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An approach to hyperbolic mixed problems by singular integral operators

By

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§1. **Introducsion.** In this paper we study the commutators of singular integral operators and then show its direct application to some mixed problems for hyperbolic equation of general order, using reflection method. In treating of hyperbolic equations, properties of commutators of singular integral operators in connection with the operator Λ (where Λ is a square root of the Laplacian) will play the most important role. In Theorem 1 given in §2, we relax the assumption on which A. P. Calderon and A. Zygmund [3] obtained the theorem for commutators. By virtue of Theorem 1, for example, we can show the existence of the solution of Cauchy problem for the first order system of regularly hyperbolic equation, under the assumption that the coefficients are continuous and piecewise smooth in $(0, \infty)\times R^n$, in other words piecewise in $C^{1+\alpha}(\alpha>0)$ relative to some hypersurfaces in $(0, \infty)\times R^n$ (cf. definition in §2). Permitting the coefficients to be piecewise smooth, has a physical meaning. Let us consider, for example, Burgers' equation

$$\text{(1.1)} \qquad \frac{\partial}{\partial t}u+u\frac{\partial u}{\partial x} = 0\,,$$

$$\text{(1.2)} \qquad u(x, 0) = \begin{cases} u^-\,, & \text{for} \quad x<0\,, \\ u^+\,, & \text{for} \quad x>0\,. \end{cases}$$

If $u^-<u^+$, as is well-known, the solution is continuous and piecewise smooth and is called rarefaction wave (cf. Gelfand [14]). A small purturbation δu satisfies the following equation with coefficient u

$$\frac{\partial}{\partial t}(\delta u)+u\frac{\partial}{\partial x}(\delta u)=0,$$

As for mixed problems for hyperbolic equation of higher order, S. Mizohata dealt with

$$(1.3)\qquad L=\prod_{i=1}^{m}\left(\frac{\partial^2}{\partial t^2}-c_i(x)a(x, D)\right)u+B_{2m-1}u=f(x, t),$$

where $a(x, D)=\sum a_{ij}(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}+\text{first order}$

$$\sum a_{ij}(x)\xi_1\xi_2\geq\delta|\xi|^2,\quad \delta>0,\quad x\in\Omega,$$
$$c_{i+1}(x)>c_i(x)>0,\qquad (i=1, \cdots, m-1)$$

$$(1.4)\qquad (1)\ \frac{\partial}{\partial n}a^k u|_s=0,\qquad k=0, 1, 2, \cdots, m-1,$$

where $\quad \frac{\partial}{\partial n}=\sum a_{ij}(x)\cos(\nu, x_i)\frac{\partial}{\partial x_j},\quad \nu$ outer normal, or

$$(2).\ a^k u|_s=0$$

$$u(x, 0)=u_0(x),\ \cdots,\ \left(\frac{\partial}{\partial t}\right)^{m-1}u(x, 0)=u_{m-1}(x),$$

in the domain $(0, \infty)\times\Omega$, Ω being the exterior or interior of a smooth and compact hypersurface in R^n, and showed well-posedness in L^2-sense. As is proved later in §7, $a(x, D)$ is transformed locally (near the boundary) to the following form in $R^n_+=\{x\,;\,x=(x', x_n)=(x_1, \cdots, x_{n-1}, x_n)\}$:

$\frac{\partial}{\partial x_n^2}+\sum_{i,j=1}^{n-1}b_{ij}(x', x_n)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}+\text{first order}$, where $b_{ij}(x', x_n)$ is positive definite. In case of $m=2$, the author [9] extended the equation (1.3) to the form

$$(1.5)\qquad \frac{\partial^4}{\partial t^4}u+(a_1(x, D)+a_2(x, D)+a_3(x, D))\frac{\partial^2}{\partial t^2}u$$
$$+a_1(x, D)a_2(x, D)u+B_3u=f,$$

where the elliptic operators $a_i(x, D)$ $(i=1, 2, 3)$ have the same conormal direction on S. After suitable local transformation (1.5) is written in the following form :

(1.5)′ $$\frac{\partial^4}{\partial t^4}u - b_2(x, D)\frac{\partial^2}{\partial t^2}u + b_4(x, D)u + B_3 = f$$

where $b_2(x, D) = b(x)\frac{\partial^2}{\partial x_n^2} + \sum_{j=1}^{n-1} b_j(x', x_n)\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_n} + \sum_{ij=1}^{n-1} b_{ij}(x', x_n)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}$

$$b_4(x, D) = c(x)\prod_{k=1}^{2}\left(\frac{\partial^2}{\partial x_n^2} + \sum c_j^k(x', x_n)\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_n} + \sum_{ij=1}^{n-1} c_{ij}^k(x', x_n)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\right)$$

$$= \sum_{\alpha=4-k,\ j\neq n} c_{k,\alpha}(x', x_n)\left(\frac{\partial}{\partial x_j}\right)^{\alpha}\left(\frac{\partial}{\partial x_n}\right)^k + \text{lower order.}$$

Then, by the assumption that $a_i(x, D)$ have the same conormal direction on S, the relations

(∗) $\quad b_j(x', 0) = 0$ and $c_{k,\alpha}(x', 0) = 0$ if k is odd

follows. (cf. (2.13), (2.14) in [9]).

Of course this assumption (*) is imposed only on the boundary. Such a type of restriction seems indispensable, if we consider the boundary condition of type (1.4) (1) (cf. [8]). After that, K. Asano-T. Shirota [2], using singular integral operator in $R_+^n = \{x = (x', x_n), x_n > 0\}$ attached to the same boundary condition as (1.4), treated the equation

(1.6) $$\frac{\partial^{2m}}{\partial t^{2m}}u + \sum_{j=1}^{2m} b_j(x, D)\left(\frac{\partial}{\partial t}\right)^j u + B_{2m-1}u = f,$$

where $b_j(x, D) = \sum c_{k,\alpha}\left(\frac{\partial}{\partial x'}\right)^{\alpha}\left(\frac{\partial}{\partial x_n}\right)^k$

$c_{k,\alpha}(x', 0) = 0$, if k is odd.

$b_{2m}(x, D)$ is uniformly elliptic,

Now we consider the following equation in $(0, \infty)\times R_+^n$

(1.7) $$\frac{\partial^m}{\partial t^m}u + \sum_{j=1}^{m} b_j(x, t, D)\left(\frac{\partial}{\partial t}\right)^{m-j} u + b_{m-1}\left(x, t, \frac{\partial}{\partial t}, D\right)u = f(x, t),$$

where $b_j(x, t, D) = \sum_{k+|\alpha|=j} c_{k,\alpha}(x', x_n, t)\left(\frac{\partial}{\partial x'}\right)^{\alpha}\left(\frac{\partial}{\partial x_n}\right)^k$ satisfies

(H) $\quad c_{k,\alpha}(x', 0, t) = 0$, if k is odd.

We may assume that boundary condition is the following simple ones (cf. Lemma 7.2).

$$(B_1) \qquad \left(\frac{\partial}{\partial x_n}\right)^{2k} u\,|_{x_n=0} = 0\,, \qquad k=0, 1, \cdots, \left[\frac{m-1}{2}\right],$$

$$(B_2) \qquad \left(\frac{\partial}{\partial x_n}\right)^{2k+1} u\,|_{x_n=0} = 0\,, \qquad k=0, 1, \cdots, \left[\frac{m-2}{2}\right],$$

The assumption that $b_m(x, t, D)$ is elliptic is not necessary, even if $m=2m'$. The detailed statement of our theorem concerning above mixed problems will be given in § 3. Here let us remark that the reflection method discussed in § 4 is closely connected with so-called Fourier's method for the wave equation. Consider

$$(1.8) \qquad \frac{\partial^2}{\partial t^2}u - \frac{\partial^2}{\partial x^2}u = 0 \qquad \text{in} \quad (0, \pi)$$

$$u(0) = u_0\,, \quad \frac{\partial}{\partial t}u(0) = u_1\,, \quad u(0) = u(\pi) = 0$$

The solution takes the form ;

$$(1.9) \qquad u = \sum_{k=1}^{\infty} a_k(t) \sin(kx)\,,$$

where $a_k(t)$ are determined by considering the initial data. We can regard (1. 9) as the restriction to $(0, \pi)$ of Fourier expansion of

$$\tilde{u}(x, t) = \begin{cases} u(x, t) & 0 \leq x \leq \pi \\ -u(-x, t) & -\pi \leq x \leq 0\,, \end{cases}$$

because of $\int_{-\pi}^{\pi} \tilde{u}(x, t) \cos(kx) dx = 0$, for every k (cf. [2]).

The outline of our argument in this paper is as follows. After extending the coefficients of (1. 7) and u into $(0, \infty) \times R^n$, we reduce the mixed problems to evolution equations in some Hilbert spaces. Then we use Friedrichs'mollifier which is suitable to those Hilbert spaces and apply the inequality given by S. Mizohata for the singular integral operator with positive definite symbol, in order to obtain the energy inequality, (§ 4). The energy inequality plays the essential role in the proof of existence of the solution, (§ 5). By virtue of Holmgren transformation at the boundary (6. 1) and a geometrical lemma, we can show that the solution has a finite speed of propagation, (§ 6). Using property and considering local transformation discussed in § 7, we can construct the solution for some mixed problems in a general domain $\Omega\times(0, \infty)$, (§ 8).

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§ 2. Commutators of singular integral operators.

At first we remember the original definition of singular integral operator in R^k given by A. P. Calderon and A. Zygmund [3]. Our argument is also based on their expansion of the symbol in spherical harmonics. This method is sometimes more powerful than any other definition of pseudo-differential operator, because of the fact that coefficients of the expansion make rapidly convergent numerical series. Let us denote by $x=(x_1, \cdots, x_k)$ a point x in R^k, by x', $x'=\frac{x}{|x|}$. The sphere $|x|=1$ in R^k will be denoted by Σ, the elements of surface area on Σ by $d\sigma$. By C^α, $\alpha\geq 0$, we denote the class of complex valued continuous bounded functions on R^k with bounded continuous derivatives up to order $[\alpha]$ and with derivative of order $[\alpha]$ satisfying a Hörder condition of order $\alpha-[\alpha]$. $h(x, \xi)\in C_\alpha^\infty$ means that $h(x, \xi)$ is in C^∞ with respect to ξ and every derivative with respect to ξ is in C^α. We shall denote $\mathcal{E}_{L^p}^r(R^k)$ simply by L_r^p and its L^p norm $\|\cdot\|_p$.

In a point of view of the theory on partial differential equations it is convenient for us to introduce the singular integral operator using so called symbol. Let $h(x, \xi)$ $x, \xi\in R^k$, be a function in C_β^∞, $\beta\geq 0$, homogeneous of degree zero in ξ, and let

$$(2.1)\qquad h(x, \xi) = a_0(x)+\sum_{n\geq 1} a_{nm}(x)Y_{nm}(\xi')$$

be its expansion in spherical harmonics. Then $a_{nm}(x)$ can be reformed in the following:

$$(2.2)\qquad a_{nm}(x) = (-1)^r n^{-r}(n+k-2)^{-r}\int_\Sigma (|\xi'|^2\Delta_\xi)^r h(x, \xi')Y_{nm}(\xi')d\sigma\,,$$

where $(|\xi|^2\Delta_\xi)^r h(x, \xi)$ is also homogeneous of degree zero in ξ. (1.2) means that the regularity in ξ of $h(x, \xi)$ make the convergence of of (1.1) more rapid than any series $\sum_{n=1}^\infty \frac{1}{n^k}$ $(k=2, 3, \cdots)$.

Corresponding each $Y_{nm}(\xi')$, a bounded operator T_{nm} in L^p is determined as follows. Fourier transformation gives the formula

(2. 3) $$Y_{nm}(\xi') = \beta_n \lim_{\varepsilon\to 0,\ \delta\to\infty} \overline{\mathcal{F}}_{x\to\xi}[Y_{nm}(\varepsilon, \delta, x)], \quad \bar{\beta}_n = (-1)^n\beta_n,$$

$$Y_{nm}(\varepsilon, \delta, x) = \begin{cases} Y_{nm}(x')|x|^{-k}, & \varepsilon \leq |x| \leq \delta \\ 0, & \text{otherwise,} \end{cases}$$

(2. 4) $|\beta_n| \leq c\, n^{1/2(k+1)}$, c depends on k,

$(T_{nm\varepsilon}f)(x)$ defined, for $f\in L^p$, $p>1$, by

(2. 5) $$T_{nm\varepsilon}f = \int_{|x-y|>\varepsilon} Y_{nm}(x-y)f(y)dy, \qquad (n=1, 2, \cdots)$$

converge almost everywhere as $\varepsilon\to 0$, and the estimate

$$\|T_{nm\varepsilon}f\|_p \leq C\|f\|_p, \qquad (n=1, 2, \cdots,)$$

hold for every ε. Therefore $T_{nm\varepsilon}f$ converge in the mean of order p to a limit $T_{nm}f$

(2. 6) $\|T_{nm}f\|_p \leq C\|f\|_p$, C depend only on p and k.

A. P. Calderón and A. Zygmund defined the singular integral operator H with symbol $\sigma(H)=h(x, \xi)$ by

(2. 7) $$Hf = a_0(x)f(x) + \sum_{n\geq 1} a_{nm}(x)\beta_n T_{nm}f \equiv \sum_{n\geq 0} a_{nm}(x)\beta_n T_{nm}f$$

Considering the number of distinct spherical harmonics Y_{nm} for each n and (2. 4), we obtain, by virtue of (2. 2),

(2. 8) $\|Hf\|_p \leq CM\|f\|_p$,

where M is a bound for the absolute value of $h(x, \xi)$ and its derivatives with respect to ξ in $|\xi|\geq 1$ of order $2k$.

Defining H^* and $H_1\circ H_2$ by

$$\sigma(H^*) = \overline{\sigma(H)}, \quad \sigma(H_1\circ H_2) = \sigma(H_1)\sigma(H_2)$$

A. P. Calderon-A. Zygmund proved Theorem 1 described below under the condition that $\sigma(H_1)$, $\sigma(H_2)$ and $\sigma(H)$ are in $C^\infty_{1+\alpha}$, $\alpha>0$. Now we extend the theorem to the case where the symbols are piecewise in $C^\infty_{1+\alpha}$, $\alpha>0$.

Definition 1. *A function $h(x)$ defined in R^k is said to be piecewise* *in $C^{1+\alpha}$ relative to given hypersurfaces S, if $h(x)$ has the properties: (i) $h(x)$ is continuous in R^k (ii) $h(x)$ is in $C^{1+\alpha}(\bar{\omega})$, where ω is any connected component of R^k-S. (iii) The derivatives of $h(x)$ have uniformly Hörder constant for every ω. Denote the class of function by $C_S^{1+\alpha}$.*

Theorem 1. *Assume that $h_i(x, \xi)=\sigma(H_i)(x, \xi)$, $(i=1, 2)$ defined in $R^k\times\{R^k-\{0\}\}$ be a C^∞ function of homogeneous degree zero in ξ and be in $C_S^{1+\alpha}$ with respect to x. Then we have*
(i) $H_1\Lambda-\Lambda H_1$, $H_1^*\Lambda-\Lambda H_1^*$ and $(H_1^*-H_1^\sharp)\Lambda$ are bounded operators in L^p with operator norms bounded by CM_1.
(ii) $\|(H_1H_2-H_1H_2)\|_p$ is bounded by CM_1M_2.
where $\Lambda=\mathcal{F}^{-1}|\xi|\mathcal{F}$. M_i is a bound for the absolute value of $h_i(x, \xi)$, $\dfrac{\partial}{\partial x_j}h_i(x, \xi)$ and their derivative with respect to ξ and their Hörder constant, $(i=1, 2)$. C depends on p, k and hypersufaces S.
Let us begin with the following lemma

Lemma 1.1. *Assume that $c(x)$ is piecewise smooth in $C^{1+\alpha}$ relative to the hyperplane $x_n=0$, $(\alpha>0)$ in R^n. Let T be a singular integral operator with the symbol independent of x and of spherical mean zero. Then for $f(x)$ in $L_1^q(R^n)$ we have*

$$\|(c(x)T-Tc(x))f_{x_i}\|_p\leq C_1C_2\|f\|_p \quad \text{for} \quad i=1, 2, \cdots, n, \tag{2.9}$$

where C_1 denote a bound for $|c(x)|$, $|c_{x_i}(x)|$ and their uniform Hörder constants in R_+^n and in R_-^n. C_2 depends on the kernel of T.

Proof.

$$\int_{|x-y|>\varepsilon}(c(x)-c(y))Y(x-y)f_{y_i}(y)dy \tag{2.10}$$

has a pointwise limite almost everywhere as ε tends to zero (cf. [3]). By virtue of Fatou's lemma it suffices to show that L^p norm of (2.10) can be estimated independent of ε by $C\|f\|_p$. By integration by parts (2.10) is equal to

$$\int_{|x-y|=\varepsilon}(c(x)-c(y))Y(x-y)f(y)\gamma_i dS_y - \int_{|x-y|>\varepsilon}c_{y_i}(y)Y(x-y)f(y)dy \tag{2.11}$$

$$+\int_{|x-y|>1}(c(x)-c(y))Y_{y_i}(x-y)f(y)dy$$
$$+\int_{1>|x-y|>\varepsilon}(c(x)-c(y))Y_{y_i}(x-y)f(y)dy\,.$$

The first and the third terms can be estimated by the same method as one used in the proof of Hausdorff-Young's inequality. In fact, concerning the first term we have the following relation using Hörder's inequality and Fubini's theorem,

$$\int\Big|\int_{|x-y|=\varepsilon}(c(x)-c(y))Y(x-y)f(y)\gamma_i dS_y\Big|^p dx$$
$$=\int\Big\{\int_{|x-y|=\varepsilon}|(c(x)-c(y))Y(x-y)|\,dS_y\Big\}^{p/q}$$
$$\times\Big\{\int|(c(x)-c(y))Y(x-y)|\,|f(y)|^p|\gamma_i|^p dS_y\Big\}dx$$
$$=\Big\{\int_{|x-y|=\varepsilon}\frac{c}{|x-y|^{n-1}}dS_y\Big\}^{p/q}\int\Big\{\int_{x-y=\varepsilon}(c(x)-c(y))Y(x-y)dS_x\Big\}|f(y)|^p dy$$
$$=C\int_{|x-y|=\varepsilon}\frac{1}{|x-y|^{n-1}}dS_y\int|f(y)|^p dy\leq C\|f(y)\|_p^p\,,\qquad\Big(\frac{1}{p}+\frac{1}{q}=1\Big)$$

The estimate of the second order is well known. Now let us decompose $f(y)$ as follows:

$$(2.12)\quad f(y)=f_1(y)+f_2(y)\,,$$
$$f_1(y)=\begin{cases}f(y)\,, & y_n\geq 0\\ 0 & y_n<0\,,\end{cases}\qquad f_2(y)=\begin{cases}0\,, & y_n\geq 0\\ f(y)\,, & y_n<0\,.\end{cases}$$

Then the fourth term of (1. 11) is equal to

$$(2.13)\quad I_1(x)+I_2(x)=\int_{1>|x-y|>\varepsilon}(c(x)-c(y))Y_{y_i}(x-y)f_1(y)dy$$
$$+\int_{1>|x-y|>\varepsilon}(c(x)-c(y))Y_{y_i}(x-y)f_2(y)dy\,.$$

Now assume that $x=(x_1,\cdots,x_{n-1},x_n)$, $x_n>0$. Then in the integrand of the first term, $c(x)-c(y)$ is written in the following form:

$$(2.14)\quad c(x)-c(y)=\sum_{i=1}^{n}(x_i-y_i)c_{x_i}(x)+b(x,y)\,,$$

where $|b(x,y)|\leq c|x-y|^{1+\alpha}$, $\alpha>0$, $y_n>0$.

Remark that $\|f_1\|_p\leq\|f\|_p$ and that the surface integral of $z_jY_{z_i}(z)$

on $|z|=1$ is equal to zero, (cf. [3], p. 915). The L^p norm of $\tilde{I}_1(x)$ defined by $I_1(x)$ in $x_n \geq 0$ and zero in $x_n < 0$, can be estimated independently of ε by $c\|f\|_p$. Concerning $I_2(x)$ we decompose

$$\begin{aligned}(2.15)\quad c(x)-c(y) &= \{c(x)-c(x^0)\}+c(x^0)-c(y)\,,\\ &= \{c(x)-c(x^0)\}+\sum_{j=1}^{n-1}(x_j-y_j)c_{x_j}(x^0)-y_n c_{x_n}(x^0)+b(x^0, y)\,,\end{aligned}$$

where $x^0=(x_1, x_2, \cdots, x_{n-1}, 0)$,

$$|b(x^0, y)| \leq c|x^0-y|^{1+\alpha} < c|x-y|^{1+\alpha}, \qquad y_n < 0\,.$$

Then it suffices to discuss the following two terms

$$J_1(x) = \int_{1>|x-y|>\varepsilon} -y_n Y_{y_i}(x-y)f_2(y)dy$$

$$J_2(x) = \int_{1>|x-y|>\varepsilon} (c(x)-c(x^0))Y_{y_i}(x-y)f_2(y)dy$$

Take the absolute value of the integrand of $J_1(x)$ and we have

$$|J_1(x)| \leq c\int_{1>|x-y|>\varepsilon} \frac{-y_n}{|x-y|^{n+1}}|f_2(y)|dy \leq c\int_{1>|x-y|>\varepsilon} \frac{x_n-y_n}{|x-y|^{n+1}}|f_2(y)|dy$$

Similarly from $|c(x)-c(x^0)| \leq cx_n$ the relation

$$|J_2(x)| \leqslant c\int_{1>|x-y|>\varepsilon} \frac{x_n}{|x-y|^{n+1}}|f_2(y)|dy \leq c\int_{1>|x-y|>\varepsilon} \frac{x_n-y_n}{|x-y|^{n+1}}|f_2(y)|dy$$

Taking account of the surface integral of $\frac{z_n}{|z|^{n+1}}$ on $|z|=1$ being zero, we can obtain the desired estimate for $\tilde{I}_2(x)$ defined by $I_2(x)$ in $x_n > 0$ and zero in $x_n < 0$. For another half space $R^n_- = \{x : x=(x_1, \cdots, x_{n-1}, x_n), x_n < 0\}$, we can follow the same argument.

q.e.d.

Now we can extend Lemma 1 to the case where $c(x)$ is piecewise smooth in $C^{1+\alpha}$ relative to smooth hypersurface S in R^n.

Lemma 2. *Assume that $c(x)$ belongs to $C_s^{1+\alpha}$, where S is a smooth hypersurface satisfying the following conditions.*

1) *For every point x^0 on S they exists a positive number δ, such that $S \cap B_\delta(x^0)$ is a connected component of S. Here $B_\delta(x^0) = \{x ; |x-x^0| < \delta\}$*

2) *$S \cap B(x^0)$ is mapped into the hyperplane $x_n=0$ and x^0 to origin by a suitable local transformation from $B_\delta(x^0)$ to a neighbourhood of origin.*
3) *In each $B(x^0)$, S is represented by $x_k=\varphi(x)$ such that $\varphi(x)$ has uniformly bounded first order derivatives. Then (2.9) holds. c_2 depends on T and S.*

Proof. Similarly to (2.11) we consider

$$\int_{\delta>|x-y|>\varepsilon}(c(x)-c(y))Y_{y_i}(x-y)f(y)dy\,. \tag{2.16}$$

By virtue of a parallel transition and a rotation of the coordinate we can assume that $x=(0, 0, \cdots, 0, a)$ $(a>0)$ and $x_n=0$ is the tangent hyperplane at $0=(0, \cdots, 0)$ of $S: x_n=\varphi(x)$. Now consider the following transformation.

$$\left\{\begin{aligned} x_j &= x'_j+\left\{\left(\frac{\partial\varphi}{\partial x_j}(x'_1, \cdots, x'_{-1})/m\right)\right\}x'_x \qquad j=1, 2, \cdots, n-1, \\ x_n &= \varphi(x'_1, \cdots, x'_{n-1})+\left\{\frac{\partial\varphi}{\partial x_n}(x'_1, \cdots, x'_{n-1})/m\right\}x'_n\,, \end{aligned}\right. \tag{2.17}$$

$$m=\left\{\sum_{j=1}^{n}\left(\frac{\partial\varphi}{\partial x_j}(x'_1, \cdots, x'_{n-1})\right)^2\right\}^{1/2}.$$

Here φ satisfies

$$\left\{\begin{aligned} &\varphi(0)=0\,, \qquad \varphi(a)>0 \\ &\frac{\partial\varphi}{\partial x_j}(0)=0\,, \quad j=1, \cdots, n-1, \quad \frac{\partial\varphi}{\partial x_n}(0)\neq 0. \end{aligned}\right. \tag{2.18}$$

Therefore in the neighbourhood of origin we have

$$|x_k-x'_k|\le c|x|^2 \tag{2.19}$$

Let us decompose $f(y)$ in (2.16)

$$f(y)=f_1(y)+f_2(y)\,, \quad f_1(y)=\begin{cases} f(y)\,, & \text{where } \varphi(y)\geqslant 0 \\ 0 & \text{elsewhere}\end{cases} \tag{2.20}$$

$$f_2(y)=\begin{cases} 0\,, & \text{where } \varphi(y)\geq 0 \\ f(y) & \text{elsewhere.}\end{cases}$$

$$I_2(x)=\int_{\delta>|x-y|>\varepsilon}(c(x)-c(y))Y_{y_i}(x-y)f_2(y)dy \tag{2.21}$$

is transformed by (2. 17) to

$$(2.22)\quad I_2(x)=\int_{\delta>|x-y|>\varepsilon}(\tilde{c}(x'))-\tilde{c}(y'))\tilde{Y}_{y_i}(x'-y')f_2(y')Jdy',$$

where Jacobian J is close to 1, and $x'=(0, 0, \cdots, 0, a)=x$. $Y_{v_i}(x'-y')$ satisfies from (2. 19)

$$(2.23)\quad |\tilde{Y}_{y_i}(x'-y')-Y_{y_i}(x-y)|\leq c|x-y|^{-n}.$$

Corresponding to (2. 15), we have

$$(2.24)\quad \begin{aligned}\tilde{c}(x')-\tilde{c}(y') &= \{\tilde{c}(x')-\tilde{c}(0)\}+\tilde{c}(0)-\tilde{c}(y')\\ &= \{\tilde{c}(x')-\tilde{c}(0)\}+\sum_{j=1}^{n-1}(x'_j-y'_j)c_{x_j}(0)-y'_n c_{xn}(0)+b(0, y')\end{aligned}$$

where $|b(0, y')|\leq c|y'|^{1+\alpha}\leq c|x'-y'|^{1+\alpha}$

$$|c(x')-c(0)|\leq c\,a$$

Substituting them into (2. 22), and considering (2. 19) we have

$$(2.25)\quad \begin{aligned}&\int_{\delta>|x-y|>\varepsilon}(x-y)\tilde{Y}_{y_i}(x'-y')f_2(y')Jdy'\\ &=\int_{\delta>|x-y|>\varepsilon}(x_j-y_j)Y_{y_i}(x-y)f_2(y)dy+c\int\frac{b(x, y)}{|x-y|^{n-1}}f_2(y)dy\end{aligned}$$

where $b(x, y)$ is a bounded function. The second term of (1. 25) can be easily estimated and the first term is a well-known one. Corresponding to $J_1(x)$ in Lemma 1 we have

$$(2.26)\quad \begin{aligned}|\tilde{J}_1(x)| &= c\left|\int_{\delta>|x-y|>\varepsilon}-y'_n Y_{y_i}(x'-y')f_2(y')Jdy'\right|\\ &\leq c\left|\int_{\delta>|x-y|>\varepsilon}-y_n Y_{y_i}(x-y)f_2(y)dy\right|\\ &\quad+\left|\int b(x, y)|x-y|^{-n+1}|f_2(y)|dy\right|,\end{aligned}$$

Here the second is an easy term and the first term is the same one as in Lemma 1. For $\tilde{J}_2(x)$ the argument is the same as $\tilde{J}_1(x)$ just like in the case of Lemma 1. q.e.d.

Remark. Even if the surface S is replaced by many hypersurfaces or piecewise smooth hypersurfaces. Lemma 2 is also true. In fact, we may decompose $f(y)$, corresponding to (2. 20) or (2. 12), into many factors.

$$f(y) = \sum_{j=1}^{p} f_j(y)$$

Now we proceed to the proof of Theorem 1. For $f(x)$ in $L_1^p(R^k)$, using $\Lambda = \sum_{l=1}^{k} R_l \frac{\partial}{\partial x_l}$ we can write

$$\begin{aligned}(\Lambda H - H\Lambda)f &= \sum_{l=1}^{k} R_l(\sum a_{nm}(x)\beta_n T_{nm} f)_{x_l} - \sum_n a_{nm}(x)\beta_n T_{nm}(\sum_l^k R_j f_{x_l}) \\ &= \sum_{l,n} \beta_n R_l(a_{nm}(x))_x T_{nm} f + \sum_{n,l} \beta_n(R_l a_{nm} - a_{nm} R_l)(T_{nm} f)_{x_i}.\end{aligned}$$

Here we have used that relation

$$T_{nm} R_l f_{x_l} = R_l T_{nm} f_{x_l} = R_l (T_{nm} f)_{x_l}.$$

$(a_{nm}(x))$, $(a_{nm}(x))_x$ and their uniform local Hörder constants c_{nm} make absolutely convergent series with sufficient rapidity, by virtue of the formula (2. 2). Now we apply (2. 9) to the second term and use (2. 6), then we can see that $H\Lambda - \Lambda H$ is a bounded operator. By the definition of H we have

$$(H^* - H^\sharp)\Lambda f = \sum_{l,n} (-1)^n \beta_n (T_{nm} \bar{a}_{nm} - \bar{a}_{nm} T_{nm})(R_l f)_x.$$

Again Lemma 2 shows that $(H^* - H^\sharp)\Lambda$ is a bounded operator. Let $H_1 = \sum b_{nm}(x)\beta_n T_{nm}$, $H_2 = \sum c_{nm}(x)\beta_n T_{nm}$, Then from (2. 2) and (2. 6), $\sum b_{nm} c_{\nu\mu} \beta_n \beta_\nu T_{nm} T_{\nu\mu}$ is absolutely convergent in operator norm in L^p. Therefore we can see

$$H_1 \circ H_2 = \sum b_{nm} c_{\nu\mu} \beta_n \beta_\nu T_{nm} T_{\nu\mu},$$

$$\begin{aligned}(H_1 H_2 - H_1 H_2)\Lambda f &= \sum \beta_n \beta_\nu b_{nm} c_{\nu\mu} T_{nm} T_{\nu\mu} - \sum \beta_n \beta_\nu b_{nm} T_{nm} c_{\nu\mu} T_{\nu\mu}(R_l f)_{x_l} \\ &= \sum \beta_n \beta_\nu b_{nm}(c_{\nu\mu} T_{nm} - T_{nm} c_{\nu\mu})(T_{\nu\mu} R_l f)_{x_l}.\end{aligned}$$

Lemma 2 and (2. 2) complete thd proof of Theorem 1.

Remark 2. If the symbol $h(x, \xi)$ is C^∞ with respect to ξ and bounded measurable in x up to all derivatives. Then the singular integral operator defined by (2. 7) is a bounded operator in L^p and satisfies (2. 8).

Remark 3. If $h(x, \xi)$ involves a parametert t and is continuous in t in the following sense.

$$(2.27)\quad \left| \sum_{|\alpha| \leq 2k} \left(\frac{\partial}{\partial \xi}\right)^\alpha \{h(x, \xi, t) - h(x, \xi, t')\} \right| \leq c|t - t'|$$

Then we have

(2.28) $\|H(t)-H(t')\|_p \leq CM|t-t'|$.

§ 3. Statement of some mixed problems

In this section we give the detailed statements of the mixed problems introduced in § 1. Consider the following regularly hyperbolic equation of general order in $(0, \infty)\times R^n_+$, $R^n_+=\{x\,;x=(x', x_n)=(x_1, \cdots, x_{n-1}, x_n), x_n>0\}$:

$$(E)\qquad L_p u+Bu = \frac{\partial^m}{\partial t^m}u+\sum_{j=1}^{m} b_j(x, t, D)\frac{\partial^{m-j}}{\partial t^{m-j}}u+B\Big(x, t, \frac{\partial}{\partial t}, D\Big)u = f(x, t)$$

$\inf\limits_{x\in R^n_+, |\xi'|=1, j\neq k} |\lambda_j(x, t, \xi')-\lambda_k(x, t, \xi')| \geq \delta>0$, where $\lambda_j(x, t, \xi)$ are characteristic roots of (E). $B\Big(x, t, \frac{\partial}{\partial t}, D\Big)$ is a lower order operator. And we impose the following assumption on every b_j

$$b_j(x, t, D) = \sum_{k+|\alpha|=j} a_{k,\alpha}(x, t)\Big(\frac{\partial}{\partial x_n}\Big)^k\Big(\frac{\partial}{\partial x'}\Big)^\alpha$$

(H) $\quad$ *$a_{k,\alpha}(x', x_n, t)$ vanish on $x_n=0$, if k is odd.*

All the coefficients are in piecewise $c^{1+\alpha}$ relative to some smooth hyperplanes in R^{n+1} ($x-t$ space). Let us takes account of the boundary conditions:

$$(B_1)\qquad \Big(\frac{\partial}{\partial x_n}\Big)^{2k}u\,|_{x_n=0} = 0, \qquad k=0, 1, \cdots, \Big[\frac{m-1}{2}\Big].$$

$$(B_2)\qquad \Big(\frac{\partial}{\partial x_n}\Big)^{2k+1}u\,|_{x_n=0} = 0, \qquad k=0, 1, \cdots, \Big[\frac{m-2}{2}\Big].$$

Corresponding to (B_1) or (B_2), we assume that the second member $f(x, t)$ is in $\mathcal{E}^1_t(\mathcal{D}^1_{L^2}(R^n_+))$[1] or in $\mathcal{E}^1_t(L^2_1(R^n_+))$.
Now we state our theorem in a half space.

Theorem 2. *For any initial data* $\Big(\frac{\partial}{\partial t}\Big)^j u(0)\in L^2_{m-j}(R)$ $(j=0, 1, \cdots,$

1) $f(t)\in\mathcal{E}^p_t(H)$ means that $f(t)$ is p times continuously differentiable in t with values in H, $(p=0, 1, 2, \cdots)$.

$m-1$) satisfying the boundary conditions (B_i) and for the second member $f(x, t)$ as above, there exists a unique solution of (E) under the assumption (H) satisfying (B_i) ($i=1, 2$). The energy inequality

$$(3.1)\quad \sum_{j=0}^{m}\left\|\left(\frac{\partial}{\partial t}\right)^j u(t)\right\|_{m-j} \leq c\left\{\sum_{j=0}^{m-1}\left\|\left(\frac{\partial}{\partial t}\right)^j u(0)\right\|_{m-j} + \int_0^t \|f(s)\|_1 ds\right\}$$

hold. $\|\cdot\|_{m-j}$ *means* $\|\cdot\|_{L^2_{m-j}}$. *Moreover the solution has a finite speed of propagation just like in the case of Cauchy problem.*

Now let us state the problems in a general domain. Let Ω be the interior or exterior of a smooth and compact hypersurface S in R^n. In $\Omega\times(0, \infty)$ we consider a regularly hyperbolic equation $(E)_g$ of type (E). Corresponding to (H), we impose on $(E)_g$ the assumption $(H)_g$ below. Our boundary conditions on $S\times(0, \infty)$ are denoted simply by

$$(B_1)_g \qquad n(x, t, D)^{2k}|_s = 0, \qquad k=0, 1, \cdots, \left[\frac{m-1}{2}\right]$$

$$(B_2)_g \qquad n(x, t, D)^{2k+1}u|_s = 0, \qquad k=0, 1, \cdots, \left[\frac{m-2}{2}\right],$$

where $n(x, t, D)=\sum_{j=1}^{n} m_j(x, t)\frac{\partial}{\partial x_j}$. *Smooth vector* $m(x, t)=(m_1(x, t), m_2(x, t), \cdots, m_n(x, t))$ *is transversal on* S *for every* t, *i.e.* $\sum_{j=1}^{n} m_j(x, t)\cos(\nu, x_j)\neq 0$ *on* S. (ν : *outer normal of* S).

Let us note that we can always rewrite an arbitrary j-th order operator as follows (By virtue of Lemma 7.3 in §7).

$$(2.2)\qquad b_j(x, t, D) = \sum_{k=0}^{j} c_{j-k}(x, t, D)n(x, t, D)^k,$$

where $(j-k)$-th order operator $c_{j-k}(x, t, D)$ is a finite sum of the product of $j-k$ first order tangential operators, (later in §7 we give a definition of tangential operator along S). Now we assume that

$(H)_g$ *All the coefficients of the principal part of $c_{j-k}(x, t, D)$ are vanished on s, if k is odd.*

Theorem 3. *For any initial data* $\left(\frac{\partial}{\partial t}\right)^j u(0)\in L^2_{m-j}(\Omega)$ $(j=0, 1, \cdots,$

$m-1$) satisfying the boundary condition $(B_1)_g$ or $(B_2)_g$ and for any second member $f(x, t)$ being in $\mathcal{E}_t^0(\mathcal{D}_{L^2}^1(\Omega))$ or in $\mathcal{E}_t^0(L_1^2(\Omega))$, there exists a unique solution of the equation $(E)_g$ satisfying (B_1) or (B_2) under the assumption $(H)_g$. The energy inequality (3.1) *holds. The solution also has a finite speed of propagation.* ($i=1, 2$).

§ 4. Reflection principle and energy inequality

In this section we show how to apply the reflection principle to the equation (E), using Theorem 1. At first we reduce the mixed problems described in the previous section to the evolution equation in certain Hilbert spaces. Then in those spaces Friedrichs' mollfier will be used in order to show the energy inequality. S. Mizohata's method in treatment of Cauchy problem is useful also in these cases. Especially we can use the inequality for the singular integral operator with positive definite symbol, an extended form of Görding's inequality.

1. *Reduction to the system*

Consider the principal part of (E)

$$(E_1) \qquad L_p u = \frac{\partial^m}{\partial t^m} u + \sum_{j=1}^m b_j(x, t, D) \frac{\partial^{m-j}}{\partial t^{m-j}} u = f(x, t)$$

Assume that $\frac{\partial^j}{\partial t^j} u(t) \in \mathcal{E}_t^0(L_{m-j}^2(R_+^n))$ ($j=0, 1, \cdots, m$) and satisfy the boundary condition (B_1) or (B_2). Let us extend the coefficients of (E_1) and u by the following rule:

$$(R_1) \qquad a_{k,\alpha}(x', -x_n, t) = -a_{k,\alpha}(x', x_n, t), \qquad \text{if } k \text{ is odd.}$$

$$(R_2) \qquad a_{k,\alpha}(x', -x_n, t) = a_{k,\alpha}(x', x_n, t), \qquad \text{if } k \text{ is even.}$$

$$(R_1) \qquad u(x', -x_k, t) = -u(x', x_n, t) \qquad \text{in case } (B_1)$$

$$(R_2) \qquad u(x', -x_n, t) = u(x', x_n, t) \qquad \text{in case } (B_2)$$

Denote by $\tilde{u}$ and $\tilde{\tilde{u}}$, the extentions of u corresponding to (B_1) and (B_2) respectively.

The extended coefficients are piecewise in $C^{1+\alpha}$ relative to hyperplane $x_n=0$ and other hypersurfaces in R^{n+1} by virtue of (H).

Lemma 4.1 *If u belongs to $L^2_m(R^n_+)$ and satisfies (B_1) or (B_2), then the extention $\tilde{u}$ or $\tilde{\tilde{u}}$ respectively belongs to $L^2_m(R^n)$.*

Proof. For $\varphi \in \mathcal{D}(R^n)$

$$(4.1)\quad \left\langle \frac{\partial}{\partial x_n}\tilde{u}, \varphi \right\rangle = -\left\langle \tilde{u}, \frac{\partial}{\partial x_n}\varphi \right\rangle = -\int_{R^n_+} \tilde{u}\frac{\partial\varphi}{\partial x_n}dx - \int_{R^n_-} \tilde{u}\frac{\partial\varphi}{\partial x_n}dx$$

from the definition of the derivative of distribution. Assume that u in $L^2_m(R^n_+)$ satisfies (B_1), then we can see that

$$(4.2)\quad \begin{cases} -\displaystyle\int_{R^n_+} \tilde{u}\frac{\partial\varphi}{\partial x_n} = \int_{R^n_+}\frac{\partial u}{\partial x_n}\varphi dx \\ -\displaystyle\int_{R^n_-} \tilde{u}\frac{\partial\varphi}{\partial x_n}dx = \int_{R^n_-}\frac{\partial\tilde{u}}{\partial x_n}\varphi dx = -\int_{R^n_-}\widetilde{\frac{\partial u}{\partial x}}\varphi dx \end{cases}$$

hold by the limit process. Therefore from (4.1) and (4.2) we have

$$(4.3)\quad \left\langle \frac{\partial}{\partial x_n}\tilde{u}, \varphi \right\rangle = \int_{R^n_+}\frac{\partial u}{\partial x_n}\varphi dx - \int_{R^n_-}\widetilde{\frac{\partial u}{\partial x_n}}\varphi dx$$

(4.3) means not only that $\dfrac{\partial}{\partial x_n}\tilde{u}$ belongs to $L^2(R^n)$ but also $\dfrac{\partial\tilde{u}}{\partial x_n}(x', -x_n) = \dfrac{\partial u}{\partial x_n}(x', x_n)$, $x_n>0$. Similarly for u in $L^2_m(R^n_+)$ satisfying (B_2),

$$(4.4)\quad \left\langle \frac{\partial}{\partial x_n}\tilde{\tilde{u}}, \varphi \right\rangle = \int_{R^n_+}\frac{\partial u}{\partial x_n}\varphi dx - \int_{R^n_-}\widetilde{\widetilde{\frac{\partial u}{\partial x_n}}}\varphi dx$$

holds, because two boundary integrals cancell each other. Therefore $\dfrac{\partial}{\partial x_n}\tilde{\tilde{u}}$ belongs to $L^2(R^n)$ and

$$\frac{\partial}{\partial x_n}\tilde{\tilde{u}}(x', -x_n) = -\frac{\partial}{\partial x_n}u(x', x_n), \qquad x_n>0$$

Putting $\dfrac{\partial u}{\partial x_n}=v$, we can repeat the above argument for v. Then $\dfrac{\partial}{\partial x_n}\tilde{v}$ and $\dfrac{\partial}{\partial x_n}\tilde{\tilde{v}}$ belong to $L^2(R^n)$, corresponding that u satisfies (B_2) and (B_1) respectively. Step by step we can show Lemma 3.1. q.e.d.

By virtue of Lemma 4.1, the extention $\tilde{u}$ of the solution u

of (E_1) satisfying the boundary condition (B_1) is the solution of the equation;

$$\frac{\partial^m}{\partial t^m}\tilde{u}+\sum \tilde{b}_j(x, t, D)\left(\frac{\partial}{\partial t}\right)^{m-j}\tilde{u} = \tilde{f}(x, t),$$

where $\tilde{f}(x, t)$ is the extended one by the rule (R_1). Hereafter for simplicity we denote $\tilde{u}$ and $\tilde{\tilde{u}}$ by u, and the extended coefficients, by $a_{k\alpha}$, so we consider (E_1) as the equation in $R^n\times(0, \infty)$.

Now let us introduce the following closed subspaces of $L^2(R^n)$:

$$L^2[B_1] = \{u\,;\ u\in L^2(R^n),\ u(x', -x_n) = -u(x', x_n)\}$$
$$L^2[B_2] = \{u\,;\ u\in L^2(R^n),\ u(x', -x_n) = u(x', x_n)\}\ .$$

Immediately we have the following Lemma that is the converse of Lemma 1.

Lemma 4. 2. *Assume that u belongs to $L^2_m\cap L^2[B_i]$ $(i=1, 2)$. Then we have*

$$\left(\frac{\partial}{\partial x_n}\right)^{2k}u\,|_{x_n=0} = 0\,, \qquad k=0, 1, 2, \cdots, \left[\frac{m-1}{2}\right] \text{ in Case } i=1,$$
$$\left(\frac{\partial}{\partial x_n}\right)^{2k+1}u\,|_{x_n=0} = 0\,, \qquad k=0, 1, \cdots, \left[\frac{m-2}{2}\right] \text{ in Case } i=2,$$

where $u|_{x_n=0}=0$ means that the trace of u to hyperplane $x_n=0$ is equal to zero.

Now we remember Friedrichs' mollifier $(\rho_\varepsilon *)$ given by the smooth function $\rho_\varepsilon(x)=\left(\frac{1}{\varepsilon}\right)^n\rho\left(\frac{x}{\varepsilon}\right)$, where

$$\text{(4. 5)}\qquad \rho(x) = \begin{cases} c\exp\left(-\dfrac{1}{1-|x|^2}\right) & (|x|<1) \\ 0 & (|x|\geq 1) \end{cases}$$

$$\int \rho(x)dx = 1$$

The following lemma plays the important role for us to show the energy inequality.

Lemma 4. 3. *For u in $L^2[B_i]$, $u_\varepsilon(x)$ defined by*

$$\text{(4. 6)}\qquad u_\varepsilon(x) = \rho_\varepsilon * u = \int \rho_\varepsilon(x-y)f(y)dy$$

belongs to $L^2_\infty \cap L^2[B_i] (\subset C^\infty \cap L^2[B_i])$.

Proof. It suffices to say that $u_\varepsilon(x)$ is in $L^2[B_i]$, $(i=1, 2)$. For $u \in L^2[B_1]$, we have from (4.5)

$$\begin{aligned} u_\varepsilon(x', -x_n) &= \iint \rho_\varepsilon(x'-y', -x_n-y_n)u(y', y_n)dy'dy_n \\ &= \iint \rho_\varepsilon(x'-y', -x_n+y_n)u(y', -y_n)dy'dy_n \\ &= -\iint \rho_\varepsilon(x'-y', x_n-y_n)u(y', y_n)dy'dy_n = -u_\varepsilon(x', x_n) \end{aligned}$$

In the same way $u_\varepsilon(x', -x_n) = u_\varepsilon(x', x_n)$ holds for u in $L^2[B_2]$. q.e.d.

Lemma 4.4. *Fourier image of* $L^2[B_i]$ *is also* $L^2[B_i]$ $(i=1, 2)$.

Proof. By virtue of Lemma 3.3 we can see that $\mathcal{D}(R^n) \cap L^2[B_i]$ is dense set in $L^2[B_i]$. Let us prove Lemma 4.4 for $\varphi(x)$ in $\mathcal{D}(R^n) \cap L^2[B_1]$

$$\begin{aligned} \mathcal{F}[\varphi](\xi', -\xi_n) &= \iint e^{-2\pi i x_n(-\xi_n)} e^{-2\pi i x'\xi'} \varphi(x', x_n)dx'dx_n \\ &= \iint e^{-2\pi i(-x_n)\xi_n} e^{-2\pi i x'\xi'} - \varphi(x', -x_n)dx'dx_n \\ &= -\iint e^{-2\pi i x_n \xi_n} e^{-2\pi i x'\xi'} \varphi(x', x_n)dx'dx_n = -\mathcal{F}[\varphi](\xi', \xi_n) \end{aligned}$$

Similarly for $\varphi(x)$ in $\mathcal{D}(R^n) \cap L^2[B_2]$, we have $\mathcal{F}[\varphi](\xi', -\xi_n) = \mathcal{F}[\varphi](\xi', \xi_n)$. q.e.d.

By Lemma 4.4 $(\Lambda+1)$ is a bounded operator from the space $L^2_k \cap L^2[B_i]$ equipped with the canonical norm $\|\cdot\|_{k \cdot L^2}$ one to one $L^2_{k-1} \cap L^2[B_i]$, $(i=1, 2)$.

Reduce (E_1) to the system by putting

$$(4.7) \qquad v_j = \{i(\Lambda+1)\}^{m-j-1} \left(\frac{\partial}{\partial t}\right)^j u, \qquad (j=0, 1, \cdots, m-1),$$

where v_j belong to $L^2_1 \cap L^2[B_i]$. *In what follows, denote* $L^2_1 \cap L^2[B_i]$ *by* $\mathcal{H}_i$ $(i=1, 2)$. From (4.7) we get

$$(4.8) \qquad \frac{\partial}{\partial t} v_j = i(\Lambda+1)v_{j+1}, \qquad (j=0, 1, \cdots, m-2).$$

Then (E_1) becomes

(4.9) $$\frac{\partial}{\partial t}v_{m-1} = i\sum_{j=0}^{m-1} H_{m-j}(t)\Lambda v_j + \sum B_j(t)v_j + f,$$

where $B_j(t)$ are bounded operators in $L^2(R^n)$ or $L^2_1(R^n)$. Let now $\mathcal{S}$ be the class of singular integral operators mapping $L^2[B_i]$ into $L^2[B_i]$ with symbols being piecewise in $C^\infty_{1+\alpha}$ relative to some given smooth hypersurfaces. By (R_1) and (R_2) and Lemma 4.4, H_j $(j=1, \cdots, m)$ in (4.9) belong to $\mathcal{S}$. Moreover $H_j \circ H_k$ also belong to $\mathcal{S}$. Put

(4.10) $$U = {}^t(v_0, v_1, \cdots, v_{m-1}), \qquad F=(0, \cdots, 0, f)$$

Then (4.8) and (4.9) is written as

(E_2) $$\frac{d}{dt}U(t) = iH(t)\Lambda U(t) + B(t)U + F(t) = A(t)U + F$$

$B(t)$ is a bounded operator in $\mathcal{H}_i$ $(i=0$ or $2)$.

$$\sigma(H(t)) = \begin{pmatrix} 0 & 1 & & & \cdot \\ & 0 & 1 & 0 & \\ 0 & & & \ddots & \\ & & & 0 & 1 \\ -b_m, & \cdots & \cdots & & -b_1 \end{pmatrix} \quad b_k = b_k(x, t, 2\pi\xi'), \quad \xi' = \frac{\xi}{|\xi|}.$$

$H(t)$ maps $\overset{m}{\Pi} L^2[B_i]$ into $\overset{m}{\Pi} L^2[B_i]$. Let us simply denote $\overset{m}{\Pi} L^2[B_i]$ by $L^2[B_i]$ and $\overset{m}{\Pi}\mathcal{H}_i$ by $\mathcal{H}_i$ $(i=1, 2)$. We can recognize (E_2) as an evolution equation in $\mathcal{H}_i$. Conversely if the solution $U(t) =(v_0(t), \cdots, v_{m-1}(t))$ of (E_2) belongs to $\mathcal{E}^1_t(L^2[B_i]) \cap \mathcal{E}^0_t(\mathcal{H}_i)$, then $u=\{i(\Lambda+1)\}^{-m+1}v_0$ belongs to $\mathcal{E}^1_t(L^2_{m-1} \cap L^2[B_i]) \cap \mathcal{E}^0_t(L^2_m \cap L^2[B_i])$ and satisfies

$$\frac{\partial^j u}{\partial t^j} = \{i(\Lambda+1)\}^{-(m-j-1)}v_j$$

and (E_1). $\frac{\partial^j}{\partial t^j}u$ is in $\mathcal{E}^1_t(L^2_{m-j-1} \cap L^2[B_i]) \cap \mathcal{E}^0_t(L^2_{m-j} \cap L^2[B_i])$, $(j =0, 1, \cdots, m-1)$.

2. *Energy inequality*

In order to show the energy inequality, we use a singular integral operator $N(t)$ whose symbol is a diagonalizer of $\sigma(H(t))$:

$\sigma(N(t))\sigma(H(t))=\sigma(\mathcal{D}(t))\sigma(N(t))$, where $\sigma(\mathcal{D}(t))=2\pi\begin{pmatrix}\lambda_1(x,t,\xi') & & 0\\ & \ddots & \\ 0 & & \lambda_m(x,t,\xi')\end{pmatrix}$

$\sigma(N(t))$ satisfies

1) $|\det\sigma(N(t))|\geqslant\delta>0$ $(x,t)\in R^n\times(0,\infty)$, $\xi\in R^n$.

2) $N\in\mathcal{S}$

3) $\sigma(N)(x,t,\xi)$ is continuous in t in the sense of Remark 3 in §2. Let us remember S. Mizohata's inequality

Lemma 4.5. *Let H be a $m\times m$ matrix whose elements $H_{j,k}$ are singular integral operators of type C^{β}_{∞} $(\beta>0)$. And assume*

(4.11) $|\sigma(H)(x,\xi)\circ\alpha|\geq\delta|\alpha|,\quad \delta>0,\quad \alpha={}^t(\alpha_1,\cdots,\alpha_m)$

being any complex vector. Then the following inequality holds

(4.12) $\|H\Lambda U\|^2\geq\delta'\|\Lambda U\|^2-\gamma\|U\|^2,\quad \delta'>0,\quad \gamma>0$

Now we proceed to

Proposition 4.1. *The solution $U(t)$ of (E_2) belonging to $\mathcal{E}^1_t(L^2[B_i])\cap\mathcal{E}^0_t(\mathcal{H}_i)$ satisfies*

(4.13) $\|U(t)\|_1\leq C(\|U(0)\|_1+\int_0^t\|F(s)\|_1 ds),\qquad 0\leq t\leq T,$

where $F(t)$ is in $\mathcal{E}^0_t(\mathcal{H}_i)$.

Let us prove Proposition 4.1 in three steps.

First step. Taking account of the properties of $\sigma(N(t))$ and Lemma 4.5, we can introduce a new norms in $L^2[B_i]$ $(i=1,2)$:

(4.14) $\|U\|^2_{L^2(B_i)(t)}=(N(t)U,N(t)U)+\beta\|(\Lambda+1)^{-1}U\|^2,$
β; large positive.

Then there exists a positive number c_1 and c_2 such that

(4.15) $c_1\|U\|\leq\|U\|^2_{L^2(B_i)(t)}\leq c_2\|U\|$.

(4.16) $\frac{d}{dt}\|U\|^2_{(B_i)(t)}=(N(t)(AU+F),NU)+(NU,N(AU+F))$
$+2\,Re\,(N'(t)U,N(t)U)+2\,Re\,\beta((\Lambda+1)^{-1}(AU+F),$
$(\Lambda+1)^{-1}U),$

where $\sigma(N'(t))=\dfrac{\partial}{\partial t}\sigma(N(t))$.

Corresponding the first and second term, let us consider

$$(4.17)\quad i\{(N(t)H(t)\Lambda U,\ N(t)U)-(N(t)U,\ H(t)H(t)\Lambda U)\}\,.$$

By virtue of Theorem 1 we know the facts

$$(4.18)\quad \begin{aligned} &N(t)H(t)\Lambda\equiv(N(t)\circ N(t))\Lambda = (\mathcal{D}(t)\circ N(t))\Lambda\,,\\ &\qquad\qquad(\equiv\text{modulo bounded operator in } L^2)\\ &\mathcal{D}^*\Lambda\equiv\mathcal{D}^\sharp\Lambda\,. \end{aligned}$$

Remark that $\mathcal{D}^\sharp=\mathcal{D}$ follows from hyperbolicity and use (4.18), then we obtain

$$(4.19)\quad \{(N(t)H(t)\Lambda U,\ N(t)U)-(N(t)U,\ N(t)H(t)\Lambda U)\}\,|\leq C\|U\|^2\,.$$

Since the other term can be easily estimated, we have

$$(4.20)\quad \frac{d}{dt}\|U(t)\|^2_{L^2(B_i)(t)}\leq C\|U\|_{L^2(B_i)(t)}(\|U\|_{L^2(B_i)(t)}+\|F\|_{L^2(B_i)(t)})\,.$$

Integrating (4.20) and using (4.15), consequently we obtain

$$(4.21)\quad \|U\|_{L^2}\leq C\left\{\|U_0\|_{L^2}+\int_0^t\|F(t)\|_{L^2}dt\right\}.$$

<u>*Second step.*</u> Now assume that the solution of (E_2) is in $\mathcal{E}_t^0(L^2_2\cap L^2[B_i])\cap\mathcal{E}_t^1(\mathcal{H}_i)$ and that $F(t)$ is in $\mathcal{E}_t^0(L^2_2\cap\mathcal{H}_i)$. We operate $\dfrac{\partial}{\partial x_j}$ on (E_2). Consider

$$(4.22)\quad \frac{d}{dt}\Big(\frac{\partial}{\partial x_n}U\Big)=(iH\Lambda)\Big(\frac{\partial}{\partial x_n}U\Big)+B\Big(\frac{\partial}{\partial x_n}U\Big)+\tilde{F}(t)\,,$$

where $\tilde{F}(t)=iH'_{x_n}(t)U+B'_{x_n}U+\dfrac{\partial}{\partial x_n}F(t)$,

$$\sigma(H'_{x_n}(t))=\frac{\partial}{\partial x_n}\sigma(H(t))\,.$$

If U is in $\mathcal{E}_t^0(L^2_2\cap L^2[B_1])\cap\mathcal{E}_t^1(\mathcal{H}_1)$, then $\dfrac{\partial}{\partial x_n}U$ belongs to $\mathcal{E}_t^0(\mathcal{H}_2)\cap\mathcal{E}_t^1(L^2[B_2])$. $\tilde{F}(t)$ is in $\mathcal{E}_t^0(\mathcal{H}_2)$, by virtue of the property of $\sigma(H'_{x_n}(t))$ and B'_{x_n} and the assumption that $F(t)$ is in $\mathcal{E}_t^0(L^2_2\cap\mathcal{H}_1)$.

Therefore we can apply the result of the first step on $\frac{\partial}{\partial x_n} U$ and $\tilde{F}(t)$, to obtain

$$(4.23)\quad \left\| \frac{\partial}{\partial x_j} U \right\|_{L^2} \leq c \left\{ \left\| \frac{\partial}{\partial x_j} U(0) \right\|_{L^2} + \int_0^t (\|(\Lambda+1)U(s)\|_{L^2} + \|F(s)\|_1) ds \right\}, \quad (j=1, \cdots, n)$$

because the same method is valid for $\frac{\partial}{\partial x_j} U$ $(j \neq n)$ and for the case $i=2$. Summation of (4.23) and (4.21) gives (4.13).

<u>*Third step.*</u> Now we can prove Proposition 1. Operate $\rho_\varepsilon *$ on (E_1), then we have

$$(4.24)\quad \frac{d^m}{dt^m} u_\varepsilon + \sum_{j=1}^m b_j(x, t, D) \frac{\partial^{m-j}}{\partial t^{m-j}} u_\varepsilon + \sum [b_j(x, t, D), \rho_\varepsilon *] \left(\frac{\partial}{\partial t}\right)^{m-j} u = f_\varepsilon,$$

where $[b_j(x, t, D), \rho_\varepsilon *]\{i(\Lambda+1)\}^{-(j-1)}$ is a bounded operator in $\mathcal{H}_i$. $U_\varepsilon(t) = \rho_\varepsilon * U(t)$ satisfies the assumptions in the second step. and

$$(4.25)\quad \frac{d}{dt} U_\varepsilon(t) = iH(t)\Lambda U_\varepsilon(t) + B(t)U_\varepsilon(t) + C_\varepsilon U(t) + F_\varepsilon(t)$$

By the result of the second step, follows

$$(4.26)\quad \|U_\varepsilon(t)\|_1 \leq C \left\{ \|U_\varepsilon(0)\|_1 + \int_0^t (\|C_\varepsilon U(t)\|_1 + \|F_\varepsilon(t)\|_1) dt \right\}.$$

Remark that $\|C_\varepsilon U\| \leq C\|U\|$ and that $\|C_\varepsilon U(t)\|$ tends to zero if ε goes to zero Tending ε to zero we can apply Lebesque's theorem to obtain (4.13). q.e.d.

§5. Extence of the solution

1. *At first we consider the case where the coefficients are independent of t.* In this case the corresponding evolution equation is

$$(E_2)_0 \quad \frac{d}{dt} U(t) = AU(t) + F(t) = iH\Lambda U(t) + BU(t) + F(t) \quad \text{in } \mathcal{H}_i \ (i=1, 2)$$

Take the definition domain of A as

$$\mathcal{D}(A)_i = \{U : U \in \mathcal{H}_i,\ AU \in \mathcal{H}_i\}\ .$$

$\mathcal{D}(A)_i$ is dense in $\mathcal{H}_i$, because $L_2^2 \cap L^2[B_i]$ contained in $D(A)_i$ is dense in $L^2[B_i]$ by Lemma 4. 3. Now introduce in each $\mathcal{H}_i$ the following new norm

$$(U, U)_{\mathcal{H}_i} = (N\Lambda U, N\Lambda U) + \beta(U, U), \qquad (i=1, 2)\ .$$

Then considering (4. 18) we get (5. 1) in the same way as (4. 16)

(5. 1) $\quad |(AU, U)_{\mathcal{H}_i} + (U, AU)_{\mathcal{H}_i}| \leq C \|U\|^2_{\mathcal{H}_i}$

From (5. 1) immediately follows

Proposition 5. 1. *For every U in $\mathcal{D}(A)_i$, a priori estimate*

(5. 2) $\quad \|(\lambda I - A)U\|_{\mathcal{H}_i} \geq (|\lambda| - \beta)\|U\|_{\mathcal{H}_i}$ *for* $|\lambda| > \beta$, λ *: real, β is a positive number.*

Proposition 5. 2. $(\lambda I - A)$ *maps $\mathcal{D}(A)_i$ in a one to one way onto $\mathcal{H}_i$.* $(i=1, 2)$

Proof. From proposition 5. 1. $(\lambda I - A)\mathcal{D}(A)_i$ is a closed subspace of $\mathcal{H}_i$. Let us prove that $(\lambda I - A)\mathcal{D}(A)_i$ is dense in $\mathcal{H}_i$. If there exist ψ in $\mathcal{H}_i$ such that

(5. 3) $\quad ((\Lambda+1)(\lambda I - A)U, (\Lambda+1)\psi) = 0 \qquad$ for every U in $\mathcal{D}(A)_i$,

then we can show in the following way that $\psi = 0$. Remark that the following relation hold.

(5. 4) $\quad (\Lambda+1)(\lambda I - A)U = (\lambda I - iH\Lambda + B_1 - B)(\Lambda+1)U$,

where $B_1 = \{i(H\Lambda - \Lambda H) + (B\Lambda - \Lambda B)(\Lambda+1)^{-1}\}$ is a bounded operator in L^2. In fact

$$(\Lambda+1)(\lambda I - iH\Lambda - B)U = (\lambda I - iH\Lambda - B)(\Lambda+1)U + i(H\Lambda - \Lambda H)\Lambda U$$
$$+ (B\Lambda - \Lambda B)U = (\lambda I - iH\Lambda - B + B_1)(\Lambda+1)U\ .$$

From (5. 4) the left-hand side of (5. 3) equals to

$$((\lambda I - iH\Lambda - B + B_1)(\Lambda+1)U, (\Lambda+1)\psi)$$
$$= ((\Lambda+1)U, (\lambda I + i\Lambda H^* - B^* + B_1^*)(\Lambda+1)\psi)$$
$$= ((\Lambda+1)U, (\lambda I + iH^\sharp\Lambda - B_2)(\Lambda+1)\psi)\ .$$

Here B_2 is a bounded operator in $L^2[B_i]$. Since $(\Lambda+1)\mathcal{D}(A)_i$ is dense in $L^2[B_i]$, it follows that

$$(\lambda I+iH^{\sharp}\Lambda-B_2)(\Lambda+1)\psi = 0\,.$$

And as in (5.4)

$$(\lambda I-iH^{\sharp}\Lambda-B_2)(\Lambda+1) = (\Lambda+1)^2(\lambda I+iH^{\sharp}\Lambda-B_3)(\Lambda+1)^{-1}\psi = 0\,.$$

Similarly to the proof of proposition 4.1, we can show that $(\Lambda+1)^{-1}\psi=0$, therefore $\psi=0$. q.e.d.

By virtue of Proposition 5.1 and 5.2, we can apply Hille-Yosida's theorem on $(E_2)_0$. For given initial data U_0 in $\mathcal{D}(A)_i$ and second member $F(t)$ in $\mathcal{E}_t^0(\mathcal{H}_i)$ such that $AF(t)$ is also in $\mathcal{E}_t^0(\mathcal{H}_i)$, there exist a unique solution of $(E)_0$:

$$U(t) = S_tU_0+\int_0^t S_{t-s}F(s)ds \tag{5.5}$$

satisfying the energy inequality

$$||U(t)||_{\mathcal{H}_i}\leq e^{\beta t}||U_0||_{\mathcal{H}_i}+C\int_0^t ||F(s)||_{\mathcal{H}_i}ds\,. \qquad \text{for} \quad t\,;\ 0\leq t\leq T. \tag{5.6}$$

For initial data U_0 in $\mathcal{H}_i$ and for $F(t)$ in $\mathcal{E}_t^0(\mathcal{H}_i)$, we can show the existence of the solution $U(t)$. Remark $\rho_\varepsilon * U_0=U_\varepsilon(0)$ and $\rho_\varepsilon * F(t)=F_\varepsilon(t)$ satisfy the above condition. Therefore we can apply (5.4) for the initial data $U_\varepsilon(0)-U_{\varepsilon'}(0)$ and second member $F_\varepsilon(t)-F_{\varepsilon'}(t)$, to obtain

$$\begin{aligned}\max_{0\leq t\leq T} ||U_\varepsilon(t)-U_{\varepsilon'}(t)||_{\mathcal{H}_i} &\leq e^{\beta t}||U_\varepsilon(0)-U_{\varepsilon'}(0)|| \\ &\quad +C\int_0^t ||F_\varepsilon(s)-F_{\varepsilon'}(s)||_{\mathcal{H}_i}ds\end{aligned} \tag{5.7}$$

Hence $\{U_\varepsilon(t)\}$ is a Cauchy sequence in $\mathcal{E}_t^0(\mathcal{H}_i)$, as ε tend to zero. Passing to the limit of

$$U_\varepsilon(t) = U_\varepsilon(0)+\int_0^t (AU_\varepsilon(s)+F_\varepsilon(s))ds$$

we can see that

$$U(t) = U(0)+\int_0^t (AU(s)+F(s))ds\,, \tag{5.8}$$

where the integral is the one in $L^2[B_i]$.

The solution $U(t)$ satisfies of course (5. 6), and

$$\begin{cases} \dfrac{d}{dt}U(t) = AU(t)+F(t) & \text{in } L^2[B_i] \\ U(0) = U_0 \end{cases}$$

2. *The case where the coefficients depend on t.*

Now we show the existence of the solution of (E), using *Cauchy's broken-line method.* S. Mizohata treated Cauchy problem for regularly hyperbolic equation, combining that method and energy inequality. Here we can follow his argument in our space $\mathcal{H}_i$ or $L^2[B_i]$. Consider

$$(5.9) \quad \begin{aligned} &\frac{d}{dt}U(t) = \tilde{A}_n(t)U(t)+F(t), \qquad 0\leq t\leq 1 \text{ in } L^2[B_j], \\ &U(0) = U_0\in\mathcal{H}_i \\ &\tilde{A}_n(t) = A\left(\frac{k}{n}\right) \text{ in } \frac{k}{n}\leq t\leq\frac{k+1}{n}, \qquad k=0, \cdots, n-1, \end{aligned}$$

The following lemma is the most important part of our argument.

Lemma 5. 1. *The solution $U_n(t)$ of* (5. 9) *has the estimate.*

(5. 10) $\|U_n(t)\|_{\mathcal{H}_i(t)}\leq M.$ *M is independed of t and n,*
$0\leq t\leq 1,\ n=1, 2, \cdots$

where

$$\|U\|^2_{\mathcal{H}_i(t)} = (N(t)\Lambda U, N(t)\Lambda U)+\beta(U, U)$$

Proof. By (5. 6) we can see

$$(5.11) \quad \left\|U_n\left(\frac{1}{n}\right)\right\|_{\mathcal{H}_i(0)}\leq e^{\beta/n}\left\{\|U_0\|_{\mathcal{H}_i(0)} + c\int_0^{1/n}\|F(t)\|_{\mathcal{H}_i(0)}dt\right\}.$$

From the continuity of $\sigma(N)(x, t, \xi)$ with respect to t and Remark 3 in §2, we can see

$$(5.12) \quad \left|\|U\|_{\mathcal{H}(t)}-\|U\|_{\mathcal{H}(t')}\right|\leq C|t-t'|, \qquad \text{for } \|U\|_{\mathcal{H}(0)}=1$$

(5. 1) and (5. 12) give

$$\left\|U_n\left(\frac{1}{n}\right)\right\|_{\mathcal{H}(1/n)}\leq e^{3c(1/n)}\left\{\|U_0\|_{\mathcal{H}(1/n)}+C\int_0^{1/n}\|F(t)\|_{\mathcal{H}(1/n)}dt\right\}$$

Step by step considering (5.6) and (5. 12) we can find M such that (5. 10) holds for every n and t, $0 \leq t \leq 1$. q.e.d.
Now introduce the following closed subspace of $L_1^2(R^n \times (0, 1))$

$$\mathcal{H}_i[0, 1)] = \{U(x, t) : U(x, t) \in L_1^2(R^n \times (0, 1)),\ U(x, t) \in L^2[B_i], \text{ for all most every } t\}$$

$U_n(t)$ are uniformely bounded in $\mathcal{H}_i[(0\ 1)]$, from (5. 10) and (5. 9). A weak limit in $\mathcal{H}_i[(0, 1)]$ belongs to $\mathcal{E}_t^0(L^2[B_i])$ and satisfies

$$(5.13)\quad \frac{d}{dt}U = A(t)U + F(t)$$

in the sense of distribution in $R^n \times (0, 1)$,

$$(5.14)\quad \operatorname*{trace}_{t \to 0} U(t) = U_0 .$$

Let us operate $\rho_\varepsilon *$ on (5. 15), then we have

$$(5.15)\quad \frac{d}{dt}U_\varepsilon(t) = A(t)U_\varepsilon(t) + F_\varepsilon(t) + C_\varepsilon(t)U(t) .$$

Where $C_\varepsilon(t)U(t)$ converge boundedly to zero in $\mathcal{H}_i$. Since $U_\varepsilon(t)$ is in $\mathcal{E}_t^r(\mathcal{H}_i) \cap \mathcal{E}_t^1(L^2[B_i])$, we can apply the energy inequality (4. 13) to $U_\varepsilon(t) - U_{\varepsilon'}(t)$

$$(5.16)\quad \max \|U(t) - U(t)\|_{\mathcal{H}_i} \leq C\Big\{\|U_\varepsilon(0) - U_{\varepsilon'}\|_{\mathcal{H}_i} + \int \|F_\varepsilon(t) - F_{\varepsilon'}(t)\| dt + \int \|(C_\varepsilon(t) - C(t))U(t)\| dt\Big\} .$$

$U_\varepsilon(t)$ converge uniformly in t to $U(t)$, so that $U(t)$ is in $\mathcal{E}_t^0(\mathcal{H}_i)$. As $U(t)$ satisfy

$$U(t) - U_0 = \int_0^t (A(S)\,U(s) + F(s))ds$$

$$U(t) \text{ is in } \mathcal{E}_t^1(L^2[B_i]) .$$

Form the argument in § 3, (E_1) has the solution satisfying the boundary conditions. Using the energy inequality we can show the existence the solution of (E) by successive approximation method.

§ 6. Finiteness of propagation speed

In this section we show that the solution given in § 5 has a

finite speed of propagation. This means essentially that the solution has a finite speed along the boundary $x_n=0$.
If the solution has this property, the grobal solution can be constructed by lacal ones. This fact is shown in the last section.
Consider the following Holmgren transformation at the boundary $x_n=0$

$$(6.1)\qquad \begin{cases} t' = t+\sum_{j=1}^{n-1}(x_j-x_j^0)^2+x_n^2 \\ y_j = x_j \qquad (j=1,\cdots,n)\ . \end{cases}$$

By (6.1) the boundary $x_n=0$ is transformed also to $y_n=0$, and the boundary conditions (B_1) and (B_2) is invariant in the following sense.

Lemma 6.1. *By* (6.1) $\left(\frac{\partial}{\partial x_n}\right)^{2k}$ *or* $\left(\frac{\partial}{\partial x_n}\right)^{2k+1}$ *is transformed to the operator whose coefficients of* $\left(\frac{\partial}{\partial y'}\right)^{\beta}\left(\frac{\partial}{\partial y_n}\right)^{2l-1}$ *or* $\left(\frac{\partial}{\partial y'}\right)^{\alpha}\left(\frac{\partial}{\partial y_n}\right)^{2l}$ *vanish respectively on* $y_n=0$. $(l<k,\ |\beta|+2l-1\leq 2k),\ |\alpha|+2l-1\leq 2k)$.

Proof. Assume that $\left(\frac{\partial}{\partial x_n}\right)^{l}$ is transformed to

$$(6.2)\qquad \left(\frac{\partial}{\partial x_n}\right)^{l} = \sum_{\substack{i\leq j\\ j=1,\cdots,l}} c_{ij}y_n^{l-j-i}\left(\frac{\partial}{\partial t'}\right)^{l-j}\left(\frac{\partial}{\partial y_n}\right)^{j-i},\quad c_{ji}\ \text{constants.}$$

This is true for $l=1$. Then

$$\begin{aligned}\left(\frac{\partial}{\partial x_n}\right)^{l+1} &= \left(\frac{\partial}{\partial y_n}+y_n\frac{\partial}{\partial t'}\right)\left(\frac{\partial}{\partial x_n}\right)^{l} \\ &= \sum_{\substack{i\leq j\\ j=1,\cdots,l}} c_{ji}y_n^{(l+1)-j-i}\left(\frac{\partial}{\partial t'}\right)^{(l+1)-j}\left(\frac{\partial}{\partial y_n}\right)^{j-i} \\ &\quad +\sum c_{ji}y_n^{l-j-i}\left(\frac{\partial}{\partial t'}\right)^{l-j}\left(\frac{\partial}{\partial y_n}\right)^{j+1-i} \\ &\quad +\sum (l-j-i)c_{ji}y_n^{(l-1)-j-i}\left(\frac{\partial}{\partial t'}\right)^{l-j}\left(\frac{\partial}{\partial y_n}\right)^{j-i} \\ &= \sum_{\substack{i\leq j\\ j=1,\cdots,l+1}} d_{ji}y_n^{(l+1)-j-i}\left(\frac{\partial}{\partial t'}\right)^{(l+1)-j}\left(\frac{\partial}{\partial y_n}\right)^{j-i},\quad d_{ji}\ \text{constants.}\end{aligned}$$

Hence (6.2) is true for all l. Remark now that

1) If l is odd and $j-i$ is even, then $l-j-i$ is odd.
2) If l is even and $j-i$ is odd, $l-j-i$ is also odd.

When $l-j-i$ is odd y_n^{l-j-i} vanishes on $y_n=0$. This complete the proof of Lemma 6.1. q.e.d.

Denote the interior of the backward cone with vertex (x_0, t_0) by

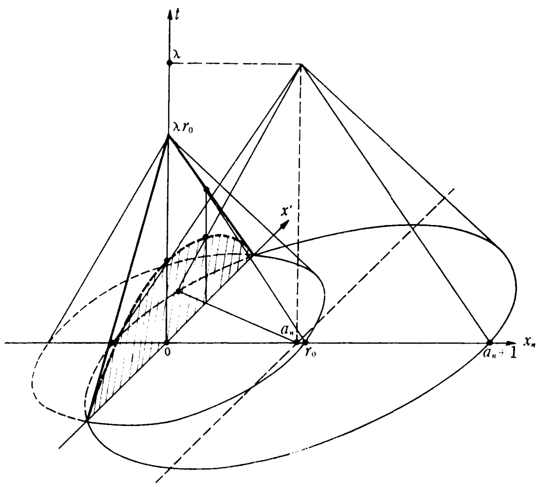
$$(6.3)\qquad C_\lambda(x_0 t_0) = \{(x, t)\,;\ -(t-t_0)\geq\lambda|x-x_0|,\ t\geq 0\}\,, \qquad \lambda>0.$$

The following geometrical lemma is used in the proof of finiteness of propagation speed.

Lemma 6.2. *Let* $a=(a_1, \cdots, a_{n-1}, a_n)$, $\sum_{i=1}^{n} a_i^2=1$, $r^0=(1-a)^{1/2}$

$$a_0 = (a_1, \cdots, a_{n-1}, 0)\,.$$

Then every point on $C_\lambda(a, \lambda)\cap\{t>0\}\cap\{x_n=0\}$ *is contained in the interior of* $C_\lambda(a_0, r^0\lambda)\cap\{t>0\}\cap\{x_n=0\}$.



The proof of this lemma will be given later. Form (6.1)

$$(6.4)\qquad \frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}+2(y_j-x_j^0)\frac{\partial}{\partial t'}\,, \qquad j\neq n.$$

$$\frac{\partial}{\partial x_n} = \frac{\partial}{\partial y_n}+2y_n\frac{\partial}{\partial t'}\,, \qquad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t}\,.$$

Therefore the transformed equation of (E) also satisfy (H). Now our discussion is divided into three parts.

First step. (local uniqueness near the boundary)
Assume that initial data equal zero in $\{x: |x-a_0|<\varepsilon\}\cap R^n_+$. By (6.1) $D_\varepsilon=\{(x, t)\in R^n_+\times[0, 1); |x-a_0|^2+t<\varepsilon, t\geq 0\}$ is mapped to

$$\tilde{D}_\varepsilon = \{(y, t)\in R^n_+\times[0, 1): \sum_{j=1}^{n-1}(y_j-a_j)^2+y\leq t'<\varepsilon\} .$$

We extend the solution u to $\{0\leq t'<\varepsilon\}\cap\{\text{outside of } \tilde{D}_t\}$ by zero. This is possible because u equals zero on $t'=y_n^2+\sum_{j=1}^{n-1}(y_j-a_j)^2$. The extended u satisfied the extended equation of (E) by keeping the coefficients constant along y_n axis outside of $\tilde{D}_\varepsilon$.
Let us apply the energy inequality, then we can see that u must be zero in $\tilde{D}_\varepsilon$.

Second step. Assume that initial data is zero in

(6.5) $\quad C_{\lambda_0}(a_0, r^0t_0)\cap\{x\in R^n_+\}\cap\{t=0\}$

We consider F. John's sweeping-out method attached to parabolic surfaces

(6.6) $\quad t = -\beta\{\sum_{j=1}^{n-1}(x_j-a_j)^2+x_n^2\}+\alpha ,\qquad$ where parameter α

moves in $0<\alpha<r^0t_0$ and $\beta>0$.
Step by step using the result of first step, we can show that the solution equals zero in

(6.7) $\quad C_{\lambda_0}(a_0, r^0t_0)\cap\{x\in R^n_+\}$.

Third step. Now assume that initial data is zero in

(6.8) $\quad C_{\lambda_0}(a, t_0)\cap\{x\in R^n_+\}\cap\{t=0\}$.

By the second step, the solution is zero in (6.7).
Considering Lemma 6.2 we can again use sweeping-out method associated with parabolic surfaces

$$t = -\beta\{\sum_{j=1}^{n-1}(x_j-a_j)^2+(x_n-a_n)^2\}+\alpha , \qquad 0\leq\alpha<t_0,\ \beta>0.$$

Consequently the solution is zero in

$$C_{\lambda_0}(a, t_0)\cap\{x\in R^n_+\} .$$

Proof of Lemma 6.2.

Without loss of generality we can assume that

$$a = (0, 0, \cdots, a_n), \quad a_0 = 0 = (0, \cdots, 0)$$

$C_\lambda(a, \lambda) \cap \{t=0\}$ and $C_\lambda(0, \lambda r_0) \cap \{t=0\}$ are respectively

$$S = \{x : |x-a| = 1\}$$

$$S_0 = \{x : |x| = r^0\}$$

$$S_0 \cap \{x_n = 0\} = S \cap \{x_n = 0\}$$

Now take a point $x=(x_1, \cdots, x_{n-1}, x_n)$ on S such that $x_n<0$. Let l be the line in $x-t$ space passing through $(x, 0)$ and (a, λ). The intersection of l and hyperplane $\{x_n = 0\}$ yields a point $(p_1, \cdots, p_{n-1}, 0, t_1)$, where

$$p_i = x_i \frac{a_n}{a_n - x_n} \tag{6.9}$$

$$t_1 = \lambda \frac{-x_n}{a_n - x_n} \tag{6.10}$$

Denote by p the point $(p_1, \cdots, p_{n-1}, 0)$ is x-plane. Let $q=(q_1, \cdots, q_{n-1}, 0)$ be the intersection of the line op and S^0. There exists a number α given by

$$\alpha q_i = p_i, \quad |q| = r^0 \tag{6.11}$$

$$\alpha = \left\{\frac{1-(a_n - x_n)^2}{1-a_n^2}\right\}^{1/2} \cdot \left(\frac{a_n}{a_n - x_n}\right), \tag{6.12}$$

because of the following relations

$$|p|^2 = \alpha^2 |x|^2 = \alpha^2 (r^0)^2$$

$$|p|^2 = \left(\sum_{j=1}^{n-1} x_i^2\right)\left(\frac{a_n}{a_n - x_n}\right)^2$$

$$\sum_{j=1}^{n-1} x_i^2 = 1-(a_n - x_n)^2$$

If the point $(p_1, \cdots, p_{n-1}, 0, t_0)$ be on the line passing through $(q, 0)$ $(0, r_0\lambda)$, t_0 must be

$$t_0 = r^0 \lambda (1-\alpha). \tag{6.13}$$

From (6.10), (6.12) and (6.13) we can show

$$t_1 < t_0. \tag{6.14}$$

This means Lema 6.2.

§7. Some Lemmas concerning local transformation

Let S be a smooth and compact hypersurface in R^n $(n\geq 2)$ and be the exterior of interior of δ. We take a suitable open finite covering $\{\Omega_p\}$ of S, Ω_p being open sets in R^n. Then we fix the following standart transformations. Denote by ω the intersection of some neighborhood of origin and R^n_+. $\Omega_p\cap\Omega$ is mapped in a one to one way into ω and S to $\{x_n=0\}$, and the outer normal direction of S to the outer normal direction of $\{x_n=0\}$, (c.f. [9] p. 289).

Now we consider a first order differential operator defined in Ω:

$$(7.1)\qquad m(x, D)=\sum_{i=1}^{n} m_i(x)\frac{\partial}{\partial x_i},$$

where $m(x)=(m_1(x), \cdots, m_n(x))$ *is sufficiently smooth and transversal on S. i.e. for* ν *outer unit normal,*

$$\sum_{j=1}^{n} m_i(x)\cos(\nu, x_i)\neq 0 \qquad on\ S.$$

Here we give the definition of the first order tangential operator

Definition We say that a first order differential operator $t(x, D)$ is *tangential* at the boundary S, if $t(x, D)=\sum_{j=1}^{n} c_j(x)\frac{\partial}{\partial x_j}$ satisfies $\sum_{j=1}^{n} c_j(x)\cos(\nu, x_j)=0$ for all $x\in S$.

$$(t(x, D)u, v)=(u, t^*(x, D)v(x)) \qquad \text{for all, } u, v\in L^2_1(\Omega).$$

Lemma 7.1. *We can find other local transformation such that* $n(x, D)$ *in* Ω_p *is transformed to* $\frac{\partial}{\partial x_n}$ *in* ω.

Proof. After a standart transformation we can assume

$$(7.2)\qquad n(x, D)=\sum_{j=1}^{n} m_j(x)\frac{\partial}{\partial x_j} \text{ in } \omega, \text{ where } m_n(x_1, \cdots, x_{n-1}, 0)\neq 0\,.$$

Let us consider the following system of ordinary differential equations with a variable y_n

$$(7.3)\qquad \frac{dx_k}{dy_n}=m_i(x_1, \cdots, x_n)\,.$$

with initial value

$$(7.4)\qquad \begin{cases} x_j(0) = y_j, & (j=1, 2, \cdots, n-1) \\ x_n(0) = 0 \end{cases}$$

The solution of (7.3) and (7.4) is sufficiently smooth with respect to y_n and the initial data y_j, $(j=1, \cdots, n-1)$,

$$(7.5)\qquad x_k = x_k(y_1, y_2, \cdots, y_n), \qquad (k=1, \cdots, n),$$

If y_n is small, Jacobian $J\binom{x}{y}$ of (7.5) is close to $m_n(x_1, \cdots, x_{n-1}, 0)$ $\neq 0$. Therefore y_k can be represented by

$$(7.6)\qquad y_k = y_k(x_1, \cdots, x_n) \qquad (k=1, \cdots, n)$$

By the transformation (7.6), $\sum^n m_i(x) \frac{\partial}{\partial x_i}$ have the form $\frac{\partial}{\partial y_n}$. q.e.d.

Lemma 7.1 means that the continuous vector field $(m_1(x), \cdots, m_n(x))$ in $\Omega_p \cap \Omega$ can be transformed to $(0, 0, \cdots, 0, 1)$ in ω.

Corollary 5.1. *There exists a family of smooth functions* $\bar{\Omega}$

$$(7.7)\qquad \begin{array}{l} \{\eta_j(x)\} \text{ such that } \sum \eta_j(x)=1 \text{ on } \bar{\Omega},\ n(x, D)\eta_j(x)=0 \\ \text{in some neighbourhood of } S. \end{array}$$

Proof. At first let us take the partitions of unity of S,

$$\sum \eta_k(s) = 1 \qquad \text{on} \quad S,$$

such that each $\eta_k(s)$ has its support in one $\bar{\Omega}_p \cap S$.

After transformation (7.6) we extend $\eta_k(s)$ defined on $\bar{\omega} \cap \{y_n=0\}$ to ω, with constant value parallel to y_n-axis. Summation of the extended form of η_k yields the desired partition of unity on $\bar{\Omega}$. q.e.d.

Now consider

$$(7.8)\qquad n(x, t, D) = \sum_{i=1}^n m_i(x, t)\frac{\partial}{\partial x_j},$$

where $m(x, t)=(m_1(x, t), \cdots, m_n(x, t))$ *is transversal on* S *for every* $t \in [0, \delta]$.

Lemma 7.2. *We can find a family of local transformation of the domain* Ω_p *to* ω, *depending smoothly on* t, *such that* $n(x, t, D)$ *is transformed to* $\frac{\partial}{\partial y_n}$ *in* ω *for every* t.

Proof. As above, we can assume that $n(x, t, D)$ is given in ω. Here we consider the system of ordinary differential equations involving parameter t.

$$(7.3)' \qquad \frac{dx_k}{dy_n} = m_i(x_1, \cdots, x_n, t)$$

$$(7.4)' \qquad \begin{cases} x_j(0) = y_j \\ x_n(0) = 0 \end{cases}$$

The solution x_k is sufficienly smooth with respect to y_i $(j=1, \cdots, n)$ and parameter t.

$$(7.5)' \qquad x_k = x_k(y_1, \cdots, y_n, t), \qquad (k=1, \cdots, n).$$

Since Jacobian of (6.5)′ is not zero in place where y_n is small.

$$(7.6)' \qquad y_k = y_k(x_1, \cdots, x_n, t)$$

Now we consider the transformation

$$(7.9) \qquad \begin{cases} y_k = y_k(x_1, \cdots, x_n, t) \\ t' = t \end{cases}$$

By (7.9) $n(x, t, D)$ is transformed to $\dfrac{\partial}{\partial y_n}$.

Lemma 5.3. *Let $b(x, t, D)$ be a k-th order partial differential operator in Ω containing parameter t. Then $b(x, t, D)$ is written as*

$$(7.10) \qquad b(x, t, D) = \sum_{j=0}^{k} c_j(x, t, D) n^{k-j}(x, t, D),$$

where $c_j(x, t, D)$ are the sum of the product of j first order tangential operators.

Proof. Take a partion of unity $\sum \eta_j(x) = 1$ on $\bar{\Omega}$, such that the support of $\eta_j(x)$ is on $\bar{\Omega}_p$, if it intersect S.

$$(7.11) \qquad b(x, t, D) = \sum \eta_j(x) b(x, t, D)$$

After a standart transformation and (7.9), each $\eta_j(x) b(x, t, D)$ is

$$(7.12) \qquad \eta_j(y) \sum \tilde{c}_j(y, t, D)\left(\frac{\partial}{\partial y_n}\right)^{k-j} = \sum \tilde{\tilde{c}}_j(y, t, D) \eta_j\left(\frac{\partial}{\partial y_n}\right)^{k-j}$$

where $\tilde{\tilde{c}}_j(y, t, D)$ does not involve $\dfrac{\partial}{\partial y_n}$.

Here $c_j(y, t, D)$ are the disired type of $c_j(x, t, D)$ in ω. Summation of (7. 12) means (7. 10).

Corollary. *Let a be a uniformly elliptic operator of second order with smooth coefficients involving parameter t in a half space.*

$$a = \sum_{ij}^{n} a_{ij}(x, t)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}, \quad a_{ij}(x, t)\xi_i\xi_j \geq \delta|\xi|^2, \qquad \delta>0$$

$$a_{ij} = a_{ji} \qquad x\in\Omega,\ t\in[0, \delta]$$

Then a is transformed to

$$a = \frac{\partial^2}{\partial x_n^2} + \sum_{i,j=1}^{n-1} b_{ij}(x, t)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \textit{first order}.$$

Proof. We may consider the transformation (7. 9)′ generated by the following ordinary differential equations.

(7. 4) $\quad \dfrac{dx_k}{dy_n} = a_{kn}(x, t)$

(7. 5)′ $\quad \begin{cases} x_k(0) = y_k \\ x_n(0) = 0\,. \end{cases}$ q.e.d.

§ 8. Problem in the general domain

Here we give the proof of Theorem 3. Consider in $\Omega\times(0, \infty)$ the following regularly hyperbolic equation.

$$(E)_g \qquad \left\{\frac{\partial^m}{\partial t^m} + \sum_{j=1}^{m} b_j(x, t, D)\frac{\partial^{m-j}}{\partial t^{m-j}} + B\left(x, t, \frac{\partial}{\partial t}, D\right)\right\}u = f(x, t)\,,$$

where each $b_j(x, t, D)$ satisfies the aassumption $(H)_g$ described in §3. Considering Lemma 7. 3 we can see that the assumption is an intrinsic one. Let us show how to contruct the solution of the equation $(E)_g$ satisfying the boundary condition:

$$(B_1)_g \qquad n(x, t, D)^{2k}u|_s = 0\,, \qquad k = 0, 1, \cdots, \left[\frac{m-1}{2}\right],$$

$$(B_2)_g \qquad n(x, t, D)^{2k+1}u|_s = 0\,, \qquad k = 0, 1, \cdots, \left[\frac{m-2}{2}\right].$$

Assume that the initial data $\left(\dfrac{\partial}{\partial t}\right)^i u(0) = \omega^i(x)$ is in L^2_{m-j} and satisfy $(B_1)_g$ or $(B_2)_g$. i.e.

$$n(x, 0, D)^{2k}\omega^i|_s = 0, \qquad k=0, 1, \cdots, \left[\frac{m-j-1}{2}\right], \quad i=0, 1, \cdots, m-1.$$

$$(8.1) \qquad n(x, 0, D)^{2k+1}\omega^i|_s = 0, \qquad k=0, 1, \cdots, \left[\frac{m-j-2}{2}\right], \quad i=0, 1, \cdots, m-1.$$

Now take the partion of unity of $\bar{\Omega}$.

$$(8.2) \qquad \sum \eta_l(x) = \sum_{p:\text{finite}} \eta_p(x) + \sum_{q:\text{finite}} \eta_q(x) = 1 \qquad \text{on } \bar{\Omega},$$

where $\eta_q(x)$ has its support in Ω and $\eta_p(x)$ has its support near the boundary of $\bar{\Omega}_p$ such that

$$(8.3) \qquad n(x, 0, D)\eta_p(x) \equiv 0 \qquad \text{in } \Omega_p \text{ for every } p.$$

By virtue of Corollary 7.1, (8.3) is possible.
Let us decompose given initial data $\{w^i(x)\}_{i+0,1,\cdots,m-1}$.

$$(8.4) \qquad w^i(x) = \sum \eta_p w^i + \sum \eta_q w^i = \sum w_p^i + \sum w_q^i$$

Because of (8.2), w_p^i satisfy (8.1) for every p.
By the local transformation (7.9) after standart one, the equation $(E)_g$ is reduced to (E) and the boundary condition $(B_i)_g$ to (B_i). Let $\tilde{u}_p(t)$ be the solution for the initial data $\{\tilde{w}_p^i\}_{i=0,\cdots,m-1}$, in a half space. By the fiiniteness of the propagation speed, $\tilde{u}_p(t)$ still has its support in ω for $0 \leq t \leq \delta_1$, if we take δ_1 sufficiently small. The inverse image $u_p(t)$ of $\tilde{u}_p(t)$ is the solution of $(E)_g$ satisfying $(B_i)_g$ ($i=1$, or 2) and initial data $\{w_p^i\}$. For $\{w_q^i\}_{i=0,\cdots,m-1}$, we may consider Cauchy problem for the equation $(E)_g$. The solution of course has a finite speed. Then for $0 \leq t < \delta_2$ the do not reach at the boundary.
The total sum $u(t) = \sum u_p(t) + \sum u_q(t)$ is the desired solution for $0 \leq t < \delta$, $\delta = \min(\delta_1, \delta_2)$. Step by step we can construct the solution $u(t)$ for $0 \leq t < \infty$. $u(t)$ has a finite speed of propagation and is the unique solution of $(E)_g$ satisfying $(B_i)_g$ and given initial data. Energy inequality hold in L^2-sense.

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REFERENCES

[1] S. Agmon : Proalème mixtes pour les équations hyperbolique d'ordre supérieur, Colloques Internationaux du. C. N. R. S. (1962), 13-18.

[2] K. Asano and T. Shirota : On certain mixed problem for hyperbolic equations of higher order. Proc. Japan Acad. **45** (3), 145-148 (1969).

[3] A. P. Carderon and A. Zygmund : Singular integral operators and differential equations. Amer. J. Math. **79**, 901-921 (1957).

[4] ―――――― : On the existence of certain singular integrals, Acta, Mathematica, vol. **88**, 85-139 (1952).

[5] A. P. Darderón : Commutators of singular integral operators, Proc. Mat. Acad. Sci. U. S. A. (5) **53** 1092-1099 (197).

[6] G. F. Duff : Mixed problem for hyperbolic equations of general order, Canada J. Math. **9**, 195-221 (1959).

[7] R. Hersh : Mixed problem in several variables, J. Math. Mech. **12** 317-334 (1963).

[8] M. Ikawa : On the mixed problem for the wave equation with an oblique boundary condition, Proc. Japan Acad. **44**, 1033-1037 (1968).

[9] S. Miyatake : On some mixed problems fourth order hyperbolic equations. J. Math. Kyoto Univ. **8** (2), 285-311 (1963).

[10] ―――――― : A remark on singular integral operator and reflection principle for some mixed problems. Proc. Japan Acad. **45** (7) 547-551 (1969).

[11] S. Mizohata : Système Hyperboliques, J. Math. Soc. Japan **11**, 205-233 (1959).

[12] ―――――― : Quelque problèmes au bord, du type mixte pour des èquations hyperboliques, Collège de France 23-60, (1966-1967).

[13] ―――――― : Theory of Partial equations, Iwanami, Tokyo, (1965). (Japanese)

[14] I. M. Gelfand : Some problems in the theory of quasilinear equations. Uspehi, Mat. Nauk, **14**, no. 2 (86) 87-158 (1959).

[15] R. Courant and D. Hilbert : Methods of mathematical physics. I. Interscience, New York. (1953).

[16] K. Yosida : Functional analysis, Sppringer-verlag Berlin (1965).