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A Hölder condition for Brownian local time

By

H. P. McKean, Jr.[1]

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Given a standard Brownian motion on R^1 beginning at 0, H. Trotter [3] proved the (simultaneous) existence of the *local times*:

1. $$\mathfrak{t}(t, a) = \lim_{b \downarrow a} \frac{\text{measure } (s : a < x(s) \leq b, s \leq t)}{b-a}$$

$$t \geq 0, \ a \in R^1$$

and derived the law

2a. $$P\left[\overline{\lim_{|b-a|=\delta \downarrow 0}} \frac{|\mathfrak{t}(t, b) - \mathfrak{t}(t, a)|}{\sqrt{\delta \, lg\, 1/\delta}} = 0\right] = 1 .$$

I give simple proofs leading to the sharper bound

2b. $$P\left[\overline{\lim_{|b-a|=\delta \downarrow 0}} \frac{|\mathfrak{t}(t, b) - \mathfrak{t}(t, a)|}{\sqrt{2\delta \, lg\, 1/\delta}} \leq 2\sqrt{\max_{R^1} \mathfrak{t}(t, \cdot)}\right] = 1 .$$

2b is proved *assuming* $\mathfrak{t}$ *exists and is continuous in space*; afterwards, I go back and prove the latter statement. H. Tanaka's (unpublished) expression for the local time as a stochastic integral:

3. $$P\left[\frac{1}{2}\mathfrak{t}(t, a) = \max[x(t) - a, 0] - \max[-a, 0] - \int_{\substack{s \leq t \\ x(s) > a}} x(ds)\right] = 1$$

and the bound[2]

4a. $$E[e^{\mathfrak{a}(t)}] \leq 1$$

for the functional

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[2] See, for example, E. B. Dynkin [1].

4b. $$\mathfrak{a}(t) = \int_0^t f[x(s)]x(ds) - \frac{1}{2}\int_0^t f^2[x(s)]ds$$

4c. $$P\left[\int_0^t f^2[x(s)]ds < +\infty\right] = 1$$

are the basic tools for this. I want to thank H. Tanaka for communicating his integral 3 and for a helpful conversation about the sample path $\mathfrak{f}$ of 17 below.

Tanaka's (unpublished) proof of 3 is as follows.

Bringing in the indicator e_{ab} of the interval $(a, b](a<b)$, an application of the formula for stochastic differential gives

5. $$\begin{aligned}&\frac{1}{2}\text{ measure }(s: a < x(s) \leq b,\ s \leq t)\\ &= \frac{1}{2}\int_0^t e_{ab}[x(s)]ds\\ &= \bar{\bar{e}}_{ab}[x(t)] - \bar{\bar{e}}_{ab}[x(0)] - \int_0^t \bar{e}_{ab}[x(s)]x(ds)\end{aligned}$$

with

6a. $$\bar{e}_{ab}(\xi) = \int_{\eta\leq\xi} e_{ab}d\eta = \begin{cases} 0 & \xi\leq a \\ \xi - a & a<\xi\leq b \\ b-a & \xi > b \end{cases}$$

6b. $$\bar{\bar{e}}_{ab}(\xi) = \int_{\eta\leq\xi} \bar{e}_{ab}d\eta = \begin{cases} 0 & \xi\leq a \\ \dfrac{(\xi-a)^2}{2} & a<\xi\leq b \\ (b-a)\left(\xi - \dfrac{a+b}{2}\right) & \xi > b, \end{cases}$$

and, using

7a. $$\lim_{b\downarrow a}(b-a)^{-1}\bar{\bar{e}}_{ab}(\xi) = \max[\xi - a, 0]$$

and

7b. $$\begin{aligned}&E\left|\int_0^t \left(\frac{\bar{e}_{ab}}{b-a} - e_{a\infty}\right)x(ds)\right|^2\\ &= E\left[\int_0^t \left(\frac{\bar{e}_{ab}}{b-a} - e_{a\infty}\right)^2 ds\right]\end{aligned}$$

$$\leq E\left[\int_0^t e_{ab}\left(\frac{b-x(s)}{b-a}\right)^2 ds\right]$$

$$\leq E\left[\int_0^t e_{ab}\, ds\right]$$

$$\leq \text{constant } (b-a),$$

3 is immediate on letting $b \downarrow a$ in 5, *assuming, as I now do, the existence of the local time* $\mathfrak{t}(t, a)$.

Given positive numbers α and β, points $a<b$, and putting $b-a=\delta$, an application of 4a gives

8a. $$P\left[2\int_0^t e_{ab}x(ds) > [\alpha+\beta \max_{R^1} \mathfrak{t}(t,\cdot)]\sqrt{2\delta\, lg\, 1/\delta}\right]$$

$$\leq P\left[\int_0^t e_{ab}x(ds) > \frac{\alpha}{2}\sqrt{2\delta\, lg\, 1/\delta} + \frac{\gamma}{2}\int_a^b \mathfrak{t}(t, \xi)d\xi\right]$$

$$\gamma \equiv \beta\sqrt{(2/\delta)\, lg\, 1/\delta}$$

$$= P\left[\int_0^t e_{ab}x(ds) - \frac{\gamma}{2}\int_0^t e_{ab}ds > \frac{\alpha}{2}\sqrt{2\delta\, lg\, 1/\delta}\right]$$

$$< E\left[e^{\gamma\int_0^t e_{ab}x(ds) - \frac{\gamma^2}{2}\int_0^t e_{ab}ds}\right] e^{-\frac{\alpha\gamma}{2}\sqrt{2\delta\, lg\, 1/\delta}}$$

$$\leq e^{-\alpha\beta\, lg\, 1/\delta}$$

$$= \delta^{\alpha\beta},$$

and since the same bound applies to $-\int_0^t e_{ab}x(ds)$ as well,

8. $$P\left[2\left|\int_0^t e_{ab}x(ds)\right| > [\alpha+\beta \max_{R^1} \mathfrak{t}(t,\cdot)]\sqrt{2\delta\, lg\, 1/\delta}\right]$$

$$\leq 2\delta^{\alpha\beta},$$

leading at once to

9. $$P\left[\max_{\substack{a=i2^{-n},\, b=j2^{-n} \\ 0<j-i\leq 2^{n\varepsilon} \\ |a|<d}} \frac{2\left|\int_0^t e_{ab}x(ds)\right|}{\sqrt{2\delta\, lg\, 1/\delta}} > \alpha+\beta \max_{R^1} \mathfrak{t}(t,\cdot)\right]$$

$$< \sum_{\substack{0<k\leq 2^{n\varepsilon} \\ |a|<d}} 2(k2^{-n})^{\alpha\beta}$$

$$< 4d2^{-n[(1-\varepsilon)\alpha\beta-1-\varepsilon]},$$

which is the general term of a convergent sum provided $d=1, 2,$

3, *etc.* is fixed, $\alpha\beta > 1$, and $\varepsilon > 0$ is so small that $(1-\varepsilon)\alpha\beta - 1 - \varepsilon < 0$. Tanaka's integral $(=3)$, the Borel-Cantelli lemma, and the fact that $\max[b-a, 0]$ is piecewise smooth can now be combined with 9 to establish

$$10. \qquad P\left[\overline{\lim_{\substack{a=i2^{-n},\, b=j2^{-n} \\ 0<k=j-i\leq 2^{n\varepsilon} \\ b-a=\delta\downarrow 0 \\ |a|<d}}} \frac{|\mathfrak{t}(t, b)-\mathfrak{t}(t, a)|}{\sqrt{2\delta\, lg\, 1/\delta}} \leq \alpha+\beta \max_{R^1} \mathfrak{t}(t, \cdot)\right] = 1$$

for each choice of $d > 1$, $\alpha\beta > 1$, and $0 < \varepsilon < \dfrac{\alpha\beta-1}{\alpha\beta+1}$.

But now, *taking into account the fact that* $\mathfrak{t}(t, a)$ *is continuous in space*, it is plain sailing over the course laid out by P. Lévy [2] for the proof of

$$11. \qquad P\left[\overline{\lim_{\substack{t-s=\delta\downarrow 0 \\ 0\leq s<t\leq 1}}} \frac{|x(t)-x(s)|}{\sqrt{2\delta\, lg\, 1/\delta}} \leq 1\right] = 1$$

to deduce from 10

$$12. \qquad P\left[\overline{\lim_{\substack{|b-a|=\delta\downarrow 0 \\ |a|<d}}} \frac{|\mathfrak{t}(t, b)-\mathfrak{t}(t, a)|}{\sqrt{2\delta\, lg\, 1/\delta}} \leq \alpha+\beta \max_{R^1} \mathfrak{t}(t, \cdot)\right] = 1$$

for each $d \geq 1$ and $\alpha\beta > 1$, and 2*b* follows on letting $d \uparrow +\infty$ (use $\mathfrak{t} \equiv 0$ near $\pm\infty$), letting $\alpha\beta \downarrow 1$, and making $\alpha+\beta \max_{R^1} \mathfrak{t}$ as small as possible subject to $\alpha\beta = 1$.

I now go back and prove that $\mathfrak{t}$ *exists and is continuous.* Beginning with the stochastic integrals $\int_0^t e_{a\infty} x(ds) \equiv \mathfrak{e}(a) (a \in R^1)$, the trick is to prove, as I now do, that $\mathfrak{e}$ can be modified so as to be continuous in space.

Because

$$13. \qquad P\left[\max_{\substack{a=(k-1)2^{-n} \\ b=k2^{-n} \\ |a|<d}} \int_0^t e_{ab}\, ds > n2^{-n}\right]$$

$$\leq d2^n P\left[\int_0^t e_{02^{-n}}\, ds > n2^{-n}\right]$$

$$\leq d(2/e)^n E\left[\exp\left(2^n \int_0^t e_{02^{-n}}\, ds\right)\right]$$

$$\leq d(2/e)^n e^t \int_0^{+\infty} e^{-\theta} d\theta E\Big[\exp\Big(2^n \int_0^t e_{02^{-n}} ds\Big)\Big]$$

$$= d(2/e)^n e^t \int_0^{+\infty} e^{-\theta} d\theta \Big[1+$$

$$+\sum_{l=1}^{\infty} 2^{nl} \int_0^{\theta} d\theta_l \cdots \int_0^{\theta_2} d\theta_1 \int_0^{2^{-n}} db_1 \cdots \int_0^{2^{-n}} db_l$$

$$\frac{e^{-b_1^2/2\theta_1}}{\sqrt{2\pi\theta_1}} \frac{e^{-(b_2-b_1)^2/2(\theta_2-\theta_1)}}{\sqrt{2\pi(\theta_2-\theta_1)}} \cdots \frac{e^{-(b_l-b_{l-1})^2/2(\theta_l-\theta_{l-1})}}{\sqrt{2\pi(\theta_l-\theta_{l-1})}}\Big]$$

$$< d(2/e)^n e^t \int_0^{+\infty} e^{-\theta} d\theta [1+\sum_{l=1}^{\infty} 1 \textit{ convoluted with } 1/\sqrt{2\pi\theta}\ l \textit{ times}]$$

$$= d(2/e)^n e^t \sum_{l=0}^{\infty} 2^{-l/2}$$

$$= d(2/e)^n e^t \frac{\sqrt{2}}{\sqrt{2}-1}$$

is the general term of a convergent sum and $\int_0^t e_{ab} ds$ is monotone in a and b, one finds

14. $$P\left[\overline{\lim_{|b-a|=\delta\downarrow 0}} \frac{\int_0^t e_{ab} ds}{\delta\, lg\, 1/\delta} < +\infty\right] = 1,$$

and so, using the obvious bound

15a. $$P\Big[\int_0^t e_{ab} x(ds) > \alpha + \frac{\beta}{2}\int_0^t e_{ab} ds\Big]$$

$$= P\Big[\beta\int_0^t e_{ab} x(ds) - \frac{\beta^2}{2}\int_0^t e_{ab} ds > \alpha\beta\Big]$$

$$\leq e^{-\alpha\beta}$$

with $\alpha\sqrt{\delta\, \lg 1/\delta}$ and $\beta/\sqrt{\delta}$ $(\alpha\beta>1)$ in place of α and β to obtain

15b. $$P\Big[\Big|\int_0^t e_{ab} x(ds)\Big| > \alpha\sqrt{\delta}\, lg\, 1/\delta + \frac{\beta}{2\sqrt{\delta}}\int_0^t e_{ab} ds\Big]$$

$$< 2\delta^{\alpha\beta} \qquad \delta = |b-a|,$$

it follows as in the proof of 2b above that

16. $$P\left[\overline{\lim_{\substack{a=i2^{-n},\, b=j2^{-n} \\ |b-a|=\delta\downarrow 0}}} \frac{\Big|\int_0^t e_{ab} x(ds)\Big|}{\sqrt{\delta}\, lg\, 1/\delta} < +\infty\right] = 1.$$

But this means that the modified sample path

17. $\mathfrak{f}(a) \equiv \lim_{b=k2^{-n}\downarrow a} \mathfrak{e}(b) = \lim_{b=k2^{-n}\downarrow a} \int_0^t e_{b\infty} x(ds) \qquad a \in R^1$

is continuous; in addition,

18. $P\left[\int_a^b \mathfrak{f}(c)dc = \int_0^t \left(\int_a^b e_{c\infty} dc\right) x(ds)\right] = 1 \qquad a < b$

because $P[\mathfrak{f}(a) = \mathfrak{e}(a)] \equiv 1$ $(a \in R^1)$, and since $\bar{\bar{e}}_{ab}$, *measure* $(s : a < x(s) \leq b, s \leq t) = \int_0^t e_{ab} ds$, and $\int_a^b \mathfrak{f} dc$ are all continuous in a and b, an application of 5 gives

19. $P\left[\frac{1}{2} \text{ measure } (s : a < x(s) \leq b, s \leq t)\right.$
$\left. = \bar{\bar{e}}_{ab}[x(t)] - \bar{\bar{e}}_{ab}(0) - \int_a^b \mathfrak{f} dc, \, a < b\right] = 1,$

leading at once to the fact that

20. $\frac{1}{2} \mathfrak{t}(t, a) = \max [x(t) - a, 0] - \max [-a, 0] - \mathfrak{f}(a)$

exists and is continuous, as was to be proved.

A second application of the above method gives the bound[3]

21. $P\left[2 \left| \int_0^t e_{0b} x(ds) \right| > [\alpha + \beta \max_{0 \leq a \leq b} \mathfrak{t}(t, a)] \sqrt{2\delta \, lg_2 \, 1/\delta}\right]$
$\leq 2(lg \, 1/\delta)^{-\alpha\beta},$

leading at once to

22. $P\left[\overline{\lim_{\delta=2^{-n}\downarrow 0}} \frac{|\mathfrak{t}(t, \delta) - \mathfrak{t}(t, 0)|}{\sqrt{2\delta \, lg_2 \, 1/\delta}} \leq 2\sqrt{\mathfrak{t}(t, 0)}\right] = 1.$

Given $t \geq 0$ and $a \in R^1$, the conditional local time $[\mathfrak{t}(t, b) : b \in R^1, P(\cdot / x(t) = a)]$ is a diffusion, and, expressing it in terms of a standard Brownian motion (via a change of scale and a time substitution), it is immediate that the bounds 2b and 22 are best possible; this beautiful result will appear in a forthcoming paper by D. B. Ray.

Massachustetts Institute of Technology
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[3] $lg_2 \, c = lg(lg \, c)$.

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