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REDUCTION OF AN INFINITE SYSTEM OF INTEGRAL EQUATIONS OF POTENTIAL TYPE ON A ONE-DIMENSIONAL LATTICE OF CLOSED CURVES IN THE PLANE TO A FINITE SYSTEM OF INDEPENDENT PSEUDODIFFERENTIAL EQUATIONS ON A CIRCLE

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ABSTRACT. An infinite system of integral equations which arises in the reduction of the Dirichlet problem for the Helmholtz equation in the plane to the boundary is considered. The boundary is formed by an infinite network of nonintersecting infinitely smooth simple closed curves obtained from a fixed one by a parallel translation by vectors belonging to a one-dimensional lattice. It is shown that if the righthand side of the system is a T-periodic function on the lattice, then the system can be reduced to a system of T independent pseudodifferential equations on the unit circle with classic elliptic pseudodifferential operators of order -1 in the Sobolev scale. Another significant outcome of this work is that this reduction allows one to apply the many known powerful methods for the numerical analysis of classic elliptic pseudodifferential equations on the unit circle to the original system.

1. Introduction. It is well known (see, for example, [**1**–**3**]) that the Dirichlet boundary value problem for the Helmholtz equation in the plane, with radiation condition at infinity and the boundary formed by a finite number N of nonintersecting infinitely smooth simple closed curves, can be reduced to a system of N integral equations. The $N \times N$ matrix integral operator of this system was proved to be an $N \times N$ matrix classic elliptic pseudodifferential operator of order -1 in the Sobolev scale of N-dimensional complex vector functions on the unit circle. The practical benefit of viewing the original system of integral equations as a system of pseudodifferential equations is that one can then apply the many known powerful methods for the

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numerical analysis of such systems of pseudodifferential equations (see, for example, [**4**] and [**5**] and references therein).

In the present paper we generalize this idea to the case in which the boundary is formed in a particular way by an infinite number of such curves. We specify two natural conditions under which we are able to reduce our original infinite system to a finite system of independent pseudodifferential equations on the unit circle with classic elliptic pseudodifferential operators of order -1 in the Sobolev scale.

Let γ be an infinitely smooth simple closed curve in the plane with coordinates (x_1, x_2). Without loss of generality, we assume the length of γ to be equal to 2π. Let $\bar{x}(\xi) = (x_1(\xi), x_2(\xi))$ define the curve γ parametrically where ξ is the natural parameter on γ. It is obvious that $\bar{x}(\xi)$ is an infinitely smooth function. We denote by Γ the following infinite network of curves in the plane:

$$\Gamma = \{\gamma_m : \bar{x}_m(\xi) = \bar{x}(\xi) + \bar{h}m, m \in Z, \xi \in [0; 2\pi]\},$$

where $\bar{h}$ is the vector $(h, 0)$ with $h > \sup\{|x_1(\xi) - x_1(\eta)| : \xi, \eta \in [0, 2\pi]\}$, $\bar{x}_m(\xi)$ defines the curve γ_m parametrically, and Z is the set of integers.

Remark. Since γ is a closed curve, $\sup\{|x_1(\xi) - x_1(\eta)| : \xi, \eta \in [0, 2\pi]\}$ is finite. This supremum is chosen as a lower bound for h to ensure that no two curves from γ intersect. Then Γ is an infinite set of nonintersecting identical curves γ_m obtained from a fixed one γ by a parallel translation by vectors $\bar{h}m$, $m \in Z$, which form a one-dimensional lattice with a lattice constant h.

We will treat a complex function on a curve γ_m in Γ as a complex function on the unit circle S, and we will identify it with the corresponding 2π-periodic function on the real line.

Let us define the following infinite system of integral equations in the unknown functions u_l on γ_l in Γ:

$$(1) \qquad \sum_{l=-\infty}^{\infty} A_{m-l} u_l = f_m, \qquad m \in Z,$$

where f_m is a given complex function on γ_m and A_{m-l} is an integral operator defined by

$$(2) \qquad (A_{m-l}u_l)(\xi) = \frac{i}{4} \int_0^{2\pi} H_0^{(2)}(k|\bar{x}(\xi) - \bar{x}(\eta) - \bar{h}(m-l)|)u_l(\eta)\, d\eta,$$

where $k > 0$, $H_0^{(2)}$ is the Hankel function and $|\cdot|$ stands for the standard Cartesian distance in the plane.

We comment that system (1) can be obtained in a standard way as a reduction of the Dirichlet boundary-value problem for the Helmholtz equation in the plane to the boundary formed by the network of curves Γ. This problem arises in radio physics, electrical engineering, acoustic and so on (see, for example, [**6**] and [**7**]).

We recall [**1**] that the Sobolev space $H_t = H_t(S)$ of complex functions on S is the completion of the space $C^\infty(S)$ of infinitely smooth complex functions on S with respect to the norm

$$||u||_t = \left(\sum_{p=-\infty}^{\infty}(1+p^2)^t|C_p(u)|^2\right)^{1/2}$$

where $C_p(u)$ are the Fourier coefficients of the function u with respect to the system $\{e^{ip\xi}, p \in Z\}$.

Also, we recall that if g is an N-periodic map from a set of positive integers $1, 2, \ldots, N$ into a Banach space, then the discrete Fourier transform of g is defined by

$$\hat{g}_s = (F_{m\to s}g)_s = \sum_{m=1}^{N} e^{-i(2\pi/N)sm} g_m, \tag{3}$$

where $\hat{g}$ stands for the discrete Fourier image of g.

The inverse discrete Fourier transform is given by

$$g_m = (F^{-1}_{s\to m}\hat{g})_m = \frac{1}{N}\sum_{s=1}^{N} e^{i(2\pi/N)ms}\hat{g}_s.$$

In this paper we will reserve the subscript s to denote the image g_s of the discrete Fourier transform of g, omitting the hat.

Let us formulate the following conditions:

Condition 1. *Periodicity.* There exists a positive integer T such that $f_{m+T} = f_m$ and $u_{m+T} = u_m$ for each m in Z. Hereafter T will stand for the least such integer.

Condition 2. *Nonresonance.* The product $khT \neq 0 \bmod 2\pi$.

Remark. Our periodicity condition is a very natural one and holds in numerous applications dealing with wave propagation, that is, with processes periodic with respect to the network of curves Γ.

Remark. The T-periodicity of the function f_m on the lattice corresponds to the Th-periodicity of an incident field (for example, an electromagnetic or acoustic field) in the x_1 direction in the plane. Since k has the physical meaning of a wave number, the nonresonance condition means that the corresponding wave length of free space $\lambda = 2\pi/k$ does not fit into the distance Th an integral number of times.

We denote by $\mathcal{A}_s$ the following operator:

$$\mathcal{A}_s = \sum_{m=1}^{T} e^{-i(2\pi/T)sl}\mathcal{A}_l, \qquad s = 1, \dots, T \tag{4}$$

where the operator $\mathcal{A}_l$ is given by

$$\mathcal{A}_l = \sum_{v=-\infty}^{\infty} A_{l+vT}, \qquad l \in Z. \tag{5}$$

We consider the following system of T independent equations in the unknown functions u_s:

$$\mathcal{A}_s u_s = f_s, \qquad s = 1, \dots, T, \tag{6}$$

where the functions f_s are given by

$$f_s = \sum_{m=1}^{T} e^{-i(2\pi/T)sm} f_m.$$

We are now ready to present the main result of this paper.

Theorem. *Let Conditions* 1 *and* 2 *hold, and let f_m belong to H_t for all $m = 1, \dots, T$. Then*:

(i) *Systems* (1) *and* (6) *are equivalent in the sense that their solutions* u_l *and* u_s *are discrete Fourier images of each other*:

$$u_l = \frac{1}{T}\sum_{s=1}^{T} e^{i(2\pi/T)ls} u_s;$$

(ii) *The operators* $\mathcal{A}_s$ *for all* $s = 1, \ldots, T$ *are classic elliptic pseudodifferential operators of order* -1 *in the Sobolev scale on* S;

(iii) *All solutions* u_l *of system* (1) *belong to* H_{t-1} *for all* $l = 1, \ldots, T$.

Remark. Under Conditions (1) and (2) the theorem reduces the problem of solving system (1) to that of system (6) which consists of T *independent* pseudodifferential equations on the unit circle with classic elliptic pseudodifferential operators of order -1 in the Sobolev scale. Also, since the kernel subspaces of such pseudodifferential operators are at most finite dimensional [**1**], the operators $\mathcal{A}_s$, for all $s = 1, \ldots, T$, are invertible except at most on some finite dimensional subspaces.

2. Proof of the Theorem. In order to prove the theorem, we need some preparation.

Lemma 1. *The function*

$$G_l(\xi,\eta) = \sum_{\substack{v=-\infty \\ v:l+vT\neq 0}}^{\infty} H_0^{(2)}(k|\bar{x}(\xi) - \bar{x}(\eta) + \bar{h}(l+vT)|)$$

is infinitely smooth on the torus $S \times S$ *whenever* $l \in Z$ *and* $khT \neq 0 \bmod 2\pi$.

Proof. In order to prove Lemma 1 it is enough to demonstrate that the series

$$(7) \qquad \sum_{\substack{v=-\infty \\ v:l+vT\neq 0}}^{\infty} \frac{\partial^i}{\partial \xi^i}\frac{\partial^j}{\partial \eta^j} H_0^{(2)}(k|\bar{x}(\xi) - \bar{x}(\eta) + \bar{h}(l+vT)|)$$

converges uniformly to a continuous function on $S \times S$, whenever $l \in Z$, $khT \neq 0 \bmod 2\pi$ and $i, j \in Z_+$ (Z_+ stands for the set of nonnegative integers). Using the asymptotic behavior of the Hankel function $H_0^{(2)}$ and its derivatives for large values of their arguments [**8**] and the infinite smoothness of the function $\bar{x}(\xi)$, it is easy to show that the summands of the above series have the following asymptote as $|v| \to \infty$:

$$\frac{e^{-ikhT|v|}}{|v|^{1/2}} C_{i,j}(\xi, \eta) + O(|v|^{-3/2}),$$

where $C_{i,j}(\xi, \eta) \in C^\infty(S \times S)$ for all $i, j \in Z_+$, $C_{i,j}(\xi, \eta)$ does not depend on v and the asymptote $O(|v|^{-3/2})$ is understood in the sense of the topology on $C^\infty(S \times S)$. Note that $C^\infty(S \times S)$ stands for the space of infinitely smooth complex functions on the torus $S \times S$ with the standard topology.

From the Dirichlet convergence test for Fourier series and from the absolute convergence of a series with summands of the form $O(|v|^{-\alpha})$ for $\alpha > 1$ as $|v| \to \infty$, we conclude that series (7) converges uniformly with respect to ξ and η to a continuous function on $S \times S$ whenever $l \in Z$, $khT \neq 0 \bmod 2\pi$ and $i, j \in Z_+$. This completes the proof of Lemma 1. □

For each $l \in Z$ we consider the following sequence of operators:

$$A_l^n = \sum_{|v| \leq n} A_{l+vT}, \qquad n \in Z_+.$$

We denote by $|| \cdot ||_{(t_1 \to t_2)}$ the norm in the Banach space $O_{(t_1 \to t_2)}$ of bounded linear operators acting from H_{t_1} to H_{t_2}.

In what follows, whenever we refer to the order of an operator we mean the order in the Sobolev scale on the unit circle S.

Lemma 2. *Let Condition* (2) *hold. Then the operator* $\mathcal{A}_l$ *defined in expression* (5) *has the following properties*:

(i) $\mathcal{A}_l$ *is a classic elliptic pseudodifferential operator* (PDO) *of order* -1 *if* $l = 0 \bmod T$ *and is an operator of order* $-\infty$ *if* $l \neq 0 \bmod T$;

(ii) *For each* $l \in Z$ *and* $t \in R$, $||\mathcal{A}_l - A_l^n||_{(t \to t+1)} \to 0$ *as* $n \to \infty$;

(iii) *For each* $l \in Z$, $\mathcal{A}_{l+T} = \mathcal{A}_l$.

Proof. From expressions (2) and (5) we obtain the following expression for the kernel of the operator $\mathcal{A}_l$:

$$(8)\qquad K_l(\xi,\eta) = \frac{i}{4}\sum_{v=-\infty}^{\infty} H_0^{(2)}(k|\bar{x}(\xi)-\bar{x}(\eta)+\bar{h}(l+vT)|), \qquad l \in Z.$$

In order to prove (i), we split the kernel (8) into two summands:

$$(9)\qquad K_l(\xi,\eta) = K_0(\xi,\eta) + K_l'(\xi,\eta)$$

where

$$K_0(\xi,\eta) = \frac{i}{4}H_0^{(2)}(k|\bar{x}(\xi)-\bar{x}(\eta)|)\sum_{v=-\infty}^{\infty}\delta(l+vT),$$

$$K_l'(\xi,\eta) = \frac{i}{4}\sum_{\substack{v=-\infty \\ v:l+vT\neq 0}}^{\infty} H_0^{(2)}(k|\bar{x}(\xi)-\bar{x}(\eta)+\bar{h}(l+vT)|),$$

and $\delta(m)$ is the Kronecker symbol.

By Lemma 1, the function $K_l'(\xi,\eta)$ belongs to $C^\infty(S\times S)$ for every $l \in Z$. In this case it is well known [**1**] that the operator with the kernel (9) is a classic elliptic PDO of order -1 if $l = 0 \bmod T$ and is an operator of order $-\infty$ if $l \neq 0 \bmod T$.

Let us prove (ii). Consider the kernel $\Delta K_l^n(\xi,\eta)$ of the operator $\Delta\mathcal{A}_l^n = \mathcal{A}_l - A_l^n$ given for $n > |l/T|$ by

$$\Delta K_l^n(\xi,\eta) = \frac{i}{4}\sum_{|v|\geq n} H_0^{(2)}(k|\bar{x}(\xi)-\bar{x}(\eta)+\bar{h}(l+vT)|).$$

To prove (ii) it is enough to show that, if $l \in Z$ and $khT \neq 0 \bmod 2\pi$, the kernel $\Delta K_l^n(\xi,\eta)$ converges to the zero function in the topology of $C^\infty(S\times S)$ as $n\to\infty$. But this follows directly from Lemma 1.

In order to finish the proof of Lemma 2, it is enough to notice that (iii) follows directly from the explicit form (8) of the kernel $K_l(\xi,\eta)$ of the operator $\mathcal{A}_l$. $\square$

Now we are ready to prove the theorem stated in the introduction.

Proof of the Theorem. Using the operator $\mathcal{A}_l$ from expression (5) and Conditions (1) and (2), we rewrite system (1) in the following way:

$$\sum_{l=1}^{T} \mathcal{A}_{m-l} u_l = f_m, \qquad m = 1, \dots, T. \tag{10}$$

Let us prove (iii) first. Let $\mathcal{A}$ be the $T \times T$ matrix operator with the elements $\mathcal{A}_{m-l}$ where $m, l = 1, \dots, T$. Then system (10) becomes

$$\mathcal{A}u = f,$$

where u and f are T-dimensional vector functions with the elements u_l and f_m, respectively. According to Lemma 2 and to reference [**1**], $\mathcal{A}$ is a matrix classic elliptic PDO of order -1 in the Sobolev scale of T-dimensional complex vector functions on S. Therefore, all solutions u_l of system (10) belong to H_{t-1} for every $l = 1, \dots, T$.

Now let us prove (i). The action of the $T \times T$ matrix operator $\mathcal{A}$ on the T-dimensional vector function u given by the lefthand side of expression (10) can be viewed as a discrete convolution over variables m and l. Diagonalizing the matrix operator $\mathcal{A}$ by applying the discrete Fourier transform to both sides of system (10), we arrive at system (6) where the functions u_s are given by the discrete Fourier transform of u_l:

$$u_s = \sum_{l=1}^{T} e^{-i(2\pi/T)sl} u_l.$$

Let us comment that use of the discrete Fourier transform is justified since $\mathcal{A}_l \in O_{(t-1\to t)}$, $f_m \in H_t$ and $u_l \in H_{t-1}$ for every $m, l = 1, \dots, T$, where $O_{(t-1\to t)}$, H_t and H_{t-1} are Banach spaces. Since the discrete Fourier transform is nondegenerate, we conclude that system (10) and system (6) are equivalent. The solution of system (10), and hence of system (1), can be found as the inverse discrete Fourier transform of the solution of system (6), as we stated in the Theorem.

In order to prove (ii), it is enough to notice that, according to expressions (4), (5) and to Lemma 2, the operator $\mathcal{A}_s$ for every $s = 1, \dots, T$ is the linear combination of T operators: $\mathcal{A}_l|_{l=T}$, which is a classic elliptic PDO of order -1, and $\mathcal{A}_l$ with $l = 1, \dots, T-1$, which are operators of the order $-\infty$. □

Conclusion. Under natural conditions of periodicity and nonresonance we reduce the infinite system (1) of integral equations on the infinite network Γ of nonintersecting infinitely smooth simple closed curves to a finite system (6) of independent pseudodifferential equations on the unit circle with classic elliptic pseudodifferential operators of order -1 in the Sobolev scale. The curves are obtained from a fixed one by parallel translation by vectors belonging to a one-dimensional lattice. System (1) is equivalent to the Dirichlet boundary-value problem for the Helmholtz equation in the plane with the boundary formed by the network of curves Γ. This problem arises in radio physics, electrical engineering, acoustic and so on. Another significant outcome of this work is that this reduction allows one to apply the many known powerful methods for the numerial analysis of classic elliptic pseudodifferential equations on the unit circle to the original system (1).

The result presented in this paper can be generalized easily to the case of a finite number N of layers of networks Γ with, generally speaking, different curves for each layer. In this case, the operators $\mathcal{A}_s$ of system (6) are $N \times N$ matrix classic elliptic pseudodifferential operators of order -1 in the Sobolev scale of N-dimensional complex vector functions on the unit circle. Also, we point out that the periodicity condition may be dropped from the theorem. In this case, system (6) becomes a one-parameter family of pseudodifferential equations whose properties we will investigate in the forthcoming papers.

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