j}$ by singular integral operator with sufficiently smooth real symbol $b(t, y, \eta)$ with homogeneous of order 1 with respect to η. If (H_2) replaced $\sum_{j=1}^{n-1} b_j\eta_j$ by $b(t, y, \eta)$ be satisfied, then Theorem 1~Theorem 4 hold.

Remark 3 As for mixed problem P, B one often says that P, B is L^2-will-posed if for some constant C_0

$$(1.24)\qquad \gamma|u|^2_{1,\gamma,+}\leq C_0\frac{1}{\gamma}|Pu|^2_{0,\gamma,+},\quad \gamma>\gamma_0$$

holds in case where $g\equiv 0$ and $u_j\equiv 0$ $(j=0, 1)$ (c.f. R. Agemi-T. Shirota [2]). It seems natural that we say that $\{P, B\}$ is L^2-well-posed if

$$(1.24)'\qquad \gamma|u|_{1,\gamma,+}\leq C_0\left\{\frac{1}{\gamma}|Pu|^2_{0,\gamma,+}+\frac{1}{\gamma}<\Lambda^{\frac{1}{2}}_{y,\gamma}Bu>^2_{0,\gamma,+}\right\}$$

Theorem 4 means that assumption (H_2) is necessary for $\{P, B\}$ to be L^2-well-posed.

§2. Analysis on (H_2)

In this section we state on some lemmas concerning the condition

(H_2) which will be used in the proofs of Theorems 1, 3, 4, 5. The proofs of lemmas given later in this section, depend on geometrical consideration of the intersections of an elilptic surface and hyperplanes.

At first let us calculate exactly the root with positive imaginary part ξ^+ of the characteristic equation at the boundary;

(2.1) $$P(t, 0, y, \sigma-i\gamma, \xi, \eta)=0, \quad (\eta=(\eta_1, \ldots, \eta_{n-1}), \sigma, \text{ real}, \gamma>0).$$

Denote simply $a(t, 0, y)=a$, etc.,.

(2.2) $$P=\{\xi+(a_n\tau+\sum_{i=1}^{n-1} a_{nj}\eta_j)\}^2-D, \text{ where}$$

(2.3) $$D=(a_n\tau+\sum_{j=1}^{n-1} a_{nj}\eta_j)^2+(a\tau^2-2\sum^{n-1} a_i\eta_i\tau-\sum^{n-1} a_{ij}\eta_i\eta_j)$$
$$\equiv D_1-2i\gamma D_2, \; D_1=ReD.$$

(2.4) $$D_2=(a+a_n^2)\sigma+a_n\sum^{n-1} a_{nj}\eta_j-\sum^{n-1} a_i\eta_i.$$

Remark that we can see the following relation

(2.5) $$D_1=(a+a_n^2)^{-1}D_2^2-a^{-1}(a+a_n^2)^{-1}d(\eta)-(a+a_n^2)\gamma^2.$$

In case of wave equation (2.4) means $D_1=\sigma^2$-η^2-γ^2. Now the root ξ^+ of (2.1) is given by

(2.6) $$\xi^+=-(a_n\tau+\sum_{i=1}^{n-1} a_{njj})+\text{sgn}(-D_2)\sqrt{\frac{|D|+D_1}{2}}+i\sqrt{\frac{|D|-D_1}{2}}.$$
$$i=\sqrt{-1}, \qquad \text{sgn } x=\begin{cases} 1, & x\geq 0 \\ -1, & x<0. \end{cases}$$

We only remark that the roots of $a+ib$ (a, b, real) is

$$\pm\left\{\text{sgn}(b)\sqrt{\frac{r+a}{2}}+i\sqrt{\frac{r-a}{2}}\right\}, \qquad r=|a+ib|.$$

From (2.4) and (2.6), $L(u, t, \tau, \eta)=|\xi^++\sum_{j=1}^{n-1} b_j\eta_j-c\tau|=0$ is equivalent to

$$(2.7)\qquad \begin{cases} \text{(i)}\ \sqrt{\dfrac{|D|-D_1}{2}}=-\rho\gamma \\ \text{(ii)}\ -\operatorname{sgn}(D_2)\sqrt{\dfrac{D+D_1}{2}}=(a+a_n^2)^{-1}\{\rho D_2+\alpha(\eta)\}, \end{cases}$$

where $\rho=c+a_n$

$$\alpha(\eta)=(a-ca_n)\sum_{j=1}^{n-1}a_{nj}\eta_j+(c+a_n)\sum_{i=1}^{n-1}a_i\eta_i-(a+a_n^2)\sum_{j=1}^{n-1}b_j\eta_j.$$

First we state the following lemma whose proof is given later.

Lemma 2.1. *Under the condition* (H_2) *it follows*

$$(2.8)\qquad \rho\geq 0.$$

Taking care of the homogenuity of L with respect to (η, σ, γ), and the fact that $\gamma=0$ follows from $L=0$, we characterize the set

$$S=\{(y, t, \eta, \sigma),\ L(y, t, \eta, \sigma, 0)=0,\ (\eta, \sigma)\in\Sigma_0\},$$
$$\Sigma_0=\{(\eta, \sigma),\ |\eta|^2+\sigma^2=1\}.$$

Lemma 2.2. *Assume the condition* (H_2), *then for* $(t_0, y_0, \sigma_0, \eta_0)\in S$ *we have*:

$$(2.9)\qquad D_1=0,\ \text{i.e.}\ \ D_2^2-a^{-1}d(\eta_0)=0$$

$$(2.10)\qquad \rho D_2+\alpha(\eta_0)=0$$

at the point $(t, x, y, \tau, \eta)=(t_0, 0, y_0, \sigma_0, \eta_0)$.

Denote

$$V\equiv\{(y, t, \eta, \sigma),\ D_2^2-a^{-1}d(\eta)=0,\ (\eta, \sigma)\in\Sigma_0\}$$
$$W=\{(y, t, \eta, \sigma),\ \rho D_2+\alpha(\eta)=0\ \ (\eta, \sigma)\in\Sigma_0\},$$

then Lemma 2.2 means $S=V\cap W$. From Lemma 2.2 we have

Corollary *There exists positive constant* δ *such that*

$$S \subset V \subset \{(y, t, \eta, \sigma): \quad (\eta, \sigma) \in \Sigma_0, \ |\eta| > \delta\}.$$

Remark that for the problem [1] *in* §0, $W = R^{n-1} \times R^1 \times \Sigma_0$ *therefore* $S = V$.

The variety V consist of the following two parts.

$$V = V_+ \cup V_-, \quad V_+ \cap V_- = \phi, \text{ where}$$

$$V_\pm = \{(y, t, \eta, \sigma): \quad D_2 = \pm a^{-\frac{1}{2}} d(\eta)^{\frac{1}{2}}, \quad (\eta, \sigma) \in \Sigma_0\}.$$

Now let us denote $\rho_\pm = \mp a^{\frac{1}{2}} \alpha(\eta) d(\eta)^{-\frac{1}{2}}$ then we have

Lemma 2.3. *Under the condition* (H_2), *in neighbourhood* $\cup_\pm$ *of* $V_\pm$ *in* $R^{n-1} \times R^1 \times \Sigma_0$, *the following inequalities follows respectively.*

(2.11) $\quad \rho \geq \rho_\pm,$

Moreover we have under the condition (H_2)

Lemma 2.4 *For every* $(y, t, \eta) \in R^{n-1} \times R^1 \times [R^{n-1} - 0]$, *we obtain*

(2.12)[2) $\quad \rho^2 d(\eta) \geq a\, \alpha(\eta)^2, \ \rho \geq 0.$

Conversely we have

Lemma 2.6 *If* (H_2) *does not hold there exists* (y_0, t_0, η_0) *such that*

(2.13) $\quad \rho - |\tilde{\rho}_\pm| \leq -\delta < 0$ at (y_0, t_0, η_0), $|\eta_0| \neq 0$,

and there exists σ_0 *such that one of the followings holds.*

(*Case I*) $\quad L(y_0, t_0, \sigma_0 - i\gamma_0, \eta_0) = 0$, *for some* $\gamma_0 > 0$.

(*Case II*) $\quad c\gamma > L(y_0, t_0, \sigma_0 - i\gamma, \eta_0)$, *for* $\gamma > 0$, *where* $c > 0$.

Remark that the condition (H_2) means that $L(y, t, \sigma, \eta) = 0$ only

2) Starting from this inequality R. Agemi [1] obtained only energy inequaly for the problem (P) with boundary data $g \equiv 0$.

at the point (y, t, σ, η) such that $P(t, 0, y, \sigma, \xi, \eta)=0$ has double real root with respect to ξ, i.e. $(y, r, \sigma', \eta')\in V$,

$$\sigma'=(\sigma^2+|\eta|^2)^{-\frac{1}{2}}\sigma,\quad \eta'=(\sigma^2+|\eta|^2)^{-\frac{1}{2}}\eta.$$

Proof of Lemma 1. Let us prove that (H_2) does not hold if $\rho=c+a_n<0$ at some point (y_0, t_0). We fix (y_0, t_0) in this proof, (τ, η) satisfying $L(\tau, \eta)=0$, i.e.

$$(2.7)'\quad \operatorname{sgn}(-D_2)\sqrt{\frac{|D|+D_1}{2}}+i\sqrt{\frac{|D|-D_2}{2}}=(c+a_n)\tau$$
$$+\{\sum_{i=1}^{n-1}a_{nj}\eta_j-\sum_{j=1}^{n-1}b_j\eta_j\},$$

satisfies the following (2.14), the esquare of (2.7)′, too.

$$(2.14)\qquad (a_n\tau+\sum_{j=1}^{n-1}a_{nj}\eta_j)^2+(a\tau^2-2\sum_{j=1}^{n-1}a_i\eta_i\tau-\sum_{i,j=1}^{n-1}a_{ij}\eta_i\eta_j)$$
$$=\{(c+a_n)\tau+(\sum_{j=1}^{n-1}a_{nj}\eta_j-\sum_{j=1}^{n-1}b_j(\eta_j))\}^2.$$

If $a_n^2+a=(c+a_n)^2$, then at $(\tau, \eta)=(1, 0)$, (H_2) does not hold. Consider the case where $a_n^2+a\neq(c+a_n)^2$ and for some $\eta_0\neq 0$, (2.14) have non-real roots. Take the root with negative imaginary part $\tau_-=\sigma_0-i\gamma_0$. Then at $(\eta_0, \sigma_0-i\gamma_0)$ (2.7)′ is satisfied. In case where the equa tion (2.14) has real roots for all η, we return to (2.7). From (2.7) (i) and $\gamma=0$ we have

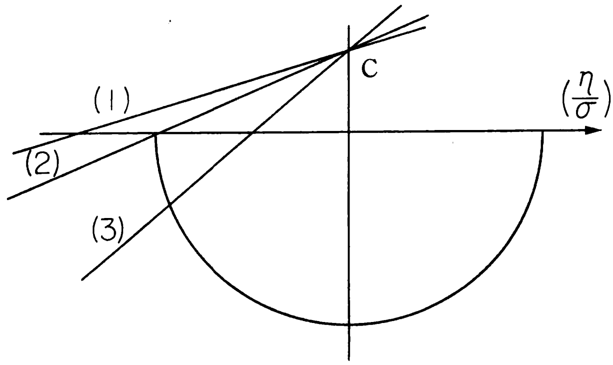
$$(2.15)\qquad D_1=(a+a_n^2)^{-1}\{D_2^2-a^{-1}d(\eta)\}=|D|\geq 0.$$

Therefore at (σ, η) satisfying (2.7), $D_2(\sigma, \eta)\neq 0$. Deviding (2.7) (ii) by $D_2=\operatorname{sgn}(D_2)\sqrt{D_2^2}$, we have

$$(2.16)\qquad -\sqrt{1-a^{-1}d(\eta')}=(a+a_n^2)^{-\frac{1}{2}}\{\rho+\operatorname{sgn}(D_2)\alpha(\eta')\}$$

where $\eta'=\eta/|D_2|$.

If $1=a^{-1}d(\eta')$ then we can see, after slight geometrical consideration



(Fig. 1)
In case where $P=D_x^2+D_y^2-D_t^2$
$B=D_x+bD_y-cD_t$, $c>0$. (1) *and* (2)
are co esponding to (H_2).

(see Fig. 1), that there exists s such that $1>|s|\geqq 0$, and

$$-\sqrt{1-s^2a^{-1}d(\eta')}=(a+a_n^2)^{-\frac{1}{2}}\{\rho+\mathrm{sgn}(D_2)s\alpha(\eta').$$

Let us take (σ^1, η^1) satisfying

$$s\eta'=\eta^1/D_2(\sigma^1, \eta^1),\ \mathrm{sgn}(D_2(\sigma^1, \eta^1))=\mathrm{sgn}(D_2(\sigma, \eta)),$$
$$(\sigma^1, \eta^1)\in\Sigma.$$

Then at $(y_0, t_0, \sigma^1, \eta^1)$ (2.7) (ii) holds and $D_2^2-a^{-1}d(\eta^1)>0$. This contradict (H_2). q.e.d.

Proof of Lemma 2.2. For $(y_0, t_0, \sigma_0, \eta_0)\in S$, $P(t_0, 0, y_0, \xi, \eta_0)=0$ has real double root. Therefore from (2.2), (2.3) and (2.5)

$$D_2^2-a^{-1}d(\eta_0)=0.$$

From (2.7) (ii) we have

$$\rho D_2+\alpha(\eta_0)=0. \qquad \text{q.e.d.}$$

Proof of Lemma 2.3 *and* 2.4. First let us prove the followings

$$(2.17) \qquad \rho+\frac{\alpha(\eta)}{D_2}\geqq 0 \quad \text{in place where } D_2^2=a^{-1}d(\eta).$$

If $\rho+\alpha(\eta_0)/D_2<0$ and $D_2^2=a^{-1}d(\eta)$ at (y, t, σ_0, η_0), then we obtain $\alpha(\eta_0)/D_2=\alpha(\eta_0/D_2)=\alpha(\eta_0')>0$ from $\rho\geqslant 0$. Taking account of $a^{-1}d(\eta_0')=1$ and $d(0)=\alpha(0)=0$, and applying 'Zwischensatz' we find $0>\delta>1$ such that

$$(2.18)\qquad -\sqrt{1-a^{-1}d(\delta\eta_0')}=(a+a_n^2)^{-1}\{\rho+\alpha(\delta\eta_0')\}$$
$$1-a^{-1}d(\delta\eta_0')>0.$$

Denote $\delta\eta_0'=\eta'$ and define σ' by

$$1=(a+a_n^2)\sigma'+(\sum_{j=1}^{n-1}a_{nj}\eta_j'-\sum_{i=1}^{n-1}a_i\eta_i')=D_2(y, t, \sigma', \eta').$$

Let us normalize (σ', η'). $(\sigma, \eta)=(r\sigma', r\eta')\in\Sigma_0$, $r>0$. Here

$$(2.19)\qquad r=(a+a_n^2)\sigma+(\sum_{j=1}^{n-1}a_{nj}\eta_j-\sum_{i=1}^{n-1}a_i\eta_i)=D_2(y, t, \sigma, \eta).$$

Hence $\eta'=\delta\eta_0'=\eta/D_2$ we obtain (2.7) (ii) from (2.18). Since $D_2^2-a^{-1}\times d(\eta)>0$, this contradict (H_2) and we obtain (2.17).

Now let us prove

$$(2.20)\qquad \rho-\tilde{\rho}_\pm\equiv\rho\pm a\alpha(\eta)d(\eta)^{-\frac{1}{2}}\geq 0$$
$$\text{for } (y, t, \eta)\in R^{n-1}\times R^1\times(R^{n-1}-\{0\}).$$

From (2.17), we have (2.10) in place where $D_2=\pm a^{-\frac{1}{2}}d(\eta)^{\frac{1}{2}}$. For arbitrary (y, t, η) we can find σ_+ and σ_- satisfying

$$D_2(y, t, \eta, \sigma_\pm)=(a+a_n^2)\sigma_\pm+a_n\sum_{j=1}^{n-1}a_{nj}\eta_j-\sum_{i=1}^{n-1}a_i\eta_i=\pm a^{-\frac{1}{2}}d(\eta)^{\frac{1}{2}}$$

respectively. Considering (2.17) at $(y, t, \eta, \sigma_\pm)$ we obtain (2.20). This complete the proof of Lemma 2.3 and 2.4. q.e.d.

Proof of Lemma 2.5. If there exists (t, y) such that $c+a_n>0$, then Lemma 2.5 follows from the proof of Lemma 2.1. If $\rho=c+a_n\geq 0$, for every (t, y), only Case II is possible. Because, since the right-hand side of (2.7) (i) is non positive, $|D|$ must be qequal to D_1 and this means $\gamma=0$. Besides $D_2^2\neq 0$ at $(y_0, t_0, \sigma_0, \eta_0)\in R^{n-1}\times R^1\times\Sigma$ satisfying $L(y_0, t_0, \sigma_0, \eta_0)=0$, follows from

$$0\leq|D|=D_1=(a+a_n^2)^{-1}\{D_2^2-a^{-1}d(\eta_0)\}.$$

Moreover the assumption of this lemma means $D_2^2-a^{-1}d(\eta_0)>0$. Therefore from (2.16) we have

$$1>a^{-1}d(\eta_0') \text{ and } \rho+\text{sgn}(D_2)\alpha(\eta_0')>0.$$

Taking $p(>1)$ such that $1=a^{-1}d(p\eta_0')$

$$\begin{aligned}0\leq\rho<-\text{sgn}(D_2)\alpha(\eta')<-\text{sgn}(D_2)\alpha(p\eta')&=\text{sgn}(D_2)\left(-\sqrt{a}\frac{\alpha(p\eta_0')}{d(p\eta_0')^{1/2}}\right)\\ &=-\text{sgn}(D_2)\sqrt{a}\,\alpha(\eta_0)d(\eta_0)^{-\frac{1}{2}}\equiv|\tilde{\rho}_\pm|.\end{aligned}$$

This complete the proof of Lemma 2.6.

§3. Localization of (P_0') and Green's formula.

In this section we consider how to localize the problem (P_0') and show how to reduce Green's formula to formal algebraic calculus. This method is used by R. Sakamoto in [22].

1. From

$$(3.1)\qquad e^{-\gamma t}D_t^k u=(D_t-i\gamma)^k e^{-\gamma t}u,\quad i=\sqrt{-1}\,,\quad \gamma\geq 0,$$

$P(t, x, y, D_t, D_x, D_y)u=f$ is reformed as follows:

$$(3.2)\qquad P(t, x, y, D_t-i\gamma, D_x, D_y)e^{-\gamma t}u=e^{-\gamma t}f.$$

Here P is regarded as homogeneous of order 2 with respect to (D_x, D_y, D_t, γ). At first we take a partition of unity on $R^1\times\overline{R_+^1}\times R^{n-1}\times\Sigma=\overline{R_+^{n+1}}\times\Sigma\ni(t, x, y, \eta, \sigma, \gamma)$, where

$$\Sigma=\{(\eta, \sigma, \gamma),\ |\eta|^2+\sigma^2+\gamma^2=1,\ \gamma\geq 0\}.$$

We consider the following C^∞ functions:

$$(3.3)\qquad \alpha_0(x)+\alpha_1(x)=1 \text{ on } \overline{R_+^1},\ \text{supp.}\ \alpha_0(x)\subset\left[\frac{\delta}{2}, \infty\right]$$

$$\text{supp } \alpha_1(x)\subset[0, \delta],\ \alpha_1(x)\equiv 1 \text{ on}\left[0, \frac{\delta}{2}\right].$$

(3.4) $$\sum_{j=1}^{N_1} \alpha_{1,j}(t, y)+\alpha_{1,0}(x, y)\equiv 1 \text{ in } R^1\times R^{n-1}$$

$$\text{supp } \alpha_{1,0}\subset R^n-K\cap R^n, \text{ supp } \alpha_{1,j}\subset \text{n.b.d. of } K\cap R^n.$$

(3.5) $$\sum_{k=1}^{N_1} \alpha_{2,k}(\eta, \sigma, \gamma)+\alpha_{2,0}(\eta, \sigma, \gamma)\equiv 1 \text{ on } \Sigma, \ \alpha_{2,0}(\eta, \sigma, \gamma)=\alpha_0(\gamma)$$

$$\alpha_{2,k}(\eta, \sigma, \gamma)=\alpha_1(\gamma)\alpha_{2,k}((1-\gamma^2)^{\frac{1}{2}}\eta, \ (1-\gamma^2)^{\frac{1}{2}}\sigma),$$

where

$$\sum_{k=1}^{N_2} \alpha_{2,k}(\eta, \sigma)\equiv 1 \text{ on } \Sigma_0=\{(\eta, \sigma), \ |\eta|^2+\sigma^2=1\}.$$

Let us denote simply

(3.6) $$\beta_{0,1}+\beta_{0,2}+\sum_{j=1}^{N=(N_1+1)N_2} \beta_i(t, x, y, \eta, \sigma, \gamma)\equiv 1 \text{ on } \overline{R_+^n}\times\Sigma,$$

where $\beta_{0,1}=\alpha_0(x)$, $\beta_{0,2}=\alpha_1(x)(\sum_{j=0}^{N_1}\alpha_{1,j})\alpha_{2,0}$.

Now we extend the definition domains of $\alpha_{2,k}$, $(k=0, 1, \ldots, N_2)$ to $(R^{n-1}\times R^1\times R_+^1)-\{0\}$ keeping homogenuity of order 0 with respect to (η, σ, γ). Then we have

(3.7) $$\sum_{j=1}^{2}\beta_{0,j}+\sum_{i=1}^{N}\beta_i\equiv 1 \text{ on } R_+^n\times R^n(\eta, \sigma) \text{ for every } \gamma>0.$$

Corresponding this partition of unity, (P_0') is localized as β being one of β_{0j} and β_i $(i=1, \ldots, N)$,

(P_0') $$\begin{cases} P(t, x, y, D_t, D_x, D_y)e^{\gamma t}\beta(D)e^{-\gamma t}u=f_\beta, \\ B(t, y, D_t, D_x, D_y)e^{\gamma t}\beta(D)e^{-\gamma t}\, u\big|_{x=0}=g_\beta, \text{ where} \end{cases}$$

(3.8) $$\beta(D)w=\iint e^{2\pi i(y\eta+t\sigma)}\beta(x, y, x, \eta, \sigma, \gamma)\mathfrak{F}_{y,t}(w(t, x, y))d\eta d\sigma.$$

(3.9) $$f_\beta=e^{\gamma t}\beta(D)e^{-\gamma t}f+[P, e^{\gamma t}\beta(D)e^{-\gamma t}]u, \text{ where } [P, Q]=PQ-QP$$
$$g_\beta=e^{\gamma t}\beta(D)e^{-\gamma t}g\big|_{x=0}+[B, e^{\gamma t}\beta(D)e^{-\gamma t}]u\big|_{x=0}.$$

Here we denote $\beta(D, \gamma)=e^{\gamma t}\beta(D)e^{-\gamma t}$. For $u\in\mathcal{H}_{m,\gamma}(R_{+}^{n+1})$

$$|\beta(D, \gamma)u|_{m,\gamma}\leq C|u|_{m,\gamma},\ |[P, \beta(D, \gamma)]u|_{m-1,\gamma}^{2}\leq C|u|_{m,\gamma}^{2}\,. \tag{3.10}$$

Since $\beta(D, \gamma)D_{x}u\,\big|_{x=0}=D_{x}\beta(D, \gamma)u\,\big|_{x=0}$, we have

$$<[B, \beta(D, \gamma)]u>_{m-1,\gamma}^{2}\leq C\langle u\rangle_{m-1,\gamma}^{2}. \tag{3.11}$$

If we can apply Theorem 1 to $(P_{0}')_{\mathrm{loc}}$, we, have

$$\gamma|\beta(D, \gamma)u|_{k,\gamma}^{2}+\gamma\sum_{j=0}^{1}<\Lambda_{y,\gamma}^{-\frac{1}{2}}D_{x}^{j}\beta(D,\gamma)u>_{k-j,\gamma}^{2}\leq\frac{C_{k}}{\gamma}\Big\{|f|_{k-1,\gamma}^{2}+<\Lambda_{y,\gamma}^{\frac{1}{2}}g>_{k-1,\gamma}^{2}+\gamma|u|_{k,\gamma}^{2}\Big\},\ \text{for } \gamma>\gamma_{k}, \tag{3.12}$$

because $<\Lambda_{y,\gamma}^{\frac{1}{2}}[B, \beta(D, \gamma)]u>_{k-1,\gamma}^{2}\leq C_{1}\langle u\rangle_{k-\frac{1}{2},\gamma}^{2}\leq C_{2}|u|_{k,\gamma}^{2}$ follows.

Conversely if (3.12) holds for every $\beta_{0,j}$, $(j=1. 2)$ and β_{i}, $(i=1, 2, \ldots, N)$, we obtain Theorem 1, taking sum of (3.12)'s for all β_{i} and $\beta_{0,j}$ and making γ_{k} and C_{k} larger if necessaary.

2. At first let P and Q be second and first order partial differential operators with constant coefficient. Consider

$$\mathcal{G}(P, Q;\ u, v)=(Pu, Qu)_{0,\gamma}-(Qu, Pu)_{0,\gamma}, \text{ where} \tag{3.13}$$

$(u, v)_{0,\gamma}=(e^{-\gamma t}u, e^{-\gamma t}u)_{L^{2}(x,y,t)}$. $P(\xi, \eta, \sigma, \gamma)$ and $Q(\xi, \eta; \sigma, \gamma)$, being characteristic polynomials of $e^{-t\gamma}P$ and $e^{-\gamma t}Q$ respectively, we associate (3.13) to

$$G(P, Q)=P(\xi, \eta, \sigma, \gamma)Q(\zeta, \eta, \sigma, \gamma)-Q(\xi, \eta, \sigma, \gamma)P(\zeta, \eta, \sigma, \gamma). \tag{3.14}$$

Now we regard that ξ, ζ, η, σ and γ are real number and denote

$$\begin{aligned} P(\xi, \eta, \sigma, \gamma)&=P_{0}(\xi, \eta, \sigma, \gamma)-i\gamma P_{1}(\xi, \eta, \sigma, \gamma) \\ &\qquad (P_{0}, P_{1}, \text{ real}) \\ Q(\zeta, \eta, \sigma, \gamma)&=Q_{0}(\zeta, \eta, \sigma, \gamma)-i\gamma Q_{1}(\zeta, \eta, \sigma, \gamma), \\ &\qquad (Q_{0}, Q_{1}, \text{ real}). \end{aligned} \tag{3.15}$$

Then $G(P, Q)$ is written in the following form:

$$G(P, Q)=(\xi-\zeta)G_x^{P,Q}(\xi, \zeta, \eta, \sigma, \gamma) -2i\gamma G_t^{P,Q}(\xi, \zeta, \eta, \sigma, \gamma), \text{ where} \tag{3.16}$$

$$G_x^{P,Q}=\frac{1}{\xi-\zeta}\{P^0(\xi)Q^0(\zeta)-P^0(\zeta)Q^0(\xi)\} +\frac{\gamma^2}{\xi-\zeta}\{P^1(\xi)Q^1(\zeta)-P^1(\zeta)Q^1(\xi)\}$$

$$G_t^{P,Q}=\frac{1}{2}\{(P^1(\xi)Q^0(\zeta)-P^0(\zeta)Q^1(\xi))+(P^1(\zeta)Q^0(\xi) -P_0(\xi)Q^1(\zeta))\}, \text{ where } P_0(\xi)=P_0(\xi, \eta, \sigma, \gamma) \text{ etc.} \tag{3.17}$$

After actual integration by parts, corresponding to G_x^{PQ} and G_t^{PQ}, we have bi-linear forms $\mathcal{G}_x(D_xu, D_xv, D_y, D_t, \gamma;\ u, v)$, $\mathcal{G}_t(D_xu, D_xv, D_0, D_t, \gamma;\ u, v)$ such that

$$\mathcal{G}(P, Q;\ u, v)=i\iint G_x(D_xu, D_xv, D_y, D_t, \gamma;\ uv)dydt -2i\gamma\iiint G_t(D_xu, D_xv, D_y, D_t, \gamma;\ uv)dydtdx. \tag{3.18}$$

We can obtain similar formula in case where P has variable coefficients and Q is a differential operator in x and t, whose coefficents are pseudo-differential operators with respect y, as follows,

$$Q=Q_0D_x+Q_1,\quad Q_iu=\int e^{i(y\eta+t\sigma)}\sigma(Q_i)(x, y, t, \eta, \sigma, \gamma)\hat{u}(\sigma, x, \eta)d\eta d\sigma.$$

(Q_i): homogeneous of order i with respect to (η, σ, γ).

Then we have formulas corresponding (3.16) and (3.17) and obtain

$$\begin{aligned}\mathcal{G}(P, Q:\ u, v)&=i\iint G_x(y, t, D_xu, D_y, D_x, \gamma;\ u, v)dydt \\ &\quad -2i\gamma\iiint G_t(D_xu, D_xv, D_y, D_t, \gamma:\ u, v)dydtdx+R(u, v) \\ &\equiv i\mathcal{G}_x\langle P, Q;\ u, v\rangle-2i\gamma\mathcal{G}_t(P, Q;\ u, v)+R(u, v),\end{aligned} \tag{3.19}$$

where $\quad |R(u, v)|\leq C|u|_{1,\gamma}|v|_{1,\gamma}.$

Remark that for $\beta=\beta_{0,1}$ we obtain (3.12) after calculating $\mathcal{G}(P, \frac{\partial P}{\partial \tau}(D);\ u, u)$ as in the case of Cauchy problem (c.f. [7]). And for

$\beta=\beta_{0,2}$ we have (3.12) by the same way as in cawe of elliptic boundary value problem since $L\neq 0$ in $\Sigma_\delta=\{(\eta, \sigma, \gamma),\ \gamma\geq\delta>0\}$.

For $\beta=\beta_i$ ($i=1, \dots, N$) such that $L=0$ for $(t, 0, y, \eta, \sigma, \gamma)\in$ supp β_i, the most complicated arguement is required. We consider this in the next section. On the other hand for β_i such that supp β_i does not contain any point of S, the estimate (3.12) has been obtained in R. Sakamoto [22]. Actually we may take Q as a first order operator with characteristic $Q=c\dfrac{\partial P}{\partial\tau}+(\xi-\xi_+(\tau\colon\ \eta))$, when ξ_+ is a simple root of $P(t, 0, y, \tau, \xi, \eta)=0$ on supp β_i. If $P=0$ have double roots on supp β_i, then replace $\xi-\xi_+(\tau, \eta)$ by $\xi-\dfrac{\xi_++\xi_-}{2}(\tau, \eta)$. c and δ must be taken sufficiently small.

§4. The estimate in the neighbourhood of S.

In this section we prove (3.12), for $(P_0')_{loc}$ with β whose support contains the point of S. From these we obtain Theorem 1. As is mentioned in previous section, it is most important how to select first order operator Q. Let us remember that the set S is characterized in Lemma 2.2 and it's cotrollary, Later the result of Lemma 2.3 is used in connection with, so-called, sharp Gärding's inequality. Speaking in conclusion, we choose the symbol of Q (that we denote the same letter Q) as

$$(4.1)\qquad Q=Q_1+Q_2=\left\{\frac{1}{2}\frac{\partial p}{\partial\tau}+(\tilde{D}_2-D_2)\right\}+(\tilde{\rho}_+-a_n)\frac{1}{2}\frac{\partial P}{\partial\xi},$$

where

$$(4.2)\qquad D_2=\frac{1}{2}(D_2+a^{-\frac{1}{2}}d(\eta)^{\frac{1}{2}})-\frac{1}{2}(a+a_n^2)^{-1}\tilde{\rho}_+(\rho D_2+\alpha(\eta))$$

$$(4.3)\qquad \tilde{\xi}=\xi+a_n\sigma+\sum^{n-1}{}_{nj}\eta_j,\ \text{then}\ \tilde{\xi}=\frac{1}{2}\frac{\partial P}{\partial\xi}+a_n\gamma_i.$$

D_2 and $\alpha(\eta)$ are defined peviously in §2. Here we have assumed that the support β contains the point of V_+ and is so small that it does not contain any point of V_-. In another case, we replace $a^{-\frac{1}{2}}d(\eta)^{\frac{1}{2}}$ and ρ_+ by $-a^{-\frac{1}{2}}d(\eta)^{\frac{1}{2}}$ and ρ_- respectively. The first term Q_1 is very close to $\frac{1}{2}\frac{\partial P}{\partial \tau}$ that keeps the interior norm positive. The boundary integrals of $G\left(P, \frac{1}{2}\frac{\partial P}{\partial \tau}; \beta u, \beta u\right)$ are canceled out by that of $G\left(P, \left(Q-\frac{1}{2}\frac{\partial P}{\partial \tau}\right); \beta u, \beta u\right)$ except cpositive semi-definite terms. The interior integrals of the latter become sufficiently small than that of $G\left(P, \frac{1}{2}\frac{\partial P}{\partial \tau}; \beta u, \beta u\right)$ by making the support of β small.

The following part of this section is devoted to actual calculus mentioned above, is several steps stated as lemmas.

We have from (2.2) and (2.3)

$$\frac{1}{2}\frac{\partial P}{\partial \tau}=a_n\tilde{\xi}-D_2+ia\gamma=Q_1-(D_2-\tilde{D}_2). \tag{4.4}$$

Remark that from (4.2), $D_2=\tilde{D}_2$ on $V_+\cap w$. We denote simply by β, one of β_j such that it's support contains a point of $V_+\cap W\subset S$. Moreover we denote pseudo-differential operator $e^{\gamma t}\beta(D)\ e^{-\gamma t}=\beta(D, \gamma)$ simply by β.

1. Tirst of all let us consider $\mathcal{G}\left(P, \frac{1}{2}\frac{\partial P}{\partial \tau}; \beta u, \beta u\right)$ and we obtain

Lemma 4.1 *There exist positive constants c, C and γ_0 such that for $\gamma>\gamma_0$ and $u\in\mathcal{H}_2$, we have*

$$\begin{aligned} i\mathcal{G}\left(P, \frac{1}{2}\frac{\partial P}{\partial t};\ \beta u.\ \beta u\right) &\geq c\gamma|\beta u|^2_{1,\gamma} \\ &-[<\sqrt{a_n}\tilde{D}_x\beta_\gamma u>^2+<a_nD_1\beta_\gamma u, \beta_\gamma u> \\ &-2Re<\tilde{D}_x\beta_\gamma u, D_2\beta_\gamma u>] \\ &-C\{<\beta g>^2_{0,\gamma}+\gamma\langle u\rangle^2_{0,\gamma}\},\ \text{where } \beta_\gamma=e^{-\gamma t}\beta(D,\gamma), \end{aligned} \tag{4.5}$$

and $\langle v\rangle^2=\langle v, v\rangle=\iint|v|^2\,dydt$. *$\tilde{D}x$, D_1 and D_2 are differential operator with symbol $\tilde{\xi}$, D_1 and D_2 defined in* (4.3), (2.5) *and* (2.4).

In the same way as Lemma 4.1 proved below, we obtain

Lemma 4.2 *There exist positive constants c, C and γ_0 such that for*

$\gamma>\gamma_0$ *and* $u\in\mathcal{H}_{2,\gamma}$ *we have.*

$$(4.5)\qquad iG(P, Q_1;\ \beta u, \beta u)\geqq c|\beta u|_{1,\gamma}^2-[<a_n\tilde{D}_x\beta_\gamma u>^2 +<a_n D_1\beta_\gamma u, \beta_\gamma u>-2Re<\tilde{D}_x\beta_\gamma u, \tilde{D}_2\beta_\gamma u>] -C\{\langle\beta g\rangle_{0,\gamma}^2+\gamma\langle u\rangle_{0,\gamma}^2\},$$

where $\tilde{D}_2$ is the first order psudo-differential operator with symbol $\tilde{D}_2$.

Proof of Lemffla 4.1 The characteristic polynomial of $P(t. x. y, D_t-i\gamma, D_x, D_y)$ is

$$(4.6)\qquad P=(\tilde{\xi}^2-D_1-a_n^2\gamma^2)-2i\gamma(a_n\tilde{\xi}-D_2).$$

Following notation (3.14) we have from (4.4) and (4.6).

$$(4.7)\quad G\left(P, \frac{1}{2}\frac{\partial P}{\partial\tau}\right)=\{(\xi^2-D_1-a_n^2\gamma^2)-2i\gamma(a_n\tilde{\xi}-D_2)\}\{(a_n\tilde{\zeta}-D_2) -i\gamma a\}-\{(a_n\tilde{\xi}-D_2)+i\gamma a\}\{(\zeta^2-D_1-a_n^2\gamma)+2i\gamma(a_n\tilde{\zeta}-D_2)\}$$

$$(4.7)'\qquad ReG\left(P, \frac{1}{2}\frac{\partial P}{\partial\tau}\right)=(\xi-\zeta)G_x\left(P, \frac{1}{2}\frac{\partial P}{\varrho\tau}\right)(\xi, \zeta, \eta, \sigma, \gamma) =(\xi-\zeta)\{a_n(\tilde{\xi}\tilde{\zeta}+D_1)-(\tilde{\xi}D_2+\tilde{\zeta}D_2) +a_n(a_n-2a)\gamma^2\}$$

$$(4.7)''\qquad i\ ImG\left(P, \frac{1}{2}\frac{\partial P}{\partial\tau}\right)=-2i\gamma G_t\left(P, \frac{1}{2}\frac{\partial P}{\partial\tau}\right)(\xi, \zeta, \eta, \sigma, \gamma) =-i\gamma\{4(a_n\tilde{\xi}-D_2)(a_n\tilde{\zeta}-D_2)+a(\tilde{\xi}^2-D_1 -a_n^2\gamma^2)+(\tilde{\zeta}^2-D_1-a_n\gamma^2)a\}.$$

Here we remark that the following inequality is known (c.f [7])

$$(4.8)\qquad G_t\left(P, \frac{1}{2}\frac{\partial P}{\partial\tau}\right)(\xi, \xi, \eta, \sigma, \gamma)\geq C(|\xi|^2+|\eta|^2+|\tau|^2).\quad C>0.$$

Corresponding to (4.7)″ the notation (3.19) gives

$$(4.9)\qquad 2\mathcal{G}_t\left(P, \frac{1}{2}\frac{\partial P}{\partial\tau}, \beta u, \beta u\right)=4\|(a_n\tilde{D}_x-D_2)\beta_\gamma u\|^2 +2Re(a\beta_\gamma u, (\tilde{D}_x^2-D_1-a_n^2\gamma^2)\beta_\gamma u).$$

Considering (4.7)″ and integrating by parts, we obtain

$$(4.10)\qquad \mathcal{J}_t\left(P, \frac{1}{2}\frac{\partial P}{\partial \tau}, \beta u, \beta u\right) - 2Im\langle a\beta_\gamma u, D_x\beta_\gamma u\rangle \geq \frac{c}{2}|\beta u|_{1,\gamma}^2 - C|\beta u|_{0,}^2.$$

The second term in the left-hand side of (4.10) is estimated as

$$(4.11)\qquad |\langle a\beta_\gamma u, D_x\beta_\gamma u\rangle| \leq C\left\{\varepsilon\langle\beta u\rangle_{\frac{1}{2},\gamma}^2 + \frac{1}{\gamma}\langle\beta g\rangle_{0,\gamma}^2 + \gamma\langle\beta u\rangle_{0,\gamma}^2 + \langle u\rangle_{0,\gamma}^2\right\}, \text{ where}$$

$c>0$, and we can take ε artbitrary small as making the support of β sufficiently small. In fact from the relation:

$$(4.12)\qquad \tilde{\xi} = B(y, t, \xi, \eta, \sigma) + (a+a_n^2)^{-1}\{\rho D_2 + \alpha(\eta)\}, \text{ at } x=0$$

we have

$$(4.13)\quad \tilde{D}_x\beta_\gamma u\big|_{x=0} = [\tilde{D}_x, \beta_\gamma]u\big|_{x=0} + \beta_\gamma\tilde{D}_x u\big|_{x=0} = [\tilde{D}_x, \beta_\gamma]u\big|_{x=0} + \beta_\gamma Bu\big|_{x=0} + \beta_\gamma(a+a_n^2)^{-1}\{\rho D_2 + \alpha(D_y)\}u\big|_{x=0}.$$

Since $\rho D_2 + \alpha(\eta)$ is small from Lemma 2.2, we have

$$(4.14)\qquad \langle\beta_\gamma(a+a_n^2)^{-1}\{\rho D_2+\alpha(D_y)\}u\rangle^2 \leq \varepsilon\langle\beta u\rangle_{1,\gamma}^2 + C\langle u\rangle_{0,\gamma}^2, \quad \exists C>0.$$

$$(4.15)\qquad \langle[\tilde{D}_x, \beta_\gamma]u\rangle^2 \leq C(\gamma^2\langle\beta u\rangle_{0,\gamma}^2 + \langle u\rangle_{0,\gamma}^2)$$

By virtue of (4.13), (4.14) and (4.15), we can prove (4.11). On the other hand we have

$$(4.16)\qquad \gamma\langle\beta u\rangle_{0,\gamma}^2 \leq \delta\langle\beta u\rangle_{\frac{1}{2},\gamma}^2 \leq C\delta|\beta u|_{1,\gamma}^2,$$

where δ is a positive number given in (3.5), which we take sufficiently smaller than $c>0$. From these it follows

$$(4.10)'\qquad \mathcal{J}_t\left(P, \frac{1}{2}\frac{\partial P}{\partial \tau}; \beta u, \beta u\right) \geq \frac{c}{4}|\beta u|_{1,\gamma}^2 - \left\{\frac{1}{\gamma}\langle\beta g\rangle_{0,\gamma}^2 + \langle u\rangle_{0,\gamma}^2\right\}$$

From (4.7)′ and (3.19) and using (4.16) again for third term of (4.7)′ we obtain (4.5)

q.e.d.

Proof of Lemma 4.2 is carried out in the same way, only remarking that

(i) $D_2-\tilde{D}_2$ is so small that (4.8) follows even if we replace $\frac{1}{2}\frac{\partial P}{\partial \tau}$ by Q_1.

(ii) $\text{Re } G(P, Q_1)=(\xi-\zeta)G_x(P, Q_1)$

$$=(\xi-\zeta)\{a_n(\tilde{\xi}\tilde{\zeta}+D_1)-(\tilde{\xi}D_2+D_2\tilde{\zeta})+a_n(a_n+2a)\gamma^2\}.$$

2. Next we consider $\mathcal{G}\left(P, \frac{1}{2}\frac{\partial P}{\partial \xi}; \beta u, \beta u\right)$ and $\mathcal{G}(P, Q_2; \beta u, \beta u)$.

Lemma 4.3 *For any $\varepsilon>0$, if we take the support of β in $R_+^{n+1}\times\Sigma$ sufficiently small, we have*

$$(4.17)\quad i\mathcal{G}\left(P, \frac{1}{2}\frac{\partial P}{\partial \xi}, \beta u, \beta u\right)\geq-\varepsilon\gamma|\beta u|_1^2-[\langle\tilde{D}_x\beta_\gamma u\rangle^2+\langle D_1\beta_\gamma u, \beta_\gamma u\rangle]-C\left\{\frac{1}{\varepsilon\gamma}|\beta f|_{0,\gamma}^2+\frac{1}{\varepsilon\gamma}|u|_{0,\gamma}^2+\frac{1}{\varepsilon\gamma}\langle\beta g\rangle_{0,\gamma}^2+\frac{\gamma}{\varepsilon}\langle\beta u\rangle_{0,\gamma}^2+C|\beta u|_{1,\gamma}^2\right\}$$

c; constant independent of ε and γ, $\gamma>0$.

Lemma 4.4 *For any $\varepsilon>0$, if we take the support of β in $R_+^{n+1}\times\Sigma$ sufficiently small, we have for $\gamma>\gamma_0$,*

$$(4.18)\quad i\mathcal{G}(P, Q_2; \beta u, \beta u)\geq-\varepsilon\gamma|\beta u|_1^2-[\langle(-\rho_+(D_y)-a_n)D_x\beta_\gamma u, D_x\beta_\gamma u\rangle+\langle(-\rho_+(D_y)-a_n)D_1\beta_\gamma u, \beta_\gamma u\rangle]-C\left\{\frac{1}{\gamma}|\beta f|_{0,\gamma}^2+\frac{1}{\gamma}|u|_{1,\gamma}^2+\langle\beta g\rangle_{0,\gamma}^2\right\}.$$

Proof of Lemma 4.3. Since $\frac{1}{2}\frac{\partial P}{\partial \xi}(t, x, y, \sigma-i\gamma, \xi, \eta)=\tilde{\xi}-ia_n\gamma$, we have

$$(4.19)\quad G\left(P, \frac{1}{2}\frac{\partial P}{\partial\xi}\right)=\{(\xi^2-D_1-a_n^2\gamma^2)-2i\gamma(a_n\xi-D_2)\}\{\zeta+ia_n\gamma\}$$
$$-\{\xi-ia_n\gamma\}\{(\zeta^2-D_1-a_n^2\gamma^2)+2i\gamma(a_n\zeta-D_2)\}$$

$$(4.19)'\quad Re\, G\left(P, \frac{1}{2}\frac{\partial P}{\partial\xi}\right)=(\xi-\zeta)G_x\left(P, \frac{1}{2}\frac{\partial P}{\partial\xi}\right)$$
$$=(\xi-\zeta)\{\xi\zeta+D_1+3a_n^2\gamma^2\}$$

$$(4.19)''\quad i\, Im\, G\left(P, \frac{1}{2}\frac{\partial P}{\partial\xi}\right)=-2i\gamma\, G_t\left(P, \frac{1}{2}\frac{\partial P}{\partial\xi}\right)$$
$$=-i\gamma\{2(a_n\xi-D_2)\zeta+2\zeta(a_n\xi-D_2)-(\xi^2-D_1-a_n^2\gamma^2)a_n$$
$$-a_n(\zeta^2-D_1-a_n^2\gamma^2).$$

Now for

$$(4.20)\quad \mathcal{G}_t\left(P, \frac{1}{2}\frac{\partial P}{\partial\xi};\ \beta u, \beta u\right)=Re(2(a_n\tilde{D}_x-D_2)\beta_\gamma u,\ \tilde{D}_x\beta_\gamma u)$$
$$+Re((\tilde{D}_x^2-D_1-a_n^2\gamma^2)\beta_\gamma u,\ a_n\beta u),$$

we can prove the following estimate in the same way as in lemma 4.3.

$$(4.21)\quad \left|\mathcal{G}_t\left(P, \frac{1}{2}\frac{\partial P}{\partial\xi};\ \beta u, \beta u\right)\right|\leq\varepsilon|\beta u|_{1,\gamma}+C\frac{1}{\varepsilon}(|\beta f|_{0,\gamma}|\beta u|_{0,\gamma}$$
$$+|u|_{1,\gamma}|\beta u|_{0,\gamma}+\frac{1}{\gamma}\langle\beta g\rangle_{0,\gamma}^2+\gamma\langle\beta u\rangle_{0,\gamma}^2).$$

The estimate of the second term follows from.

$$(4.22)\quad (\tilde{D}_x^2-D_1-a_n^2\gamma^2)\beta_\gamma u=e^{-\gamma t}P\beta u+2i\gamma(a_n\tilde{D}_x-D_2)\beta_\gamma u.$$

and

$$|(\gamma(a_n\tilde{D}_x-D_2)\beta_\gamma u,\ \beta_\gamma u)|\leq\varepsilon|\beta u|_{1,\gamma}^2.$$

The estimate of the first term of (4.20) follows essentially from

$$(4.23)\quad |(\tilde{D}_x\beta_\gamma u,\ \tilde{D}_x\beta_\gamma u)|\leq\varepsilon^2|\beta u|_{1,\gamma}^2+C(|\beta f|_{0,\gamma}^2|\beta u|_{0,\gamma}+|u|_{1,\gamma}|\beta u|_{0,\gamma})$$

and

$$|(D_2\beta_\gamma u,\ \tilde{D}_x\beta_\gamma u)|\leq\varepsilon|\beta u|_{1,\gamma}^2+\frac{C}{\varepsilon}(\tilde{D}_x\beta_\gamma u,\ \tilde{D}_x\beta_\gamma u).$$

(4.23) is decomposed to

$$(4.24)\quad (\tilde{D}_x\beta_\gamma u,\ D_x\beta_\gamma u)=Re(\tilde{D}_x^2\beta_\gamma u,\ \beta_\gamma u)+Im\langle\tilde{D}_x\beta_\gamma u,\ \beta_\gamma\rangle$$
$$+R(\beta_\gamma u,\ \beta_\gamma u),$$

where $|R(\beta_\gamma u, \beta_\gamma u)| \leq C|\beta u|_{1,\gamma}|\beta u|_{0,\gamma}$.

$(D_x^2 \rho_\gamma u, \beta_\gamma u)$ is estimated in the sam eway as (4.22), if we remark that

$$(4.25) \qquad |(D_1\beta_\gamma u, \beta_\gamma u)| \leq \varepsilon^2 |\beta u|_{1,\gamma}^2$$

follows from Lemma 2.2. The estimate of $Im\langle D_x\beta_\gamma u, \beta_\gamma u\rangle$ can be carried out in the same way as (4.11). Therefore remarking $\langle \beta u\rangle_{\frac{1}{2},\gamma} \leq C|\beta u|_{1,\gamma}$ we obtain (4.21). Finaly remark $\gamma|\beta u|_{0,\gamma} \leq \varepsilon^2|\beta u|_{1,\gamma}$ then we have Lemma 4.3 from (4.21) and (3.19) in §3, q.e.d.

Proof of Lemma 4.4 is the same as that of Lemma 4.3.

Combining Lemma 4.2 and Lemma 4.4, we obtain

$$(4.26) \quad i\mathcal{J}(P, Q; \beta u, \beta u) \geq \frac{c}{2}\gamma|\beta u|_{1,\gamma}^2 - \mathcal{B} - C\left\{\frac{1}{\gamma}|\beta f|_{0,\gamma}^2 + \langle \beta g\rangle_{0,\gamma}^2 + \frac{1}{\gamma}|u|_{1,\gamma}^2\right\},$$

$$(4.27) \qquad \mathcal{B} \equiv -2Re\langle \tilde{D}_x\beta_\gamma u, \tilde{D}_2\beta_\gamma u\rangle - \langle \tilde{\rho}_+(D_y)\tilde{D}_x\beta_\gamma u, \tilde{D}_x\beta_\gamma u\rangle - \langle \tilde{\rho}_+(D_y)D_1\beta_\gamma u, \beta_\gamma u\rangle.$$

Now we can show the following (4.28) proved later:

$$(4.28) \qquad |\mathcal{B} - \mathcal{H}\langle \beta_\gamma u, \beta_\gamma u\rangle| \leq \varepsilon\gamma|\beta u|_{1,\gamma}^2 + C\left(|u|_{1,\gamma}^2 + \frac{1}{\gamma}\langle \beta g\rangle_{\frac{1}{2},\gamma}^2\right),$$

where

$$(4.29) \qquad \mathcal{H}\langle \beta_\gamma u, \beta_\gamma u\rangle = Re\langle (a+a_n^2)^{-1}(\rho-\tilde{\rho}_+)D_2\beta_\gamma u, (D_2+\sqrt{a}\, d^{\frac{1}{2}}(D_y))\beta_\gamma u\rangle.$$

Now remark that on the support of β it holds

$$(4.30) \qquad (\rho-\tilde{\rho}_+)D_2(D_2+\sqrt{a}\, d^{\frac{1}{2}}(\eta)) \geq 0$$

because of $\rho-\rho_+ \geq 0$ and that D_2 is close to $\sqrt{a}\, d^{\frac{1}{2}}(\eta)$ and $d^{\frac{1}{2}}(\eta) \neq 0$.

Using (4.29) and *sharp Garding's inequality* we have

$$(4.31) \qquad \mathcal{H}\langle \beta_\gamma u, \beta_\gamma u\rangle \geq -C\langle \beta_\gamma u\rangle_{\frac{1}{2}}^2 = -C\langle \beta u\rangle_{\frac{1}{2},\gamma}^2, \text{ where}$$

$$\langle v\rangle_{\frac{1}{2}}^2 = \iint (|\eta|^2+\sigma^2)^{\frac{1}{2}}|v(\eta, \sigma)|^2 d\sigma d\eta.$$

From the corollary of Lemma 2.2 we have

$$(|\eta|^2+|\tau|^2)\beta\leq C(|\eta|^2+\gamma^2)\beta. \tag{4.32}$$

Therefore it follows

$$<\beta g>^2_{\frac{1}{2},\gamma}\leq C<\Lambda^{\frac{1}{2}}_{y,\gamma}\beta g>^2_{0,\gamma}, \quad <\Lambda^{-\frac{1}{2}}_{y,\gamma}\beta u>^2_{1,\gamma}\leq C|\beta u|^2_{1,\gamma}. \tag{4.33}$$

From (4.26), (4.28) (4.31) and (4.32) we have

Lemma 4.5 *There exist positive constants C, c and γ_0 such that for $\gamma>\gamma_0$ and for $u\in\mathcal{H}_{2,\gamma}$*

$$i\mathcal{J}(P, Q;\ \beta u, \beta u)\geq\frac{c}{4}\gamma|\beta u|^2_{1,\gamma}-C\Big(\frac{1}{\gamma}|\beta f|^2_{0,\gamma} + \frac{1}{\gamma}<\Lambda^{\frac{1}{2}}_{y,\gamma}\beta g>^2_{0,\gamma}+|u|^2_{1,\gamma}\Big). \tag{4.34}$$

Remark to Lemma 4.5. Changing point of view in the process of the proof of Lemma 4.5, we have

$$i\mathcal{J}(P, G;\ \beta u, \beta u)\leq 2c\gamma|\beta u|^2_{1,\gamma}+C\Big(\frac{1}{\gamma}|\beta f|^2_{1,\gamma}+<\Lambda^{\frac{1}{2}}\beta g>^2_{0,\gamma} +|u|^2_{1,\gamma}\Big)+\mathcal{H}<\beta_\gamma u, \beta_\gamma u>, \text{ for } \gamma>\gamma_0. \tag{4.35}$$

(4.35) is used in the proof of Theorem 4 given in §5.

Taking account of

$$|\mathcal{J}(P, Q;\ \beta u, \beta u)|\leq C|P\beta u|_{0,\gamma}|\beta u|_{1,\gamma}\leq\frac{1}{4\gamma\varepsilon}|P\beta u|^2_{0,\gamma} +C^2\gamma\varepsilon|\beta u|^2_{1,\gamma}, \tag{4.36}$$

choosing ε sufficiently small, we can see from (4.34) and (4.36) that there exist constant C_1 and γ_1 such that we obtain (3.12) which is our purpose in this section. end of remark.

Proof of (4.28). First remark that

$$\tilde{D}_x\beta_\gamma u\,\Big|_{x=0}=[\tilde{D}_x, \beta_\gamma]u\,\Big|_{x=0}+\beta_\gamma Bu\,\Big|_{x=0} \tag{4.37}$$

$$+\beta_\gamma(a+a_n^2)^{-1}\{\rho D_2+\alpha(D_y)-c\gamma i\}u\big|_{x=0}$$
$$=\beta_\gamma g\big|_{x=0}+[B,\beta_\gamma]u\big|_{x=0}+(a+a_n^2)^{-1}\{\rho D_2$$
$$+\alpha(D_y)-c\gamma i\}\beta_\gamma u\big|_{x=0}.$$

Let $F(D)$ be first order pseudo-differential operator, then we have

$$(4.38)\qquad |\langle F(d)\beta_\gamma u,\ \tilde{D}_x\beta_\gamma u\rangle-\langle F(D)\beta_\gamma u,\ (a+a_n^2)^{-1}\{\rho D_2$$
$$+\alpha(D_y)-c\gamma i\}\beta_\gamma u\rangle|\leq C\{\langle\beta u\rangle_{\frac{1}{2},\gamma}\langle\beta g\rangle_{\frac{1}{2},\gamma}$$
$$+\langle\beta u\rangle_{\frac{1}{2},\gamma}\langle u\rangle_{\frac{1}{2},\gamma}\}\leq C\{\varepsilon\gamma|\beta u|_{1,\gamma}^2+\frac{1}{\varepsilon\gamma}\langle\beta g\rangle_{\frac{1}{2},\gamma}^2$$
$$+|u|_{1,\gamma}^2\}\ \text{for},\ \gamma>{}^\exists\gamma_0>0.$$

Now we calculate the symbol in (4.27) replacing $\tilde{\xi}$ by

$$(4.37)\qquad h=(a+a_n^2)^{-1}\{\rho D_2+\alpha(\eta)-c\gamma i\}=h_0-ih_1\gamma,\quad h_1=(a+a_n^2)^{-1}c$$

$$(4.27)'\qquad I\equiv-(h\tilde{D}_2+\tilde{D}_2\bar{h})-\tilde{\rho}_+(h\bar{h}+D_1)$$
$$=-2h_0\tilde{D}_2-\tilde{\rho}_+(h_0^2+D_1)-\tilde{\rho}_+h_1^2\gamma^2\equiv I_0-\tilde{\rho}_+h_1^2\gamma^2.$$

From $\tilde{\rho}_+=-\sqrt{a}\,ad^{-\frac{1}{2}}$ we have

$$(4.40)\qquad h_0=(a+a_n^2)^{-1}(\rho-\tilde{\rho}_+)D_2+(a+a_n^2)^{-1}(\tilde{\rho}_+D_2+\alpha)$$
$$=(a+a_n^2)^{-1}(\rho-\tilde{\rho}_+)D_2+(a+a_n^2)^{-1}\tilde{\rho}_+(D_2-a^{-\frac{1}{2}}d(\eta)^{\frac{1}{2}}).$$

Since $D_1=(a+a_n^2)^{-1}(D_2^2-a^{-1}d(\eta))-(a+a_n^2)\gamma^2$ and

$$\tilde{D}_2=\frac{1}{2}(D_2+\sqrt{a}\,d^{\frac{1}{2}})+\frac{1}{2}\tilde{\rho}_+h_0$$

it holds

$$(4.41)\qquad I_0=-2h_0\left\{\frac{1}{2}(D_2+a^{-\frac{1}{2}}d^{\frac{1}{2}})+\frac{1}{2}\tilde{\rho}_+h_0\right\}+\tilde{\rho}_+(h_0^2+D_1)$$
$$=-h_0(D_2+a^{-\frac{1}{2}}d^{\frac{1}{2}})+\tilde{\rho}_+D_1=-(a+a_n^2)^{-1}\{(\rho-\tilde{\rho}_+)D_2$$
$$+\tilde{\rho}_+(D_2-a^{-\frac{1}{2}}d^{\frac{1}{2}})\}\{D_2+a^{-\frac{1}{2}}d^{\frac{1}{2}}\}+(a+a_n^2)^{-1}\tilde{\rho}_+(D_2$$
$$-a^{-\frac{1}{2}}d^{\frac{1}{2}})+\tilde{\rho}_+(a+a_n^2)\gamma^2$$
$$=-(a+a_n^2)^{-1}(\rho-\tilde{\rho}_+)D_2(D_2+a^{-\frac{1}{2}}d^{\frac{1}{2}})+(a+a_n^2)\tilde{\rho}_+\gamma^2.$$

From (4.37) and (4.40) we can obtain (4.28).

q.e.d.

Summing up (3.12)'s for β_j such that supp β_j contain a point of S, we can prove Theorem 1 with $k=1$, For $k\geq 2$ considering

$$PA^k u = A^k f + (PA^k - A^k P)u$$
$$BA^k u = A^k g + (BA^k - A^k B)u. \quad A^k = \mathcal{F}(\sigma^2 + |\eta|^2 + \gamma^2)^{\frac{k}{2}}\mathcal{F},$$

and using
$$|(PA^k - A^k P)u|_{0,\gamma} \leq C(|u|_{k+1,\gamma} + |f|_{k-1,\gamma})$$
$$<(BA^k - A^k B)u>_{0,\gamma} \leq C(<u>_{k,\gamma} + <g>_{k-1,\gamma})$$

we obtain Theorem 1.

§5. Proof of Theore 4.

Now assume that (H_2) is not satisfied at $(t, y, \eta, \sigma) = (t_0, y_0, \eta_0, \sigma_0)$. Without loss of generality we can assume $y_0 = 0$, $(\eta_0, \sigma_0) \in \Sigma_0$, $t_0 > 0$ (from Lemma 2.6).

(5.1) $$\rho - |\tilde{\rho}_+| \leq -\delta < 0,$$

in a neighbourhood of X_0 and that Case I or II described in Lemma 2.6 holds. Let $\beta(x, y, t, \eta, \sigma, \gamma)$ be a C^∞ function defined in §3 with its support in neighbourhood of $(t_0, 0, y_0, \sigma_0, \eta_0, 0)$. By the inequality Gärdings type we have from (5.1)

(5.2) $$\frac{\delta}{2} <\beta u>^2_{1,\gamma} \leq -\mathcal{H} <\beta_\gamma u, \beta_\gamma u> + c <\beta u>^2_{\frac{1}{2},\gamma}.$$

By virtue of (4.34), (4.35) and (5.2). we have for $\gamma > \gamma_1$

(5.3) $$<\beta u>^2_{1,\gamma} \leq C\{\gamma|\beta u|^2_{1,\gamma} + \frac{1}{\gamma}|\beta f|^2_{0,\gamma} + \frac{1}{\gamma} <A^{\frac{1}{2}}_{y,\gamma}\beta g>^2_{0,\gamma} + |u|^2_{1,\gamma}\}.$$

If (1.14) holds, (5.3) and the localization of (1.14)' yields

(5.4) $$<\beta u>^2_{1,\gamma} \leq C\left\{\frac{1}{\gamma}|\beta f|^2_{0,\gamma} + \frac{1}{\gamma} <A^{\frac{1}{2}}\beta g<^2_{0,\gamma} + |u|^2_{1,\gamma}\right\}, \text{ for } \gamma > \gamma_1.$$

Now we can regard $t_0 = 0$ after parallel transition. We can find a sequence of function $\{v_k\}_{k=1,2,\cdots}$, with its support in small neighbourhood of origin such that (5.4) does not hold for every v_n with any constant C.

Let $\xi_+(\delta)$ be a root with positive inaginary part of

$$P(0, 0, 0, \sigma_0-i\delta, \xi, \eta_0)=0.$$

Take

$$u_{\delta,\varepsilon}(x, y, t)=e^{\delta t}e^{ix\xi_+(\delta)}\mathcal{F}_{y,t}(w_\varepsilon(\eta, \sigma)),\ 0<\delta<\delta_0, \tag{5.5}$$

where $w_\varepsilon(\eta, \sigma)$ is a non-negative C^∞ function such that

$$\operatorname{supp} w_\varepsilon(\eta, \sigma)\subset U_\varepsilon=\{(\eta, \sigma);\ |\eta-\eta_0|^2+|\sigma-\sigma_0|^2<\varepsilon^2\}$$

$$\iint w_\varepsilon(\eta, \sigma)^2 d\sigma d\eta=1,\ w_\varepsilon(\eta, \sigma)=\varepsilon^{-n/2}w(\eta-\eta_0)/\varepsilon,\ (\sigma-\sigma_0)/\varepsilon).$$

Here $w(\eta, \sigma)$ belongs to $\mathcal{D}$.

$$|u|_{0,\delta}^2=\iiint e^{-2\delta t}|u_{\delta\varepsilon}|^2 dydtdx=\frac{1}{2\operatorname{I}m\xi_+(\delta)}. \tag{5.6}$$

Now let us put

$$\begin{aligned} u_{m,\delta,\varepsilon}(x, y, t)&=u_{\delta,\varepsilon}(mx, my, mt)m^n \\ &=e^{m\delta t}e^{imx\xi_+(\delta)}\mathcal{F}_{y,t}\left(w_\varepsilon\left(\frac{\eta}{m}, \frac{\sigma}{m}\right)\right). \end{aligned} \tag{5.7}$$

Then we have $|u_{m,\delta,\varepsilon}|_{0,m\delta}^2=m^{n-1}|u_{\delta,\varepsilon}|_{0,\delta}$ Let

$$\beta=\alpha_1(x)\alpha_2(y, t)\alpha_3(\eta, \sigma)\alpha_4(\gamma), \tag{5.8}$$

where $\alpha_i\in C_0^\infty$ such that $\alpha_4(\gamma)=1$ for $\gamma<2\delta_0$, $\alpha_3(\eta, \sigma)=1$ on U_ε, and $\alpha_3\alpha_4$ is homogeneous with respect to (σ, η, γ), $\alpha_1\alpha_2=1$ in some neighbourhood of origin.

If we denote $\beta u=e^{\gamma t}\overline{\mathcal{F}}_{\eta,\sigma}\beta\mathcal{F}_{y,t}e^{-\gamma t}u$, we have

$$\begin{cases} 1)\ \ \langle\beta u_{m,\delta,\varepsilon}\rangle_{0,m\delta}^2\geq\frac{3}{4}\langle u_{m,\delta,\varepsilon}\rangle_{0,m\delta}^2, & \text{for } m>{}^\exists m_0(\delta, \varepsilon) \\ 2)\ \ \langle\beta u_{m,\delta,\varepsilon}\rangle_{0,m\delta}^2\geq\frac{1}{4}\langle u_{m,\delta,\varepsilon}\rangle_{1,m\delta}^2, & m\geqslant{}^\exists m_1(\delta, \varepsilon), \end{cases} \tag{5.9}$$

Since this proof will be given in the same way as that of Lemma 5.1 below, we omit it here.

Now let us decompose P and B;

$$(5.10)\quad \begin{cases} P(t, x, y, D_t, D_x, D_y)=P(0, 0, 0, D_t, D_x, D_y) \\ \qquad +\{(P(t, x, y, D_t, D_x, D_y)-P(0, D))\} \\ \qquad =P_0+(P-P_0) \\ B(y, t, D_x, D_y, D_t)=B(0, 0, D_x, D_y, D_t) \\ \qquad +\{(B(y, t, D_x, D_y, D_t)-B(0, 0, D_x, D_y, D_t))\} \\ \qquad =B_0+(B-B_0). \end{cases}$$

Now we state following lemmas whose proof will be given later.

Lemma 5.1. *For any* $\varepsilon>0$ *there exists* $m_0=m_0(\delta, \varepsilon)$ *such that*

$$(5.11)_1\qquad |(P-P_0)u_{m,\delta,\varepsilon}|^2_{0,\delta m}\leq\varepsilon^2|u_{m,\delta,\varepsilon}|^2_{2,\delta m}\qquad \text{for } m>m_0$$

$$(5.11)_2\qquad <(B-B_0)u_{m,\delta,\varepsilon}>^2_{0,\delta m}\leq\varepsilon^2<u_{m,\delta,\varepsilon}>^2_{1,\delta m}\qquad \text{for } m>m_0.$$

Lemma 5.2. *There exists positive constant* C *such that*

$$(5.12)\qquad |P_0u_{m,\delta,\varepsilon}|^2\leq C\varepsilon^2|u_{m,\delta,\varepsilon}|^2_{2,\delta m}.$$

Now consider Case I and Case II.

(Case I) $|B_0(\xi_+(\delta), \eta_0, \sigma_0-i\delta)|<\text{const. }\delta$

(Case II) $B_0(\xi_+(\delta_0), \eta_0, \sigma_0-i\delta_0)=0$, for $\delta_0>0$.

Lemma 5.3. *There exists positive constant* C *such that in Case I*

$$(5.13)_1\qquad <B_0u_{m,\delta,\varepsilon}>^2_{0,\delta m}\leq C(\varepsilon^2+\delta^2)<u_{m,\delta,\varepsilon}>^2_{1,\delta m}$$

and in Case II

$$(5.13)_2\qquad <B_0u_{m,\delta_0,\varepsilon}>^2_{0,\delta_0 m}\leq C\varepsilon^2<u_{m,\delta_0,\varepsilon}>^2_{1,\delta_0 m}.$$

From (5.4), (5.9) and above lemmas we have

(Case I)

$$(5.14)_1 \quad \langle u_{m,\delta,\varepsilon}\rangle^2_{1,\delta m} \leq \frac{C}{\delta m}\{\varepsilon^2|u_{m,\delta,\varepsilon}|^2_{2,\delta m}+(\varepsilon^2+\delta^2)m\langle u_{m,\delta,\varepsilon}\rangle^2_{1,\delta m}$$
$$+\delta m|u_{m,\delta,\varepsilon}|^2_{1,\delta m}\}, \qquad \text{for } m \geqslant m_0(\delta, \varepsilon),$$

(Case II)

$$(5.14)_2 \quad \langle u_{m,\delta_0,\varepsilon}\rangle^2_{1,\delta_0 m} \leq \frac{C}{\delta_0 m}\{\varepsilon^2|u_{m,\delta_0,\varepsilon}|^2_{2,\delta_0 m}+\varepsilon^2 m\langle u_{m,\delta,\varepsilon}\rangle^2_{1,\delta m}$$
$$+\delta_0 m|u_{m,\delta_0,\varepsilon}|^2_{1,\delta_0 m}\}, \qquad \text{for } m < m_0(\varepsilon).$$

On the other hand we have

$$(5.15) \quad \langle u_{m,\delta,\varepsilon}\rangle^2_{1,\delta m} = \iint \{\eta\}^2+|\sigma-i\delta m|^2\} \left|w_\varepsilon\left(\frac{\eta}{m}, \frac{\sigma}{m}\right)\right|^2 d\eta d\sigma$$
$$= \iint \{|\eta|^2+\sigma^2+\delta^2\}|w_\varepsilon(\eta, \sigma)|^2 m^{n+2} d\eta d\sigma$$
$$\geqslant C m^{n+2}, \ (C>0),$$

$$(5.16) \quad 1) \ |u_{m,\delta,\varepsilon}|^2_{2,\delta m} \leq C_2 \delta^{-1} m^{n+3},$$
$$2) \ |u_{m,\delta,\varepsilon}|^2_{1,\delta m} \leq C_1 \delta^{-1} m^{n+1}, \ C_1, C_2>0,$$

because of $|u_m|^2_{2,\delta m} \leq C\{|D^2_x u_m|^2_{0,\delta m}+|\Lambda^2 u_m|^2_{0,\delta m}\}$ and

$$|D^2_x u_{m,\delta,\varepsilon}|^2_{0,\delta m} = \iiint |m\xi_+(\delta)|^4 e^{-2mx\, Im\xi_+(\delta)} \left|w\left(\frac{\eta}{m}, \frac{\sigma}{m}\right)\right|^2 dx d\eta d\sigma$$
$$= m^{n+3}\iiint e^{-2\, Im\xi_+(\delta)x}|w(\eta, \sigma)|^2 dx d\eta d\sigma$$
$$= \frac{|\xi_+(\delta)|^4}{2Im\xi_+(\delta)} m^{n+3} \leq \text{const. } \delta^{-1} m^{n+3}.$$

Therefore we obtain (5.16) 1) and in the same way (5.16) 2)

(*Case II*) Choose $\varepsilon=\frac{1}{k}$, $m(k)>m_0(\delta_0, k^{-1})$, $m(k)\ \delta_0>\gamma_1$ and

$$w_k = u_{m(k),\delta_0,k^{-1}}$$

and tend k to ∞, then we can see from (5.15) and (5.16) that $(5.14)_2$ does not hold for any constant C.

(*Case I*) Choose $\varepsilon=\frac{1}{k^2}$, $\delta=\frac{1}{k}$ and $m(k)>m_0(k^{-1}, k^{-2})$, $m(k)k^{-1}>\gamma_1$, and put $w_k=u_{m(k),k^{-1},k^{-2}}$.
Then $(5.14)_1$ does not hold for every k even if we choose any constant C. This arguement is true if we replace w_k by $v_k=\alpha(x, y, t)w_k$, where $\alpha(x, y, t)\in C_0^\infty$, $\alpha(0, 0, 0)=1$.

This means that (1.14) does not hold. As for (1.18) the arguement proceeds in the same way.

Proof of Lemma 5.1. For given $\varepsilon>0$ denote $u_{m,\delta,\varepsilon}$ simply by u_m. Let us consider for example

$$(5.17)\qquad |(a(t, x, y)-a(0, 0, 0))D_t^2u_m|_{0,\delta m}^2$$
$$=\iiint |a(t, x, y)-a(0, 0, 0)|^2|(D_t-i\delta m)^2e^{-\delta mt}u(mt, mx, my)m^n|^2dtdxdy$$
$$=m^{n+3}\iiint \left|a\left(\frac{t}{m}, \frac{x}{m}, \frac{y}{m}\right)-a(0, 0, 0)\right|^2|(D_t-i\delta)^2e^{-\delta t}u(t, x, y)|^2dtdxdy.$$

On the other hand

$$(5.18)\qquad |D_t^2u_m|_{0,\delta m}^2=m^{n+3}\iiint |(D_t-i\delta)^2e^{-\delta t}u(t, x, y)|dtdxdy.$$

Compare (5.17) with (5.18), then we can see $(5.11)_1$, because the coefficient are uniformly bounded and that $\left|a\left(\frac{t}{m}, \frac{x}{m}, \frac{y}{m}\right)-a(0, 0, 0)\right|$ tend to zero uniformly on a compact set $K(t, x, y)$.

We obtain $(5.11)_2$ in the same way. q.e.d.

Proof of Lemma 5.2. From (5.7) and $P_0(\sigma_0-i\delta, \xi_+(\delta), \eta_0)=0$ we have

$$(5.19)\qquad |P_0u_m|_{0,\delta m}^2=|P_0(\sigma-i\delta m, D_x)\mathfrak{F}e^{-\delta mt}u_m|_{L_2}^2$$
$$=\iiint\left|P_0(\sigma-i\delta m, m\xi_+(\delta), \eta)e^{im\xi_+(\delta)x}w\left(\frac{\eta}{m}, \frac{\sigma}{m}\right)\right|^2dxd\eta d\sigma$$
$$=\iiint|P_0(\sigma-i\delta, \xi_+(\delta), \eta)e^{i\xi_+(\delta)x}w(\eta, \sigma)|^2m^4m^{n-1}dxd\eta d\sigma$$
$$=\iiint|\{P_0(\sigma-i\delta, \xi_+(\delta), \eta)-P_0(\sigma_0-i\delta, \xi_+(\delta), \eta_0)\}$$
$$\times e^{i\xi_+(\delta)x}w(\eta, \sigma)|^2m^{n+3}dxd\eta d\sigma.$$

Considering

$$|P_0(\sigma-i\delta,\ \xi_+(\delta),\ \eta)-P_0(\sigma_0-i\delta,\ \xi_+(\delta),\ \eta_0)|<C\varepsilon, \tag{5.20}$$

as ε tends to zero, we obtain

$$|P_0 u_m|^2_{0,\delta m}\leq C\varepsilon|^2\iiint|e^{i\xi_+(\delta)x}w(\eta,\ \sigma)|^2 m^{n+3}dxd\eta d\sigma\leq C\varepsilon^2|u_m|^2_{2,\delta m}.$$

This completes the proof of Lemma 5.2.

Proof of Lemma 5.3 The proof of $(5.13)_2$ is the same as that of Lemma 5.2. except replacing the integral domain R^{n+1}_+ by R^n.

The proof of $(5.13)_1$ in Case I is given by virtue of

$$<B_0 u_m>^2_{0,\delta m}= \tag{5.21}$$
$$\iint|\{B_0(\sigma-i\delta,\ \xi_+(\delta),\ \eta)-B_0(\sigma_0,\ \xi_+(0),\ \eta_0)\}w(\eta,\ \sigma)|^2 m^{n+2}d\eta d\sigma$$

$$|B_0(\sigma_0-i\delta,\ \xi_+(\delta),\ \eta)-B_0(\sigma_0,\ \xi_+(0),\ \eta_0)|^2\leq C(\varepsilon^2+\delta^2). \tag{5.22}$$

§6. existence theorem with zero initial data

1. Combining Dirichlet considion with our boundary condition we have Dirichlet set $\{B,\ 1\}$. (i.e, arbitrary first order polinomial is written in the linear combination of B and 1.).

Let P^* be the formal adjoint of homogeneous operator P and P_0^* be the principal part of P^*, then.

Then there exist a Dirichlet set $\{B',\ C'\}$ such that

$$(Pu,\ v)-(u,\ P^*v)=<Bu,\ C'v>+<Cu,\ B'v> \tag{6.1}$$

for $u\in\mathcal{H}_{2,\gamma}(R^{n+1}_+)$ and $v\in\mathcal{H}_{2,-\gamma}(R^{n+1}_+)$.

After actual calculation we can see $C'=1$.

Lemma 6.1. *If* $\{P,\ B\}$ *satisfies* (H_2), $\{\tilde{P}_0^*,\ \tilde{B}'\}$ *also does so, where* $\tilde{P}_0^*$ $=P_0^*(-t,\ x,\ y,\ -D_t,\ D_x,\ D_y)$, $\tilde{B}=B'(-t,\ y,\ -D_t,\ D_x,\ D_y)$

Proof. Let B_0' be the principal part of B', and $\tilde{\xi}_+^*(t, x, y, \eta, \sigma-i\gamma)$ be a root with positive imaginary part of $\tilde{P}_0^*(t, x, y, \sigma-i\gamma, \xi, \eta)=0, \gamma>0$.

$$\begin{aligned}(6.2)\qquad \tilde{\xi}_+^*(t, x, y, \sigma-i\gamma, \eta) &= \xi_+^*(-t, x, y, -\sigma+i\gamma, \eta)\\ &= \overline{\xi_-(-t, x, y, -\sigma-i\gamma, \eta)}.\end{aligned}$$

Now freeze the coefficients at any point $(t_0, 0, y_0)$ and let us denote $\xi_+=\xi_+(t_0, 0, y_0, \tau, \eta)$, $\xi_-=\xi_-(t_0, 0, y_0, \tau, \eta)$.

Then we have not only (6.1) but also

$$\begin{aligned}(6.1)'\qquad ((D_x-\xi_+)(D_x-\xi_-)e^{-\gamma t}u, e^{\gamma t}v)-(e^{-\gamma t}u, (D_x-\bar{\xi}_+)(D_x-\bar{\xi}_-)e^{\gamma t}v)\\ =\langle B_0(t_0, y_0, D_t-i\gamma, D_x, D_y)e^{-\gamma t}u, e^{\gamma t}v\rangle\\ +\langle e^{-\gamma t}u, B_0'(t_0, y_0, D_t-i\gamma, D_x, D_y)e^{\gamma t}v\rangle.\end{aligned}$$

Now we take

$$e^{-\gamma t}u = e^{i\xi_+(t_0, y_0, \sigma-i\gamma, \eta)x}\mathscr{F}[\psi(\eta, \sigma)],\quad e^{\gamma t}v=e^{i\xi_-(t_0, y_0, \sigma-i\gamma, \eta)x}\mathscr{F}[\psi(\eta, \sigma)],$$

where $\psi(\eta, \sigma)\in\mathscr{D}(\eta, \sigma)$.

Then from (6.1)' we have

$$(6.2)\qquad B_0(t_0, y_0, \sigma-i\gamma, \xi_+, \eta)+\overline{B_0'(t_0, y_0, \sigma+i\gamma, \xi_-, \eta)}=0.$$

From (6.2) we obtain

$$\begin{aligned}(6.3)\qquad &|\tilde{B}_0'(-t_0, y_0, \sigma-i\gamma, \tilde{\xi}_+^*(-t_0, y_0, \sigma-i\gamma, \eta), \eta)|\\ &=|B_0'(t_0, y_0, -\sigma+i\gamma, \overline{\xi_-(t_0, y_0, -\sigma-i\gamma, \eta)}, \eta)|\\ &=|B_0(t_0, y_0, -\sigma-i\gamma, \xi_+(t_0, y_0, -\sigma-i\gamma, \eta), \eta)|.\end{aligned}$$

This complete the proof of Lemma 6.1.

From Theorem 1 follows

$$\begin{aligned}(6.4)\qquad \gamma|u|_{1,\gamma}^2+\gamma\sum_{j=0}^{1}\langle\Lambda_{y,\gamma}^{-\frac{1}{2}}D_x^j u\rangle_{1-j,\gamma}^2\leq\frac{C}{\gamma}\{|\tilde{P}^*u|_{0,\gamma}^2\\ +\langle\Lambda_{y,\gamma}^{\frac{1}{2}}\tilde{B}'u\rangle_{0,\gamma}^2\},\qquad \text{for } u\in\mathscr{H}_{2,\gamma},\ \gamma>\gamma_0.\end{aligned}$$

By putting $v(t, x, y)=u(-t, x, y)$ for $u\in\mathscr{H}_{2,-\gamma}$, we have

$$(6.5)\qquad (P^*v)(t, x, y)=(\tilde{P}^*u)(-t, x, y),\ (B'v)(t, x, y)(\tilde{B}'u)(-t, x, y),$$

and

(6.6) $\quad |u|_{k,\gamma}=|v|_{k,-\gamma},\ \langle u\rangle_{k,\gamma}=\langle v\rangle_{k,-\gamma},$

and from (6.4)

$$(6.7)\qquad \gamma|v|^2_{1,-\gamma}+\gamma\sum_{j=0}^{1}<\Lambda_{y,\gamma}^{-\frac{1}{2}}D_x^j v>^2_{1-j,-\gamma}\leq\frac{C}{\gamma}\{|P^*v|^2_{0,-\gamma}+<\Lambda_{y,\gamma}^{-\frac{1}{2}}B'v>^2_{0,-\gamma}\}.$$

Now we define $\Lambda'=\Lambda'(\gamma,\ D_t,\ D_y)$ by

$$\Lambda'(\gamma,\ D_t\ D_y)v=e^{-\gamma t}\overline{\mathcal{F}}(\gamma^2+|\eta|^2+\sigma^2)^{\frac{1}{2}}\mathcal{F}(e^{\gamma t}v).$$

Then from (6.7) in the same way as in §4 we have

$$(6.8)\qquad \gamma|\Lambda'^s v|^2_{1,-\gamma}+\gamma\sum_{j=0}^{1}<\Lambda_{y,\gamma}^{-\frac{1}{2}}D_x^j v>^2_{1+s-j,-\gamma}\leq\frac{C}{\gamma}\{|\Lambda'^s P^*v|^2_{0,\gamma}+<\Lambda_{y,\gamma}^{\frac{1}{2}}\Lambda'^s B'v>^2_{0,\gamma}\}.$$

Let us introduce Hilbert space $h_{s,-\gamma}$ defined by the completion of $C_0^\infty(\overline{R_+^{n+1}})$ with the following norm.

$$(6.9)\qquad |v|^2_{h_{s,-\gamma}}=\Lambda'^s P^*v|^2_{0,-\gamma}+<\Lambda_{y,\gamma}^{\frac{1}{2}}\Lambda'^s B'v>^2_{0,-\gamma}.$$

Then we have

$$(6.10)\qquad \gamma|\Lambda'^s v|^2_{1,-\gamma}+\gamma\sum_{j=0}^{1}<\Lambda_{y,\gamma}^{-\frac{1}{2}}D_x^j v>^2_{1-j,-\gamma}\leq\frac{C}{\gamma}|v|_{h_{s,-\gamma}}.$$

Proposition 6.1 *Let f be in $\mathcal{H}_{k,\gamma}(R_+^{n+1})$ and $\Lambda_{y,\gamma}^{\frac{1}{2}}\ g$ in $\mathcal{H}_{k,\gamma}(R^n)$, then there exists a unique solution u of (P_0') belonging to $\mathcal{H}_{k+1,\gamma}(R_+^{n+1})$ for $\gamma>\gamma_k$.* $\quad (k=0, 1, 2, \ldots)$.

Proof. Denote $h_{-(k+1),-\gamma}$ simply by h. There exists w in h such that

(6.11) $(w, v)_h=(f, v)-\langle g, C'v\rangle$

by virtue of Riesz's Theorem and

$$|(f, v)|=(\Lambda^k f, \Lambda^{-k}v)|\leq C|f|_{k,\gamma}|v|_h$$
$$|\langle g, C'v\rangle|\leq|\langle \Lambda_{y,\gamma}^{\frac{1}{2}}\Lambda^k g, \Lambda_{y,\gamma}^{-\frac{1}{2}}(\Lambda')^{-k}C'v\rangle|\leq C\langle \Lambda_{y,\gamma}^{\frac{1}{2}}g\rangle_{k,\gamma}|v|_h .$$

Now put

$$u=\Lambda^{-(1+k)}e^{2\gamma t}(\Lambda')^{-(1+k)}P^*w.$$

Then we have $Pu=f$ in distribution sence. By usual interpolation theorem used in case where elliptic boundary value problem, we can prove that u is in $\mathcal{H}_{k,\gamma}(R_+^{n+1})$, For $k\geq 2$, $Pu=f$ holds in L^2-sence.

From (6.9) and (6.11) we have

(6.12) $$(Pu, v)-\langle g, C'v\rangle$$
$$=(u, P^*v)+\langle \Lambda_{y,\gamma}^{\frac{1}{2}}\Lambda'^{-(1+k)}B'w, \Lambda_{y,\gamma}^{\frac{1}{2}}\Lambda'^{-(1+k)}B'v\rangle.$$

On the other hand

(6.13) $(Pu, v)-(u, P^*v)=\langle Bu, C'v\rangle+\langle Cu, B'v\rangle$

holds by definition, therefore we have

$$\langle Bu, C'v\rangle=\langle g, C'v\rangle$$

for all $v\in\mathcal{H}_{2,-\gamma}(\subset h)$ such that $B'v=0$.

Hence $Bu=g$. Taking v in $\mathcal{D}(R_+^n\times(-\infty, \infty))$, we can see $u=\Lambda'^{-(1+k)}\Lambda_{y,\gamma}^{\frac{1}{2}}\ e^{2\gamma t}\Lambda_{y,\gamma}^{\frac{1}{2}}\Lambda'^{-(1+k)}B'w$. $e^{-\gamma t}\Lambda_{y,\gamma}^{-\frac{1}{2}}\Lambda'^{(1+k)}u=e^{\gamma t}\Lambda_{y,\gamma}^{\frac{1}{2}}\Lambda'^{-(1+k)}B'w$ belongs to $L^2(R^n)$, i.e. $\Lambda_{y,\gamma}^{-\frac{1}{2}}u$ belongs to $\mathcal{H}_{1+k,\gamma}(R^n)$ on the boundary. Using energy inequality we can see $u\in\mathcal{H}_{k+1,\gamma}$.

If the supports of f and g be in $R_+^n\times(0, \infty)$, then considering evergy inequality and

(6.14) $|f|_{0,\gamma}\leq|f|_{0,\gamma'}, \ \langle\Lambda_{y,\gamma}^{\frac{1}{2}}g\rangle_{0,\gamma}\leq\langle\Lambda_{y,\gamma}^{\frac{1}{2}}g\rangle_{0,\gamma'},$

for $\gamma>\gamma'$. we have $\gamma|u|^2_{1,\gamma}\leq C\frac{1}{\gamma}$ for $\gamma>\gamma_0$.

This means $\operatorname{supp}_t [u]\subset[0, \infty)$. It follows that the solution is uniquely determined independently of γ, such that $\gamma>\gamma_0$ (*c, f* [23]).

This complete the proof of Theorem 2.

§7. Energy inequlity with intial data and existence theorem.

In this section we prove Theorem 3 and it's corollaries. For that purpose we need not use the partition of unity in §3, but apply the results of Theorem 1 and 2. In same way as in §4 we consider

(7.1) $$\mathcal{J}(P, \tilde{Q})=(Pu, \tilde{Q}u)_{0,\gamma,I}-(\tilde{Q}u, Pu)_{0,\gamma,I}$$

where $\tilde{Q}$ is a first order differential operator defined grobally and is equal to Q given in §4 on the set S.

(7.2) $$\tilde{Q}=\left\{\frac{1}{2}\frac{\partial P}{\partial\tau}-\frac{1}{2}(a+a_n^2)^{-1}\rho(\rho D_2+\alpha(D_y))\right\} +\left\{(\rho-a_n)\frac{1}{2}\frac{\partial P}{\partial\xi}\right\}=\tilde{Q}_1+\tilde{Q}_2.$$

First by actual caculation of (7.1) we can show

Lemma 7.1 *There exist positive constants γ_0 and C such that for $\gamma>\gamma_0$ and u in $\mathcal{H}_{2,\gamma,I}$,*

(7.3) $$\gamma|u|^2_{1,\gamma,I}+[u(t)]^2_{1,\gamma}\leq C\left\{\frac{1}{\gamma}|Pu|^2_{0,\gamma,I}+[u(0)]^2_{1,\gamma}+\langle Bu\rangle^2_{0,\gamma,I} +|\langle Bu, \Lambda_{y,\gamma}u\rangle_{0,\gamma,I}|^2+|\langle Bu, D_t u\rangle_{0,\gamma,I}|^2\right\}.$$

Proof. Corresponding to (7.1), let us consider

(7.4) $$G(P, Q)=P(\tau, \xi, \eta)\overline{\tilde{Q}(\tau, \zeta, \eta)}-P(\tau, \zeta, \eta)\overline{\tilde{Q}(\tau, \xi, \eta)}.$$

In what follows we use the following notations, replacing σ by τ, in the previous notations:

$$(7.5)\quad \begin{cases} \tilde{\xi}=\xi+a_n\tau+\sum_{j-1}^{n-1}a_{nj}\eta_j,\ \tilde{\zeta}=\zeta+a_n\tau+\sum_{j=1}^{n-1}a_{nj}\eta_j \\ D=D(\tau,\eta)=D_1(\tau,\eta).\ \tilde{D}_2=D_2+\frac{1}{2}(a+a_n^2)^{-1}\rho(\rho D_2+\alpha(\eta)). \end{cases}$$

Then it follows

$$(7.6)\quad \begin{cases} P(\tau,\xi,\eta)=\xi^2-D=\xi^2-(a+a_n^2)(D_2^2-a^{-1}d(\eta)), \\ \frac{1}{2}\frac{\partial P}{\partial\tau}=a_n\tilde{\xi}-D_2,\ \frac{1}{2}\frac{\partial P}{\partial\xi}=\tilde{\xi}. \end{cases}$$

Now we can prove

$$(7.7)\qquad G(P,\tilde{Q})$$

$$\begin{aligned}
&=(\xi-\zeta)[\rho\{\tilde{\zeta}\tilde{\xi}+(a+a_n^2)^{-1}D_2\bar{D}_2-a^{-1}(a+a_n^2)^{-1}d(\eta)\}-(\tilde{\xi}\bar{\tilde{D}}_2+\tilde{D}_2\zeta)] \\
&+(\tau-\bar{\tau})[\left(1+\frac{1}{2}(a+a_n^2)^{-1}\rho^2\right)\{D_2\bar{D}_2+a^{-1}d(\eta)+(a+a_n^2)\tilde{\xi}\tilde{\zeta}\} \\
&\quad -a_n\left\{\left(1+\frac{1}{2}(a+a_n^2)^{-1}\rho^2\right)(\tilde{\xi}\bar{D}_2+D_2\zeta)+\frac{1}{2}(a+a_n^2)^{-1}\rho(\tilde{\xi}\alpha(\eta)\right. \\
&\quad \left.+\alpha(\eta)\tilde{\zeta})\right\}+\rho\{a_n\tilde{\xi}\tilde{\zeta}+a_n(a+a_n^2)^{-1}(D_2\bar{D}_2-a^{-1}d(\eta)) \\
&\quad -(\tilde{\xi}\bar{D}_2+D_2\tilde{\zeta})+\frac{1}{2}(a+a_n^2)(D_2\alpha(\eta)+\alpha(\eta)\bar{D}_2\}] \\
&\quad \equiv(\xi-\zeta)G_x+(\tau-\bar{\tau})G_t,
\end{aligned}$$

by using (7.6) and the following relations;

$$(7.8)\qquad \text{i)}\ \ \tilde{\xi}-\tilde{\zeta}=(\xi-\zeta)+a_n(\tau-\bar{\tau})$$

$$\begin{aligned}
\text{ii)}\ \ &D_2-\bar{D}_2=(a+a_n^2)(\tau-\bar{\tau}), \\
&\tilde{D}_2-\bar{\tilde{D}}_2=(a+a_n^2)\left\{1+\frac{1}{2}(a+a_n^2)\rho^2\right\}(\tau-\bar{\tau}) \\
\text{iii)}\ \ &\tilde{\xi}\bar{D}-D\tilde{\zeta}=(\xi-\zeta)\{(a+a_n^2)^{-1}(D_2\bar{D}_2-a^{-1}d(\eta)\} \\
&\quad+(\tau-\bar{\tau})\{a_n(a+a_n^2)^{-1}(D_2\bar{D}_2-a^{-1}d(\eta)) \\
&\quad-(\tilde{\xi}\bar{D}_2+D_2\tilde{\zeta}) \\
\text{iv)}\ \ &D\bar{\tilde{D}}_2-\tilde{D}_2\bar{D}=\left\{1+\frac{1}{2}(a+a_n^2)^{-1}\rho^2\right\}(\tau-\bar{\tau})\{D_2\bar{D}_2+a^{-1}d(\eta) \\
&\qquad+\frac{1}{2}(a+a_n^2)^{-1}\rho(\tau-\bar{\tau})(D_2\alpha(\eta)+\alpha(\eta)\bar{D}_2).
\end{aligned}$$

Next we prove the following enequality as guadratic form:

(7.9) $G_t \geq C\{\xi\zeta + D_2 D_2 + d(\eta),\ C > 0.$

Denote simply

(7.10) $\begin{cases} k=(a+a_n^2)^{\frac{1}{2}},\ X=k\xi,\ \bar{X}=k\zeta,\ Y=a^{-\frac{1}{2}}d(\eta)^{\frac{1}{2}},\ z=D_2 \\ \rho k^{-1}=\lambda,\ a_n k^{-1}=s \text{ and } \alpha'=k^{-1}Y^{-1}\alpha. \end{cases}$

Then from Lemma 2.1 and Lemma 2.4 we have

(7.11) 1) $-1 < s < 1$

2) $\lambda \geq 0$

3) $\lambda^2 - \alpha'^2 = k^{-2}(\rho^2 - ad(\eta)^{-1}\alpha(\eta)) \geq 0,$

(7.12)
$$\begin{aligned} G_t &= \left(1+\lambda s+\frac{\lambda^2}{2}\right)X\bar{X} - \frac{1}{2}s\lambda\alpha'(X\bar{Y}+Y\bar{X}) \\ &\quad -\left(\lambda+s+\frac{1}{2}s\lambda^2\right)(X\bar{Z}+Z\bar{X}) + \left(1-\lambda s+\frac{\lambda^2}{2}\right)Y\bar{Y} \\ &\quad +\frac{1}{2}\lambda\alpha'(Y\bar{Z}+Z\bar{Y}) + \left(1+\lambda s+\frac{\lambda^2}{2}\right)Z\bar{Z} \\ &= (X,\ Y,\ Z)\begin{pmatrix} \left(1+\lambda s+\frac{\lambda^2}{2}\right), & -\frac{1}{2}\lambda s, & -\left(\lambda+s+\frac{1}{2}s\lambda^2\right) \\ -\frac{1}{2}\lambda s\alpha', & \left(1-\lambda s+\frac{\lambda^2}{2}\right), & \frac{1}{2}\lambda\alpha' \\ -\left(\lambda+s+\frac{1}{2}s\lambda^2\right), & \frac{1}{2}\lambda\alpha', & \left(1+\lambda s+\frac{\lambda^2}{2}\right) \end{pmatrix}\begin{pmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{pmatrix} \\ &= (X\,Y\,Z)A\,{}^t(X\,Y\,Z). \end{aligned}$$

We can prove that symmetric matrix A is positive definite from the followings:

(7.13) 1) $\left(1+\lambda s+\frac{\lambda^2}{2}\right) > 0$

2) $|A_1| = \begin{vmatrix} 1+s+\frac{\lambda^2}{2}, & -\left(\lambda+s+\frac{1}{2}s\lambda^2\right) \\ -\left(\lambda+s+\frac{1}{2}s\lambda^2\right), & \left(1+\lambda s+\frac{\lambda^2}{2}\right) \end{vmatrix} = (1-s^2)\left(\frac{\lambda^2}{4}+1\right) > 0$

3) $|A| = \left(1+\lambda s+\frac{\lambda^2}{2}\right)^2\left(1-\lambda s+\frac{\lambda^2}{2}\right)$

$$-\left(\lambda+s+\frac{1}{2}s\lambda^2\right)^2\left(1-\lambda s+\frac{\lambda^2}{2}\right)-\left(1+\lambda s\right.$$
$$\left.+\frac{\lambda^2}{2}\right)\left(\frac{s^2+1}{4}\right)\lambda^2\alpha'^2+2\left(\lambda+s+\frac{1}{2}s\lambda^2\right)\frac{\lambda^2 s}{4}\alpha'^2$$
$$=\left(1-\lambda s+\frac{\lambda^2}{2}\right)\left\{(1-s^2)\left(\frac{\lambda^2}{4}+1\right)\right\}$$
$$-\frac{\lambda^2\alpha'^2}{4}(1-s^2)\left(1-\lambda s+\frac{\lambda^2}{2}\right)$$
$$=(1-s^2)\left(1-\lambda s+\frac{\lambda^2}{2}\right)\left\{\frac{\lambda^2}{4}(\lambda^2-\alpha'^2)+1\right\}$$
$$\geq\frac{1}{2}(1-s^2)>0.$$

This means (7.9), Hence we have

$$(7.9)' \qquad G_t\geq C\{\xi\bar{\zeta}+\tau\bar{\tau}+|\eta|^2\}, \quad (C>0).$$

Next we show

$$(7.14) \qquad G_x\leq C\{B\bar{B}+(B\bar{Z}+Z\bar{B})+(B\bar{Y}+Y\bar{B})\}, \quad C>0.$$

From (4.12) and the definition of D_2 ((7.5)), we have

$$(7.15) \qquad \begin{cases} X=\tilde{B}+(\lambda Z+\alpha' Y) \\ \tilde{D}_2=Z+\dfrac{\lambda}{2}(\lambda Z+\alpha' Y), \text{ where } \tilde{B}=kB. \end{cases}$$

$$(7.16) \qquad kG_x=\lambda(X\bar{X}+Z\bar{Z}-Y\bar{Y})-(X\bar{\tilde{D}}_2+\tilde{D}_2\bar{X})$$
$$=[\lambda\left\{\tilde{B}\bar{\tilde{B}}+\frac{1}{2}\tilde{B}(\lambda\bar{Z}+\alpha'\bar{Y})+\frac{1}{2}(\lambda Z+\alpha' Y)\bar{\tilde{B}}+\tilde{B}\bar{Z}+Z\bar{\tilde{B}}\right]$$
$$=[\lambda(Z\bar{Z}-Y\bar{Y})-(\lambda Z+\alpha' Y)\bar{Z}-Z(\lambda\bar{Z}+\alpha'\bar{Y})]\equiv\mathcal{H}_1+\mathcal{H}_2.$$

Here since we have

$$\mathcal{H}_2=-\lambda Z\bar{Z}-\lambda Y\bar{Y}-\alpha' Y\bar{Z}-\alpha' Z\bar{Y}$$
$$=-\lambda\left|Z+\frac{\alpha'}{\lambda}Y\right|^2-\frac{1}{\lambda}(\lambda^2-\alpha'^2)|Y|^2<0 \text{ (from (7.11), 3))},$$

(7.14) follows. From (7.9)′ and (7.14) we obtain (7.3) by integration of (7.1) by parts.

q.e.d.

If we can prove that there exists positive constant C such that

$$(7.17)\qquad \gamma<\Lambda_{y,\gamma}^{-\frac{1}{2}}D_t u>_{0,\gamma,I}^2 \leq C\left(\frac{1}{\gamma}|Pu|_{0,\gamma,I}^2+\frac{1}{\gamma}<\Lambda_{y,\gamma}^{\frac{1}{2}} Bu>_{0,\gamma,I}^2+[u(0)]_{1,\gamma}^2\right)$$

then from (7.3) and (7.17) we obtain (1.20).

To prove (7.17), we must consider dual problem $\{P^*, B'\}$.

Concerning the following problem

$$(P_0^*)' \begin{cases} P^*v=0 \\ B'v_{x=0}=\varphi(t, y), \ \varphi(t, y)\in C_0^\infty(R^{n-1}\times R'_+), \end{cases}$$

we have

Lemma 7.2 *There exists the solution v of $(P_0^*)'$ in $\mathcal{H}_{m,-\gamma}$ such that for $\gamma>{}^\exists\gamma_0$*

$$(7.18)\qquad \gamma|v|_{1,-\gamma,+}^2+\gamma\sum_{j=0}^{1}<\Lambda_{y,\gamma}^{-\frac{1}{2}}D_x^j v>_{1-j,-\gamma,+}^2+[v(0)]_{1,-\gamma}^2 \leq\frac{C}{\gamma}<\Lambda_{y,\gamma}^{\frac{1}{2}}\varphi>_{0,-\gamma,+}^2.$$

Proof. Since $(\tilde{P}^*, \tilde{B}')$ satisfies (H_2), we have the following estimate for the solution of

$$(\tilde{P}_0^*) \begin{cases} \tilde{P}^*u=0 \\ \tilde{B}'u=\varphi, \end{cases}\qquad \tilde{\varphi}(t, y)=\begin{cases} \varphi(-t, y) & t\geqslant 0 \\ 0 & , t<0 \end{cases}$$

$$(7.19)\qquad \gamma|u|_{1,\gamma}^2+\gamma\sum_{j=0}^{1}<\Lambda_{y,\gamma}^{-\frac{1}{2}}D_x^j u>_{1-j,\gamma}^2\leq\frac{C}{\gamma}<\Lambda_{y,\gamma}^{\frac{1}{2}}\tilde{\varphi}>_{0,\gamma,-}^2,$$

and similarly to (7.3) we have

$$(7.20)\qquad \gamma|u|_{1,\gamma,-}^2+[u(0)]_{1,\gamma}^2\leq C(<B'u>_{0,\gamma-}^2+|<B'u, \Lambda_{y,\gamma}u>_{0,\gamma,-}| +|<B'u, D_t u>_{0,\gamma,-}.$$

We obtain from (7.19) and (7.20)

$$(7.21)\qquad \gamma|u|^2_{1,\gamma,-}+\gamma\sum<\Lambda^{\frac{1}{2}}_{y,\gamma}D^j_x u>^2_{1-j,\gamma,-}+[u(0)]^2_{1,\gamma} \leq\frac{C}{\gamma}<\Lambda^{\frac{1}{2}}_{y,\gamma}\tilde{\varphi}>^2_{0,\gamma,-}.$$

Here we have used

$$|<B'u,\ D_t u>_{0,\gamma,-}|\leq\frac{1}{\varepsilon\gamma}<\Lambda^{\frac{1}{2}}_{y,\gamma}B'u>^2_{0,\gamma,-}+\frac{\varepsilon\gamma}{4}<\Lambda^{-\frac{1}{2}}_{y,\gamma}u>^2_{1,\gamma,-}.$$

Let us put $v(t, x, y)=u(-t, x, y)$. Then (7.19) follows from

$$|u|_{1,\gamma,-}=|v|_{1,-\gamma,+} \text{ and } \langle u\rangle_{1,\gamma,-}=\langle v\rangle_{1,-\gamma,+}.$$

We must prepare another lemma concerning Green's formula. Concerning the identity (6.1), we have following lemma.

Lemma 7.3 *We have*

$$(7.22)\qquad (Pu,\ D_x v)_+-(D_t u,\ P^*v)_+=<Bu,\ D_t v>_++<D_t u,\ B'u>_+ +R(u,\ v)+I[u,\ v]+r<u,\ v> \text{ for } u\in\mathcal{H}_{2,\gamma},\ v\in\mathcal{H}_{2,-\gamma}$$

where $|R(u, v)|\leq C|u|_{1,\gamma,+}|v|_{1,-\gamma,+}$, $|r<u, v>|\leq C|u|_{1,\gamma,+}|v|_{1,\gamma,+}$
$|I[u, v]|\leq C[u(0)]_{1,\gamma}[v(0)]_{1,-\gamma}$.

Here $(u, v)_+=(u, v)_{0,0+}$.

Proof. It suffices to notice the followint facts. Corresponding to (6.1),

$$(7.23)\qquad P_0(\tau,\xi,\eta)-\overline{P^0_*(\tau,\zeta,\eta)}=(\xi-\zeta)(B_0(\tau,\xi,\eta)+\overline{B'_0(\tau,\zeta,\eta)}) +(\tau-\bar{\tau})G_t(\xi,\zeta,\tau,\bar{\tau},\eta).$$

holds. Then we have.

$$(7.24)\qquad P_0\bar{\tau}-\tau\overline{P^*_0}=(P_0-\overline{P^*_0})\bar{\tau}+(\tau-\bar{\tau})\overline{P^*_0} =(\xi-\zeta)(B_0(\tau,\xi,\eta)\bar{\tau}+\tau\overline{B'_0(\tau,\zeta,\eta)}) +(\tau-\bar{\tau})\{-(\xi-\zeta)\overline{B'_0(\tau,\zeta,\eta)}+\overline{P^*_0}+G_t(\xi,\zeta,\tau,\bar{\tau},\eta)\bar{\tau}\}.$$

Put the second term $(\tau-\bar{\tau})\tilde{G}$, then we can see that $\tilde{G}$ is a bi-linear form. From (7.24) we have (7.22). q.e.d.

Now we can prove (7.17) in the following way;

$$(7.25)\qquad \gamma^{\frac{1}{2}}<\Lambda_{y,\gamma}^{-\frac{1}{2}}D_t u>_{0,\gamma,+}=\sup_{\varphi\in\mathcal{D}(R_+^n)}\frac{<\gamma^{\frac{1}{2}}\Lambda_{y,\gamma}^{-\frac{1}{2}}D_t u,\ \gamma^{\frac{1}{2}}\Lambda_{y,\gamma}^{\frac{1}{2}}\varphi>_+}{\gamma^{-\frac{1}{2}}<\Lambda_{y,\gamma}^{\frac{1}{2}}\varphi>_{0,-\gamma,+}}$$

$$=\sup_{\varphi\in\mathcal{D}(R_n^+)}\frac{<D_t u,\ \varphi>_+}{<\gamma^{-\frac{1}{2}}\Lambda_{y,\gamma}^{\frac{1}{2}}\varphi>_{0,-\gamma,+}}.$$

From Lemma 7.3 we have for the solution v of $(P_0^*)'$

$$(7.26)\qquad |<D_t u,\ B'v>_+|\leq C\{|f|_{0,\gamma,+}|v|_{1,-\gamma,+}$$
$$+<\Lambda_{y,\gamma}^{-\frac{1}{2}}D_t v>_{0,-\gamma,+}<\Lambda_{y,\gamma}^{\frac{1}{2}}g>_{0,\gamma,+}+|u|_{1,\gamma,+}|v|_{1,-\gamma,+}$$
$$+[u(0)]_{1,\gamma}[v(0)]_{1,-\gamma}\}.$$

Take v as the solution of (P_0^*), then from Lemma 7.2 we have

$$(7.27)\qquad |<D_t u,\ \varphi>_+|\leq CF^{\frac{1}{2}}\{\gamma^{\frac{1}{2}}(|v|_{1,-\gamma,+}+<\Lambda_{y,\gamma}^{-\frac{1}{2}}D_t v>_{0,-\gamma,+})$$
$$+[v(0)]_{1,-\gamma}\}\leq CF^{\frac{1}{2}}\gamma^{-\frac{1}{2}}<\Lambda_{y,\gamma}^{\frac{1}{2}}\ \varphi>_{0,-\gamma,+},\quad \text{where}$$

$$(7.28)\qquad F=\frac{1}{\gamma}(|f|_{0,\gamma,+}^2+<\Lambda_{y,\gamma}^{\frac{1}{2}}g>_{0,\gamma,+}^2)+[u(0)]_{1,\gamma}^2.$$

Therefor we have

$$\gamma<\Lambda_{y,\gamma}^{-\frac{1}{2}},D_t u>_{0,\gamma,+}^2\leq CF.$$

Considering Theorem 2 we have (7.17) if f and g satisfy $\operatorname{supp}_t[f]$, $\operatorname{supp}_t[g]\subset[0,\ t]$. This completes the proof of Theorem 3. with $k=0$. For $k\geq 1$, we can prove in the same way.

Proof of corollary of Theorem **3**. (existence theorem for the mixed problem (P))

The existence of the solution of $\{P\}$ is obtained from Theorem 3 and Theorem 1 by the aid of Cauchy problem. Let the coefficients of P be defined in $R^n\times R_+^1$. Consider Cauchy problem for regularly hyperbolic equation;

$$(C)_m \quad \begin{cases} Pu=f_m \\ D_t^j u \big|_{t=0} = u_{j,m}, \qquad (j=0, 1), \end{cases}$$

where $f_m(t, x, y)$ and $u_{j,m}$ are in $C_0^\infty(\overline{R^n \times R_+^1})$ and in $C_0^\infty(\overline{R_+^n})$ respectively such that the followings hold

(7.29) 1) $\|u_{j,m}-u_j\|_{H^{k+1-j}(R_+^n)}$ and $|f_m-f|_{\mathcal{H}_{k,\gamma}(R_+^n \times R_0^+)}$

converges to zero as m tends to ∞.

2) There exist $g_m(t, y)$ satisfying $\langle \Lambda_{y,\lambda}^{\frac{1}{2}}(g_m-g)\rangle_{k,\gamma,+} \to 0$ and $B_l(f_m, u_{1,m}, u_{2,m})=(D_t^{l-1}g_m)(0, y)$ $(l=1, 2, \dots, k+1)$,

The solution u_m of $(C)_m$ belongs to $\mathcal{H}_{k+2,\gamma}$. Then we consider

$$(P_0)_m \begin{cases} Pv=0 \\ Bv \big|_{x=0} = g_m - Bu_m = \tilde{g}_m, \quad \Lambda_{y,\gamma}^{\frac{1}{2}} \tilde{g} \in \mathcal{H}_{k+1,\gamma}(R^{n-1} \times (0, \infty)) \\ D_t^j v \big|_{t=0} = 0, \quad (j=0, 1). \end{cases}$$

Let us extend g_m by

$$\tilde{\tilde{g}}_m = \begin{cases} \tilde{g}_m & t \geq 0 \\ 0 & t < 0 \end{cases}$$

then we can see from (7.29), 2) that $\Lambda_{y,\gamma}^{\frac{1}{2}} \tilde{\tilde{g}}_m$ belongs to $\mathcal{H}_{k+1,\gamma}(R^n)$. Therefore we can apply Theorem 2 to the solution v_m of $(P_0)_m$, then $w_m = u_m + v_m$ satisfies

$$(P)_m \begin{cases} Pw_m = f_m \\ Bw_m \big|_{x=0} = g_m \\ D_t^j w_m \big|_{t=0} = u_{j,m}, \qquad (j=0, 1). \end{cases}$$

Using the energy inequality of Theorem 3, we can show the the limit u of w_m satifies (P) and belongs to $\mathcal{E}_t^0(H^{1+k}(R_+^n)) \cap \mathcal{E}_t^1(H^k(R_+^n))$.

§8. Finiteness of propagation speed and problems in general domain

1. Consider Holmgren transformation:

$$(8.1)\qquad \begin{cases} x'=x,\ y'_j=y_j, & (j=1, \dots, n-1) \\ t'=t+\dfrac{1}{2}\{(x-x^0)^2+\sum\limits_{j=1}^{n-1}(y_j-y_j^0)^2\}. \end{cases}$$

Let $\tilde{P}(t', x', y', D'_t, D'_x, D'_y)$ and $\tilde{B}(t', y', D'_t, D'_x, D'_y)$ be the transforms of P and B by (8.1) respectively, Condition (H_1) in kept by (8.1) in neighbourbhood of (x_0, y_0). Let us prove

Lemma 8.1 *$\{P, B\}$ satisfies (H_2) in place where (H_1) is satified.*

Proof. It suffices to conside P and B the operator with constant coe fficient by freezing the coefficient at any point.

Since (8.1) makes

$$(8.1)\qquad D_t=D'_t,\ D_x=D'_x+(x-x^0)D'_t,\ D_{y_j}=D'_{y_j}+(y_j-y_j^0)D'_t,$$

then we have

$$(8.2)\qquad \begin{cases} \tilde{P}(\tau', \xi', \eta')=P(\tau', \xi'+(x-x^0)\tau', \eta'+(y-y^0)\tau') \\ \tilde{B}(\tau', \xi', \eta')=B(\tau', \xi'+(x-x^0)\tau', \eta'+(y-y^0)\tau'). \end{cases}$$

Let $\tilde{\xi}'_+(\tau', \eta')$ be a root of $\tilde{P}(\tau', \xi', \eta')=0$ with positive imaginary part when $-Im\ \tau'=\gamma'>0$. Then from (8.2) we have.

$$(8.3)\qquad \tilde{\xi}'_+(\tau', \eta')=\xi_+(\tau', \eta'+(y-y^0)\tau')-(x-x^0)\tau'$$

$$(8.4)\qquad \begin{aligned} \tilde{B}(\tau', \tilde{\xi}'_+(\tau', \eta'), \eta') &= B(\tau', \tilde{\xi}'_+(\tau', \eta')+(x-x^0)\tau', \eta'+(y-y^0)\tau') \\ &= B(\tau', \xi_+(\tau', \eta'+(y-y^0)\tau'), \eta'+(y-y^0)\tau'). \end{aligned}$$

From (8.4) we can see that Case II in Lemma 2.6 never appears. And this is also true if we replace (8.1) by

$$(8.1)_s\qquad \begin{cases} x'=x,\ y'_j=y_j, & (j=1, \dots, n-1) \\ t'=t+\dfrac{1}{2}s\{(x-x^0)^2+|y-y^0|^2\}, & 0\leq s\leq 1. \end{cases}$$

Now we prove that $\tilde{B}(\tau', \tilde{\xi}'_+(\tau', \eta'), \eta'\neq 0$ for $-Im\ \tau'=\gamma'>0$

$|\eta'|^2+|\tau'|^2=1$. For this purpose we use the following lemma whose proof is same as that of Lemma 2.6.

Lemma 1) *If case I in Lemma* **2.6** *appear, then* ρ *must be negative.*
2) *Case II happens if and only if there exists* η', *such that* $-\sqrt{1-a^{-1}d(\eta')}=(a+a_n^2)^{-1}(\rho+\alpha(\eta'))<0$.
By geometrical consideration we can see that "there exists a positive constant δ *such that if* $-\delta\leq\rho<0$ *then Case II must appear."*

If $\tilde{B}(\sigma_0-i\gamma_0, \tilde{\xi}_+(\sigma_0-i\gamma_0, \eta_0), \eta_0)=0$ for $\gamma_0>0$ then

$$(8.5)\qquad \rho_s=\rho+s(\sum_{j=1}^{n-1}b_j(y_j-y_j^0)+\sum_{j=1}^{n-1}a_{nj}(y_j-y_j^0))\equiv\rho+sh.$$

From above lemma, (1), $\rho_1<0$, therefore $h>0$. Hence we can find $s_0(0\leq S_0<1)$ such that $\rho_{s_0}=\rho+s_0h=0$. If we take ε sufficiently small, then transformation of $\{P, B\}$ by $(8.1)_{s_0+\varepsilon}$: $\{\tilde{P}_{s_0+\varepsilon}, \tilde{B}_{s_0+\varepsilon}\}$ must satisfy Case II. This contradict the above assertion. q.e.d.

Consider grobal transformation:

$$(8.6)\qquad \begin{cases} x'=x,\ y'_j=y_j \\ t'=(t-t_0)+\lambda_{\max}^{-1}(|x-x^0|^2+|y-y^0|^2+\theta)^{\frac{1}{2}}, \text{ where} \end{cases}$$

$$\lambda_{\max}=\max_{j=1,2,\xi^2+|\eta|^2=1}\lambda_j(\xi, \eta)\qquad 0<\theta<t_0\lambda_{\max}.$$

Lemma 8.1 holds for (8.6) instead of (8.1). Therefore we can apply F. John's sweeping out method to the mixed problem, considering Theorem 3, and obtain Theorem 5.

2. Now we consider the mixed problem in general cylindrical domain $\Omega\times(0, \infty)$ which is corresponding to that considered before in $R_+^n\times(0, \infty)$:

$$(P)_\Omega\quad \begin{cases} P(t, x, D_t, D_x)u=f(t, x) & \Omega\times(0, \infty) \\ B(t, s, D_t, D_x)u=g(t, s) & \partial\Omega\times(0, \infty) \\ D_t^j u\,\big|_{t=0}=u(t, x), & (j=0, 1). \end{cases}$$

P is a regularly hyperbolic operator of second order with respect to t, and B is a first order operator with real coefficients. In order to describe the condition (H_1) and (H_2), it need to prepare some consideration about local transformation.

First consider a transformation.

$$(8.7) \qquad x'_i=\varphi_i(x), \ (i=1, \dots, n), \ (x_j=\psi_j(x'))$$

which maps some neighbourhood U in R^n of x_0 on $\partial\,\Omega$ in one to one way onto a neighhbourhood of origin such that, $\psi_i(x^0)=0$. $\partial\Omega \cap U$ is transformed to $x'_n=0$, and $\Omega\cap U$ to $W\cap\{x'=(x'_1\dots x'_n), \ x'_n>0\}$. Here we take $\varphi_n(x)$ as the distance from $\partial\Omega$ to x measured along inner normal direction (c.f. [18], p. 289). From (8.1) we have

$$(8.8) \qquad \frac{\partial}{\partial x_i}=\sum_{j=1}^{n}\frac{\partial\varphi_j}{\partial x_i}\frac{\partial}{\partial x'_j}.$$

The differential operator $P(t, x, D_t, Dx)$ is transformed to $\tilde{P}(t, x', D_t, D_{x'})=P(t, \psi(x'), D_t, T(x')D_{x'})$, where

$$T(x)=\left(\frac{\partial\varphi_j}{\partial x_i}\right)(x)=\left(\frac{\partial\varphi_j}{\partial x_i}\right)(\psi(x')).$$
$$i, j=1, \dots, n,$$

The principal part of $\tilde{P}$ is determined from P_0 as its characteristic polybomial being

$$(8.9) \qquad \tilde{P}_0(t, x', D_T, \xi')=P_0(t, \varphi(x'), \tau, \ T(x')\xi').$$

Corresponding to (8.8), we denote

$$(8.8)' \qquad \xi=T.\xi' \qquad \text{i.e } {}^t(\xi_1, \dots, \xi_n)=T^t(\xi'_1, \dots, \xi'_n).$$

Now let us notice the following relations whose proof in omitted:

Lemma 8.2

$$\sum_{i=1}^{n}\frac{\partial\varphi_j}{\partial x_i}\frac{\partial\varphi_n}{\partial x_i}=0 \text{ on } \partial\Omega, \ (j=1, \dots, n-1).$$

Denote the natural frame on $\bar{\Omega}$ by $(e_1, \ldots, e_n)$, and put

(8.10) $$(e_1, \ldots, e_n)T=(\mu_1, \ldots, \mu_{n-1}, \nu).$$

Here ν means inner normal direction and μ_j $(j=1, \ldots, n\text{-}1)$ tangential direction on $\partial\Omega$, It is more natural that we denote the original characteristic polynomial by

(8.11) $$P(t, x, \tau, e\xi) \text{ where } e\xi=(e_1, \ldots, e_n)^t(\xi_1, \ldots, \xi_n).$$

Then the principal part P_0 of P is given from (8.9) by

$$\begin{aligned} P_0(t, x, \tau, eT\xi')&=P_0(t, x, \tau, (\mu_1, \ldots, \mu_{n-1}, \nu)\cdot\xi') \\ &=P_0(t, \psi(x'), \tau, \eta\boldsymbol{\mu}+\xi'_n\nu), \quad \eta=(\xi_1, \ldots, \xi'_{n-}). \end{aligned}$$

The condition (H_1) and (H_2) is described as

(H_1) 1) $P(t, x, \tau, \nu)>0$ on $\partial\Omega$

2) $B(t, x, \tau, \nu)=1$

(H_2) Let $z^+(t, x, \tau, \eta)$ be a root with positive imaginary part of $P(t, x, \tau, \eta+z\nu)=0$, where η is real tangential vector on $\partial\Omega$. Then

$$\begin{aligned} &|B(t, x, \tau, \boldsymbol{\mu}+z^+(t, x, \tau, \boldsymbol{\mu}))|\geq C(t, x, \boldsymbol{\mu}, \sigma)\gamma^{\frac{1}{2}} \\ &\qquad c(t, x, \boldsymbol{\mu}, \sigma)>0, \quad \tau=\sigma-i\gamma, \quad |\boldsymbol{\mu}|^2+\sigma^2+\gamma^2=1. \end{aligned}$$

Let Λ_η be a positive root of Laplace-Vertrami's operator on $\partial\Omega$ Denote $\Lambda^{\frac{1}{2}}_{\eta,\gamma}=(\Lambda_\eta+\gamma)^{\frac{1}{2}}$, $(\gamma>0)$. Then we can state our theorem.

Theorem *Assume* (H_1). *There exist positive constants* γ_k *and* C_k, and β_k, $(k=1, 2, \ldots,)$ such that for $u\in\mathcal{H}_{k+1,\gamma}$ $(\Omega\times(0, \infty))$,

$$\begin{aligned} &\gamma|u|_{\mathcal{H}_{k,\gamma}(\Omega\times(0,t))}+\gamma\sum_{j=1}^{1}\langle\Lambda^{-\frac{1}{2}}_{\eta,\gamma}D^j_n u\rangle^2_{\mathcal{H}_{k-j,\gamma}(\partial\Omega\times(0,t))} \\ &\quad+[u(t)]^2_{k,\gamma,\Omega}\leq C_k e^{\beta_k t}\Big\{\frac{1}{\gamma}|Pu|^2_{\mathcal{H}_{k-1,\gamma}(\Omega\times(0,t))} \\ &\quad+\frac{1}{\gamma}\langle\Lambda^{\frac{1}{2}}_{\eta,\gamma}Bu\rangle^2_{\mathcal{H}_{k-1,\gamma}(\partial\Omega\times(0,t))}+u[u(0)]^2_{k,\gamma,\Omega}\Big\}, \\ &\quad\text{for } \gamma>\gamma_k, \ t>0, \end{aligned}$$

if and only if (H_2) *holds. Under the assumption* (H_2), *the existence theorem follows in the same way as corollary of Theorem* **3**. *And the propagation speea is the same as that of Cauchy problem.*

Proof. By the transformation of type (8.1), we reduce the problem to that in $R_+^n \times (0, \infty)$, In $R_+^n \times (0, \infty)$ the solution exists and it's propagation speed is finite, therefore as in [19]. we can use the method of partition of unity for initial data. Taking the summation of the solution for each part of initial data, we construct the solution in $\Omega \times (0, \infty)$.

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