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On The Existence of Solutions of Stochastic Differential Equations with Boundary Conditions

By

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§0. Introduction.

One-dimensional diffusion processes were studied by Feller, Itô, McKean and Dynkin. On the other hand multi-dimensional diffusion processes have been studied in various points of view. Ventcel' [15] pointed out that under suitable regularity conditions, a diffusion process on a smooth manifold $\bar{D}=D\cup\partial D$ of n dimensions with a smooth boundary is determined by the following (A, L, ρ). Suppose (ψ, U) is a coordinate mapping with the following property,

$$\psi^1(x)>0 \Leftrightarrow x\in D\cap U,$$

$$\psi^1(x)=0 \Leftrightarrow x\in \partial D\cap U.$$

A is an elliptic differential operator of second order which is expressed in the form,

$$Af(x)=\sum_{i,j=1}^{n} a^{ij}(x)D_{ij}f(x)+\sum_{i=1}^{n} b^i(x)D_i f(x)+c(x)f(x),$$

where $(a^{ij}(x))$ is symmetric and positive semi-definite and $c(x)\leqq 0$. L is an operator which maps a smooth function on $\bar{D}$ to a function on ∂D given in the form,

$$Lg(x)=\sum_{i,j=2}^{n}\alpha^{ij}(x)D_{ij}g(x)+\sum_{i=2}^{n}\beta^{i}(x)D_{i}g(x)+\gamma(x)g(x)+\delta(x)D_{1}g(x)$$

$$+\int_{\bar{D}\setminus\{x\}}[g(y)-g(x)-I_{U}(y)\sum_{i=2}^{n}(y^{i}-x^{i})D_{i}g(x)]\nu_{x}(dy),$$

where $(\alpha^{ij}(x))$ is symmetric and positive semi-definite, $\gamma(x)\leqq 0$, $\delta(x)\geqq 0$ and $\nu_x(dy)$ is a σ-finite measure on $\bar{D}\setminus\{x\}$ satisfying a usual convergence condition. ρ is a non-negative function on ∂D. Ventcel' showed that if $\mathfrak{G}$ is the infinitesimal generator with domain $\mathscr{D}(\mathfrak{G})$ of the semigroup of the diffusion, then, for $f\in\mathscr{D}(\mathfrak{G})\cap\mathbf{C}^2(\bar{D})$, $\mathfrak{G}f=Af$ and f satisfies $Lf=\rho Af$ on ∂D (Ventcel's boundary condition).

Now it is an important problem to find regularity conditions of (A, L, ρ) under which the diffusion process corresponding to it exists. Roughly speaking, there have been two ways of attacking this problem; analytic way and probabilistic way. In analytic way, such a problem has been discussed by Sato-Ueno [7] and Bony-Courrège-Priouret [1]. In probabilistic way, using stochastic differential equations, Skorohod [8] studied one-dimensional reflecting diffusion processes and Ikeda [2] studied two-dimensional diffusion processes. S. Watanabe [13] [14] constructed, combining the methods of [8] and [2], the class of diffusion processes on the upper half space of R^n corresponding to (A, L, ρ) in case that $c=\gamma=\nu_x=0$ and $\delta(x)$ is positive. Stroock-Varadhan [10] formulated this problem as a submartingale problem and showed the existence and uniqueness of solutions by using several results on differential equations. The aim of this paper is, following the formulation of [13] [14], to prove the existence of solutions of stochastic differential equations with boundary conditions for continuous coefficients. The uniqueness fails in general and it is important to obtain the conditions of coefficients which guarantee the uniqueness of solutions. It should be remarked that Stroock-Varadhan [10] proved the uniqueness for a general class of coefficients.

§1 is devoted to give the precise formulation of stochastic differential equations with boundary conditions. In §2 we shall prove the existence of solutions in case that each coefficient is bounded and continuous and ρ is identically zero (the non-sticky case). In §3 we shall prove the existence of solutions in case that ρ is not identically zero (the sticky case).

Finally the author wishes to express his hearty thanks to Professors M. Nisio and S. Watanabe for their valuable advices.

§1. The formulation of stochastic differential equations with boundary conditions.

Let $n \geqq 2$, $\bar{D}=\{x=(x^1,\cdots, x^n)\in R^n;\ x^1\geqq 0\}$, $D=\{x\in\bar{D};\ x^1>0\}$ and $\partial D=\{x\in\bar{D};\ x^1=0\}$. For $x=(x^1,\cdots, x^n)\in\bar{D}$, we define $\tilde{x}=(0, x^2,\cdots, x^n)\in\partial D$. We will be given the following quantities;

$$
\begin{aligned}
&\sigma=(\sigma^i_j(t, x))^n_{i,j=1} &&: [0, \infty)\times\bar{D} \longrightarrow R^n\otimes R^n,\\
&b=(b^i(t, x))^n_{i=1} &&: [0, \infty)\times\bar{D} \longrightarrow R^n,\\
&\tau=(\tau^i_j(t, x))^n_{i,j=2} &&: [0, \infty)\times\partial D \longrightarrow R^{n-1}\otimes R^{n-1},\\
&\beta=(\beta^i(t, x))^n_{i=2} &&: [0, \infty)\times\partial D \longrightarrow R^{n-1},\\
&\rho=\rho(t, x) &&: [0, \infty)\times\partial D \longrightarrow [0, \infty),
\end{aligned}
$$

where $R^n\otimes R^n$ (resp. $R^{n-1}\otimes R^{n-1}$) is the class of linear applications of R^n into R^n (resp. R^{n-1} into R^{n-1}). In this paper, we shall assume that each component is bounded and Borel measurable. If coefficients are time independent, they are denoted by $\sigma(x)$, $b(x)$, $\tau(x)$, $\beta(x)$ and $\rho(x)$.

If $\rho\equiv 0$, it is called the non-sticky case and if $\rho\not\equiv 0$, it is called the sticky case. We shall consider a stochastic differential equation with boundary conditions in the non-sticky case, in the following form;

$$
(1.1)\qquad \begin{cases} dx^1_t=\sigma^1(t, x_t)dB_t+b^1(t, x_t)dt+d\varphi_t,\\ dx^i_t=\sigma^i(t, x_t)dB_t+b^i(t, x_t)dt+\tau^i(t, \tilde{x}_t)dM_t+\beta^i(t, \tilde{x}_t)d\varphi_t \\ \hfill i=2,\cdots, n. \end{cases}
$$

We shall consider a stochastic differential equation with boundary conditions in the sticky case, in the following form;

$$
(1.2)\quad \begin{cases} dx_t^1 = \sigma^1(t, x_t) I_D(x_t) dB_t + b^1(t, x_t) I_D(x_t) dt + d\varphi_t, \\ dx_t^i = \sigma^i(t, x_t) I_D(x_t) dB_t + b^i(t, x_t) I_D(x_t) dt + \tau^i(t, \tilde{x}_t) dM_t \\ \qquad + \beta^i(t, \tilde{x}_t) d\varphi_t \qquad\qquad i = 2, \cdots, n, \\ I_{\partial D}(x_t) dt = \rho(t, \tilde{x}_t) d\varphi_t. \end{cases}
$$

Now we shall discuss the meaning of the equation (1.1) and (1.2). In this paper, we shall understand that a quadruplet written by $(\Omega, \mathscr{F}, P; \{\mathscr{F}_t\}_{t\in[t_0,\infty)})$ satisfies the following conditions;

(i) $(\Omega, \mathscr{F}, P)$ is a standard probability space (cf. Itô [3]),

(ii) t_0 is non-negative and $\{\mathscr{F}_t\}_{t\in[t_0,\infty)}$ is a right continuous and increasing system of sub-σ-fields of $\mathscr{F}$. I_A is an indicator function of a set A. The following definitions are due to S. Watanabe [13] [14]. Let μ be a probability law on $\bar{D}$.

Definition 1.1. By a solution of (1.1) with initial distribution μ at time $t_0 \geqq 0$, we mean a stochastic process $\mathfrak{x} = \{x_t = (x_t^1, \cdots, x_t^n), B_t = (B_t^1, \cdots, B_t^n), M_t = (M_t^2, \cdots, M_t^n), \varphi_t\}_{t\in[t_0,\infty)}$, defined on a quadruplet $(\Omega, \mathscr{F}, P; \{\mathscr{F}_t\}_{t\in[t_0,\infty)})$, satisfying the following conditions (i)$\sim$(iv);

(i) with probability one, x_t, B_t, M_t and φ_t are continuous in $t \in [t_0, \infty)$ such that $B_{t_0} = (0, \cdots, 0)$, $M_{t_0} = (0, \cdots, 0)$, $\varphi_{t_0} = 0$ and $P(x_{t_0} \in dx) = \mu(dx)$,

(ii) with probability one, $x_t \in \bar{D}$ for all $t \in [t_0, \infty)$ and φ_t is non-decreasing; furthermore,

$$
\int_{t_0}^t I_{\partial D}(x_s) d\varphi_s = \varphi_t \qquad t \geqq t_0,
$$

(iii) x_t and φ_t are adapted to $\{\mathscr{F}_t\}$ and (B_t, M_t) is a system of $\{\mathscr{F}_t\}$-martingales such that

$$
\langle B^i, B^j \rangle_t = \delta_{ij}(t - t_0) \qquad i, j = 1, \cdots, n, \qquad t \geqq t_0,
$$

$$
\langle B^i, M^j \rangle_t = 0 \qquad i = 1, \cdots, n,\ j = 2, \cdots, n, \qquad t \geqq t_0,
$$

$$
\langle M^i, M^j \rangle_t = \delta_{ij}\varphi_t \qquad i, j = 2, \cdots, n, \qquad t \geqq t_0,
$$

where $< \ , \ >_t$ is the usual notation ([4] pp. 211),

(iv) $\mathfrak{x}=\{x_t, B_t, M_t, \varphi_t\}$ satisfies

$$
(1.3)\quad \begin{cases}
x_t^1 = x_{t_0}^1 + \sum_{j=1}^{n} \int_{t_0}^{t} \sigma_j^1(s, x_s)\, dB_s^j + \int_{t_0}^{t} b^1(s, x_s)\, ds + \varphi_t, \\[2ex]
x_t^i = x_{t_0}^i + \sum_{j=1}^{n} \int_{t_0}^{t} \sigma_j^i(s, x_s)\, dB_s^j + \int_{t_0}^{t} b^i(s, x_s)\, ds \\[1ex]
\qquad + \sum_{j=2}^{n} \int_{t_0}^{t} \tau_j^i(s, \tilde{x}_s)\, dM_s^j + \int_{t_0}^{t} \beta^i(s, \tilde{x}_s)\, d\varphi_s \\[1ex]
\hfill i=2,\cdots, n,
\end{cases}
$$

where the integrals by dB and dM are understood in the sense of stochastic integrals, cf. [4].

Definition 1.2. By a solution of (1.2) with initial distribution μ at time $t_0 \geqq 0$, we mean a stochastic process $\mathfrak{x}=\{x_t=(x_t^1,\cdots, x_t^n), B_t=(B_t^1,\cdots, B_t^n), M_t=(M_t^2,\cdots, M_t^n), \varphi_t\}_{t\in[t_0,\infty)}$, defined on a quadruplet $(\Omega, \mathscr{F}, P; \{\mathscr{F}_t\}_{t\in[t_0,\infty)})$, satisfying the following conditions (i)~(iv);

(i), (ii) and (iii) are same as (i), (ii) and (iii) of Definition 1.1,

(iv) $\mathfrak{x}=\{x_t, B_t, M_t, \varphi_t\}$ satisfies

$$
(1.4)\quad \begin{cases}
x_t^1 = x_{t_0}^1 + \sum_{j=1}^{n} \int_{t_0}^{t} \sigma_j^1(s, x_s) I_D(x_s)\, dB_s^j + \int_{t_0}^{t} b^1(s, x_s)\, I_D(x_s)\, ds + \varphi_t, \\[2ex]
x_t^i = x_{t_0}^i + \sum_{j=1}^{n} \int_{t_0}^{t} \sigma_j^i(s, x_s) I_D(x_s)\, dB_s^j + \int_{t_0}^{t} b^i(s, x_s)\, I_D(x_s)\, ds \\[1ex]
\qquad + \sum_{j=2}^{n} \int_{t_0}^{t} \tau_j^i(s, \tilde{x}_s)\, dM_s^j + \int_{t_0}^{t} \beta^i(s, \tilde{x}_s)\, d\varphi_s \qquad i=2,\cdots, n, \\[2ex]
\int_{t_0}^{t} I_{\partial D}(x_s)\, ds = \int_{t_0}^{t} \rho(s, \tilde{x}_s)\, d\varphi_s,
\end{cases}
$$

where the integrals by dB and dM are understood in the sense of stochastic integrals.

When σ, b, τ, β and ρ are time independent, the solutions of (1.1) and (1.2) are defined in a similar way. In this case we may take always $t_0=0$.

When ρ is identically zero, a solution of (1.2) is a solution of (1.1), but the converse is not always true.

§2. The existence of solutions in the non-sticky case.

In this section we shall prove the existence of solutions of (1.1) for bounded and continuous coefficients. The following theorem is due to S. Watanabe [13].

Theorem 2.1. *Suppose σ, b, τ and β are all time independent and bounded Lipschitz functions. Further, suppose a constant $c>0$ exists such that*

$$|\sigma^1(x)|=(\sum_{j=1}^n \sigma_j^1(x)^2)^{1/2} \geqq c \qquad \textit{for all } x.$$

Then, for any probability law μ on $\bar{D}$, there exists a solution of (1.1) *with initial distribution μ and the uniqueness holds in the sense of probability law.*

In order to obtain a solution of (1.1) under the weaker conditions, namely, all coefficients are bounded and continuous, we shall prepare several fundamental lemmas. We set

$$||\sigma(x)||=(\sum_{i,j=1}^n \sigma_j^i(x)^2)^{1/2}, \qquad ||b(x)||=(\sum_{i=1}^n b^i(x)^2)^{1/2},$$

$$||\tau(x)||=(\sum_{i,j=2}^n \tau_j^i(x)^2)^{1/2}, \qquad ||\beta(x)||=(\sum_{i=2}^n \beta^i(x)^2)^{1/2}.$$

We shall assume that σ, b, τ and β satisfy the conditions of Theorem 2.1 and that

$$||\sigma(x)|| \leqq c_1,\ ||b(x)|| \leqq c_2,\ ||\tau(x)|| \leqq c_3,\ ||\beta(x)|| \leqq c_4.$$

Let $\mathfrak{x}=\{x_t, B_t, M_t, \varphi_t\}$ be a solution of (1.1) for coefficients σ, b, τ and β stating from $x_0\in\bar{D}$ defined on $(\Omega, \mathscr{F}, P; \{\mathscr{F}_t\})$. Following Lemma 2.1–2.5 are concerned with this process $\mathfrak{x}$.

Lemma 2.1.

$$E[(x_t^1)^2]\leqq\{(x_0^1)^2+(c_1^2+2c_2)t\}\exp(2c_2t)\qquad t\geqq 0, \tag{2.1}$$

$$E[\varphi_t^2]\leqq 4[(x_0^1)^2+c_1^2t+c_2^2t^2+\{(x_0^1)^2+(c_1^2+2c_2)t\}\exp(2c_2t)]\qquad t\geqq 0. \tag{2.2}$$

Proof. Set for $N>0$,

$$T_N=\begin{cases}\inf\{s;\ |x_s^1|\geqq N\} & \\ +\infty & \text{if } \{\quad\}=\phi.\end{cases}$$

As T_N is an $\{\mathscr{F}_t\}$-stopping time, using the generalized Itô's formula on stochastic integral [4], we have

$$\begin{aligned}(x_{t\wedge T_N}^1)^2&=(x_0^1)^2+2\sum_{j=1}^n\int_0^{t\wedge T_N}x_s^1\sigma_j^1(x_s)dB_s^j+2\int_0^{t\wedge T_N}x_s^1b^1(x_s)ds\\&\quad+2\int_0^{t\wedge T_N}x_s^1d\varphi_s+\sum_{j=1}^n\int_0^{t\wedge T_N}\sigma_j^1(x_s)^2ds\\&=(x_0^1)^2+2\sum_{j=1}^n\int_0^t x_s^1\sigma_j^1(x_s)I_{\{T_N>s\}}dB_s^j+2\int_0^t x_s^1b^1(x_s)I_{\{T_N>s\}}ds\\&\quad+2\int_0^t x_s^1I_{\{T_N>s\}}d\varphi_s+\sum_{j=1}^n\int_0^t\sigma_j^1(x_s)^2I_{\{T_N>s\}}ds.\end{aligned}$$

$\mathfrak{x}$ is a solution of (1.1). Hence the condition (ii) of Definition 1.1 implies that

$$\int_0^t x_s^1I_{\{T_N>s\}}d\varphi_s=0\qquad t\geqq 0.$$

Since $\sum_{j=1}^n\int_0^t x_s^1\sigma_j^1(x_s)I_{\{T_N>s\}}dB_s^j$ is a martingale with mean 0, we find that

$$E[(x^1_{t\wedge T_N})^2]=(x^1_0)^2+E[2\int_0^t x^1_s b^1(x_s)I_{\{T_N>s\}}\,ds+\sum_{j=1}^n\int_0^t \sigma^1_j(x_s)^2 I_{\{T_N>s\}}\,ds]$$

$$\leqq (x^1_0)^2+c_1^2 t+2c_2\int_0^t E[x^1_{s\wedge T_N}]ds$$

$$\leqq (x^1_0)^2+c_1^2 t+2c_2\int_0^t E[(x^1_{s\wedge T_N})^2]^{1/2}ds$$

$$\leqq (x^1_0)^2+c_1^2 t+2c_2\int_0^t \{1+E[(x^1_{s\wedge T_N})^2]\}\,ds.$$

This functional inequality provides us with the estimate

$$E[(x^1_{t\wedge T_N})^2]\leqq\{(x^1_0)^2+(c_1^2+2c_2)t\}\exp(2c_2 t).$$

Letting $N\to+\infty$, we obtain (2.1).

It follows from (1.3) that

$$\varphi_t^2\leqq 4\left\{(x^1_t)^2+(x^1_0)^2+\left(\sum_{j=1}^n\int_0^t \sigma^1_j(x_s)dB^j_s\right)^2+\left(\int_0^t b^1(x_s)ds\right)^2\right\}.$$

Taking the expectation with respect to P, we have

$$E[\varphi_t^2]\leqq 4\left\{E[(x^1_t)^2]+(x^1_0)^2+\sum_{j=1}^n E\left[\int_0^t \sigma^1_j(x_s)^2 ds\right]+E\left[\left(\int_0^t b^1(x_s)ds\right)^2\right]\right\}$$

$$\leqq 4\,\{E[(x^1_t)^2]+(x^1_0)+c_1^2 t+c_2^2 t^2\}.$$

Hence (2.2) follows from (2.1). Q.E.D.

Let

$$K_1(x^1_0,\,t)=\{(x^1_0)^2+(c_1^2+2c_2)t\}\exp(2c_2 t),$$

$$K_2(x^1_0,\,t)=4\{(x^1_0)^2+c_1^2 t+c_2^2 t^2+K_1(x^1_0,\,t)\}.$$

Recalling the condition (iii) of Definition 1.1, we get

(2.3) $$E[(M_t^i)^2]=E[\varphi_t]\leqq K_2(x_0^1,\ t)^{1/2} \qquad i=2,\cdots,n, \quad t\geqq 0.$$

Lemma 2.2.

(2.4) $$E[|M_t^i|^3]\leqq 6K_2(x_0^1,\ t)^{3/4} \qquad i=2,\cdots,n, \quad t\geqq 0.$$

Proof. Set, for $N>0$,

$$T_N=\begin{cases} \inf\ \{s;\quad |M_s^i|\geqq N\} & \\ +\infty & \text{if } \{\quad\}=\phi. \end{cases}$$

The formula on stochastic integral states that

$$|M_{t\wedge T_N}^i|^3=3\int_0^{t\wedge T_N} g(M_s^i)dM_s^i+3\int_0^{t\wedge T_N}|M_s^i|\,d\varphi_s$$

$$=3\int_0^t g(M_s^i)I_{\{T_N>s\}}dM_s^i+3\int_0^t|M_s^i|\,I_{\{T_N>s\}}d\varphi_s,$$

where $g(x)=\begin{cases} x^2 & \text{for } x\geqq 0 \\ -x^2 & \text{for } x<0. \end{cases}$

Now $\int_0^t g(M_s^i)I_{\{T_N>s\}}dM_s^i$ is a martingale with mean 0. Therefore, taking the expectation, we have

$$E[|M_{t\wedge T_N}^i|^3]=3E\left[\int_0^t|M_s^i|\,I_{\{T_N>s\}}d\varphi_s\right]$$

$$\leqq 3E\left[\int_0^t|M_s^i|\,d\varphi_s\right].$$

Since $|M_t^i|$ is a non-negative continuous submartingale, we can apply a Doob's inequality (cf. [6] pp. 94) to $|M_t^i|$ and get

$$E[\sup_{0\leqq s\leqq t}|M_s^i|^2]^{1/2}\leqq 2E[|M_t^i|^2]^{1/2}.$$

Because of this, we have

$$E\left[\int_0^t |M_s^i|\, d\varphi_s\right] \leqq E[\sup_{0\leqq s\leqq t} |M_s^i| \varphi_t] \leqq E[\sup_{0\leqq s\leqq t} |M_s^i|^2]^{1/2} E[\varphi_t^2]^{1/2}$$

$$\leqq 2E[|M_t^i|^2]^{1/2} E[\varphi_t^2]^{1/2} \leqq 2E[\varphi_t^2]^{3/4}.$$

Thus we see from (2.2) that

$$E[|M_{t\wedge T_N}^i|^3] \leqq 6K_2(x_0^1, t)^{3/4}.$$

Letting $N \to +\infty$, we obtain the inequality (2.4). Q.E.D.

Lemma 2.3. *There exist positive constants K_3, K_4, K_5, h_1 and h_2 depending only on c_1 and c_2 such that*

(2.5) $E[(x_t^1 - x_s^1)^4] \leqq K_3 |t-s|^2$ *for all* $t, s \geqq 0$ *such that* $|t-s| \leqq h_1$,

(2.6) $E[(\varphi_t - \varphi_s)^4] \leqq K_4 |t-s|^2$ *for all* $t, s \geqq 0$ *such that* $|t-s| \leqq h_2$,

(2.7) $E[|M_t^i - M_s^i|^5] \leqq K_5 |t-s|^{5/4}$ *for all* $t, s \geqq 0$ *such that* $|t-s| \leqq h_2$, $i = 2, \dots, n$.

Proof. We begin by proving the estimate (2.5). According to the formula on stochastic integral, for $t \geqq s \geqq 0$, $\mathfrak{x}$ satisfies the equation

$$(x_t^1 - x_s^1)^4 = 4\sum_{j=1}^n \int_s^t (x_u^1 - x_s^1)^3 \sigma_j^1(x_u)\, dB_u^j + 4\int_s^t (x_u^1 - x_s^1)^3 b^1(x_u)\, du$$

$$+ 4\int_s^t (x_u^1 - x_s^1)^3\, d\varphi_u + 6\sum_{j=1}^n \int_s^t (x_u^1 - x_s^1)^2 \sigma_j^1(x_u)^2\, du.$$

From the same argument given in Lemma 2.1, we see

$$E\left[\sum_{j=1}^n \int_s^t (x_u^1 - x_s^1)^3 \sigma_j^1(x_u)\, dB_u^j\right] = 0.$$

Since $\mathfrak{x}$ is a solution of (1.1), we have by the condition (ii) of Definition

1.1 that

$$\int_s^t (x_u^1 - x_s^1)^3\, d\varphi_u = -\int_s^t (x_s^1)^3\, d\varphi_u \leqq 0.$$

Taking the expectation, we have

$$E[(x_t^1 - x_s^1)^4] \leqq 4E\left[\int_s^t (x_u^1 - x_s^1)^3 b^1(x_u)\, du\right]$$

$$+6\sum_{j=1}^n E\left[\int_s^t (x_u^1 - x_s^1)^2 \sigma_j^1(x_u)^2\, du\right]$$

$$\leqq 4c_2\int_s^t E[(x_u^1 - x_s^1)^4]^{3/4}\, du + 6c_1^2\int_s^t E[(x_u^1 - x_s^1)^4]^{1/2}\, du.$$

This functional inequality provides us with the estimate (2.5).

Next we shall prove (2.6). It holds from (1.3) that

$$(\varphi_t - \varphi_s)^4 = \left\{x_t^1 - x_s^1 + \sum_{j=1}^n \int_s^t \sigma_j^1(x_u)\, dB_u^j + \int_s^t b^1(x_u)\, du\right\}^4$$

$$\leqq (n+2)^3\left\{(x_t^1 - x_s^1)^4 + \sum_{j=1}^n \left(\int_s^t \sigma_j^1(x_u)\, dB_u^j\right)^4 + \left(\int_s^t b^1(x_u)\, du\right)^4\right\}.$$

Taking the expectation, we have

$$E[(\varphi_t - \varphi_s)^4] \leqq (n+2)^3\left\{E[(x_t^1 - x_s^1)^4] + \sum_{j=1}^n E\left[\left(\int_s^t \sigma_j^1(x_u)\, dB_u^j\right)^4\right]\right.$$

$$\left. + E\left[\left(\int_s^t b^1(x_u)\, du\right)^4\right]\right\}$$

$$\leqq (n+2)^3\left\{E[(x_t^1 - x_s^1)^4] + 36\sum_{j=1}^n E\left[\left(\int_s^t \sigma_j^1(x_u)^2\, du\right)^2\right]\right.$$

$$\left. + c_2^4(t-s)^4\right\}$$

$$\leqq (n+2)^3\{E[(x_t^1 - x_s^1)^4] + 36c_1^4(t-s)^2 + c_2^4(t-s)^4\}.$$

Combining this inequality with (2.5), we obtain the estimate (2.6).

Finally we shall prove the estimate (2.7). It follows by using the formula on stochastic integral that

$$(M_t^i - M_s^i)^4 = 4\int_s^t (M_u^i - M_s^i)^3 dM_u^i + 6\int_s^t (M_u^i - M_s^i)^2 d\varphi_u.$$

$\int_s^t (M_u^i - M_s^i)^3 dM_u^i$ is a martingale with mean 0. Hence, taking the expectation, we have

$$\begin{aligned} E[(M_t^i - M_s^i)^4] &= 6E\left[\int_s^t (M_u^i - M_s^i)^2 d\varphi_u\right] \\ &\leqq 6E[\sup_{s\leqq u\leqq t} |M_u^i - M_s^i|^2(\varphi_t - \varphi_s)] \\ &\leqq 12E[(M_t^i - M_s^i)^4]^{1/2} E[(\varphi_t - \varphi_s)^4]^{1/4}. \end{aligned}$$

Using the formula on stochastic integral and taking the expectation, we see

$$\begin{aligned} E[|M_t^i - M_s^i|^5] &= 10\, E\left[\int_s^t |M_u^i - M_s^i|^3 d\varphi_u\right] \\ &\leqq 40E[(M_t^i - M_s^i)^4]^{3/4} E[(\varphi_t - \varphi_s)^4]^{1/4} \\ &\leqq 40\cdot 12^{3/2} E[(\varphi_t - \varphi_s)^4]^{5/8}. \end{aligned}$$

Hence we obtain (2.7) from (2.6). Q.E.D.

Let

$$A_t = t + \varphi_t, \qquad \gamma(t) = A_t^{-1} = \inf\{s: A_s > t\}.$$

Obviously $\gamma(t)$ is an $\{\mathscr{F}_t\}$-stopping time.

Lemma 2.4. *There exist positive constants K_6 and h_3 depending only on c_1, c_2, c_3 and c_4 such that*

$$E[(x_{\gamma(t)}^i - x_{\gamma(s)}^i)^4] \leqq K_6 |t-s|^2 \tag{2.8}$$

for all $t, s \geqq 0$ such that $|t-s| \leqq h_3$, $i = 2, \cdots, n$.

Proof. The formula on stochastic integral states that, for $t \geqq s \geqq 0$,

$$(x^i_{\gamma(t)} - x^i_{\gamma(s)})^4 = 4\sum_{j=1}^n \int_{\gamma(s)}^{\gamma(t)} (x^i_u - x^i_{\gamma(s)})^3 \sigma^i_j(x_u) dB^j_u$$

$$+ 4\int_{\gamma(s)}^{\gamma(t)} (x^i_u - x^i_{\gamma(s)})^3 b^i(x_u) du$$

$$+ 4\sum_{j=2}^n \int_{\gamma(s)}^{\gamma(t)} (x^i_u - x^i_{\gamma(s)})^3 \tau^i_j(\tilde{x}_u) dM^j_u$$

$$+ 4\int_{\gamma(s)}^{\gamma(t)} (x^i_u - x^i_{\gamma(s)})^3 \beta^i(\tilde{x}_u) d\varphi_u$$

$$+ 6\sum_{j=1}^n \int_{\gamma(s)}^{\gamma(t)} (x^i_u - x^i_{\gamma(s)})^2 \sigma^i_j(x_u)^2 du$$

$$+ 6\sum_{j=2}^n \int_{\gamma(s)}^{\gamma(t)} (x^i_u - x^i_{\gamma(s)})^2 \tau^i_j(\tilde{x}_u)^2 d\varphi_u.$$

By an argument similar to Lemma 2.1, we see

$$E\left[\sum_{j=1}^n \int_{\gamma(s)}^{\gamma(t)} (x^i_u - x^i_{\gamma(s)})^3 \sigma^i_j(x_u) dB^j_u\right]$$

$$= E\left[\sum_{j=2}^n \int_{\gamma(s)}^{\gamma(t)} (x^i_u - x^i_{\gamma(s)})^3 \tau^i_j(\tilde{x}_u) dM^j_u\right] = 0.$$

Taking the expectation, we have

$$E[(x^i_{\gamma(t)} - x^i_{\gamma(s)})^4] \leqq 4c_2 E\left[\int_{\gamma(s)}^{\gamma(t)} |x^i_u - x^i_{\gamma(s)}|^3 du\right]$$

$$+ 4c_4 E\left[\int_{\gamma(s)}^{\gamma(t)} |x^i_u - x^i_{\gamma(s)}|^3 d\varphi_u\right]$$

$$+ 6c_1^2 E\left[\int_{\gamma(s)}^{\gamma(t)} |x^i_u - x^i_{\gamma(s)}|^2 du\right]$$

$$+ 6c_3^2 E\left[\int_{\gamma(s)}^{\gamma(t)} |x^i_u - x^i_{\gamma(s)}|^2 d\varphi_u\right]$$

$$\leqq 4(c_2 + c_4) E\left[\int_{\gamma(s)}^{\gamma(t)} |x^i_u - x^i_{\gamma(s)}|^3 dA_u\right]$$

$$+6(c_1^2+c_3^2)E\Big[\int_{\gamma(s)}^{\gamma(t)}|x_u^i-x_{\gamma(s)}^i|^2 dA_u\Big]$$

$$\leqq 4(c_2+c_4)\int_s^t E[(x_{\gamma(u)}^i-x_{\gamma(s)}^i)^4]^{3/4}du$$

$$+6(c_1^2+c_3^2)\int_s^t E[(x_{\gamma(u)}^i-x_{\gamma(s)}^i)^4]^{1/2}du.$$

This functional inequality provides us with the estimate (2.8).

Q.E.D.

Remark 2.1. Lemma 2.3 and Lemma 2.4 hold for a solution of (1.1) with any initial distribution.

Lemma 2.5. *Let N and T be arbitrary positive numbers. Then we have*

$$\text{(2.9)}\qquad P\{\max_{0\leq t\leq T}|\varphi_t|>N\}\leqq\frac{K_2(x_0^1,\ T)}{N^2},$$

$$\text{(2.10)}\qquad P\{\max_{0\leq t\leq T}|x_t^1|>3N\}\leqq I_{\{|x_0^1|+c_2T>N\}}+\frac{K_2(x_0^1,\ T)+c_1^2T}{N^2},$$

$$\text{(2.11)}\qquad P\{\max_{0\leq t\leq T}|M_t^i|>N\}\leqq\frac{K_2(x_0^1,\ T)^{1/2}}{N^2}\qquad i=2,\cdots,n,$$

$$\text{(2.12)}\qquad P\{\max_{0\leq t\leq T}|x_t^i|>4N\}\leqq I_{\{|x_0^1|+c_2T>N\}}$$

$$+\frac{c_4^2K_2(x_0^1,\ T)+c_1^2T+c_3^2K_2(x_0^1,\ T)^{1/2}}{N^2}$$

$$i=2,\cdots,n.$$

Proof. By (2.2) and Čebyšev's inequality, we obtain the estimate (2.9).

The estimate (2.11) is an immediate consequence of (2.2), (2.3) and the martingale inequality.

Now we shall prove (2.10). We have

$$\max_{0\le t\le T}|x_t^1|\leqq|x_0^1|+\max_{0\le t\le T}\left|\sum_{j=1}^n\int_0^t\sigma_j^1(x_s)\,dB_s^j\right|$$

$$+\max_{0\le t\le T}\left|\int_0^t b^1(x_s)ds\right|+\varphi_T$$

$$\leqq|x_0^1|+\max_{0\le t\le T}\left|\sum_{j=1}^n\int_0^t\sigma_j^1(x_s)dB_s^j\right|+c_2T+\varphi_T.$$

Since $\sum_{j=1}^n\int_0^t\sigma_j^1(x_s)dB_s^j$ is a martingale, an application of the martingale inequality gives

$$P\left\{\max_{0\le t\le T}\left|\sum_{j=1}^n\int_0^t\sigma_j^1(x_s)dB_s^j\right|>N\right\}\leqq\frac{1}{N^2}E\left[\left(\sum_{j=1}^n\int_0^T\sigma_j^1(x_s)dB_s^j\right)^2\right]$$

$$=\frac{1}{N^2}\sum_{j=1}^n E\left[\int_0^T\sigma_j^1(x_s)^2ds\right]\leqq\frac{c_1^2T}{N^2}.$$

Hence (2.10) follows from (2.9).

Finally we shall prove (2.12). In the same way as (2.10), namely, noting that the following inequality, we obtain (2.12),

$$P\left\{\max_{0\le t\le T}\left|\sum_{j=2}^n\int_0^t\tau_j^i(\tilde{x}_s)dM_s^j\right|>N\right\}\leqq\frac{1}{N^2}E[(\sum_{j=2}^n\int_0^T\tau_j^i(\tilde{x}_s)dM_s^j)^2]$$

$$=\frac{1}{N^2}\sum_{j=2}^n E\left[\int_0^T\tau_j^i(\tilde{x}_s)^2d\varphi_s\right]$$

$$\leqq\frac{c_3^2}{N^2}E[\varphi_T].$$

Q.E.D.

Now we shall prove the following theorem by appealing the above lemmas.

Theorem 2.2. *Suppose σ, b, τ and β are all time independent,* *bounded and continuous. Then, for any probability law μ on $\bar{D}$, there exists a solution of* (1.1) *for coefficients σ, b, τ and β with initial distribution μ.*

Proof. Our method is similar to Skorohod [9] (cf. [11]). First we shall prove the existence of a solution starting from $x_0 \in \bar{D}$. We can take an approximate sequence $(\sigma^{(m)}, b^{(m)}, \tau^{(m)}, \beta^{(m)})$ such that

(i) $(\sigma^{(m)}, b^{(m)}, \tau^{(m)}, \beta^{(m)})_{m=1,2,\dots}$ satisfy the conditions of Theorem 2.1,

(ii) $(\sigma^{(m)}, b^{(m)}, \tau^{(m)}, \beta^{(m)})_{m=1,2,\dots}$ converges uniformly on every compact set to (σ, b, τ, β).

Let $\mathfrak{X}^{(m)} = \{x_t^{(m)}, B_t^{(m)}, M_t^{(m)}, \varphi_t^{(m)}\}$ be a solution of (1.1) for coefficients $\sigma^{(m)}$, $b^{(m)}$, $\tau^{(m)}$ and $\beta^{(m)}$ starting from x_0 defined on $(\Omega^{(m)}, \mathscr{F}^{(m)}, P^{(m)}; \{\mathscr{F}_t^{(m)}\})$.
Let

$$A_t^{(m)} = t + \varphi_t^{(m)}, \qquad \gamma^{(m)}(t) = \inf\{s\colon A_s^{(m)} > t\}.$$

Lemma 2.5 implies that

$$\lim_{N\to+\infty} \sup_{m\geqq 1} P^{(m)}\{\max_{0\leqq t\leqq T} |\varphi_t^{(m)}| > N\} = 0 \qquad T \geqq 0,$$

$$\lim_{N\to+\infty} \sup_{m\geqq 1} P^{(m)}\{\max_{0\leqq t\leqq T} |x_t^{(m)}| > N\} = 0 \qquad T \geqq 0,$$

$$\lim_{N\to+\infty} \sup_{m\geqq 1} P^{(m)}\{\max_{0\leqq t\leqq T} |M_t^{(m)}| > N\} = 0 \qquad T \geqq 0.$$

Lemma 2.3 implies that, for any $\varepsilon > 0$,

$$\lim_{h\downarrow 0} \sup_{m\geqq 1} P^{(m)}\{\max_{\substack{|t-s|\leqq h \\ 0\leqq t,s\leqq T}} |x^{(m)1}_t - x^{(m)1}_s| > \varepsilon\} = 0 \qquad T \geqq 0,$$

$$\lim_{h\downarrow 0} \sup_{m\geqq 1} P^{(m)}\{\max_{\substack{|t-s|\leqq h \\ 0\leqq t,s\leqq T}} |\varphi_t^{(m)} - \varphi_s^{(m)}| > \varepsilon\} = 0 \qquad T \geqq 0,$$

$$\lim_{h\downarrow 0} \sup_{m\geqq 1} P^{(m)}\{\max_{\substack{|t-s|\leqq h \\ 0\leqq t,s\leqq T}} |M_t^{(m)} - M_s^{(m)}| > \varepsilon\} = 0 \qquad T \geqq 0.$$

It follows from (2.9) that, for any $\varepsilon'>0$, a constant $T'>0$ exists such that

$$\sup_{m\geqq 1} P^{(m)}\{\gamma^{(m)}(T')<T\}<\frac{\varepsilon'}{2}.$$

From (2.6), an application of Kolmogorov's theorem (cf. [12] pp. 32) assures us the existence of a constant K such that

$$\sup_{m\geqq 1} P^{(m)}\{|A_t^{(m)}-A_s^{(m)}|\leqq K|t-s|^{1/8}$$
$$\text{for all } 0\leqq t,\ s\leqq T' \text{ such that } |t-s|\leqq 1\}\geqq 1-\frac{\varepsilon'}{2}.$$

Since $\gamma^{(m)}$ is the inverse function of $A^{(m)}$, we see

$$\sup_{m\geqq 1} P^{(m)}\left\{|\gamma^{(m)}(t)-\gamma^{(m)}(s)|\geqq\left(\frac{|t-s|}{K}\right)^8\right.$$
$$\left.\text{for all } 0\leqq t,\ s\leqq T' \text{ such that } |t-s|\leqq 1\right\}\geqq 1-\frac{\varepsilon'}{2}.$$

Therefore

$$\sup_{m\geqq 1} P^{(m)}\{\max_{\substack{|t-s|\leqq h\\ 0\leqq t,s\leqq T}}|x^{(m)}{}_t^i-x^{(m)}{}_s^i|>\varepsilon\}$$
$$\leqq\sup_{m\geqq 1} P^{(m)}\{\gamma^{(m)}(T')\geqq T \text{ and } \max_{\substack{|t-s|\leqq h\\ 0\leqq t,s\leqq T}}|x^{(m)}{}_t^i-x^{(m)}{}_s^i|>\varepsilon\}$$
$$+\sup_{m\geqq 1} P^{(m)}\{\gamma^{(m)}(T')<T\}$$
$$\leqq\sup_{m\geqq 1} P^{(m)}\{\gamma^{(m)}(T')\geqq T \text{ and}$$
$$\max_{\substack{|t-s|\leqq Kh^{1/8}\\ 0\leqq t,s\leqq T'}}|x^{(m)}{}_{\gamma^{(m)}(t)}^i-x^{(m)}{}_{\gamma^{(m)}(s)}^i|>\varepsilon\}+\varepsilon'.$$

Hence it follows by Lemma 2.4 that, for any $\varepsilon>0$,

$$\lim_{h\downarrow 0}\sup_{m\geqq 1} P^{(m)}\{\max_{\substack{|t-s|\leqq h\\ 0\leqq t,s\leqq T}}|x^{(m)}{}_t^i-x^{(m)}{}_s^i|>\varepsilon\}=0 \quad i=2,\cdots n, \quad T\geqq 0.$$

Consequently we see that $\{\mathfrak{x}^{(m)}\}$ is conditionally compact in Prohorov topology (cf. [9]). Therefore, for a suitable subsequence $\{m_k\}$, we can construct a family of stochastic processes $\{\hat{\mathfrak{x}}^{(m_k)}\}$ and a stochastic process $\mathfrak{x}$ on Wiener space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$ which satisfy the following properties;

(i) for $k=1, 2,\cdots$, $\hat{\mathfrak{x}}^{(m_k)}$ has the same finite dimensional distribution as $\mathfrak{x}^{(m_k)}$ and is continuous almost surely,

(ii) $\mathfrak{x}$ is continuous almost surely,

(iii) for any $\varepsilon>0$,

$$\lim_{k\to+\infty} \hat{P}\{\max_{0\leq t\leq T} |\hat{\mathfrak{x}}_t^{(m_k)}-\mathfrak{x}_t|>\varepsilon\}=0 \qquad T\geqq 0.$$

For simplicity we write $\{k\}$ instead of $\{m_k\}$.
Let

$$\hat{\mathfrak{x}}^{(k)}=\{\hat{x}_t^{(k)}, \hat{B}_t^{(k)}, \hat{M}_t^{(k)}, \hat{\varphi}_t^{(k)}\} \qquad k=1, 2,\cdots, \quad \mathfrak{x}=\{x_t, B_t, M_t, \varphi_t\},$$

$$\hat{\mathscr{G}}_t^{(k)}=\sigma\{\hat{\mathfrak{x}}_s^{(k)}; s\leqq t\} \qquad k=1, 2,\cdots, \quad \mathscr{G}_t=\sigma\{\mathfrak{x}_s; s\leqq t\},$$

$$\hat{\mathscr{F}}_t^{(k)}=\bigcap_{s>t}\hat{\mathscr{G}}_s^{(k)} \qquad k=1, 2,\cdots, \quad \mathscr{F}_t=\bigcap_{s>t}\mathscr{G}_s,$$

where $\sigma\{\hat{\mathfrak{x}}_s^{(k)}; s\leqq t\}$(resp. $\sigma\{\mathfrak{x}_s; s\leqq t\}$) denotes the smallest σ-field relative to which $\{\hat{\mathfrak{x}}_s^{(k)}; s\leqq t\}$ (resp. $\{\mathfrak{x}_s; s\leqq t\}$) are all measurable. Then, noting that $\hat{\mathfrak{x}}^{(k)}$ has the same finite dimensional distribution as $\mathfrak{x}^{(k)}$, it is simple to check that $\hat{\mathfrak{x}}^{(k)}$ is a solution of (1.1) for coefficients $\sigma^{(k)}$, $b^{(k)}$, $\tau^{(k)}$ and $\beta^{(k)}$ starting from x_0 defined on $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P}: \{\hat{\mathscr{F}}_t^{(k)}\})$. Since $\{(\hat{B}^{(k)})^2\}$ is uniformly integrable, it is easily seen that B_t is an $\{\mathscr{F}_t\}$-martingale such that

$$<B^i, B^j>_t=\delta_{ij}t \qquad i, j=1,\cdots, n.$$

Lemma 2.1 and Lemma 2.2 imply that $\{\hat{\varphi}^{(k)}\}$ and $\{(\hat{M}^{(k)})^2\}$ are uniformly integrable. Therefore we can easily verify that M_t is an $\{\mathscr{F}_t\}$-martingale and

$$< B^i,\ M^j >_t = 0 \qquad i=1,\cdots,n,\ j=2,\cdots,n,$$

$$< M^i,\ M^j >_t = \delta_{ij}\varphi_t \qquad i, j=2,\cdots,n.$$

From a result of Skorohod [9] (cf. [11]), for $i=1,\cdots,n$, $\sum_{j=1}^{n}\int_0^t \sigma^{(k)i}_j(\hat{x}^{(k)}_s)\, d\hat{B}^{(k)j}_s$ (resp. $\int_0^t b^{(k)i}(\hat{x}^{(k)}_s)ds$) converges in probability to $\sum_{j=1}^{n}\int_0^t \sigma^i_j(x_s)dB^j_s$ (resp. $\int_0^t b^i(x_s)ds$).

Now we shall verify that

$$\text{(2.13)} \qquad \sum_{j=2}^{n}\int_0^t \tau^{(k)i}_j(\tilde{x}^{(k)}_s)dM^{(k)j}_s \xrightarrow[k\to+\infty]{} \sum_{j=2}^{n}\int_0^t \tau^i_j(\tilde{x}_s)dM^j_s$$

in probability $\qquad i=2,\cdots,n.$

Let

$$y^{(k)}_t = \int_0^t \tau^{(k)i}_j(\tilde{x}^{(k)}_s)d\hat{M}^{(k)j}_s, \qquad a^k_s = \tau^{(k)i}_j(\tilde{x}^{(k)}_s) \qquad k=1, 2,\cdots,$$

$$y_t = \int_0^t \tau^i_j(\tilde{x}_s)dM^j_s, \qquad a_s = \tau^i_j(\tilde{x}_s).$$

For any positive numbers ε, ε' and any partition of $[0, t]$, $0=t_0\leqq t_1\leqq\cdots\leqq t_l=t$,

$$\hat{P}\{|y^{(k)}_t - y_t| > \varepsilon\} \leqq \hat{P}\left\{|y^{(k)}_t - \sum_{p=0}^{l-1} a^{(k)}_{t_p}(\hat{M}^{(k)j}_{t_{p+1}} - \hat{M}^{(k)j}_{t_p})| > \frac{\varepsilon}{3}\right\}$$

$$+\hat{P}\left\{|y_t - \sum_{p=0}^{l-1} a_{t_p}(M^j_{t_{p+1}} - M^j_{t_p})| > \frac{\varepsilon}{3}\right\}$$

$$+\hat{P}\left\{|\sum_{p=0}^{l-1} a^{(k)}_{t_p}(\hat{M}^{(k)j}_{t_{p+1}} - \hat{M}^{(k)j}_{t_p}) - \sum_{p=0}^{l-1} a_{t_p}(M^j_{t_{p+1}} - M^j_{t_p})| > \frac{\varepsilon}{3}\right\}$$

$$= I_1 + I_2 + I_3.$$

We put $\delta = \max_p |t_{p+1} - t_p|$. An application of the martingale inequality gives

$$I_1 \leqq \frac{9}{\varepsilon^2} \hat{E}\left[\sum_{p=0}^{l-1} \int_{t_p}^{t_{p+1}} |a_s^{(k)} - a_{t_p}^{(k)}|^2 d\hat{\varphi}_s^{(k)}\right]$$

$$\leqq \frac{9}{\varepsilon^2} \hat{E}[\sup_{\substack{|u-s|\leqq\delta \\ 0\leqq u,s\leqq t}} |a_u^{(k)} - a_s^{(k)}|^2 \hat{\varphi}_t^{(k)}]$$

$$\leqq \frac{9}{\varepsilon^2} \hat{E}[\sup_{\substack{|u-s|\leqq\delta \\ 0\leqq u,s\leqq t}} |a_u^{(k)} - a_s^{(k)}|^4]^{1/2} E[(\hat{\varphi}_t^{(k)})^2]^{1/2}.$$

The equicontinuity and uniform boundedness of $\{\tau^{(k)}{}_j^i\}$ assure us the estimate

$$\lim_{\delta\downarrow 0} \sup_{k\geqq 1} \hat{E}[\sup_{\substack{|u-s|\leqq\delta \\ 0\leqq u,s\leqq t}} |a_u^{(k)} - a_s^{(k)}|^4] = 0.$$

From (2.2), $\hat{E}[(\hat{\varphi}_t^{(k)})^2]$ is bounded in k. Hence there exists a constant $\delta_1 > 0$, which is independent of k, such that $I_1 < \frac{\varepsilon'}{3}$ for all $\delta \leqq \delta_1$. By the same argument, a constant $\delta_2 > 0$ exists such that $I_2 < \frac{\varepsilon'}{3}$ for all $\delta \leqq \delta_2$. When we fix a partition satisfying $\delta \leqq \delta_1 \wedge \delta_2$, it is clear that a positive integer k_1 exists such that $I_3 < \frac{\varepsilon'}{3}$ for all $k \geqq k_1$. Hence it follows that

$$\hat{P}\{|y_t^{(k)} - y_t| > \varepsilon\} < \varepsilon' \qquad k \geqq k_1$$

and (2.13) is proved. In the same way as (2.13), we can verify that $\int_0^t \beta^{(k)i}(\tilde{x}_s^{(k)}) d\hat{\varphi}_s^{(k)} \xrightarrow[k\to+\infty]{} \int_0^t \beta^i(\tilde{x}_s) d\varphi_s$ in probability $i = 2, \cdots, n$. Therefore, with probability one,

$$x_t^1 = x_0^1 + \sum_{j=1}^n \int_0^t \sigma_j^1(x_s) dB_s^j + \int_0^t b^1(x_s) ds + \varphi_t,$$

$$x_t^i = x_0^i + \sum_{j=1}^n \int_0^t \sigma_j^i(x_s) dB_s^j + \int_0^t b^i(x_s) ds + \sum_{j=2}^n \int_0^t \tau_j^i(\tilde{x}_s) dM_s^j$$

$$+ \int_0^t \beta^i(\tilde{x}_s) d\varphi_s \qquad i = 2, \cdots, n.$$

Since the convergence of $\{\hat{x}_t^{(k)1}\}$ and $\{\hat{\varphi}_t^{(k)}\}$ is locally uniform in t, we find that

$$\int_0^t I_{\partial D}(x_s)d\varphi_s=\varphi_t.$$

Hence $\mathfrak{x}$ is a solution of (1.1) for coefficients σ, b, τ and β starting from x_0.

The proof in the case of a general μ is similar to above, if we note that we have similar estimates by Remark 2.1 and Lemma 2.5.

Q.E.D.

Now we shall state the theorem in the case of time dependent coefficients corresponding to Theorem 2.2.

Theorem 2.3. *Suppose σ, b, τ and β are all time dependent, bounded and continuous. Then, for any probability law μ on $\bar{D}$ and any $t_0\geqq 0$, there exists a solution of* (1.1) *for coefficients σ, b, τ and β with initial distribution μ at time t_0.*

Proof. By setting

$$\sigma_j^{n+1}=0 \qquad j=1,\cdots,n+1, \qquad \sigma_{n+1}^i=0 \qquad i=1,\cdots,n,$$

$$b^{n+1}=1,\ \tau_j^{n+1}=0 \qquad j=2,\cdots,n+1, \qquad \tau_{n+1}^i=0 \qquad i=2,\cdots,n,$$

$$\beta^{n+1}=0, \qquad x=(x^1,\cdots x^{n+1}),$$

$$\sigma_j^i(x)=\sigma_j^i(x^{n+1},x^1,\cdots,x^n) \qquad i,j=1,\cdots,n+1,$$

$$b^i(x)=b^i(x^{n+1},x^1,\cdots,x^n) \qquad i=1,\cdots,n+1,$$

$$\tau_j^i(\tilde{x})=\tau_j^i(x^{n+1},0,x^2,\cdots,x^n) \qquad i,j=2,\cdots,n+1,$$

$$\beta^i(\tilde{x})=\beta^i(x^{n+1},0,x^2,\cdots,x^n) \qquad i=2,\cdots,n+1,$$

the case of Theorem 2.3 is reduced to the case of Theorem 2.2.

Q.E.D.

§ 3. The existence of solutions in the sticky case.

In this section we shall discuss the existence of solution of (1.2).

Theorem 3.1. *Suppose σ, b, τ and β are all time independent, bounded and continuous and $\mathfrak{x}=\{x_t, B_t, M_t, \varphi_t\}$ is a solution of* (1.1) *for coefficients σ, b, τ and β defined on $(\Omega, \mathscr{F}, P; \{\mathscr{F}_t\})$. Further, suppose $|\sigma^1(x)|$ is positive for all $x \in \bar{D}$. Then*

$$E\left[\int_0^t I_{\partial D}(x_s)ds = 0\right] \qquad t \geqq 0. \tag{3.1}$$

Proof. First we shall prove in case that a constant $c>0$ exists such that $|\sigma^1(x)| \geqq c$ for all $x \in \bar{D}$. From a result of S. Watanabe [13], there exists a continuous orthogonal matrix Q such that

$$\sigma Q^{-1} = \begin{pmatrix} |\sigma^1(x)|, 0, \cdots, 0 \\ * \qquad * \end{pmatrix}.$$

Let

$$\sigma^{(1)} = \sigma Q^{-1}, \quad \sigma^{(2)} = \frac{1}{|\sigma^1(x)|}\sigma^{(1)}, \qquad b^{(2)} = \frac{1}{|\sigma^1(x)|^2} b,$$

$$d = \left(-\frac{b^1}{|\sigma^1(x)|^2}, 0, \cdots, 0,\right), \qquad b^{(3)} = b^{(2)} + \sigma^{(2)} d.$$

Then $\sigma^{(2)} = \begin{pmatrix} 1, 0, \cdots, 0 \\ * \qquad * \end{pmatrix}$ and $b^{(3)1} = 0$.

Let $\mathfrak{x}^{(1)} = \{x_t^{(1)}, B_t^{(1)}, M_t^{(1)}, \varphi_t^{(1)}\}$ be a solution of (1.1) for coefficients $\sigma^{(2)}$, $b^{(3)}$, τ and β defined on a quadruplet $(\Omega, \mathscr{F}, P^{(1)}; \{\mathscr{F}_t^{(1)}\})$, where $\mathscr{F} = \mathscr{F}_\infty^{(1)}$.

Since $x_t^{(1)1}$ is a one-dimensional reflecting Brownian motion, it is well known that

$$E^{(1)}\left[\int_0^t I_{\partial D}(x_s^{(1)})\, ds\right] = 0 \qquad t \geqq 0. \tag{3.2}$$

Let $P^{(2)}$ be a probability measure on $(\Omega, \mathscr{F})$ such that, for $B \in \mathscr{F}_t^{(1)}$,

$$P^{(2)}(B)=\int_B \exp\left[-\sum_{i=1}^n \int_0^t d^i(x_s^{(1)})dB_s^{(1)i} - \frac{1}{2}\sum_{i=1}^n \int_0^t d^i(x_s^{(1)})^2 ds\right] dP^{(1)}$$

and define

$$B_t^{(2)i}=B_t^{(1)i}+\int_0^t d^i(x_s^{(1)})ds \qquad i=1,\dots, n,$$

$$x_t^{(2)}=x_t^{(1)},\ M_t^{(2)}=M_t^{(1)},\ \varphi_t^{(2)}=\varphi_t^{(1)},\ \mathscr{F}_t^{(2)}=\mathscr{F}_t^{(1)}.$$

By the Cameron-Martin's formula (cf. [5]), $\mathfrak{x}^{(2)}=\{x_t^{(2)}, B_t^{(2)}, M_t^{(2)}, \varphi_t^{(2)}\}$ is a solution of (1.1) for coefficients $\sigma^{(2)}$, $b^{(2)}$, τ and β defined on $(\Omega, \mathscr{F}, P^{(2)}; \{\mathscr{F}_t^{(2)}\})$. Then we have

$$E^{(2)}\left[\int_0^t I_{\partial D}(x_s^{(2)})ds\right]=E^{(1)}\left[\left(\int_0^t I_{\partial D}(x_s^{(1)})ds\right)\times\right.$$

$$\left.\exp\left[-\sum_{i=1}^n \int_0^t d^i(x_s^{(1)})\,dB_s^{(1)i} - \frac{1}{2}\sum_{i=1}^n \int_0^t d^i(x_s^{(1)})^2 ds\right]\right]$$

$$\leqq E^{(1)}\left[\left(\int_0^t I_{\partial D}(x_s^{(1)})ds\right)^2\right]^{1/2} E^{(1)}\left[\exp\left[-2\sum_{i=1}^n \int_0^t d^i(x_s^{(1)})dB_s^{(1)i}\right.\right.$$

$$\left.\left.-\sum_{i=1}^n \int_0^t d^i(x_s^{(1)})^2 ds\right]\right]^{1/2}.$$

The boundedness of $d(x)$ assures us that $E^{(1)}\left[\exp\left[-2\sum_{i=1}^n \int_0^t d^i(x_s^{(1)})dB_s^{(1)i}-\sum_{i=1}^n \int_0^t d^i(x_s^{(1)})^2 ds\right]\right]$ is finite for all $t \geqq 0$. Hence (3.2) shows that

$$E^{(2)}\left[\int_0^t I_{\partial D}(x_s^{(2)})ds\right]=0 \qquad t \geqq 0. \tag{3.3}$$

Define

$$A_t=\int_0^t \frac{1}{|\sigma^{(1)}(x_s^{(2)})|^2}\,ds, \qquad A_t^{-1}=\inf\{s;\ A_s>t\},$$

$$P^{(3)}=P^{(2)},\ \mathscr{F}_t^{(3)}=\mathscr{F}_{A_t^{-1}}^{(2)},\ x_t^{(3)}=x_{A_t^{-1}}^{(2)},\ M_t^{(3)}=M_{A_t^{-1}}^{(2)},\ \varphi_t^{(3)}=\varphi_{A_t^{-1}}^{(2)},$$

$$B_t^{(3)}=\int_0^t \frac{1}{|\sigma^1(x_s^{(3)})|}\, dB_{A_s^{-1}}^{(2)}.$$

By the theory of time change (Doob's optional sampling theorem), $\mathfrak{x}^{(3)}=\{x_t^{(3)}, B_t^{(3)}, M_t^{(3)}, \varphi_t^{(3)}\}$ is a solution of (1.1) for coefficients $\sigma^{(1)}$, b, τ and β defined on $(\Omega, \mathscr{F}, P^{(3)}; \{\mathscr{F}_t^{(3)}\})$. Therefore it follows that

$$E^{(3)}\left[\int_0^t I_{\partial D}(x_s^{(3)})\, ds\right]=E^{(2)}\left[\int_0^t I_{\partial D}(x_{A_s^{-1}}^{(2)})\, ds\right]$$

$$=E^{(2)}\left[\int_0^{A_t^{-1}} I_{\partial D}(x_s^{(2)})\, dA_s\right].$$

If $c\leqq|\sigma^1(x)|\leqq c_1$, then we have

$$E^{(2)}\left[\int_0^{A_t^{-1}} I_{\partial D}(x_s^{(2)})\, dA_s\right]\leqq E^{(2)}\left[\int_0^{c_1^2 t} I_{\partial D}(x_s^{(2)})\frac{ds}{c^2}\right].$$

Therefore we see from (3.3) that

$$E^{(3)}\left[\int_0^t I_{\partial D}(x_s^{(3)})\, ds\right]=0 \qquad t\geqq 0. \tag{3.4}$$

Define

$$P^{(4)}=P^{(3)}, \mathscr{F}_t^{(4)}=\mathscr{F}_t^{(3)},\ x_t^{(4)}=x_t^{(3)},\ M_t^{(4)}=M_t^{(3)},\ \varphi_t^{(4)}=\varphi_t^{(3)},$$

$$B_t^{(4)i}=\sum_{j=1}^n \int_0^t Q^{-1}{}_j^i(x_s^{(3)})\, dB_s^{(3)j} \qquad i=1,\cdots, n.$$

As is well known, $\mathfrak{x}^{(4)}=\{x_t^{(4)}, B_t^{(4)}, M_t^{(4)}, \varphi_t^{(4)}\}$ is a solution of (1.1) for coefficients σ, b, τ and β defined on $(\Omega, \mathscr{F}, P^{(4)}; \{\mathscr{F}_t^{(4)}\})$. Hence it follows from (3.4) that

$$E^{(4)}\left[\int_0^t I_{\partial D}(x_s^{(4)})\, ds\right]=0 \qquad t\geqq 0.$$

Since any solution of (1.1) can be constructed in this way, the assertion in this case is proved.

Now we shall prove (3.1) in the general case. There exists a sequence $\{\sigma^{(m)}\}_{m=1,2,\dots}$ such that, for $m=1, 2,\cdots$, $\sigma^{(m)}$ satisfies the conditions in the above case and $\sigma^{(m)}(x)=\sigma(x)$ for $|x|\leqq m$. Set, for $m=1, 2,\cdots$,

$$S_m=\begin{cases} \inf\{s;\ |x_s|\geqq m\} & \\ +\infty & \text{if } \{\quad\}=\phi. \end{cases}$$

Then the above result shows that

$$E\left[\int_0^{t\wedge S_m} I_{\partial D}(x_s)\,ds\right]=0 \qquad m=1, 2,\cdots$$

Noting that we $\lim_{m\to+\infty} S_m=+\infty$, we obtain (3.1). Q.E.D.

Theorem 3.2. *Suppose σ, b, τ and β are all time independent, bounded and continuous and ρ is time independent, bounded and Borel measurable. Further, suppose $|\sigma^1(x)|$ is positive for all $x\in\bar{D}$. Then, for any probability law μ on $\bar{D}$, there exists a solution of* (1.2) *for coefficients σ,b, τ, β and ρ with initial distribution μ.*

Proof. By Theorem 2.2, there exists a solution $\bar{\mathfrak{x}}$ of (1.1) for coefficients σ, b, τ and β and by Theorem 3.1, it satisfies (3.1). In the same way as [14], we can get a solution $\mathfrak{x}$ of (1.2) from $\bar{\mathfrak{x}}$.

Q.E.D.

Now we shall state the theorem in the case of time dependent coefficients corresponding to Theorem 3.2.

Theorem 3.3. *Suppose σ, b, τ, β and ρ are all time dependent, bounded and continuous and $|\sigma^1(t, x)|$ is positive for all $(t, x)\in[0, \infty)\times\bar{D}$. Then, for any probability law μ on $\bar{D}$ and any $t_0\geqq 0$, there exists a solution of* (1.2) *for coefficients σ, b, τ, β and ρ with initial distribution*

μ at time t_0.

Proof. By setting

$$\sigma^i_{n+1}=0 \qquad i=1,\cdots,n, \qquad \sigma^{n+1}_i=0 \qquad i=1,\cdots,n+1,$$

$$b^{n+1}=1,\ \tau^i_{n+1}=0 \qquad i=2,\cdots,n, \qquad \tau^{n+1}_i=0 \qquad i=2,\cdots,n+1,$$

$$\beta^{n+1}=\rho,$$

the case of Theorem 3.3 is reduced to the case of Theorem 3.2.

Q.E.D.

In Theorem 3.3, we cannot remove the continuity of ρ. This situation is different from Theorem 3.2.

Example. Let $n=2$, $x_0=(0, 0)$, $b=(0, 1)$, $\tau=0$, $\beta=0$, $\sigma=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ and $\rho(t, x)=\begin{cases} 1 & \text{for} \quad x^2\geqq t\\ 0 & \text{for} \quad x^2<t.\end{cases}$

Then, by Theorem 2.2 and Theorem 3.1, there exists a solution of (1.1) for coefficients σ, b, τ and β starting from x_0 and it satisfies the property (3.1). But there exists no solution of (1.2) for coefficients σ, b, τ, β and ρ starting from x_0 at time $t_0=0$.

Proof. Suppose that there exists a solution $\mathfrak{x}=\{x_t, B_t, M_t, \varphi_t\}$ of (1.2) starting from x_0 at time $t_0=0$ defined on $(\Omega, \mathscr{F}, P; \{\mathscr{F}_t\})$. Let

$$T=\begin{cases} \inf\{s;\ x_s^2<s\} & \\ +\infty & \text{if } \{\quad\}=\phi.\end{cases}$$

Then T is an $\{\mathscr{F}_t\}$-stopping time and we put $\Omega_1=\{\omega;\ T(\omega)>0\}$. The condition (iv) of Definition 1.2 implies that

$$x_t^2-x_s^2=\int_s^t I_D(x_u)du \leqq t-s \qquad t\geqq s\geqq 0.$$

Since $x_t^2=t$ for $t\leqq T$, we have

$$\int_0^t I_{\partial D}(x_s)ds=0 \qquad t\leqq T.$$

Therefore

$$\int_0^t \rho(s,\ \tilde{x}_s)d\varphi_s=\varphi_t=0 \qquad t\leqq T.$$

By the definition of stochastic integral, this implies that

$$x_t^1=\int_0^t I_D(x_s)dB_s^1=B_t^1 \qquad t\leqq T.$$

Therefore, $B_t^1\geqq 0$ for $t\leqq T$. Hence $P(\Omega_1)=0$. Because of this, we have

$$\begin{aligned} x_t^2&=\int_0^t I_D(x_s)ds=t-\int_0^t I_{\partial D}(x_s)ds=t-\int_0^t \rho(s,\ \tilde{x}_s)d\varphi_s \\ &=t \qquad\qquad t\geqq 0, \end{aligned}$$

and this is a contradiction. Q.E.D.

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