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On automorphisms of G-structures

By

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Introduction

The purpose of the present paper is to investigate the properties concerning automorphisms of G-structures. § 1 contains definitions and notations. In § 2, we shall prove some lemmas which will be used in the remaining sections. In § 3, we shall study infinitesimal automorphisms of G-structures. Conditions that a vector field to be an infinitesimal automorphism of G-structure will be given. We consider the set $\mathcal{A}$ of all infinitesimal automorphisms of G-structure. Under certain coudition, $\mathcal{A}$ is a finite dimensional Lie algebra. In § 4, we shall give a condition that an infinitesimal automorphism of G-connexion is also an infinitesimal automorphism of G-structure. The last section is devoted to the study of invariant G-structures on reductive homogeneous spaces.

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§ 1. Preliminaries and notations

1. Throughout this paper, all manifolds, mappings, vector fields and differential forms are understood to be of class C^∞.

Let M be a differentiable manifold. We shall denote by $T(M)$ the tangent bundle of M and by $T_u(M)$ the tangent space of M at $u \in M$. Suppose $f: M \rightarrow N$ to be a differentiable mapping of M into a differentiable manifold N. Then f induces a mapping $f_*: T(M) \rightarrow T(N)$. Let F be a vector space over reals. We denote by $\phi_F(M)$ the set of all F-valued differential forms on M. The dual of f_* gives a mapping $f_*: \phi_F(N) \to \phi_F(M)$.

Let G be a Lie subgroup of the general linear group $GL(n, R)$ in n variables, and $\mathcal{G}$ be its Lie algbra. Let E be an n-dimensional vector space over reals, and E^* be the dual space of E. We define the representation (ρ, E) of $GL(n, \mathrm{R})$ on a vector space E as follows:

$$\rho(g)e_i = \sum_j g_i^j e_j \qquad \text{for} \quad g = (g_i^j) \in GL(n, \mathrm{R}),$$

where $e_1, \cdots, e_n$ is a base of E. If we consider the restriction of ρ to the subgroup G, then we obtain a representation of G which we denote by the same notation (ρ, E). Then we obtain two representation $(\rho^* \otimes \mathrm{ad}, E^* \otimes \mathcal{G})$ and $((\rho^* \wedge \rho^*) \otimes \rho, (E^* \wedge E^*) \otimes E)$ of G, where (ρ^*, E^*) is the dual representation of (ρ, E) and $(\mathrm{ad}, \mathcal{G})$ is the adjoint representation of G. For brevity, we shall denote these representations by $(\alpha_1, E^* \otimes \mathcal{G})$ and $(\alpha_2, (E^* \wedge E^*) \otimes E)$ respectively.

Let $(\tilde{e}_\sigma)$ $(\sigma = 1, \cdots, \dim. G)$ be a base of $\mathcal{G}$, and let $(e_1, \cdots, e_n)$ be a base of E and $(e^1, \cdots, e^n)$ its dual base. The representaion (ρ, E) of G induces the representation $(\bar{\rho}, E)$ of the Lie algebra $\mathcal{G}$. Then $\bar{\rho}(\tilde{e}_\sigma)$ can be represented by a matrix $\|a_{\sigma j}^i\|$:

$$\bar{\rho}(\tilde{e}_\sigma) \cdot e_j = \sum_i a_{\sigma j}^i e_i .$$

In the following we shall write $\bar{\rho}(A)\xi = A \cdot \xi$ for $A \in \mathfrak{gl}(n, \mathrm{R})$, $\xi \in E$.

We define the linear map $\partial: E^* \otimes \mathcal{G} \to (E^* \wedge E^*) \otimes E$ as follows:

$$\partial\Big(\sum_{\sigma, k} \xi_k^\sigma e^k \otimes \tilde{e}_\sigma\Big) = \sum_{\sigma, i, j, k} (a_{\sigma j}^i \xi_k^\sigma - a_{\sigma k}^i \xi_j^\sigma) e^j \wedge e^k \otimes e_i$$

for any $g \in G$, we see immediately that $\alpha_2(g)$ leaves $Im\, \partial$ invariant, and hence we obtain an automorphism $\alpha_3(g)$ of $Coker\, \partial$. Thus we obtain the representation $(\alpha_3, Coker\, \partial)$ of G.

DEFINITION 1.1. We say that the group G has the *property* $(\mathcal{P})$ if the following conditions are satisfied

1. $Ker\, \partial = 0$.
2. There exists a linear map $k: Coker\, \partial \to (E^* \wedge E^*) \otimes E$ such that

   (i) $q \circ k = 1$.

   (ii) $k \circ \alpha_3(g) = \alpha_2(g) \circ k \qquad$ for any $\quad g \in G$, where q denotes the natural projection $(E^* \wedge E^*) \otimes E \rightarrow Coker\ \alpha$.

2. Let M be an n-dimensional differentiable manifold and $\mathscr{F}(M)$ be the frame bundle of M whose projection is π. Let E be an n-dimensional vector space over reals. Recalling that every element x of $\mathscr{F}(M)$ is considered as a linear isomorphism of E onto $T_{\pi(x)}(M)$, we define a tensorial 1-form θ of type (ρ, E) on $\mathscr{F}(M)$[1], called basic form, as follows [7]

$$\theta_x(Z) = x^{-1} \cdot \pi_* Z \qquad \text{for any} \quad Z \in T_x(\mathscr{F}).$$

Suppose that a connexion Γ is given in $\mathscr{F}(M)$ and denote by ω the connexion form of Γ. For any vector field Z on $\mathscr{F}(M)$, we denote by hZ (resp. vZ) the horizontal (resp. vertical) component of Z with respect to Γ. We denote by X^* the lift of $X \in T(M)$ with respect to Γ. For each point x of $\mathscr{F}(M)$, we denote by $\mathfrak{H}_x$ the set of all points which can be joined to x by horizontal curves. These $\mathfrak{H}$'s are submanifold of $\mathscr{F}(M)$ which we call *horizontal manifolds.*

The covariant differential of an l-form Ξ on $\mathscr{F}(M)$ is defined by

$$D\Xi(Z_1, \cdots, Z_{l+1}) = d\Xi(hZ_1, \cdots, hZ_{l+1}) \tag{1.1}$$

for any vector fields $Z_1, \cdots, Z_{l+1}$ on $\mathscr{F}(M)$. Moreover, if Ξ is a tensorial form of type (r, F), then $D\Xi$ is given by ([10])

$$D\Xi = d\Xi + \bar{r}(\omega) \cdot \Xi\,, \tag{1.2}$$

where $(\bar{r}, F)$ denotes the induced representation of the Lie algebra $\mathfrak{gl}(n, R)$.

We shall denote by Ω and Θ the curvature form and torsion form of a given connexion Γ respectively, that is, $\Omega = D\omega$ and $\Theta = D\theta$. Concernig the curvature form and the torsion form, we have the following structure equations ([7]):

$$d\omega = -\frac{1}{2}[\omega, \omega] + \Omega\,. \tag{1.3}$$

$$d\theta = -\bar{\rho}(\omega) \cdot \theta + \Theta\,. \tag{1.4}$$

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1) As to the definition of tensorial forms, see [10].

Let X be a vector field on M. For a differential form Ξ, the Lie derivative $\mathcal{L}_X\Xi$ of Ξ with respect to X is defined by

$$\mathcal{L}_X\Xi = \lim_{t\to 0}\frac{1}{t}\{\varphi_t^*\Xi - \Xi\}, \tag{1.5}$$

where φ_t denote the local transformations generated by X ([8]).

For X and $Y\in T(M)$, we have ([8])

$$X\cdot\Xi(Y) = (\mathcal{L}_X\Xi)(Y)+\Xi([X, Y]). \tag{1.6}$$

$$2d\Xi(X, Y) = X\cdot\Xi(Y)-Y\cdot\Xi(X)-\Xi([X, Y]). \tag{1.7}$$

$$2d\Xi(X, Y) = (\mathcal{L}_X\Xi)(Y)-(\mathcal{L}_Y\Xi)(X)+\Xi([X, Y]). \tag{1.8}$$

A vector field X on M induces a vector field $\tilde{X}$ on $\mathcal{F}(M)$ in the following manner ([7]). For each $x\in\mathcal{F}(M)$ and $u=\pi(x)$, X generates a local 1-parameter group of local transformations φ_t in a neighboriood U of u. Each φ_t induces a local 1-parameter group of transformations $\tilde{\varphi}_t$ in $\pi^{-1}(U)$ and $\tilde{\varphi}_t$ induce a vector field $\tilde{X}$ on $\pi^{-1}(U)$. Since $\tilde{\varphi}_t$ commute with right translation R_g ($g\in GL(n, R)$), the induced vector field $\tilde{X}$ is invariant under right translations;

$$R_{g*}\tilde{X} = \tilde{X}. \tag{1.9}$$

It can be shown by straightfoward calculation that $\tilde{\varphi}_t$ leave the basic form θ invariant. Hence we have from (1.5)

$$\mathcal{L}_{\tilde{X}}\theta = 0. \tag{1.10}$$

§ 2. Several lemmas

3. Keeping the notation of the preceding section, we shall prove several lemmas which will be used in the following.

LEMMA 2.1. *Let f be a tensor on $\mathcal{F}(M)$ of type (r, F), then for any $A\in\mathfrak{gl}(n, R)$ and $x\in\mathcal{F}(M)$, we have*

$$\sigma(A)_x f = -\bar{r}(A)f(x),$$

where $\sigma(A)$ is the fundamental vector field[2] *corresponding to A and*

2) Cf. [7].

$(\bar{r}, F)$ *is the induced representation of* (r, F).

Proof.

$$\sigma(A)_x f = \lim_{t\to 0} \frac{1}{t} \{f(R_{\exp tA}\cdot x) - f(x)\}$$
$$= \lim_{t\to 0} \frac{1}{t} \{r((\exp tA)^{-1}) f(x) - f(x)\} = -\bar{r}(A) f(x). \quad \text{q.e.d.}$$

For any vector field X on M, we define the differentiable functions β_X and γ_X on $\mathscr{F}(M)$ as follows

$$\beta_X(x) = \omega_x(\tilde{X}) \qquad \text{for} \quad x \in \mathscr{F}(M),$$

and

$$\gamma_X(x) = \theta_x(\tilde{X}) \qquad \text{for} \quad x \in \mathscr{F}(M).$$

From (1.9) we see that

(2.1) β_X *is a tensor of type* $(ad, \mathfrak{gl}(n, R))$ *on* $\mathscr{F}(M)$.

(2.2) γ_X *is a tensor of type* (ρ, E) *on* $\mathscr{F}(M)$.

Let Z be any vector field on $\mathscr{F}(M)$ and A be the element of $\mathfrak{gl}(n, R)$ such that $\sigma(A)_x = vZ_x$. From (2.2) and Lemma 2.1, it follows that

$$\sigma(A)_x \cdot \theta(\tilde{X}) = -\bar{\rho}(A)\theta(\tilde{X}) = -A\cdot\theta(\tilde{X}),$$

and hence

(2.3) $$vZ\cdot\theta(\tilde{X}) = -\omega(Z)\cdot\theta(\tilde{X}).$$

Since $\overline{ad}(A)B = [A, B]$ for $A, B \in \mathfrak{gl}(n, R)$, we have similarly

(2.4) $$vZ\cdot\omega(\tilde{X}) = -[\omega(Z), \omega(\tilde{X})].$$

LEMMA 2.2. *For any vector fields X and Y on M,*

$$\theta([\tilde{X}, \tilde{Y}]) = 2d\theta(\tilde{X}, \tilde{Y}) = \omega(\tilde{Y})\cdot\theta(\tilde{X}) - \omega(\tilde{X})\cdot\theta(\tilde{Y}) + 2\Theta(\tilde{X}, \tilde{Y}).$$

Proof. From (1.10) it follows that $\mathcal{L}_{\tilde{X}}\theta = 0$ and $\mathcal{L}_{\tilde{Y}}\theta = 0$. Therefore, making use of (1.6) and (1.7), we have

$$2d\theta(\tilde{X}, \tilde{Y}) = \theta([\tilde{X}, \tilde{Y}]).$$

The right hand side is nothing but the structure equation (1.4).

LEMMA 2.3. *For any vector fields X and Y on M,*

$$\omega([\tilde{X}, \tilde{Y}]) = h\tilde{X}\cdot\omega(\tilde{Y}) - h\tilde{Y}\cdot\omega(\tilde{X}) - [\omega(\tilde{X}), \omega(\tilde{Y})] - 2\Omega(\tilde{X}, \tilde{Y}).$$

Proof. From (1.3), (1.7) and (2.4), we have

$$\begin{aligned} 2\Omega(\tilde{X}, \tilde{Y}) &= 2d\omega(\tilde{X}, \tilde{Y}) + [\omega(\tilde{X}), \omega(\tilde{Y})] \\ &= \tilde{X}\cdot\omega(\tilde{Y}) - \tilde{Y}\cdot\omega(\tilde{X}) - \omega([\tilde{X}, \tilde{Y}]) + [\omega(\tilde{X}), \omega(\tilde{Y})] \\ &= h\tilde{X}\cdot\omega(\tilde{Y}) - h\tilde{Y}\cdot\omega(\tilde{X}) - \omega([\tilde{X}, \tilde{Y}]) + v\tilde{X}\cdot\omega(\tilde{Y}) - v\tilde{Y}\cdot\omega(\tilde{X}) \\ &\qquad + [\omega(\tilde{X}), \omega(\tilde{Y})] \\ &= h\tilde{X}\cdot\omega(\tilde{Y}) - h\tilde{Y}\cdot\omega(\tilde{X}) - ([\tilde{X}, \tilde{Y}]) - [\omega(\tilde{X}), \omega(\tilde{Y})]. \end{aligned}$$

LEMMA 2.4. *Let X be a vector field on M and Z be a vector field on $\mathcal{F}(M)$, then we have*

$$hZ\cdot\theta(\tilde{X}) = \omega(\tilde{X})\cdot\theta(Z) - 2\Theta(\tilde{X}, Z).$$

Proof. Making use of (1.10), (1.6), (1.7), (1.4) and (2.3), we obtain

$$\begin{aligned} 0 = (\mathcal{L}_{\tilde{X}}\theta)(Z) &= \tilde{X}\cdot\theta(Z) - \theta([\tilde{X}, Z]) \\ &= 2d\theta(\tilde{X}, Z) + Z\cdot\theta(\tilde{X}) \\ &= \omega(Z)\cdot\theta(\tilde{X}) - \omega(\tilde{X})\cdot\theta(Z) + 2\Theta(\tilde{X}, Z) + hZ\cdot\theta(\tilde{X}) \\ &\qquad - \omega(Z)\cdot\theta(\tilde{X}) \\ &= -\omega(\tilde{X})\cdot\theta(Z) + 2\theta(\tilde{X}, Z) + hZ\cdot\theta(\tilde{X}), \quad \text{q.e,d.} \end{aligned}$$

We say that a vector field X on M is an *infinitesimal automorphism of a given connexion* ω, if the local transformations φ_t generated by X are all local automorphisms of the given connexion ω ([7]). Concerning infinitesimal automorphisms of a connexion, we shall prove the following two lemmas.

LEMMA 2.5. *If X is an infinitesimal automorphism of a connexion ω, then it holds that, for any vector field Z on $\mathcal{F}(M)$,*

$$hZ\cdot\omega(\tilde{X}) = 2\Omega(Z, \tilde{X}),$$

where Ω denotes the curvature from of ω.

Proof. Since the local transformations $\tilde{\varphi}_t$ induced by $\tilde{X}$ leave the connexion form ω invariant, we see from (1.5) that $\mathcal{L}_{\tilde{X}}\omega = 0$. Hence, by virtue of (1.6), (1.7). (1.3) and (2.4), we obtain

$$\begin{aligned}0 = (\mathcal{L}_{\tilde{X}}\omega)(Z) &= \tilde{X}\cdot\omega(Z)-\omega([\tilde{X}, Z]) \\ &= 2d\omega(\tilde{X}, Z)+Z\cdot\omega(\tilde{X}) \\ &= -[\omega(\tilde{X}), \omega(Z)]+2\Omega(\tilde{X}, Z)+hZ\cdot\omega(\tilde{X})+vZ\cdot\omega(\tilde{X}) \\ &= 2\Omega(\tilde{X}, Z)+hZ\cdot\omega(\tilde{X}).\end{aligned}$$

LEMMA 2.6. *If X and Y are infinitesimal automorphisms of a connexion ω, then it holds that*

$$\omega([\tilde{X}, \tilde{Y}]) = 2\Omega(\tilde{X}, \tilde{Y})-[\omega(\tilde{X}), \omega(\tilde{Y})].$$

Proof. From the fact that $(\mathcal{L}_{\tilde{X}}\omega)(\tilde{Y})=0$ and from (1.6), it follows that

$$\omega([\tilde{X}, \tilde{Y}]) = \tilde{X}\cdot\omega(\tilde{Y}).$$

Using (2.4) and Lemma 2.5, we have

$$\tilde{X}\cdot\omega(\tilde{Y}) = h\tilde{X}\cdot\omega(\tilde{Y})+v\tilde{X}\cdot\omega(\tilde{Y}) = 2\Omega(\tilde{X}, \tilde{Y})-[\omega(\tilde{X}), \omega(\tilde{Y})].$$

This proves our lemma.

Now we shall study the tensor β_X.

LEMMA 2.7. *There is a one-to-one correspondence between the set of tensors κ of type $(ad, \mathfrak{gl}(n, R))$ on $\mathcal{F}(M)$ and the set of $(1,1)$-tensor fields*[3] *K on M. The correspondence is given by*

$$\kappa(x)\cdot\theta(X^*) = x^{-1}\cdot K_{\pi(x)}(X),$$

where X is a tangent vector at $u=\pi(x)$ and X^* is the lift of X. Moreover it holds that

$$(\nabla_X K)_{\pi(x)}(Y) = x\cdot(X^*\kappa)\cdot x^{-1}Y, \qquad X, Y\in T_{\pi(x)}(M),$$

where $\nabla_X K$ denotes the covariant derivative of K with respect to X.

Proof. The first half of lemma is obvious. We shall prove the second part. Using the formula for the definition of $\nabla_X Y$ ([7]):

$$(\nabla_X Y)_u = x\cdot(X^*\cdot\theta(Y^*)), \qquad \pi(x) = u,$$

we obtain

$$x^{-1}\cdot[\nabla_X(K(Y))] = X^*_x\cdot\{\kappa\cdot\theta(Y^*)\} = (X^*_x)\theta(Y^*)+\kappa(x)\cdot X^*_x\theta(Y^*).$$

3) As to the definition of tensor fields, see [7].

On the other hand, it is known[4] that

$$(\nabla_X K)(Y) = \nabla_X(K(Y)) - K(\nabla_X Y) .$$

Hence we have

$$\begin{aligned}(\nabla_X K)(Y) &= x\cdot(X_x^*\kappa)\cdot x^{-1}Y + x\cdot\kappa(x)\cdot x^{-1}(\nabla_X Y) - K(\nabla_X Y)\\ &= x\cdot(X_x^*\kappa)\cdot x^{-1}Y ,\end{aligned}$$

because $x\cdot\kappa(x)\cdot x^{-1}(\nabla_X Y) = K(\nabla_X Y)$. Thus we have proved the lemma.

Now, for any vector field X on M, we define the $(1, 1)$-tensor fields B_X and T_X on M as follows:

$$B_X(Y) = -\nabla_Y X ,$$

and

$$T_X(Y) = T(X, Y) ,$$

where T denotes the torsion tensor field. We define the $(1, 1)$-tensor field A_X[5] on M by

$$A_X = T_X - B_X .$$

Then A_X *corresponds to the tensor* β_X *of type* $(ad, \mathfrak{gl}(n, R))$ *in the sense of Lemma* 2.7. In fact, according to Lemma 2.4, we see that $\omega(\tilde{X})\cdot\theta(Y^*) = Y^*\cdot\theta(X^*) + 2\Theta(X^*, Y^*)$, for any vector fields X and Y on M. From the definition of torsion tensor field T, it follows that $2x\cdot\Theta_x(X^*, Y^*) = T_{\pi(x)}(X, Y)$. On the other hand, we have

$$x\cdot(Y^*\cdot\theta(X^*)) = \nabla_Y X = -B_X(Y) .$$

Thus we conclude that

$$x\cdot\omega_x(\tilde{X})\cdot x^{-1}Y = A_{X,\pi(x)}(Y) .$$

Suppose X to be an infinitesimal automorphism of a connexion. By Lemma 2.7, the formula in Lemma 2.5 is written in the following form

$$\nabla_Y A_X = R(Y, X) ,$$

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4) Cf. [7].

5) A_X is defined by Kostant [4] in Riemannian case.

where R is the curvature tensor fiield, that is, $R_{\pi(x)}(X_1, X_2)X_3=2x\cdot\Omega_x(X_1^*, X_2^*)x^{-1}X_3^*$, $X_1, X_2, X_3 \in T_{\pi(x)}(M)$. Thus we have the well-known formulae

$$\text{(2.5)} \qquad \begin{cases} \nabla_Y X = A_X(Y) - T(X, Y). \\ \nabla_Y A_X = R(Y, X). \end{cases}$$

§ 3. Infinitesimal automorphisms of G-structures

4. We say that an n--dimensional differentiable manifold M possesses a *G-structure* when the structure group of the frame bundle $\mathscr{F}(M)$ of M is reducible to a Lie subgroup G of $GL(n, \mathrm{R})$.

Suppose that M possesses a G-structure and denote by $H(G)$ the reduced bundle. From the definition of the reduced bundle, there is the injection $\iota: H(G) \to \mathscr{F}(M)$. We call a connexion Γ' in $H(G)$ a *reduced G-connexion.* Given a reduced G-connexion Γ' in $H(G)$, the injection ι maps Γ' into a connexion Γ in $\mathscr{F}(M)$ (see [7]). The linear connexion thus obtained is called a *G-connexion* in $\mathscr{F}(M)$.

The following proposition follows immediately from the definition of G-connexion.

PROPOSITION 3.1. *A G-connexion Γ has the following properties:*
(I) *The holonomy group Ψ_b with reference point $b \in H(G)$ of Γ is contained in G.*
(II) *Each Γ-horizontal manifold through $b \in H(G)$ is a submanifold of the reduced bundle $H(G)$.*
(III) *If ω is the connexion from of Γ, then the connexion form of reduced G-connexion Γ' is $\iota^*\omega$.*
(IV) *If Ξ is a differential form on $\mathscr{F}(M)$, then*

$$\iota^*(D\Xi) = D'(\iota^*\Xi),$$

where D (resp. D') denotes the covariant differentiation operator with respect to Γ (resp. Γ'). In particular, if Ω is the curvature form of Γ, then the curvature form of Γ' is $\iota^*\Omega$.

Thus the curvature form Ω of a G-connexion restricted to $H(G)$ has its values in $\mathcal{G}$, the Lie algebra of G.

Let φ be a differentiable transformation of M onto itself. Then φ induces naturally a differentiable transformation $\tilde{\varphi}$ of $\mathscr{F}(M)$ in the following manner. Any frame $x=(t_1, \cdots, t_n)$ at $u=\pi(x)$ is mapped into the frame $\tilde{\varphi}(x)=(\varphi_* t_1, \cdots, \varphi_* t_n)$. The induced transformation $\tilde{\varphi}$ is an automorphism of $\mathscr{F}(M)$, that is, $\tilde{\varphi}$ satisfies the conditions: $\pi\circ\tilde{\varphi}=\varphi\circ\pi$ and $\tilde{\varphi}\circ R_g=R_g\circ\tilde{\varphi}$ for every $g\in GL(n, \mathrm{R})$.

Given a G-structure on a differentiable manifold M, a differentiable transformation φ of M is called an *automorphism of the G-structure* if the induced transformation $\tilde{\varphi}$ maps each element of $H(G)$ into an element of $H(G)$.

DEFINITION 3.1. We say that a vector field X on M is an *infinitesimal automorphism of a given G-structure*, if the local transfomations φ_t generated by X are all local automorphisms of the given G-structure.

PROPOSITION 3.2. *A vector field X on M is an infinitesimal automorphism of a G-structure if and only if the vertical component of $\tilde{X}$ with respect to any G-connxion is tangent to $H(G)$ at every point of $H(G)$.*

Proof. X is an infinitesimal automorphism of the G-structure if and only if $\tilde{X}_b\in T_b(H(G))$ for every $b\in H(G)$. On the other hand, according to Propositon 3.1, (II), the horizontal component $h\tilde{X}$ of $\tilde{X}$ with respect to any G-connexion is tangent to $H(G)$ at $b\in H(G)$. Hence we have proved the proposition.

From Proposition 3.2, it follows immediately

PROPOSITION 3.3. *A vector field X on M is an infinitesimal automorphism of a G-structure if and only if*

$$\omega_b(\tilde{X})\in \mathcal{G}$$

for every point b of $H(G)$, where ω is the connexion form of a G-connexion.

PROPOSITION 3.4. *The set of all infinitesimal automorphisms of G-structure forms a Lie algebra under the usual bracket operation for vector fields.*

Proof. Let X and Y be infinitesimal automorphisms of G-structure and let $\tilde{X}$ and $\tilde{Y}$ be the induced vector fields on $\mathscr{F}(M)$. Let ω be the connexion from of a G-connexion and Ω its curvature form. We first remark that $\widetilde{[X, Y]}=[\tilde{X}, \tilde{Y}]$. It follows from Proposition 3. 3 that $\omega_b(\tilde{X})\in \mathcal{G}$ and $\omega_b(\tilde{Y})\in \mathcal{G}$ for every $b\in H(G)$, and hence

(i) $\quad [\omega_b(\tilde{X})\omega_b(\tilde{Y})]\in \mathcal{G} \quad$ for every $b\in H(G)$.

Since the horizontal component $h\tilde{X}$ of $\tilde{X}$ is tangent to $H(G)$ at $b\in H(G)$, we see that

(ii) $\quad h\tilde{X}_b\cdot\omega(\tilde{Y})\in \mathcal{G} \quad$ for every $b\in H(G)$.

Finally, from Proposition 3. 1, (IV), we have

(iii) $\quad \Omega_b(\tilde{X}, \tilde{Y})\in \mathcal{G} \quad$ for every $b\in H(G)$.

Thus from (i), (ii), (iii) and Lemma 2. 3, we conclude that

$$\omega_b(\widetilde{[X, Y]})\in \mathcal{G} \quad \text{for every } b\in H(G),$$

which proves our Proposition 3. 4.

5. In the rest of this section, we shall confine ourselves to the case where G has property $(\mathscr{P})$ and we shall consider the canonical G-connexion whose existence has been proved in [1]. We shall prove the following

LEMMA 3. 1. *If an automorphism φ of G-structure is an automorphism of reduced G-connexion, then φ is an automorphism of G-connexion.*

Proof. Let Q_x be the horizontal subspace at $x\in\mathscr{F}(M)$ with respect to the G-connexion. Every element x of $\mathscr{F}(M)$ can be written as $x=R_g\cdot b$ with $g\in GL(n, R)$, $b\in H(G)$. Taking account of the fact that $\tilde{\varphi}\circ R_g=R_g\circ\tilde{\varphi}$, we have

$$\begin{aligned}\tilde{\varphi}_*Q_x &= \tilde{\varphi}_*Q_{bg} = \tilde{\varphi}_*R_gQ_b = R_{g*}\tilde{\varphi}_*Q_b = R_{g*}Q_{\tilde{\varphi}(b)} = Q_{\tilde{\varphi}(b)g}\\ &= Q_{\tilde{\varphi}(x)}, \quad \text{q.e.d.}\end{aligned}$$

In the previous paper [1] we have proved that when G has property $(\mathscr{P})$ an automorphism of G-structure is an automorphism of the canonical reduced G-connexion. Hence, from the above Lemma 3.1, we have

PROPOSITION 3.5. *Assume that G has the property* ($\mathscr{P}$). *Then an automorphism of the G-structure is an automorphism of the canonical G-connexion.*

For a moment, we shall use the symbol $\mathscr{T}_u$ to denote the tangent space of M at $u \in M$. Let b be an element of $H(G)$ such that $\pi(b)=u$. Taking account of the fact that b gives a linear isomorphism of E onto $\mathscr{T}_u$, we see that $b\cdot\bar{\rho}(\mathscr{G})\cdot b^{-1}$ is a Lie algebra of endomorphisms of $\mathscr{T}_u$. We put $\mathscr{G}(\mathscr{T}_u)=b\cdot\bar{\rho}(\mathscr{G})\cdot b^{-1}$, which is independent of the choice of $b \in H(G)$ such that $\pi(b)=u$.

We introduce into $\mathscr{G}(\mathscr{T}_u)+\mathscr{T}_u$ a bracket operation by setting

$$[A_1, A_2] = A_1\cdot A_2 - A_2\cdot A_1, \qquad [A, t] = -[t, A] = A(t),$$
$$[t_1, t_2] = 2b\cdot\Omega_b(t_2^*, t_1^*)\cdot b^{-1} + 2b\cdot\Theta_b(t_2^*, t_1^*),$$

for $A, A_1, A_2 \in \mathscr{G}(\mathscr{T}_u)$ and $t, t_1, t_2 \in \mathscr{T}_u$, where Ω and Θ denote the curvature form and torsion form of the canonical G-connexion and t_1^*, t_2^* denote the horizontal vector at b such that $\pi_* t_1^* = t_1$, $\pi_* t_2^* = t_2$.

We note that $\mathscr{G}(\mathscr{T}_u)+\mathscr{T}_u$ is not in general a Lie algebra under this bracket.

Let $\mathscr{A}$ be the Lie algebra of infinitesimal automorphisms of G-structure and let $\mathscr{B}$ be the set of all infinitesiml automorphisms of the canonical G-connexion. $\mathscr{B}$ is a Lie algebra under the usual bracket operation for vector fields [7]. According to Proposition 3.5, if G has property ($\mathscr{P}$), then $\mathscr{A}$ is a subalgebra of $\mathscr{B}$.

PROPOSITION 3.6. *Let M be a connected differentiable manifold with a G-structure. Assume that G has the property* ($\mathscr{P}$). *Let $\Delta : \mathscr{A} \to \mathscr{G}(\mathscr{T}_u)+\mathscr{T}_u$ be the mapping defined by*

$$\Delta(X) = -b\cdot\omega_b(\tilde{X})\cdot b^{-1} - b\cdot\theta_b(\tilde{X}) \qquad \textit{for} \quad X \in \mathscr{A},$$

where ω denotes the connexion form of the canonical G-connexion and θ denotes the basic form. Then Δ is an isomorphism of $\mathscr{A}$ onto $\Delta(\mathscr{A})$.

Proof. We first remark that, under our assumption, an infinitesimal automorphism of the G-structure is an infinitesimal automorphism of the canonical G connexion. Let $\tau(t)$ be a differentiable curve in M and $Y(t)$ be the tangent vector to the curve at $\tau(t)$. According to (2. 5), an infinitesimal automorphism X of the G-structure satisfies the following system of differential equations along the curve $\tau(t)$:

$$\begin{cases} \nabla_{Y(t)}X(t) = A(t)Y(t) - T(X(t), Y(t)) \\ \nabla_{Y(t)}A(t) = R(Y(t), X(t)), \end{cases}$$

where $X(t)=X_{\tau(t)}, A(t)=A_{X_{\tau(t)}}$ and ∇ denotes the covariant differential with respect to the canonical G-connexion. Therefore, an infinitesimal automorphism X of the G-structure is uniquely determined by the values of X and A_X at any single point of M. This implies that Δ is one-to-one.

We shall show that Δ is a homomorphism. Let $X, Y \in \mathcal{A}$. Since $[\tilde{X}, \tilde{Y}]=\widetilde{[X, Y]}$, we have

$$\Delta([X, Y] = -b\cdot\omega_b([\tilde{X}, \tilde{Y}])\cdot b^{-1} - b\cdot\theta([\tilde{X}, \tilde{Y}]) .$$

But since X and Y are infinitesimal automorphisms of the canonical G-connexion, we have by Lemma 2. 6 that

$$\omega([\tilde{X}, \tilde{Y}]) = 2\Omega(\tilde{X}, \tilde{Y}) - [\omega(\tilde{X}), \omega(\tilde{Y})] .$$

Moreover, by Lemma 2. 2 we have

$$\theta([\tilde{X}, \tilde{Y}]) = \omega(\tilde{Y})\cdot\theta(\tilde{X}) - \omega(\tilde{X})\cdot\theta(\tilde{Y}) + 2\Theta(\tilde{X}, \tilde{Y}) .$$

Thus we have

$$\begin{aligned} \Delta([Y, Y]) = & -2b\cdot\Omega(\tilde{X}, \tilde{Y})\cdot b^{-1} + b\cdot\{[\omega(\tilde{X}), \omega(\tilde{Y})]\}\cdot b^{-1} \\ & - b\cdot\omega(\tilde{Y})\cdot\theta(\tilde{X}) + b\cdot\omega(\tilde{X})\theta(\tilde{Y}) - 2b\cdot\Theta(\tilde{X}, \tilde{Y}). \end{aligned}$$

On the other hand, from the definition of the bracket operation in $\mathcal{G}(\mathcal{T}_u^-) + \mathcal{T}_u^-$, it follows that

$$\begin{aligned} [\Delta(X), \Delta(\tilde{Y})] &= [-b\cdot\omega(\tilde{X})\cdot b^{-1} - b\cdot\theta(\tilde{X}),\ -b\cdot\omega(\tilde{Y})\cdot b^{-1} - b\cdot\theta(\tilde{Y})] \\ &= b\cdot\{[\omega(\tilde{X}), \omega(\tilde{Y})]\cdot b^{-1} + b\cdot\omega(\tilde{X})\cdot\theta(\tilde{Y}) - b\cdot\omega(\tilde{Y})\cdot\theta(\tilde{X}) \\ &\quad - 2b\cdot\Omega(X^*, Y^*)\cdot b^{-1} - 2b\cdot\Theta(X^*, Y^*) . \end{aligned}$$

Since $h\tilde{X}=X^*$ and $hY=Y^*$, we see that $\Omega(X^*, Y^*)=\Omega(\tilde{X}, \tilde{Y})$ and $\Theta(X^*, Y^*)=\Theta(\tilde{X}, \tilde{Y})$. Hence we conclude that

$$\Delta([X, Y]) = [\Delta(X), \Delta(Y)].$$

COROLLARY 1[6]. *Under the same assumption as in Proposition* 3. 6, *it holds that*

$$dim\ \mathcal{A} \leqq dim\ G + dim\ M.$$

Making use of Palais's theorem [8, Theorem VII, Chap. IV], we have from the finite dimensionality of $\mathcal{A}$

COROLLARY 2[6]. *Under the same assumption as in Proposition* 3. 6, *the group of all automorphisms of a G-structure is a Lie group.*

§ 4. Holonomy and infinitesimal automorphisms

6. We consider the holonomy group Ψ_x with reference point x of a connexion. The holonomy theorem [7] states: The holonomy algebra σ_x, the Lie algebra of Ψ_x, is the subalgebra of $\mathfrak{gl}(n, R)$ which is generated by all elements of the form $\Omega_y(X^*, Y^*)$, $y\in\mathfrak{H}_x$, where Ω denotes the curvature form and X^* and Y^* are arbitrary horizontal vectors at y.

We shall prove the following

LEMMA 4. 1. *Let M be a simply-connected differentiable manifold of dimension n. If the holonomy algebra* σ_x *with reference point* $x\in\mathcal{F}(M)$ *of a connexion* ω *in* $\mathcal{F}(M)$ *is weakly reductive in* $\mathfrak{gl}(n, R)$, *then, for any infinitesimal automorphism X of the connexion, we have*

$$\omega_y(\tilde{X})\in N(\sigma_x) \qquad for \quad y\in\mathfrak{H}_x,$$

where $N(\sigma_x)$ *denotes the normalizer of* σ_x *in* $\mathfrak{gl}(n, R)$ *and* $\mathfrak{H}_x$ *denotes the holonomy manifold through x.*

Proof. Since σ_x is weakly reductive in $\mathfrak{gl}(n, R)$, there exists a subspace $\mathfrak{n}$ of $\mathfrak{gl}(n, R)$ such that $\mathfrak{gl}(n, R)=\sigma_x+\mathfrak{n}$ (direct sum) and $[\sigma_x, \mathfrak{n}]\subset\mathfrak{n}$. For any element A of $\mathfrak{gl}(n, R)$, we denote by $A_{\mathfrak{n}}$ (resp.

6) Cf. [5].

A_{σ_x}) the $\mathfrak{n}$-component (resp. σ_x-component) of A. It is well known that the structural group of the prinicipal bundle $\mathfrak{H}_x$ is the holonomy group Ψ_x with reference point x. The simply-connectedness of M implies that Ψ_x is connected. Therefore from the weak reductivity of σ_x, it follows that $\omega(\tilde{X})_{\mathfrak{n}}$ is a tensor of type $(ad, \mathfrak{n})$ on $\mathfrak{H}_x$. From the holonomy theorem, we know that $\Omega_y(X^*, Y^*)\in\sigma_x$, where $y\in\mathfrak{H}_x$ and $X,Y\in T(M)$. Since $Y^*\cdot\omega(\tilde{X})_{\mathfrak{n}}\in\mathfrak{n}$ and $Y^*\cdot\omega(\tilde{X})_{\sigma_x}\in\sigma_x$, it follows from Lemma 2. 5 that $Y^*\cdot\omega(\tilde{X})_{\mathfrak{n}}=0$, that is, $\omega(\tilde{X})_{\mathfrak{n}}$ is constant on $\mathfrak{H}_x$. Consequently we have for any $a\in\Psi_x$

$$\omega_y(\tilde{X})_{\mathfrak{n}}=\omega_{ya}(\tilde{X})_{\mathfrak{n}}=ad(a^{-1})\omega_y(\tilde{X})_{\mathfrak{n}},$$

and hence, for any $A\in\sigma_x$, $[A,\omega_y(\tilde{X})_{\mathfrak{n}}]=0$. Thus we have

$$[\omega_y(\tilde{X}), A]=[\omega_y(\tilde{X})_{\sigma_x}, A]\in\sigma_x \qquad \text{for} \quad A\in\sigma_x,$$

which proves our assertion.

We shall seek for the condition that an infinitesimal automorphism of a G-connexion is an infinitesimal automorphism of a G-structure.

We recall that $\mathfrak{H}_b\subset H(G)$ and $\sigma_b\subset\mathcal{G}$ for $b\in H(G)$. Every point z of $H(G)$ can be written in the form $z=y\cdot g$, where $g\in G$ and $y\in\mathfrak{H}_b$. Then from Lemma 4. 1 we see that $\omega_{yg}(\tilde{X})=ad(g^{-1})\omega_y(\tilde{X})\in ad(g^{-1})N(\sigma_b)$. Thus if $\sigma_b=N(\sigma_b)$ for a single point $b\in H(G)$, then $\omega_z(\tilde{X})\in\mathcal{G}$ for every $z\in H(G)$.

On the other hand, S. Kobayashi [3] has proved the following

(4. 1) *Suppose that the subalgebra σ_b of $\mathfrak{gl}(n, R)$ satisfies the following conditions*: (i) *$\overline{ad}(\sigma_b)$ is irreducible,* (ii) *σ_b is reductive in* $\mathfrak{gl}(n, R)$ *in the sense of Koszul,* (iii) *σ_b does not contain any non-trivial ideal of* $\mathfrak{gl}(n, R)$. *Then $N(\sigma_b)=\sigma_b$.*

If σ_b is reductive in $\mathfrak{gl}(n, \mathrm{R})$ in the sense of Koszul, then σ_b is weakly reductive in $\mathfrak{gl}(n, R)$. Consequently, we have, combining these facts with Proposition 3. 3

PROPOSITION 4. 1. *Let M be a simply-connected differentiable manifold with a G-structure. Suppose that the holonomy group Ψ_b*

with reference point $b \in H(G)$ *of a G connexion satisfies the following conditions*: (i) $\overline{ad}(\sigma_b)$ *is irreducible*, (ii) *the holonomy algebra* σ_b *is reductive in* $\mathfrak{gl}(n, R)$ *in the sense of Koszul*, (iii) σ_b *does not contain any non-trival ideal. Then an infinitesimal automorphism of the G-connexion is also an infinitesimal automorphism of the G-structure.*

§ 5. Homogeneous *G*-structures

7. Let K be a connected Lie group, L be a closed subgroup of K. Denote by $\mathfrak{K}$ and $\mathfrak{L}$ the Lie algebra of K and L respectively. Let K/L be a reductive homogeneous space of dimension n. Namely there exists a subspece $\mathfrak{m}$ of $\mathfrak{K}$ such that $\mathfrak{K}=\mathfrak{m}+\mathfrak{L}$ and $ad(L)\mathfrak{m}\subset\mathfrak{m}$. Let p be the natural projection $K\rightarrow K/L$ and $p(e)=u_0$. Each element k of K defines a differentiable transformation $\tau(k)$ of K/L. Since $\tau(l)u_0=u_0$ for $l\in L$, $\tau(l)$ induces a linear transformation $\tau(l)_*$ of the tangent space at u_0 onto itself, which is the same as $ad(l)$ on $\mathfrak{m}$. Thus we obtain the so-called linear isotopy representation α of L, $\alpha: L\rightarrow GL(n, R)$. We shall denote by $\tilde{L}$ the linear isotropy group, that is $\tilde{L}=\alpha(L)$. Each differentiable transformation $\tau(k)$ induces an automorphism $\widetilde{\tau(k)}$ of the frame bundle $\mathscr{F}(K/L)$ of K/L. Thus it holds that

$$R_a\circ\widetilde{\tau(k)} = \widetilde{\tau(k)}\circ R_a\,, \quad a\in GL(n, R)\,. \tag{5.1}$$

$$\tau(k)\circ\pi = \pi\circ\widetilde{\tau(k)}\,, \tag{5.2}$$

where π is the projection of the frame bundle $\mathscr{F}(K/L)$.

DEFINITION 5.1. A G-structure on a reductive homogeneous space K/L is called an *invariant G structure* if every $\tau(k)$, $k\in K$, is an automorphism of G-structure.

Let x_0 be the frame at $u_0=p(e)$ such that $x_0\cdot\xi=p_*\xi$ for $\xi\in\mathfrak{m}$. If we fix a base $\xi_1, \cdots, \xi_n$ of $\mathfrak{m}$, then x_0 may be identified with $(u_0, p_*\xi_1, \cdots, p_*\xi_n)$. It is easily verified that $\tau(l)x_0=R_{\alpha(l)}x_0$ for $l\in L$. Now we define the map $\chi: K\rightarrow\mathscr{F}(K/L)$ as follows

$$\chi(k) = \widetilde{\tau(k)}\,x_0 \qquad \text{for any} \quad k\in K\,. \tag{5.3}$$

Then we see that

(5. 4) $\quad \chi(kl) = R_{\alpha(l)}\chi(k) \quad$ for $\quad k\in K \quad$ and $\quad l\in L$,

and

(5. 5) $\quad \chi\circ\Phi_k = \widetilde{\tau(k)}\circ\chi \quad$ for $\quad k\in K$,

where Φ_k denotes the left translation of K corresponding to $k\in K$. Therefore χ is a homomorphism of the principal bundle $K(K/L, L)$ into the frame bundle $\mathscr{F}(K/L)$. It can be readily verified that $\chi(K)$ is a reduced bundle of $\mathscr{F}(K/L)$ and has the structural group $\tilde{L}$. Moreover, from (5. 5), we see that each $\widetilde{\tau(k)}$ leaves $\chi(K)$ invariant. Thus we obtain

(5. 6)[7] *A reductive homogenous space K/L possesses an invariant $\tilde{L}$-structure.*

Now we shall prove the following

PROPOSITION 5. 1. *In order that a reductive homogeneous space K/L admits an invariant G-structure it is necessary and sufficient that there exists an element a of $GL(n, R)$ such that $aGa^{-1}\supset\tilde{L}$.*

Proof. Suppose that K/L admits an invariant G--structure and denote by $H(G)$ its reduced bundle. Take a frame b_0 of $H(G)$ at u_0. Then there exists an element a of $GL(n, R)$ such that $b_0=R_a x_0$. Denote by $\tilde{L}_u$ (resp. G_u) the fibre over u of $\chi(K)$ (resp. $H(G)$). Then $R_a\tilde{L}_{u_0}\subset G_{u_0}$. In fact, any element x of $\tilde{L}_{u_0}$ can be written as $x=R_{\alpha(l)}x_0=\widetilde{\tau(l)}\,x_0$. Hence

$$R_a x = R_a\widetilde{\tau(l)}\,x_0 = \widetilde{\tau(l)}\,R_a x_0 = \widetilde{\tau(l)}\,b_0\in H(G)\,.$$

Since $\tilde{L}_{\tau(k)u_0}=\widetilde{\tau(k)}\,\tilde{L}_{u_0}$ and $G_{\tau(k)u_0}=\widetilde{\tau(k)}\,G_{u_0}$, we see that

$$R_a\tilde{L}_{\tau(k)u_0} = R_a\widetilde{\tau(k)}\,\tilde{L}_{u_0} = \widetilde{\tau(k)}\,R_a\tilde{L}_{u_0}\subset\widetilde{\tau(k)}\,G_{u_0} = G_{\tau(k)u_0}\,.$$

Therefore $R_a\chi(K)\subset H(G)$. This implies that $G\supset a^{-1}\tilde{L}a$.

Conversely, let G be a Lie subgroup of $GL(n, R)$ such that $aGa^{-1}\supset\tilde{L}$, $a\in GL(n, R)$. $R_a\chi(K)$ is a principal bundle over K/L

7) Added in Proof. A similar result was obtained independently by D. Bernard. (See D. Bernard: Sur la géométrie différentielle des G-structures, Thèse, (1960)).

with structural group $a^{-1}\tilde{L}a$. Moreover, from (5.2) and (5.5) it follows that $R_a\chi(K)$ is invariant by $\tau(k)$, $k\in K$. Let $H(G)$ be the principal bundle over K/L which is obtained from $R_a\chi(K)$ by enlarging the structural group from $a^{-1}\tilde{L}a$ to G. Clearly $H(G)$ is invariant by all $\tau(k)$, $k\in K$. Hence K/L admits an invariant G-structure. Thus we have proved the proposition.

We consider next invariant connexion on K/L. Since K/L is reductive, there exists an invariant connexion $\Gamma_{\mathfrak{m}}$ in the principal bundle $K(K/L, L)$ (see [7]). Namely $\Gamma_{\mathfrak{m}}$-horizontal subspace at $k\in K$ is $\Phi_{k*}\mathfrak{m}$. The homomorphism χ of $K(K/L, L)$ into $\mathscr{F}(K/L)$ maps the above invariant connexion $\Gamma_{\mathfrak{m}}$ in $K(K/L, L)$ into Γ_0 in $\mathscr{F}(K/L)$. Thus the horizontal subspace at $R_a\chi(k)$ with respect to Γ_0 is $R_{a*}\chi_*\Phi_{k*}\mathfrak{m}$. This connexion Γ_0 is nothing but the canonical connexion of the second kind in the sense of Nomizu [6]. From the construction of Γ_0, we easily see that Γ_0 is reducible to a connexion in $R_a\chi(K)$ which is invariant connexion. Consequently we have the following two propositions.

PROPOSITION 5.2. *The canonical connexion of the second kind on a reductive homogeneous space is an invariant $\tilde{L}$-connexion.*

PROPOSITION 5.3. *Suppose that a reductive homogeneous space admits an invariant G-structure. Then the canonical connexion of the second kind is an invariant G-connexion.*

It is well known [6] that the canonical connexion of the second kind on a symmetric homogeneous space is without torsion. On the other hand, if there exists a G-connexion without torsion, then the structure tensor of G-structure vanishes (see. [1]). Hence we have

COROLLARY 1. *If a symmetric homogeneous space admits an invariant G-structure, then the structure tensor of the G-structure vanishes.*

Finally we shall prove the following

PROPOSITION 5.4. *Let K/L be a reductive homogeneous space with a fixed decomposition of the Lie algebra $\mathfrak{K}=\mathfrak{m}+\mathfrak{L}$, $ad(L)\mathfrak{m}\subset\mathfrak{m}$.*

Suppose that K/L admits an invariant G-structure. Then there exists a one-to-one correspondence between the set of all invariant G-connexions and the set of all linear maps Λ of $\mathfrak{m}$ into $\mathcal{G}$ such that

$$\Lambda \circ ad(l) = ad(l) \circ \Lambda \circ ad(l^{-1}) \qquad \textit{for} \quad l \in L .$$

Proof. Let Γ_0 be the canonical connexion of the second kind and let ω_0 be the restriction of the connexion form of Γ_0 to the reduced buudle $H(G)$. Take any invariant G-connexion Γ. We denote by ω the restriction of the connexion form of Γ to $H(G)$. Put $\lambda = \omega - \omega_0$. Since ω_0 and ω are both invariant $\mathcal{G}$-valued forms on $H(G)$, we see that λ is an invariant tensorial 1-form of type $(ad, \mathcal{G})$ on $H(G)$. Conversely, given an invariant tensorial 1-form λ of type $(ad, \mathcal{G})$ on $H(G)$, then $\omega_0 + \lambda$ gives rise to an invariant reduced G-connexion. Thus we see that there exists a one-to-one correspondence between the set of all invariant G-connexions and the set of all invariant tensorial 1-forms of type $(ad, \mathcal{G})$ on $H(G)$.

For an invariant 1-form λ of type $(ad, \mathcal{G})$ on $H(G)$, we define

$$\Lambda_u(X)\, Y = b \cdot \lambda_b(X^*) \cdot b^{-1} Y ,$$

where $\pi(b) = u$, and $X, Y \in T_u(K/L)$ and X^* denotes the lift of X with respect to Γ_0. Clearly this definition is independent of the choice of $b \in H(G)$ such that $\pi(b) = u$. Then Λ is an invariant (1, 2)-tensor field on K/L:

$$[\Lambda_{\tau(k)u}(\tau(k)_* X)](\tau(k)_* Y) = \tau(k)_*[\Lambda_u(X) \cdot Y] .$$

In fact, since $\widetilde{\tau(k)}_*$ maps each Γ_0-horizontal subspace onto a Γ_0-horizontal subspace and $\tau(k) \circ \pi = \pi \circ \widetilde{\tau(k)}$, we obtain $(\tau(k)_* X)^*_{\widetilde{\tau(k)}b} = \widetilde{\tau(k)}_* X^*_b$ by the uniqueness of a lift. Therefore we obtain

$$\lambda_{\widetilde{\tau(k)}b}((\tau(k)_* X)^*) = \lambda_{\widetilde{\tau(k)}b}(\widetilde{\tau(k)}_* X^*) = \lambda_b(X^*)$$

because λ is an invariant tensorial 1-form. On the other hand, for any $\xi \in \mathfrak{m}$, we have $[\widetilde{\tau(k)}b] \cdot \xi = \tau(k)_*(b \cdot \xi)$, $b \in H(G)$, $k \in K$, and hence $[\widetilde{\tau(k)}b]^{-1} \tau(k)_* Y = b^{-1} Y$. Thus we have

$$\begin{aligned}[\Lambda_{\tau(k)u}(\tau(k)_* X)](\tau(k)_* Y) &= [\widetilde{\tau(k)}b] \cdot \lambda_{\widetilde{\tau(k)}b}((\tau(k)_* X)^*) [\widetilde{\tau(k)}b]^{-1} \tau(k)_* Y \\ &= \tau(k)_*[b \cdot \lambda_b(X^*) \cdot b^{-1} Y] = \tau(k)_*[\Lambda_u(X) Y] .\end{aligned}$$

In particular, the invariance of Λ at $p(e)$ by $\tau(l)$ implies

$$\Lambda_{u_0}(\tau(l)_*X)Y = \tau(l)_*\Lambda_{u_0}(X)\tau(l^{-1})_*Y. \tag{5.7}$$

Conversely, given a (1, 2)-tensor Λ_{u_0} on $T_{u_0}(K/L)$ which satisfies the relation (5.7) and such that $\Lambda_{u_0}(X)\in b_0\mathcal{G}b_0^{-1}$, $\pi(b_0)=p(e)$, then by the transitivity of K we can define the invariant (1, 2)-tensor field Λ on K/L such that $\Lambda_u(X)\in b\mathcal{G}b^{-1}$, $\pi(b)=u$. Hence we obtain the tensorial 1-form of type $(ad, \mathcal{G})$ on $H(G)$. Remarking that $b\cdot\mathcal{G}\cdot b^{-1}$ is isomorphic to $\mathcal{G}$ and $T_{u_0}(K/L)$ is isomorphic to $\mathfrak{m}$, we have proved the proposition.

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