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On the traces of Hecke operators for a normalizer of $\Gamma_0(N)$

By

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For a positive integer N, let $\Gamma_0(N)$ be the congruence subgroup of level N, i.e. $\Gamma_0(N)=\left\{\begin{pmatrix} a & c \\ b & d \end{pmatrix}\in \mathrm{SL}_2(\mathbf{Z}) \,\middle|\, c\equiv 0 \bmod N\right\}$ and $S_k(N)$ denote the space of cusp forms f of weight k for $\Gamma_0(N)$. Let q be a prime divisor of N and $q^\nu \| N$. Let W_{q^ν} denote an element of the order $R=\begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ N\mathbf{Z} & \mathbf{Z} \end{pmatrix}$ of the form $W_{q^\nu}=\begin{pmatrix} q^\nu x & y \\ Nz & q^\nu w \end{pmatrix}$ such that det $W_{q^\nu}=q^\nu$ with integers x, y, z and w. In other words, putting $R_q=\underset{\mathbf{Z}}{\otimes}\mathbf{Z}_q$, W_{q^ν} denote any one of the elements of $R^\times \cap \begin{pmatrix} 0 & -1 \\ q^\nu & 0 \end{pmatrix} R_q^\times$, where $R^\times$(resp. $R_q^\times$) is the group of all units in R(resp. R_q). Then W_{q^ν} normalizes $\Gamma_0(N)$ and keeps $S_k(N)$ invariant under the usual operation $f\mapsto f|W_{q^\nu}$ (see 1.1). Recently, H. Hijikata [2] has given the trace formula of the Hecke operator T_n acting on $S_k(N)$ for arbitrary N. Now the purpose of this paper is to give the trace of the operator $T_n\cdot W_{q^\nu}$ on the space $S_k(N)$. Then using the result of [2], we are able to give the traces of Hecke operators on the space of cusp forms for a normalizer of $\Gamma_0(N)$, generated by $\Gamma_0(N)$ and W_{q^ν}. We mainly follow the terminology and notation by [2].

1.1. For a complex valued function $f(z)$, difine the operator

$$f|\sigma=(\det\sigma)^{k/2}\cdot(cz+d)^{-k}f\left(\frac{az+b}{cz+d}\right)$$

where $\sigma=\begin{pmatrix} a & c \\ b & d \end{pmatrix}\in GL_2(\mathbf{R})$, and k is a positive integer. We denote the space of cusp forms of weight k for $\Gamma_0(N)$ by $S_k(N)$, i.e. the set of all holomorphic function $f(z)$ on the complex upper half plane satisfying $f|\sigma=f$ for any $\sigma\in\Gamma_0(N)$ and vanishes at every cusps of $\Gamma_0(N)$. Let n be a positive integer prime to N, then the Hecke operator T_n is acting on $S_k(N)$ by

$$f\circ T_n=n^{k/2-1}\,\Sigma f|\beta_j$$

where T_n corresponds to the union of the double cosets $\Xi=\cup\Gamma\alpha_i\Gamma$ and we put its left cosets $\Xi=\cup\Gamma\beta_j$ with $\Gamma=\Gamma_0(N)$, or explicitly

$$f\circ T_n=n^{k-1}\sum_{\substack{ad=n\\ a>0}}\ \sum_{b \bmod d} f\left(\frac{az+b}{d}\right)d^{-k}.$$

1.2. Let $N=\prod_{i=1}^{t} q_i^{\nu_i}$ where q_i are distinct primes and let $N_0=\prod_{i=1}^{u} q_i^{\nu_i}$ where $0<u\leqq t$. For $q_i|N_0$, we define $W_{q_i^{\nu_i}}=\begin{pmatrix} q_i^{\nu_i}x & y \\ Nz & q_i^{\nu_i}w \end{pmatrix}$ where x, y, z and w are integers satisfying $\det W_{q_i^{\nu_i}}=q_i^{\nu_i}$. Then it is known that $W_{q_i^{\nu_i}}$ defines a $\mathbf{C}$-linear automorphism of order 2: $f\mapsto f|W_{q_i^{\nu_i}}$ on $S_k(N)$, moreover $W_{q_i^{\nu_i}}$ commutes with T_n ([1, Lemma 17]). Define

$$W_{N_0}=\prod_{q|N_0} W_{q^\nu}.$$

1.3. Since $\Gamma W_{N_0}\Gamma=\Gamma W_{N_0}=W_{N_0}\Gamma$ and T_n corresponds to the union of the double cosets $\Xi=\cup\Gamma\alpha_i\Gamma$, the operator $T_n\cdot W_{N_0}$ corresponds to $\Xi W_{N_0}=\cup\Gamma\alpha_i W_{N_0}\Gamma$. Thus the trace $\operatorname{tr} T_n W_{N_0}$ is given as follows by the formula of Hijikata ([2, Theorem 0.1, 5.4])

$$\operatorname{tr} T_n W_{N_0}=-\sum_s a(s)\sum_f b(s,f)\prod_{q|N} c^*(s,f,q)+\delta(k)\cdot\sum_{d|n} d,$$

where $\delta(k)=1$ or 0 according as $k=2$ or not. The meaning of the symbols $a(s)$, $b(s,f)$ etc. shall be given in the following. Let s run over all integers such that

$$s^2-4nN_0=\begin{cases} 0 & \text{case } (p) \\ t^2 & \text{case } (h) \end{cases}$$

$$\left.\begin{array}{lll} t^2 m & m \equiv 1 & \bmod 4 \\ t^2 \cdot 4m & m \equiv 2, 3 & \bmod 4 \end{array}\right\} \text{case } (e)$$

with a positive integer t and a negative square free m. Corresponding to the above case of s, $a(s)$ is given by

$$a(s) = \begin{cases} \dfrac{|x|}{4} & (\mathrm{p}) \\ \min\{|x|, |y|\}^{k-1} \cdot (nN_0)^{1-k/2} & (h) \\ \dfrac{1}{2}(x^{k-1} - y^{k-1})(x-y)^{-1} \cdot (nN_0)^{1-k/2} & (e) \end{cases}$$

where x and y are the solutions of the equation $\Phi_s(X) = X^2 - sX + nN_0 = 0$. For each s fixed, let f run over all positive divisors of t in the cases (h) and (e), and $f=1$ in the case (p). Let K be the quotient ring $\mathbf{Q}[X]/\Phi_s(X)$ and g denote the canonical image of X in K. K is a commutative $\mathbf{Q}$-algebra of rank 2, and g generates an order $\mathbf{Z}+\mathbf{Z}g$ in K. Put $\Delta = (s^2 - 4nN_0)/f^2$, then for each f, there is a uniquely determined order $\Lambda = \Lambda_\Delta$ of K containing $\mathbf{Z}+\mathbf{Z}g$ as a submodule of index f, $[\Lambda : \mathbf{Z}+\mathbf{Z}g] = f$. Let $h(\Delta)$ denote the class number of locally principal ideals of Λ, and let $w(\Delta)$ denote the half of the cardinality of the unit group $\Lambda^\times$. Then

$$b(s, f) = h(\Delta)/w(\Delta).$$

Finally $c^*(s, f, q)$ is the number of non-equivalent embeddings of K into $M_2(\mathbf{Q}_q)$ optimal with respect to R_q/Λ_q such that $\psi(g) \in W_{N_0} R_q^\times$, i.e.

$$c^*(s, f, q) = |\mathrm{Emb}(g, W_{N_0} R_q, R_q/\Lambda_q)/R_q^\times|$$

in the notation of [2, 2.0].

1.4. For any $q|N$, $\mathrm{Emb}(g, R_q, R_q/\Lambda_q)/R_q^\times$ can be given as follows ([2, Theorem 2.3]). Let $\nu = \mathrm{ord}_q(N)$ and $\rho = \mathrm{ord}_q(f)$. Put $\tilde{F} = \{\xi \in Z | \Phi_s(\xi) \equiv 0 \ (q^{\nu+2\rho}) \text{ and } 2\xi \equiv s \ (q^\rho)\}$. Let F be a complete system of representatives of F modulo $q^{\nu+\rho}$, and put $F' = \{\xi \in F | \Phi_s(\eta) \equiv 0 \ (q^{\nu+2\rho+1})\}$. Define $\psi_\xi : K \to M_2(\mathbf{Q}_q)$, by $\psi_\xi(g) = \begin{pmatrix} \xi & q^\rho \\ -q^{-\rho} f(\xi) & s-\xi \end{pmatrix}$ and $\psi'_\eta : K \to M_2(\mathbf{Q}_q)$, by $\psi'_\eta(g) = \begin{pmatrix} s-\eta & -q^{-(\nu+\rho)} f(\eta) \\ q^{\nu+\rho} & \eta \end{pmatrix}$ respectively. If $(q^{-2\rho}(s^2 - 4nN_0)$ is prime to q (resp. divisible by q), the set $\{\psi_\xi \mid \xi \in F\}$ (resp. $\{\psi_\xi \mid \xi \in F\} \cup \{\psi'_\eta \mid \eta \in F'\}$) is a complete system of representatives of $\mathrm{Emb}(g, R_q, R_q/\Lambda_q)/\tilde{R}_q^\times$. Let $c(s, f, q)$ denote its cardinality i.e. $c(s, f, q) = |F|$ (resp. $|F| + |F'|$) if $q^{-2\rho}(s^2 - 4nN_0) \not\equiv 0$ (resp. $\equiv 0$) mod q. If $(q, N_0) = 1$, then $W_{N_0}R_q = R_q$. Hence $c^*(s, f, q) = c(s, f, q)$. We claim

Lemma. *If* $(q, N_0) > 1$, *and* $q^\nu \| N_0$, *then*

$$c^*(s, f, q) = \begin{cases} 1, & \text{if } s \equiv 0 \ (q^\nu) \text{ and } (f, q) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since $R_q^\times$ normalizes $W_{N_0}R_q = \begin{pmatrix} q^\nu Z_q & Z_q \\ q^\nu Z_q & q^\nu Z_q \end{pmatrix}$, $E = \mathrm{Emb}(g, W_{N_0}R_q, R_q/\Lambda_q)/R_q^\times$ can be considered as a set of $\psi \in \mathrm{Emb}(g, R_q/\Lambda_q)/R_q^\times$ such that $\psi(g) \in W_{N_0}R_q$. If $\psi_\xi \in E$, i.e. $\psi_\xi(g) \in W_{N_0}R_q$, then $\xi \equiv s - \xi \equiv 0(q^\nu)$. Since $\det \psi_\xi(g) = \xi(s-\xi) - f(\xi) = nN_0$, and $\mathrm{ord}_q(nN_0) = \nu$, we get $\mathrm{ord}_q(f(\xi)) = \nu$. Since $f(\xi) \equiv 0$ $(q^{\nu+2\rho})$, ρ should be equal to 0. Hence we can take $F = \{0\}$. Similarly if $\psi'_\eta \in E$, we get $\mathrm{ord}_q(f(\eta)) = \nu$. Then the condition $f(\eta) \equiv 0$ $(q^{\nu+1})$ for F', implies that F' is empty. Thus

$$E = \begin{cases} \{0\}, & \text{if } s \equiv 0\,(q^\nu) \text{ and } \rho = \mathrm{ord}_q(f) = 0 \\ \text{empty, otherwise}. \end{cases}$$

Hence $c^*(s, f, q) = |E| = 1$ or 0.

1.5. As is easily seen the case (p) occurs if and only if $N_0 = 4$. The case (h) occurs if and only if N_0 is a square. Hence in the formula of $\mathrm{tr}T_n W_{N_0}$, the partial sum $\sum_s$ over the s's of type (h) is given by

$$-\sum_{s^2 - 4nN_0 = t^2} a(s) \Sigma\, b(s, f) \prod_{q \mid N} c^*(s, f, q)$$

$$= -\varphi(\sqrt{N_0}) \cdot N_0^{k/2-1} \sum_a a^{k-1} \prod_{q \mid NN_0^{-1}} c(s_0, f_0, q)$$

where the sum $\sum_a$ is extended over $0 < a < \sqrt{n}$, $a \mid n$ and $a^2 + n \equiv 0$ (mod $\sqrt{N_0}$), and $s_0 = \sqrt{N_0}(a + n/a) > 0$, $f_0 = n/a - a$, $\varphi(m)$ denotes the Euler function. We remark that the volume part of the trace does not appear since the intersection of the center $GL_2(\mathbf{Q})$ with ΞW_{N_0} is empty.

1.6. Summing up, we have obtained the following

Theorem. *If* $(n, N)=1$, *the trace* $\mathrm{tr}\ T_n W_{N_0}$ *of the operator* $T_n W_{N_0}$ *on* $S_k(N)$ *is given as follows:*

$$\mathrm{tr}\ T_n W_{N_0} = -\frac{1}{2}\Sigma_1 \prod_{q|NN_0^{-1}} c(s, f, q)\cdot\frac{h((s^2-4nN_0)f^{-2})}{w((s^2-4nN_0)f^{-2})}\cdot\frac{x^{k-1}-y^{k-1}}{x-y}$$

$$-\delta_1\cdot\varphi(\sqrt{N_0})\cdot N_0^{k/2-1}\cdot\Sigma_2 a^{k-1}\cdot\prod_{q|NN_0^{-1}} c(s_0, f_0, q)$$

$$-\delta_2\cdot\frac{n^{\frac{k-1}{2}}}{2}\cdot\prod_{q^\nu\|N/4}\left(q^{[\frac{\nu}{2}]}+q^{[\frac{\nu-1}{2}]}\right)$$

$$+\delta_3\cdot\sum_{d|n} d,$$

where $\delta_1=\begin{cases}1, & \textit{if } N_0 \textit{ is a square,}\\ 0, & \textit{otherwise}\end{cases}$ $\delta_2=\begin{cases}1, & \textit{if } N_0=4 \textit{ and } n \textit{ is a square}\\ 0, & \textit{otherwise}\end{cases}$

and $\delta_3=\begin{cases}1, & \textit{if } k=2\\ 0, & \textit{otherwise.}\end{cases}$ *The sum* Σ_1 *is extended over the pairs of integers* (s, f) *such that* $s^2<4nN_0$, $s\equiv 0 \bmod N_0$, $f>0$, $(s^2-4nN_0)f^{-2}\equiv 0$, 1 mod 4 *and* $(f, N_0)=1$. *The sum* Σ_2 *is extended over the integers* a *such that* $0<a<\sqrt{n}$, $a|n$, $a^2+n\equiv 0 \bmod \sqrt{N_0}$ *and we put* $s_0=\sqrt{N_0}(a+n/a)$, $f_0=n/a-a$. *The number* $c(s, f, q)=|F|$ *or* $|F|+|F'|$ *hasb een explained in* 1.4.

2.1. Let $S_k(N, N_0)$ denote the space of cusp forms of weight k for the group generated by $\Gamma_0(N)$ and W_{N_0}. Then $S_k(N, N_0)$ can be viewed as a subspace of $S_k(N)$. Let $S_k^0(N)$ be the essential part of $S_k(N)$, i.e. the space spanned by the new forms in the sense of [1] and put $S_k^0(N, N_0)=S_k^0(N)\cup S_k(N, N_0)$. Let tr_N(resp. tr_{N,N_0}, tr_N^0, tr_{N,N_0}^0) denote the trace on $S_k(N)$ (resp. $S_k(N, N_0)$, $S_k^0(N)$, $S_k^0(N, N_0)$). It is known (cf. [1]) that there exists a basis $\{f_\lambda\}$ of $S_k(N)$ such that each f_λ is an eigenform of all the operators T_n and $W_{q^\nu}(q|N_0)$, and W_{N_0} maps $S_k^0(N)$ into itself. Since

$$f_\lambda | W_{N_0} = \begin{cases} f_\lambda, & \text{if } f_\lambda \in S_k(N, N_0) \\ -f_\lambda, & \text{if } f_\lambda \notin S_k(N, N_0). \end{cases}$$

Hence we obtain the following

Proposition 1.

$$\mathrm{tr}_N T_n - 2\mathrm{tr}_{N,N_0} T_n = -N_0{}^{k/2-1} \cdot \mathrm{tr}_N T_n W_{N_0}$$

$$\mathrm{tr}_N^0 T_n - 2\mathrm{tr}_{N,N_0}^0 T_n = -N_0{}^{k/2-1} \cdot \mathrm{tr}_N^0 T_n W_{N_0}$$

Remark. Let g (resp. g^*) denote the gneus of the group $\Gamma_0(N)$ (resp. $\Gamma^*(N) = <\Gamma_0(N), W_N>$). Specializing our Theorem and Proposition 1 to the simplest case where $N_0 = N$, $k=2$ and $n=1$, we get the well known formula of Fricke;

$$g - 2g^* = \begin{cases} \frac{1}{2}(h(-4N) + h(-N)) - 1, & \text{if } -N \equiv 1 \bmod 4 \text{ and } N \neq 3, \\ \frac{1}{2}h(-4N) - 1, & \text{if } -N \equiv 0, 2, 3 \bmod 4 \text{ and } \\ & N \neq 2, 4 \\ 0, & \text{if } N = 2, 3, 4. \end{cases}$$

2.2. For the $\mathrm{tr}_N^0 T_n W_{N_0}$ given in Proposition 1, we have the following formula;

Proposition 2.

$$\mathrm{tr}_N^0 T_n W_{N_0} = \sum_{M | NN_0^{-1}} \beta(NN_0^{-1}M^{-1}) \cdot \sum_{Q^2 | N_0} \mu(Q) \mathrm{tr}_{MN_0Q^{-2}} T_n W_{N_0Q^{-2}}$$

where $\mu(m)$ *denotes the Möbius function, and* $\beta(n) = \sum_{d|n} \mu(d)\mu\left(\frac{n}{d}\right)$.

Proof. Since $S_k(N) = \bigoplus_{MN|} \bigoplus_{d|MN^{-1}} S_k^0(M)^d$ (direct sum), where $S_k^0(M)^d = \{f(dz) | f \in S_k^0(M)\}$. We have

$$\mathrm{tr}_N T_n W_{N_0} = \sum_{\substack{M|N \\ d|NM^{-1}}} \mathrm{tr} S_k^0(M)^d T_n W_{N_0}.$$

Now for a divisor M of N, assume $q^{\mu}\|M$ where q is a fixed prime such that $q^{\nu}\|N_0$. Then by [1, Lemma 26], we get for any divisor L of NM^{-1} prime to q, $tr\,W_{q^{\nu}}=0$ on $S_k^0(M)^{Lq^{\delta}}\oplus S_k^0(M)^{Lq^{\nu-\mu-\delta}}$ if $0\leqq 2\delta<\nu-\mu$, and for any other prime $q_1^{\nu_1}\|N_0$ distinct from q, $tr\,W_{q_1^{\nu_1}}$ on $S_k^0(M)^{Lq^{\delta}}=\mathrm{tr}W_{q_1^{\nu_1}}$ on $S_k^0(M)^{Lq^{\nu-\mu-\delta}}$. Hence we get

$$\mathrm{tr}_N T_n W_{N_0}=\sum_{M|NN_0^{-1}} d\left(\frac{N}{N_0M}\right)\Sigma_{\mu}\mathrm{tr}^0_{MQ_{\mu}}T_n W_{Q_{\mu}}$$

where $d(m)$ is the number of divisors of m, and the sum Σ_{μ} is extended over all $\mu=(\mu_1, \ldots, \mu_u)$ such that $0\leqq\mu_i\leqq\nu_i$, $\mu_i\equiv\nu_i \mod 2\,(1\leqq i\leqq u)$, and we put $Q_{\mu}=\prod_{i=1}^{u}q_i^{\mu_i}$ (u: the number of prime factors of N_0). Put $N'=NN_0^{-1}$, $\lambda(N')=\mathrm{tr}_N T_n W_{N_0}$ and $\lambda^0(M)=\Sigma_{\mu}\mathrm{tr}^0_{MQ_{\mu}}T_n W_{Q_{\mu}}$ then

$$\lambda(N')=\sum_{M|N'} d\left(\frac{N'}{M}\right)\lambda^0(M)$$

hence

$$\lambda^0(N')=\sum_{M|N'} \beta\left(\frac{N'}{M}\right)\lambda(M)$$

with $\beta(m)=\sum_{d|m}\mu(d)\mu\left(\frac{m}{d}\right)$.

While

$$\lambda^0(N')=\Sigma_{\mu}\mathrm{tr}^0_{N'Q_{\mu}}T_n W_{Q_{\mu}}$$

$$=\sum_{Q^2|N_0}\mathrm{tr}_{NQ^{-2}}T_n W_{N_0Q^{-2}}$$

hence

$$\mathrm{tr}^0_N T_n W_{N_0}=\sum_{M|N'}\beta\left(\frac{N'}{M}\right)\sum_{Q^2|N_0}\mu(Q)\mathrm{tr}_{MN_0Q^{-2}}T_n W_{N_0Q^{-2}}.$$

3.1. Using our Theorem and Propositions we give here some numerical examples of the eigen-values of T_n or W_{N_0}, exclusively for weight $k=2$.

(1) The case $N=3\cdot 5\cdot 7=105$, $\dim S_2(105)=13$.

	eigen-values													
	I		II		III						IV		
W_3	1	1	−1	−1	1	1	1	−1	−1	−1	1	1	−1
W_5	−1	−1	1	−1	1	−1	−1	1	−1	−1	1	1	−1
W_7	1	−1	1	1	−1	1	1	−1	1	1	−1	−1	−1
T_2	−1	−1	−1	−1	0	α	β	0	α	β	$\sqrt{5}$	$-\sqrt{5}$	1

$\left(\alpha,\ \beta=\frac{-1\pm\sqrt{17}}{2}\right)$

I, II and III denote the old forms obtained from the forms of level $3\cdot 5$, $3\cdot 7$ and $5\cdot 7$, respectively. IV denotes the new forms. Each column corresponds to an eigenform. From the above table, we can see, for example, $\dim S_2(\Gamma')=3$, for $\Gamma'=<\Gamma_0(105),\ W_3,\ W_5>$.

(2) The case $N=7^3=343$, $\dim S_2^0(7^3)=24$, $\dim S_2^0(7^3,\ 7^3)=9$. The space $S_2^0(7^3)$ is divided into five blocks I, II, II_χ, III and IV of the eigen-spaces of Hecke operators and characteristic polynomial of T_p in each block is as follows (The part left open in the table is not computed yet).

	I	II
T_2	$x^3+4x^2+3x-1=0$	$x^6+2x^5-6x^4-10x^3+10x^2+11x-1=0$
T_3	$x^3=0$	$x^6+5x^5-x^4-34x^3-28x^2+49x+49=0$
T_5	$x^3=0$	
T_{11}	$x^3+9x^2+20x+13=0$	

	II_χ	III	IV
T_2	$x^6+2x^5-6x^4-10x^3$ $+10x^2+11x-1=0$	$(x^3-2x^2-x+1)^2=0$	$x^3-3x^2-4x+13=0$
T_3	$x^6-5x^5-x^4+34x^3$ $-28x^2-49x+49=0$	$x^6-20x^4+124x^2-232=0$	$x^3=0$
T_5		$x^6-24x^4+164x^2-232=0$	$x^3=0$
T_{11}		$(x^3-x^2-2x+1)^2=0$	$x^3-5x^2-36x+167=0$

I and II form the space $S_2^0(7^3, 7^3)$, I and IV are corresponding to the Grössen-character of conductor 7 of $\mathbf{Q}(\sqrt{-7})$ (see [4]). Let $f(z)=\Sigma a_n e^{2\pi inz}\in S_2^0(7^3)$, then $f_\chi(z)=\Sigma a_n\left(\frac{n}{7}\right)e^{2\pi inz}$ is also contained in $S_2^0(7^3)$ ([3] Prop. 3.64.) II_χ is obtained from II in this way, i.e. if $f(z)$ is contained in II then $f_\chi(z)$ is contained in II_χ. Now we give some remarks for III. It seems that the part III is closely related to the theory of [3, 7.7] (Construction of class fields over real quadratic fields). If $f(z)$ is contained in III, $f_\chi(z)$ is also contained in III, and let F (resp. K) be the field generated over $\mathbf{Q}$ by the eigen-values of T_n for all $\left(\frac{n}{7}\right)=1$ (resp. for all $\left(\frac{n}{7}\right)=-1$) then we see $F=\mathbf{Q}(\zeta_7+\zeta_7^{-1})$, where $\zeta_7=e^{\frac{2\pi i}{7}}$ and K is a quadratic extension of F with relative discriminant $N_{F/Q}\mathfrak{d}(K/F)=29$. For $\left(\frac{p}{7}\right)=1$ (p: a prime), let $p^7=\alpha_p\alpha_p'$ where α_p is an element of the order of conductor 7 in $\mathbf{Q}(\sqrt{-7})$ normalized as $\left(\frac{\alpha_p+\alpha_p'}{7}\right)=1$ (α_p': the conjugate of α_p), for example $\alpha_2=\frac{13+7\sqrt{-7}}{2}$. Let a_p be an eigenvalue of T_p and π_p, π_p' be the solutions of $x^2-a_px+p\equiv 0 \pmod{29}$ in the prime field, then we can see

$$\pi_p^7+\pi'^7_p\equiv\alpha_p+\alpha_p' \pmod{29}$$

for $p=2$, 11. Furthermore we may expect that $N_{F/Q}\mathfrak{d}(K/F)=29$ is related to the fact $N_{F/Q}(\varepsilon^7-1)=8\cdot 29$ with $\varepsilon=\zeta_7+\zeta_7^{-1}$.

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