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PSEUDOSPECTRA OF WIENER-HOPF INTEGRAL OPERATORS AND CONSTANT-COEFFICIENT DIFFERENTIAL OPERATORS

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ABSTRACT. A number $z \in \mathbf{C}$ is in the ε-pseudospectrum of a linear operator A if $||(zI - A)^{-1}|| \geq \varepsilon^{-1}$. In this paper, we investigate the ε-pseudospectra of Volterra Wiener-Hopf integral operators and constant-coefficient differential operators with boundary conditions at one endpoint for the interval $[0, b]$. We show that although the spectra of these operators are not continuous in the limit $b \to \infty$, the ε-pseudospectra are continuous as $b \to \infty$ for all $\varepsilon > 0$. These results are an extension of previous work on the pseudospectra of Toeplitz matrices.

1. Introduction. Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot, \cdot)$ and norm $||\cdot||$. Let $T : \mathcal{H} \to \mathcal{H}$ be a closed linear operator with domain $\mathcal{D}(T)$, spectrum $\Lambda(T)$, and resolvent set $\rho(T)$ [**9**]. For each $\varepsilon \geq 0$, the ε-pseudospectrum of T, which we denote by $\Lambda_\varepsilon(T)$, can be defined in the following manner [**21, 22**]:

Definition. For each $\varepsilon \geq 0$, a number $z \in \mathbf{C}$ is in the ε-pseudospectrum of T if

$$z \in \{\lambda \in \rho(T) : ||(\lambda I - T)^{-1}|| \geq \varepsilon^{-1}\} \cup \Lambda(T). \tag{1.1}$$

This definition is essentially equivalent to that for the set of ε-*approximate eigenvalues* introduced by Landau [**11**]. Similar sets have also been considered by other researchers; see [**21, 22**] for a discussion.

As the definition shows, the sets $\Lambda_\varepsilon(T)$ are nested and $\Lambda_0(T)$ is the spectrum. Pseudospectra were introduced by Trefethen [**20**] to analyze the behavior of *non-normal* matrices. A normal matrix satisfies $A^+A = AA^+$, where A^+ is the adjoint, and has orthogonal eigenfunctions. The ε-pseudospectrum of a normal matrix is simply the union of the closed

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disks of radius ε centered at the eigenvalues. In most applications, the behavior of a normal matrix is governed by the eigenvalues. This need not be the case for a non-normal matrix. The ε-pseudospectra of a non-normal matrix may be much larger than the spectrum, even if $\varepsilon \ll 1$. In applications involving non-normal matrices, it may be more appropriate to examine pseudospectra instead of spectra alone; see [**21**] for a discussion of various applications in numerical analysis. These ideas extend to operators.

It is straightforward to determine the pseudospectra of a matrix numerically: compute $||(zI - T)^{-1}||$ on a grid and then send the data to a contour plotter. Computations of this kind for operators arising in hydrodynamic stability are presented in [**17, 23**]. The pseudospectra of various non-normal matrices arising in numerical analysis are presented in [**21**].

It is more difficult to determine the pseudospectra of an operator or matrix analytically. In [**18**], asymptotic formulas for the resolvent norm of a convection-diffusion operator are derived. In [**17**], upper and lower bounds for the pseudospectra of a model operator arising in hydrodynamic stability are computed. There are many results in the literature giving analytical bounds on the resolvent norm in various applications [**9**]. For the most part, these analyses give either upper or lower bounds for the resolvent norm, so the sharpness of the results is not clear.

In recent work, Reichel and Trefethen have obtained analytical results on the pseudospectra of Toeplitz matrices [**19**]. We illustrate these results with an example. Consider the family of matrices $\{T_N\}$ defined by

$$T_N = \begin{bmatrix} 0 & 1 & 1 & & \\ & 0 & 1 & 1 & \\ & & 0 & 1 & 1 \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}_{N\times N}, \tag{1.2}$$

where N is the dimension. (The entries are non-zero on the first two super-diagonals only.)

A plot of the pseudospectra of T_N, which are computed numerically, is presented in Figure 1. (Here the l^2 norm is used.) For finite N, the

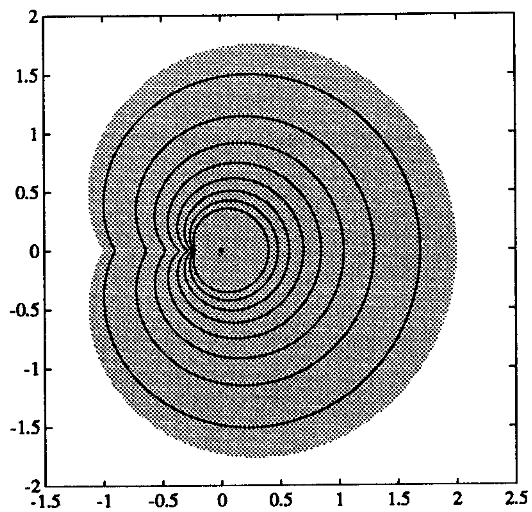

FIGURE 1. Pseudospectra of T_N for $N = 32$. The shaded region is $\Lambda(T_\infty)$. The ε-pseudospectrum of T_∞ is the shaded region plus a strip of thickness ε. The dot is $\Lambda(T_N)$. The contours (from inner to outer) are the boundaries of the pseudospectra of T_N for $\varepsilon = 10^{-8}, 10^{-7}, \ldots, 10^{-1}$.

spectrum of T_N is the origin, and the curves are the boundaries of the sets $\Lambda_\varepsilon(T_N)$ for $\varepsilon = 10^{-8}, 10^{-7}, \ldots, 10^{-1}$. The shaded region is the spectrum of T_∞, the semi-infinite version of T_N. For any $\varepsilon > 0$, the ε-pseudospectrum of T_∞ is the spectrum plus a border of thickness ε.

The analysis of upper-triangular Toeplitz matrices is based on the *symbol*, which for the example is the function $f(z) = z^2 + z$, the discrete Fourier transform of the doubly-infinite sequence $\{\ldots, 0, 1, 1, 0, \ldots\}$. Let Δ_ε be the closed disk of radius ε centered at the origin and let Δ be the closed unit disk. We assume that the sum of sets is defined by $U_1 + U_2 = \{z : z = z_1 + z_2,\ z_1 \in U_1,\ z_2 \in U_2\}$. The main results on the pseudospectra of the family $\{T_N\}$ are:

(I) $\Lambda_\varepsilon(T_\infty) = f(\Delta) + \Delta_\varepsilon$ for all $\varepsilon \geq 0$;

(II) if $\lambda \in \operatorname{int}(\Lambda(T_\infty))$, then $||(\lambda I - T_N)^{-1}|| \to \infty$ exponentially as $N \to \infty$;

(III) if $N \leq N' \leq \infty$, then $\Lambda_\varepsilon(T_N) \subseteq \Lambda_\varepsilon(T_{N'})$ for all $\varepsilon \geq 0$;

(IV) $\lim_{N\to\infty} \Lambda_\varepsilon(T_N) \to \Lambda_\varepsilon(T_\infty)$ for each $\varepsilon \geq 0$.

Setting $\varepsilon = 0$ in (I), we have $\Lambda(T_\infty) = f(\Delta)$, and this result follows from the fact that the vector $[1, z, z^2, \dots]^T$, with $|z| < 1$, is an eigenvector of T_∞ with eigenvalue z^2+z. Properties (III) and (IV) show that the pseudospectra are growing functions of N, and that as $N \to \infty$, the ε-pseudospectra of T_N converge to those of T_∞. Here, for each $\varepsilon > 0$ we define $\lim_{N\to\infty} \Lambda_\varepsilon(T_N) = \{z : z_N \to z \text{ for some } z_N \in \Lambda_\varepsilon(T_N)\}$. As our example shows, the sets $\Lambda(T_N)$ do not converge to $\Lambda(T_\infty)$. The results (I)–(IV) are valid for general triangular Toeplitz matrices. (Formula (I) must be modified for lower-triangular matrices.) The convergence result (IV) applies to non-triangular Toeplitz matrices as well [**24**].

The purpose of this paper is to extend the above results on the pseudospectra of Toeplitz matrices to Wiener-Hopf (W-H) integral operators and constant-coefficient (C-C) differential operators. Our focus is on "triangular" operators, a term that we define below.

Wiener-Hopf integral operators are defined in terms of a kernel function $\kappa(x)$ and are the continuous analogs of Toeplitz matrices. Let us assume that $\kappa \in L^1(-\infty,\infty) \cap L^2(-\infty,\infty)$ and suppose that $u \in L^2[0,b]$. We define the W-H operator W_b by

$$[W_b u](x) = \int_0^b \kappa(x-y)u(y)\,dy \tag{1.3}$$

and consider W_b as an operator from the space $L^2[0,b]$ to itself. The parameter b plays the same role as the dimension N for Toeplitz matrices. We say that W_b is triangular if $\kappa(x) \equiv 0$ for $x < 0$ or $x > 0$.

We also study constant-coefficient differential operators. Suppose that $u \in L^2[0,b]$ is sufficiently differentiable. We define the C-C operator A_b by

$$A_b u = a_n \frac{d^n u}{dx^n} + a_{n-1}\frac{d^{n-1}u}{dx^{n-1}} + \cdots + a_1 \frac{du}{dx} + a_0 u, \tag{1.4}$$

where the numbers $\{a_i\}$ are complex constants and it is assumed that $a_n \neq 0$. We combine (1.4) with $n = s_0 + s_b$ homogeneous boundary conditions of the form

$$u^{(i)}(0) = 0, \quad i = 0,1,\dots,s_0-1, \qquad u^{(i)}(b) = 0, \quad i = 0,1,\dots,s_b-1. \tag{1.5}$$

We say that A_b is triangular if either $s_0 = 0$ or $s_b = 0$.

As in the case of Toeplitz matrices, the analysis of W-H and C-C operators is based on a symbol, defined by a Fourier transform. The symbol of W_b is

$$\hat{\kappa}(\omega) = \int_{-\infty}^{\infty} \kappa(x) e^{i\omega x}\, dx, \tag{1.6}$$

and the symbol of A_b is

$$\hat{a}(\omega) = a_n(-i\omega)^n + a_{n-1}(-i\omega)^{n-1} + \ldots + a_1(-i\omega) + a_0. \tag{1.7}$$

Our main results show that conditions (I)–(IV) are valid for triangular W-H integral operators and C-C differential operators if f is replaced by $\hat{\kappa}$ or $\hat{a}$, the dimension N is replaced by the length b, and the unit disk Δ is replaced by the closed lower half-plane.

Our results for general W-H and C-C operators are not as complete, but we do present results on the *spectrum of the family* [**2, 4**] of the operators $\{W_b\}$ and $\{A_b\}$. Here is the definition

Definition. Let $\{T_\nu\}$ be a family of closed operators. A number $z \in \mathbf{C}$ is in the spectrum of the family, which we denote by $P(\{T_\nu\})$, if

$$\limsup_{\nu\to\infty} ||(zI - T_\nu)^{-1}|| = \infty. \tag{1.8}$$

We show that for the families $\{W_b\}$ and $\{A_b\}$, $P(\{W_b\}) = \Lambda(W_\infty)$ and $P(\{A_b\}) = \Lambda(A_\infty)$. This result can be considered as a weaker version of (IV). Since (IV) holds for Toeplitz matrices, we believe that it may hold for W-H and C-C operators, but we have not been able to establish this result.

Our results for general W-H operators are closely related to recent work by Anselone and Sloan [**1**]. Their definition of W_b is slightly different from ours and they use the $||\cdot||_\infty$ norm. They show that the spectrum of the family $P(\{W_b\})$ is the spectrum of W_∞ and that each open neighborhood of $\Lambda(W_\infty)$ contains the spectrum of W_b for all sufficiently large b.

Throughout this paper we let L^1 and L^2 denote the spaces $L^1(-\infty,\infty)$ and $L^2(-\infty,\infty)$, respectively. We let $||\cdot||$ denote the L^2 norm on the finite or infinite interval. To avoid ambiguity, we use the notation $||\cdot||_{[0,b]}$

to denote the L^2 norm for the interval $[0, b]$ when necessary. In addition we let $||\cdot||_1$ denote the L^1 norm on the infinite interval.

This paper is organized as follows. Section 2 briefly examines Wiener-Hopf integral operators and constant-coefficient differential operators defined on $(-\infty, \infty)$. Section 3 proves our main results for triangular W-H operators. Section 4 presents examples illustrating these results. Section 5 examines general W-H operators. Section 6 presents our main results for triangular C-C differential operators. Section 7 examines general constant-coefficient differential operators. This paper is adapted in part from [**16**], where the results on the pseudospectra of constant-coefficient differential operators were originally presented.

2. Pseudospectra for operators defined on the infinite interval. Before turning to the analysis of the pseudospectra of Wiener-Hopf and constant-coefficient differential operators defined on finite and semi-infinite intervals, we first examine the pseudospectra for operators defined on the infinite interval. Throughout this section we assume that the underlying Hilbert space is $\mathcal{H} = L^2$.

The infinite interval version of the W-H operator corresponding to $\kappa(x)$ is defined by

$$[\tilde{W}u](x) = \int_{-\infty}^{\infty} \kappa(x-y)u(y)\,dy. \tag{2.1}$$

The operator $\tilde{W}$ is bounded and maps $\mathcal{H}$ to itself. The expression (2.1) is the formula for the convolution $\kappa * u$, and this fact greatly simplifies the analysis. For example, Fourier analysis and an application of Parseval's relation imply

$$||\tilde{W}|| = \sup_{\omega\in\mathbf{R}} |\hat{\kappa}(\omega)|. \tag{2.2}$$

Using this last formula, it can be show that $||\tilde{W}|| \leq ||\kappa||_1$.

Here is our main result.

Theorem 2.1. *Let $\tilde{W}$ be the Wiener-Hopf integral operator defined by* (2.1)*. Let $\hat{\kappa}$ denote the symbol. The pseudospectra of $\tilde{W}$ are given by*

$$\Lambda_\varepsilon(\tilde{W}) = \hat{\kappa}(\mathbf{R}) + \Delta_\varepsilon \quad \forall\, \varepsilon \geq 0. \tag{2.3}$$

Proof. Consider the inhomogeneous problem $\lambda u - \tilde{W}u = f$, where $f \in \mathcal{H}$. Fourier transforming both sides, we obtain

$$(\lambda - \hat{\kappa}(\omega))\hat{u}(\omega) = \hat{f}(\omega), \tag{2.4}$$

where $\hat{u}$ and $\hat{f}$ are the Fourier transforms of u and f, respectively. If $\lambda \in \hat{\kappa}(\mathbf{R})$, then it can be shown that λ lies in the continuous spectrum of $\Lambda(\tilde{W})$ by considering a sequence of band-limited functions f. Hence, (2.3) holds for $\varepsilon = 0$.

Now, suppose $\lambda \notin \hat{\kappa}(\mathbf{R})$. We have

$$\hat{u}(\omega) = \frac{\hat{f}(\omega)}{\lambda - \hat{\kappa}(\omega)}. \tag{2.5}$$

Since $u = (\lambda I - \tilde{W})^{-1} f$, it follows from (2.5) that $(\lambda I - \tilde{W})^{-1}$ is the W-H integral operator with a kernel defined by the symbol $1/(\lambda - \hat{\kappa})$. Hence, by (2.2), it follows that

$$||(\lambda I - \tilde{W})^{-1}|| = \sup_{\omega \in \mathbf{R}} \frac{1}{|\hat{\kappa}(w) - \lambda|} = \frac{1}{\operatorname{dist}(\lambda, \hat{\kappa}(\mathbf{R}))}. \tag{2.6}$$

Here $\operatorname{dist}(\lambda, \hat{\kappa}(\mathbf{R}))$ denotes the distance from λ to the curve $\hat{\kappa}(\mathbf{R})$. It follows that $(\lambda I - \tilde{W})^{-1}$ is a bounded linear operator. The result (2.3) for $\varepsilon > 0$ follows from (2.6) by the connection between resolvents and pseudospectra. $\square$

The assumption that $\kappa \in L^1$ implies that $\hat{\kappa}$ is continuous for $\omega \in R$ and that $|\hat{\kappa}(\omega)| \to 0$ as $|\omega| \to \infty$. Hence, the spectrum of $\tilde{W}$ is a continuous closed curve containing the origin. The above theorem shows that the ε-pseudospectrum of $\tilde{W}$ consists of the spectrum plus a strip of thickness ε on each side of the spectrum. Finally, we note that the solution of $\lambda u - \tilde{W}u = f$ can be written in the form

$$u(x) = \frac{f(x)}{\lambda} + \frac{1}{\lambda} \int_{-\infty}^{\infty} g_\lambda(x-y) f(y)\, dy, \tag{2.7}$$

where $g_\lambda(x)$ is the inverse transform of $\hat{\kappa}/(\lambda - \hat{\kappa})$.

We now turn our attention to C-C differential operators. Let $\tilde{A}$ be a C-C operator without any boundary conditions defined on $(-\infty, \infty)$. The domain of $\tilde{A}$ is $\mathcal{D}(\tilde{A}) = Q_n(-\infty, \infty)$, where

$$Q_n[b_1, b_2] = \{u \in L^2[b_1, b_2] : u^{(n-1)} \text{ absolutely continuous in } [b_1, b_2],\ u^{(n)} \in L^2[b_1, b_2]\} \tag{2.8}$$

is the *maximal domain* [**9**].

The pseudospectra of $\tilde{A}$ satisfy a formula similar to (2.3), and the proof is similar to that given above for W-H operators.

Theorem 2.2. *Let $\tilde{A}$ be the constant-coefficient differential operator defined above. Let $\hat{a}$ be the symbol. The pseudospectra of $\tilde{A}$ are given by*

$$\Lambda_\varepsilon(\tilde{A}) = \hat{a}(\mathbf{R}) + \Delta_\varepsilon \quad \forall\, \varepsilon \geq 0. \tag{2.9}$$

The symbol $\hat{a}(\omega)$ is a polynomial. Hence, the spectrum and the pseudospectra are unbounded sets. The spectrum can be determined by examining the differential equation $\tilde{A}u - \lambda u = 0$; $\lambda \in \Lambda(\tilde{A})$ if and only if the differential equation has a solution of the form $e^{i\gamma x}$, where γ is a real constant.

The solution of the inhomogeneous problem $\lambda u - \tilde{A}u = f$, where $f \in L^2$, is given by a formula similar to (2.7). For later use, we note that if $||f|| = 1$ and $\lambda \notin \Lambda(\tilde{A})$, then $|u^{(j)}(x)| < C$ for some constant C independent of $x \in \mathbf{R}$ for $j = 0, 1, \ldots, n-1$. The proof is straightforward.

The formulas (2.3) and (2.9) suggest that there may be a simple formula relating the pseudospectra of C-C differential operators and W-H integral operators on the infinite interval. Suppose that $0 \notin \Lambda(\tilde{A})$. It is straightforward to show that $\tilde{A}^{-1}$ is the W-H operator $\tilde{W}$ with the symbol $\hat{\kappa} = 1/\hat{a}$. If we define the spectrum of $\tilde{A}$ so that it includes the point at ∞ and assume that $1/\infty \equiv 0$, then by the spectral mapping theorem [**9**], we have

$$\Lambda(\tilde{W}) = \{z : z = 1/\lambda,\ \lambda \in \Lambda(\tilde{A})\} = \hat{\kappa}(\mathbf{R}). \tag{2.10}$$

There does not appear to be a simple formula relating the sets $\Lambda_\varepsilon(\tilde{A})$ and $\Lambda_\varepsilon(\tilde{W})$ for $\varepsilon > 0$.

The formulas for the spectra of $\tilde{W}$ and $\tilde{A}$, particularly in the case of $\tilde{W}$, are well known [**5, 10**]. The formula for the resolvent norm, (2.6), appears to be less well known.

3. Pseudospectra of triangular Wiener-Hopf integral operators. We now examine the pseudospectra of triangular W-H operators. For definiteness we assume that $\kappa(x) = 0$ for $x > 0$. In this case we have

$$[W_b f](x) = \int_x^b \kappa(x-y) f(y)\, dy, \tag{3.1}$$

where we assume that $0 < b_0 \le b \le \infty$. The symbol is

$$\hat{\kappa}(\omega) = \int_{-\infty}^0 \kappa(x) e^{i\omega x}\, dx. \tag{3.2}$$

The condition $\kappa \in L^1$ guarantees that $\hat{\kappa}(\omega)$ is continuous in the closed lower half-plane, $\mathbf{C}^-$, and that it is analytic in the interior of $\mathbf{C}^-$ [**10**]. For each b, we consider W_b to be a map from $L^2[0,b]$ to itself. As we will see, the behavior of the pseudospectra of W_b is similar to that of the pseudospectra of an upper-triangular Toeplitz matrix.

The following theorem characterizes the pseudospectra of W_∞.

Theorem 3.1. *Let W_∞ be the Wiener-Hopf integral operator defined by* (3.1). *Let $\hat{\kappa}$ denote the symbol. The pseudospectra of W_∞ are given by*

$$\Lambda_\varepsilon(W_\infty) = \hat{\kappa}(\mathbf{C}^-) + \Delta_\varepsilon \quad \forall\, \varepsilon \ge 0. \tag{3.3}$$

Proof. We first derive a lower bound for the pseudospectra. For each $\gamma \in \mathrm{int}(\mathbf{C}^-)$, $\hat{\kappa}(\gamma)$ is an eigenvalue of W_∞ with the associated decaying eigenfunction $u = e^{-i\gamma x}$, and this is easy to show. We have

$$W_\infty u = \int_x^\infty \kappa(x-y) e^{-i\gamma y}\, dy = e^{-i\gamma x} \int_{-\infty}^0 \kappa(s) e^{i\gamma s}\, ds = \hat{\kappa}(\gamma) u. \tag{3.4}$$

It follows that $\hat{\kappa}(\text{int}(\mathbf{C}^-))\subseteq\Lambda(W_\infty)$. Since the spectrum is closed and $\hat{\kappa}(\omega)$ is continuous for $\omega \in \mathbf{C}^-$, it follows that $\hat{\kappa}(\mathbf{C}^-)\subseteq\Lambda(W_\infty)$.

For any closed operator T and number $\lambda \notin \Lambda(T)$, the resolvent satisfies [**9**]

$$||(zI - T)^{-1}|| \geq \frac{1}{\text{dist}(z, \Lambda(T))}. \tag{3.5}$$

Translated into the language of pseudospectra, (3.5) implies that $\Lambda(T)+\Delta_\varepsilon\subseteq\Lambda_\varepsilon(T)$ for all $\varepsilon \geq 0$. Using the lower bound for the spectrum derived above, it follows that $\hat{\kappa}(\mathbf{C}^-) + \Delta_\varepsilon\subseteq\Lambda_\varepsilon(W_\infty)$ for all $\varepsilon \geq 0$.

We now derive an upper bound for the pseudospectra. Suppose that $\lambda \notin \hat{\kappa}(\mathbf{C}^-)$. Consider the inhomogeneous equation

$$\lambda u - \int_x^\infty \kappa(x-y)u(y)\,dy = f, \quad x \in [0,\infty), \tag{3.6}$$

where $f \in L^2[0,\infty)$. Let us define $\tilde{f}$ such that $\tilde{f}(x) = f(x)$ for $x \geq 0$ and $\tilde{f}(x) = 0$ for $x < 0$ and consider the inhomogeneous problem $\lambda\tilde{u} - \tilde{W}\tilde{u} = \tilde{f}$ for the infinite interval. The solution, given in (2.7), is

$$\tilde{u}(x) = \frac{\tilde{f}(x)}{\lambda} + \frac{1}{\lambda}\int_{-\infty}^\infty g_\lambda(x-y)\tilde{f}(y)\,dy, \tag{3.7}$$

where $g_\lambda(x)$ is the inverse transform of $\hat{g}_\lambda(\omega) = \hat{\kappa}/(\lambda-\hat{\kappa})$. If $\lambda \notin \hat{\kappa}(\mathbf{C}^-)$, then $\hat{g}_\lambda(\omega)$ is analytic for ω in the open lower half-plane and $\hat{g}_\lambda(\omega) \to 0$ as $\omega \to \infty$ in this half-plane since $\hat{\kappa}$ satisfies these properties. Using standard results on contour integration, it can be shown that $g_\lambda(x) = 0$ for $x > 0$ [**10**], so (3.7) becomes

$$\tilde{u}(x) = \frac{\tilde{f}(x)}{\lambda} + \frac{1}{\lambda}\int_x^\infty g_\lambda(x-y)\tilde{f}(y)\,dy. \tag{3.8}$$

Note that $\tilde{u}(x)$ depends on $\tilde{f}(y)$ for $y \geq x$ only. This last remark implies that $u(x) = \tilde{u}(x)$ for $x \geq 0$. It follows that $||u||_{[0,\infty)} \leq ||\tilde{u}||_{(-\infty,\infty)}$. Since $||f||_{[0,\infty)]} = ||\tilde{f}||_{(-\infty,\infty)}$, we obtain

$$\begin{aligned} ||(\lambda I - W_\infty)^{-1}||_{[0,\infty)} &\leq ||(\lambda I - \tilde{W})^{-1}||_{(-\infty,\infty)} \\ &= \frac{1}{\text{dist}(\lambda, \hat{\kappa}(\mathbf{R}))} = \frac{1}{\text{dist}(\lambda, \hat{\kappa}(\mathbf{C}^-))}. \end{aligned} \tag{3.9}$$

The last result implies that $(\lambda - W_\infty)^{-1}$ is a bounded operator, so $\lambda \notin \Lambda(W_\infty)$. Combined with the lower bound derived in the first part of the proof, it follows that $\Lambda(W_\infty) = \hat{\kappa}(\mathbf{C}^-)$. Hence, (3.3) holds for $\varepsilon = 0$. The bound (3.9) also implies that $\Lambda_\varepsilon(W_\infty) \subseteq \hat{\kappa}(\mathbf{C}^-) + \Delta_\varepsilon$ for all $\varepsilon > 0$. Hence, (3.3) holds for all $\varepsilon > 0$ as well. □

This result shows that $\Lambda_\varepsilon(W_\infty)$ consists of $\hat{\kappa}(\mathbf{C}^-)$ plus a strip of thickness ε. Formula (3.3) is analogous to the formula (I) in the Introduction, with the unit disk replaced by the lower half-plane and with a geometrically decaying eigenvector $[1, z, z^2, \dots]^T$ replaced by an exponentially decaying eigenfunction $e^{-i\gamma x}$.

Let us now turn our attention to W_b for finite b. This operator is a compact Volterra operator. The compactness of W_b follows from the fact that

$$\int_0^b \int_0^b |\kappa(x-y)|^2 \, dx \, dy < \infty. \tag{3.10}$$

Compactness implies that the spectrum of W_b is discrete and nonempty [**9**]. The fact that W_b is a Volterra operator implies that W_b does not have any non-zero eigenvalues [**7**]. Hence, the spectrum is necessarily the singleton $\{0\}$.

What about the pseudospectra of W_b? We are not able to give a precise formula for the pseudospectra, but we can derive lower and upper bound sets $L_\varepsilon(W_b)$ and $U_\varepsilon(W_b)$ satisfying $L_\varepsilon(W_b) \subseteq \Lambda_\varepsilon(W_b) \subseteq U_\varepsilon(W_b)$. We start with the following identity [**9**]: if $\lambda \notin \Lambda(W_b)$, then

$$||(\lambda I - W_b)^{-1}|| = \left[\inf_{u \in \mathcal{D}(W_b)} \frac{||W_b u - \lambda u||}{||u||} \right]^{-1}. \tag{3.11}$$

A standard technique for determining a lower bound for $\Lambda_\varepsilon(W_b)$ is to search for functions that are close to achieving the optimum in (3.11). If there exists $u \in \mathcal{D}(W_b)$ such that $||u|| = 1$ and

$$||W_b u - \lambda u|| \le \varepsilon, \tag{3.12}$$

then $\lambda \in \Lambda_\varepsilon(W_b)$. The function u is called an *ε-pseudo-eigenfunction* [**17, 20**].

Motivated by the results in [**19**], we will see that the exponentials $e^{-i\gamma x}$ are appropriate choices as pseudo-eigenfunctions. Using these functions we can verify that the resolvent norm of W_b grows exponentially as in condition (II) in the Introduction. Suppose that $\gamma \in \operatorname{int}(\Lambda(W_\infty))$ and that $u = e^{-i\gamma x}$ is the associated eigenfunction. Then we have

(3.13)
$$\begin{aligned} W_b u - \lambda u &= \int_x^b \kappa(x-y) e^{-i\gamma y}\,dy - \int_x^\infty \kappa(x-y)e^{-i\gamma y}\,dy \\ &= -\int_b^\infty \kappa(x-y)e^{-i\gamma y}\,dy. \end{aligned}$$

Bounding the right-hand side of (3.13) we obtain $|W_b u - \lambda u| \leq ||\kappa||_1 e^{b \operatorname{Im} \gamma}$, which in turn implies that

$$||W_b u - \lambda u||_{[0,b]} \leq b||\kappa_1||e^{b \operatorname{Im}\gamma}. \tag{3.14}$$

Using the fact that $||u||_{[0,b]} = O(1)$ as $b \to \infty$, it follows that $||(\lambda I - W_b)^{-1}|| \to \infty$ exponentially as $b \to \infty$.

It is also straightforward to show that the pseudospectra of W_b satisfy a nesting property analogous to (III). Suppose that $b \leq b' \leq \infty$. Theorem 3.1 and the discussion of the spectra of W_b imply that $\Lambda_\varepsilon(W_b) \subseteq \Lambda_\varepsilon(W_{b'})$ for $\varepsilon = 0$. Now, suppose that $\lambda \notin 0$ and assume that $u \in L^2[0,b] = \mathcal{D}(W_b)$. Let us define $\tilde{u}$ such that $\tilde{u}(x) = u(x)$ for $x \in [0,b]$ and $\tilde{u}(x) = 0$ for $x \in (b, b']$. This function satisfies $\tilde{u} \in \mathcal{D}(W_{b'})$. A straightforward calculation shows that $||W_b u - \lambda u||_{[0,b]} = ||W_{b'}\tilde{u} - \lambda \tilde{u}||_{[0,b']}$. Since $||\tilde{u}||_{[0,b]} = ||\tilde{u}||_{[0,b']}$, it follows that $||(\lambda I - W_b)^{-1}||_{[0,b]} \leq ||(\lambda I - W_{b'})^{-1}||_{[0,b']}$ by (3.11). Hence, $\Lambda_\varepsilon(W_b) \subseteq \Lambda(W_{b'})$ for $\varepsilon \geq 0$ as well.

If we set $b' = \infty$ in this last argument, we obtain

$$\Lambda_\varepsilon(W_b) \subseteq \Lambda_\varepsilon(W_\infty) = \hat{\kappa}(\mathbf{C}^-) + \Delta_\varepsilon \quad \forall\, \varepsilon \geq 0. \tag{3.15}$$

As we will see in the next section, this upper bound may be far from sharp.

To obtain a clean formula for the lower bound for the pseudospectra it is convenient to introduce the truncated symbol [**19**]

$$\hat{\kappa}_b(\omega) = \int_{-b}^0 k(x) e^{i\omega x}\,dx. \tag{3.16}$$

The function $\hat{\kappa}_b$ is the symbol of $\kappa_b(x)$, which satisfies $\kappa_b(x) = \kappa(x)$ for $x \in [-b, 0]$ and $\kappa_b(x) = 0$ for $x \notin [0, b]$. The function $\hat{\kappa}_b(\omega)$ is entire since κ_b has compact support. In addition, we define $\mathbf{C}_t^-$ to be the half-plane $\{z : \operatorname{Im} z \leq t\}$.

The following theorem is analogous to Theorem 2.2 in [**19**].

Theorem 3.2. *Let W_b be the Wiener-Hopf integral operator defined by* (3.1). *The pseudospectra of W_b are given by*

$$\hat{\kappa}_b(\mathbf{C}_t^-) \subseteq \Lambda_\varepsilon(W_b) \subseteq \hat{\kappa}_b(\mathbf{C}^-) + \Delta_\varepsilon \quad \forall\, \varepsilon \geq 0, \tag{3.17}$$

where $t = \log(\varepsilon/||\kappa||_1)/b$.

Proof. First, let us consider the lower bound. Let $u = e^{-i\gamma x}$ and $\lambda = \hat{\kappa}_b(\gamma)$. We have

$$\begin{aligned} W_b u - \lambda u &= \int_{x-b}^{-b} \kappa_b(s) e^{-i\gamma(x-s)}\, ds \\ &= e^{-i\gamma b} \int_0^x \kappa_b(x-b-s) e^{-i\gamma s}\, ds. \end{aligned} \tag{3.18}$$

For $x \in [0, b]$ the integral term on the right-hand side of (3.18) is equal to the convolution $\kappa_b(x-b) * \tilde{u}$, where $\tilde{u} = u$ for $x \in [0, b]$ and $\tilde{u} = 0$ for $x \notin [0, b]$. It follows that

$$\begin{aligned} ||(W_b - \lambda)u||_{[0,b]} &= e^{b \operatorname{Im} \gamma} ||\kappa(x-b) * \tilde{u}||_{[0,b]} \\ &\leq e^{b \operatorname{Im} \gamma} ||\kappa_b||_1 ||\tilde{u}||_{(-\infty,\infty)} \\ &\leq e^{b \operatorname{Im} \gamma} ||\kappa||_1 ||u||_{[0,b]}. \end{aligned} \tag{3.19}$$

By (3.12), this last expression implies that $\lambda \in \Lambda_\varepsilon(W_b)$ for $\varepsilon = ||\kappa||_1 e^{b \operatorname{Im} \gamma}$. Setting $t = \log(\varepsilon/||\kappa||_1)/b$, we obtain the lower bound in (3.17).

The upper bounds in (3.17) can be verified using the arguments leading to (3.15). The key observation is that W_b can be defined as a finite interval version of an operator W_∞ defined by the kernel κ_b. $\square$

Finally, we verify that property (IV) also holds for upper-triangular W-H operators.

Theorem 3.3. *Let W_b be the Wiener-Hopf integral operator defined by* (3.1)*. Let $\hat{\kappa}$ denote the symbol. For $b \le b' \le \infty$, we have*

$$\Lambda_\varepsilon(W_b) \subseteq \Lambda_\varepsilon(W_{b'}) \quad \forall\, \varepsilon \ge 0. \tag{3.20}$$

In addition, for each $\varepsilon > 0$, the pseudospectra satisfy

$$\lim_{b\to\infty} \Lambda_\varepsilon(W_b) \to \Lambda_\varepsilon(W_\infty). \tag{3.21}$$

Proof. The inclusion (3.20) was proved above.

Let us consider (3.21) and let $Z_\varepsilon = \lim_{b\to\infty} \Lambda_\varepsilon(W_b)$. The limit exists because the sets $\Lambda_\varepsilon(W_b)$ are nested and are bounded from above. As $b \to \infty$, we have $t \to 0$ and $\hat{\kappa}_b(\omega) \to \hat{\kappa}(\omega)$ uniformly for all $\omega \in \mathbf{C}^-$. Hence, taking the limit of (3.17) as $b \to \infty$, we have

$$\Lambda(W_\infty) = \hat{\kappa}(\mathbf{C}^-) \subseteq Z_\varepsilon \subseteq \hat{\kappa}(\mathbf{C}^-) + \Delta_\varepsilon = \Delta_\varepsilon(W_\infty) \quad \forall\, \varepsilon > 0. \tag{3.22}$$

To prove that the right hand inclusion is an inequality we start with the following property of pseudospectra, valid for all operators T [**22**]:

$$\Lambda_\varepsilon(T) + \Delta_\delta \subseteq \Lambda_{\varepsilon+\delta}(T) \quad \forall\, \varepsilon \ge 0. \tag{3.23}$$

Substituting W_b in (3.23) and taking the limit $b \to \infty$, we obtain

$$Z_\varepsilon + \Delta_\delta \subseteq Z_{\varepsilon+\delta}, \quad \varepsilon > 0\ \delta \ge 0. \tag{3.24}$$

Suppose that $\delta > 0$. Taking the limit of (3.24) as $\varepsilon \to 0$ and using the fact that $\lim_{\varepsilon\to 0} Z_\varepsilon = \Lambda(W_\infty) = \hat{\kappa}(\mathbf{C}^-)$, which follows from (3.22), we obtain

$$\Lambda(W_\infty) + \Delta_\delta \subseteq Z_\delta \quad \forall\, \delta > 0, \tag{3.25}$$

and the theorem is proved. $\square$

Finally, we note that the results in this section can be extended to lower-triangular W-H operators, defined by kernels satisfying $\kappa(x) = 0$

for $x < 0$. The adjoint, W_b^+, of a lower-triangular W-H operator is an upper-triangular W-H operator with kernel $\kappa^*(-x)$, where $*$ denotes complex conjugation [**9**]. The pseudospectra of W_b can then be determined using the formula $\Lambda_\varepsilon(W_b) = \Lambda_\varepsilon^*(W_b^+)$ [**22**].

4. Examples. We illustrate the results for triangular W-H operators with two examples.

We first note that the upper bounds (3.15) and (3.17) may not be sharp. We derive an alternative upper bound. If W_b is upper-triangular, then the arguments leading to (3.7) can be used to show that the solution of the problem $\lambda u - W_b u = f$ is

$$u(x) = (\lambda I - W_b)^{-1} f = \frac{f(x)}{\lambda} + \frac{1}{\lambda}\int_x^b g_\lambda(x-y) f(y)\, dy, \tag{4.1}$$

where $g_\lambda(x)$ is the inverse transform of $\hat{g}_\lambda = \hat{\kappa}/(\lambda - \hat{\kappa})$. Equation (4.1) yields the bound [**9**]

$$||(\lambda I - W_b)^{-1}|| \leq \frac{1}{|\lambda|}\left[1 + \left(\int_0^b \int_x^b |g_\lambda(x-y)|^2\, dy\, dx\right)^{1/2}\right] = B(\lambda). \tag{4.2}$$

The bound (4.2) and the definition of pseudospectra imply that the set

$$U_\varepsilon(W_b) = \{z \in \rho(W_b) : B(z) \geq \varepsilon^{-1}\} \cup \Lambda(W_b) \tag{4.3}$$

satisfies $\Lambda_\varepsilon(W_b) \subseteq U_\varepsilon(W_b)$ for all $\varepsilon \geq 0$. We find that (4.2) is sharp if λ lies in the interior of $\Lambda(W_\infty)$ but may not be sharp for $\lambda \notin \Lambda(W_\infty)$. When (4.3) is poor we use (3.15).

Let us first consider the upper-triangular W-H operator defined by $\kappa(x) = e^x$ for $x \leq 0$. Straightforward calculations show that $||\kappa||_1 = 1$,

$$\hat{\kappa}(\omega) = \frac{1}{1 + i\omega}, \tag{4.4}$$

$$\hat{\kappa}_b(\omega) = \frac{1 - e^{(1+i\omega)b}}{1 + i\omega}, \tag{4.5}$$

and

$$g_\lambda(x) = \frac{e^{(1-1/\lambda)x}}{\lambda}. \tag{4.6}$$

The results are shown in Figure 2. The spectrum of W_∞ is the shaded disk, and $\Lambda_\varepsilon(W_\infty)$ is the disk plus a border of thickness ε. The spectrum of W_b is the origin. The dashed lines are lower bounds for the pseudospectra, computed using (3.17), and the solid lines are upper bounds for the pseudospectra, computed using (3.15) and (4.3).

The figure reveals several important features about the bounds and the behavior of the pseudospectra of W-H operators. First, the upper and lower bounds are close, particularly for $\varepsilon \ll 1$. The upper bound predicted by Theorem 3.2 is poor; for example, for $\varepsilon \ll 1$ and $b = 10$, $\hat{\kappa}_b(\mathbf{C}^-) + \Delta_\varepsilon$ is approximately the shaded disk. Second, the ε-pseudospectra essentially lie to one side of the spectrum only, and this is due to the unusual behavior of resolvent in the neighborhood of the origin. The function $(\lambda I - W_b)^{-1}$ is singular at $\lambda = 0$. However, as the figure shows, $||(\lambda I - W_b)^{-1}||$, for $|\lambda| \ll 1$, depends crucially on the position of λ. For example, if λ is real and negative, then

$$||(\lambda I - W_b)^{-1}|| = \frac{1}{|\lambda|}. \tag{4.7}$$

This result follows from (3.15) and (3.5). On the other hand, using (3.19) and (4.5), it can be shown that

$$||(\lambda I - W_b)^{-1}|| \geq C_1 e^{C_2/\lambda}, \quad \lambda \to 0^+, \tag{4.8}$$

where C_1 and C_2 are constants. This type of behavior does not occur for matrices, where the resolvent norm can only have an algebraic singularity at an eigenvalue. The exponential behavior of the resolvent norm at the origin is a characteristic feature of triangular W-H operators.

For our second example we consider the upper-triangular W-H operator defined by $\kappa(x) = -xe^x$ for $x \leq 0$. We have $||\kappa||_1 = 1$,

$$\hat{\kappa}(\omega) = \frac{1}{(1+i\omega)^2}, \tag{4.9}$$

$$\hat{\kappa}_b(\omega) = \frac{1}{(1+i\omega)^2} - \frac{be^{-(1+i\omega)b}}{1+i\omega} - \frac{e^{-(1+i\omega)b}}{(1+i\omega)^2}, \tag{4.10}$$

and

$$g_\lambda(x) = -\frac{e^x \sinh(x/\lambda^{1/2})}{\lambda^{1/2}}. \tag{4.11}$$

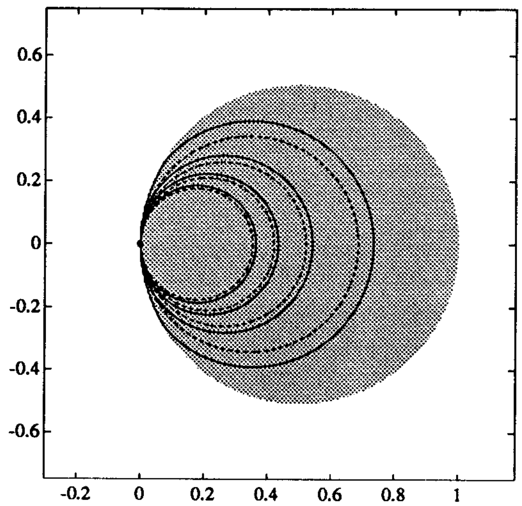

FIGURE 2. Pseudospectra of the upper-triangular W-H operator defined by the kernel $k(x) = e^x$ for $x \leq 0$. $\Lambda(W_\infty)$ is the shaded disk. The spectrum of W_b is the origin. The dashed and solid lines (from inner to outer) are lower and upper bounds for the pseudospectra of W_b for $\varepsilon = 10^{-8}, 10^{-6}, 10^{-4}, 10^{-2}$. Here $b = 10$.

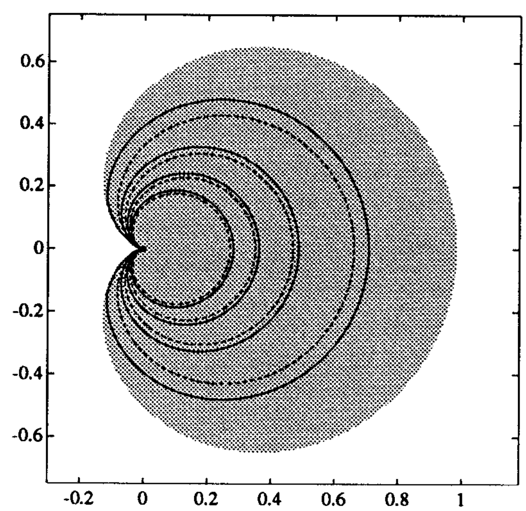

FIGURE 3. Same as Figure 2 for the upper-triangular W-H operator defined by $k(x) = -xe^x$ for $x \leq 0$. Here $b = 20$.

The results are shown in Figure 3.

5. General Wiener-Hopf integral operators. We now turn our attention to general Wiener-Hopf integral operators for finite and semi-infinite intervals. Our focus in this section is on the spectrum of the family of the operators $\{W_b\}$.

The analysis of the operator W_∞ is based on a winding number defined by

$$I(\hat{\kappa}(\mathbf{R}),\lambda) = -\frac{1}{2\pi}\arg[\lambda - \hat{\kappa}(\omega)]_{-\infty}^{\infty}, \quad \lambda \notin \hat{\kappa}(\mathbf{R}). \tag{5.1}$$

If $\lambda \in \hat{\kappa}(\mathbf{R})$, then $I(\hat{\kappa}(\mathbf{R}),\lambda)$ is not defined. The function $I(\hat{\kappa}(\mathbf{R}),\lambda)$ is equal to the number of times the curve $\hat{\kappa}(\mathbf{R})$ encircles λ in the counter-clockwise direction. The following theorem characterizes the spectrum.

Theorem 5.1. *Let W_∞ be a Wiener-Hopf integral operator with symbol $\hat{\kappa}$ and consider W_∞ as an operator from the space $L^2[0,\infty)$ to itself. Let $I(\hat{\kappa}(\mathbf{R}),\lambda)$ denote the winding number. We have*

$$\Lambda(W_\infty) = \{\lambda \notin \hat{\kappa}(\mathbf{R}) : I(\hat{\kappa}(\mathbf{R}),\lambda) \notin 0\} \cup \hat{\kappa}(\mathbf{R}). \tag{5.2}$$

In words, the theorem states that the spectrum consists of the curve $\hat{\kappa}(\mathbf{R})$ plus all points enclosed by this curve. If $\nu = I(\hat{\kappa}(\mathbf{R}),\lambda) > 0$, then λ is an eigenvalue with ν associated linearly independent eigenfunctions. If $\nu < 0$, then λ lies in the residual spectrum; the co-dimension of the space $(\lambda I - W_\infty)L^2[0,\infty)$ is $-\nu$. Finally, if $\lambda \in \hat{\kappa}(\mathbf{R})$, then λ lies in the continuous spectrum. A proof of the theorem for the case $\kappa \in L^1$ is given by Krein [**10**]. According to Krein, the central role of the winding number was first recognized by Rapaport, who proved a result similar to Theorem 5.1 for a more restrictive class of kernels.

The proof of Theorem 5.1 is based in part on a factorization of the operator $\lambda I - \tilde{W}$ [**10**]. If $\lambda \notin \Lambda(W_\infty)$, then there exist bounded upper- and lower-triangular W-H operators $\tilde{V}_+$ and $\tilde{V}_-$ such that $\lambda I - \tilde{W} = \tilde{V}_-\tilde{V}_+ = \tilde{V}_+\tilde{V}_-$. These factors are unique up to a constant. The decomposition leads to a second formula for the solution of the inhomogeneous problem $\lambda\tilde{u} - \tilde{W}\tilde{u} = \tilde{f}$. We have $\tilde{u} = \tilde{V}_+^{-1}\tilde{V}_-^{-1}\tilde{f} = \tilde{V}_-^{-1}\tilde{V}_+^{-1}\tilde{f}$. The factorization can also be used to solve the semi-infinite inhomogeneous problem $\lambda u - W_\infty u = f$ [**10**].

Except in certain special cases, there does not appear to be a simple formula for the pseudospectra of W_∞. One such case is the class of normal W-H operators. For these operators, we have $\Lambda_\varepsilon(W_\infty) = \Lambda(W_\infty) + \Delta_\varepsilon$ for all $\varepsilon \geq 0$. One class of normal operators is the class of self-adjoint W-H operators, defined by kernels satisfying $\kappa(x) = \kappa^*(-x)$. For these operators $\Lambda(W_\infty) = \hat{\kappa}(\mathbf{R})$.

We now turn our attention to W_b for finite b. This operator is compact, so its spectrum consists of discrete points plus possibly the origin. There is no general formula for the spectrum. Much of the previous work on W_b has focused on the behavior of the spectrum as $b \to \infty$. A classical result, first proved by Kac, Murdock and Szegö [**8**] and then extended by Landau [**11**] to higher dimensional integral operators, concerns the asymptotic distribution of the eigenvalues when W_b is self-adjoint. Let $n(a_1, a_2)$ denote the number of eigenvalues of W_b lying in the interval (a_1, a_2), which we assume does not contain the origin. Let Γ denote the set of points $\omega \in \mathbf{R}$ where $\hat{\kappa}(\omega) \in (a_1, a_2)$, and let $M(a_1, a_2)$ be the Lebesgue measure of Γ. If the set of points for which $\hat{\kappa}(\omega)$ is equal to a_1 or a_2 has measure zero, then

$$\lim_{b\to\infty} \frac{n(a_1, a_2)}{b} = \frac{M(a_1, a_2)}{2\pi}. \tag{5.3}$$

This last formula is analogous to a asymptotic formula for the eigenvalues of Toeplitz matrices [**8**].

In other work, Gohberg and Fel'dman derived formulas for the solution to the inhomogeneous problem $\lambda u - W_b u = f$ [**5**]. They showed that if $\lambda \notin \Lambda(W_\infty)$ and $\tilde{u}$ is the solution of $\lambda\tilde{u} - W_\infty \tilde{u} = f$ for the same function $f \in L^2[0, \infty)$, then $||u - \tilde{u}||_{[0,b]} \to 0$ as $b \to \infty$.

For the remainder of this section we focus on the spectrum of the family of the operators $\{W_b\}$, where the interval length satisfies $0 < b_0 < b < \infty$. Recall that a number λ is in the spectrum of the family, which we denote by $P(\{W_b\})$, if $\limsup_{b\to\infty} ||(\lambda I - W_b)^{-1}|| = \infty$. Here is our main result.

Theorem 5.2. *The spectrum of the family of the operators* $\{W_b\}$ *satisfies*

$$P(\{W_b\}) = \Lambda(W_\infty). \tag{5.4}$$

As mentioned in the Introduction, the theorem is similar to a result proved by Anselone and Sloan [**1**]. However, there are several differences. Let $\tilde{W}_b$ denote the operator in [**1**] analogous to our W_b. The formula for $\tilde{W}_b$ is the same as that given in (1.3). However, the underlying space for $\tilde{W}_b$, for both finite and infinite b, is the Banach space of bounded, continuous, complex-valued functions on $[0,\infty)$ with the $||\cdot||_\infty$ norm. It turns out that $\Lambda(\tilde{W}_\infty) = \Lambda(W_\infty)$ and that $\Lambda(\tilde{W}_b)$ is discrete for finite b. The main results are that $P(\{\tilde{W}_b\}) = \Lambda(\tilde{W}_\infty)$ and that for any $\delta > 0$, $\Lambda(\tilde{W}_b)\subseteq\Lambda(\tilde{W}_\infty) + \Delta_\delta$ for all sufficiently large b.

Theorem 5.2 is also related work by Landau on a family of Wiener-Hopf integral operators arising in the study of lasers [**12**]. For this family, the parameter b appears in the kernel and the length of the interval is fixed. It is shown that the spectrum of the family contains the unit circle. The main focus of this work is on the number of pseudo-eigenfunctions associated with each point on the unit circle as $b \to \infty$. Results on the number of orthogonal pseudo-eigenfunctions for general W-H operators are presented in [**11**].

Because of the differences in the definition of the operators and the underlying spaces, we cannot directly apply the arguments in [**1**] to prove Theorem 5.2. We present a different proof, consisting of two parts. The first part shows that $\Lambda(W_\infty)\subseteq P(\{W_b\})$.

Lemma 5.3. *If $\lambda \in \Lambda(W_\infty)$, then $||(\lambda I - W_b)^{-1}||_{[0,b]} \to \infty$ as $b \to \infty$.*

Proof. By (3.11) we are done if we can show that for each $\delta > 0$ there exists $\phi \in L^2[0,b]$ such that $||W_b\phi - \lambda\phi||_{[0,b]}/||\phi||_{[0,b]} < \delta$ for all sufficiently large b.

First, let us suppose that λ is in the continuous spectrum or the point spectrum of W_∞ and let $\delta > 0$ be given. Then there exists $\phi \in L^2[0,\infty)$ such that $||W_\infty\phi - \lambda\phi||_{[0,b]}/||\phi||_{[0,b]} \leq \delta/2$ for all sufficiently large b. A straightforward calculation shows that

(5.5)

$$\frac{||W_b\phi - \lambda\phi||_{[0,b]}}{||\phi||_{[0,b]}} \leq \frac{||W_\infty\phi - \lambda\phi||_{[0,b]}}{||\phi||_{[0,b]}} + \frac{||\int_b^\infty \kappa(x-y)\phi(y)dy||_{[0,b]}}{||\phi||_{[0,b]}}.$$

We need to estimate the second term on the right-hand side. Let us define $\tilde{\phi}$ such that $\tilde{\phi}(x) = \phi(x)$ for $x \in [b,\infty)$ and $\tilde{\phi}(x) = 0$ for $x < b$.

The integral in the second term is the convolution $\kappa * \tilde{\phi}$. Hence, we have

(5.6)
$$\frac{|| \int_b^\infty \kappa(x-y)\phi(y)dy||_{[0,b]}}{||\phi||_{[0,b]}} \leq \frac{||\kappa||_1||\tilde{\phi}||_{(-\infty,\infty)}}{||\phi||_{[0,b]}} \leq \frac{||\kappa||_1||\phi||_{[b,\infty)}}{||\phi||_{[0,b]}}.$$

We can make this term to be $\leq \delta/2$ by choosing b to be sufficiently large.

Second, suppose that λ is in the residual spectrum of W_∞. We prove the lemma by considering the adjoint W_b^+, which is the W-H operator with kernel $\kappa^*(-x)$ [**9**]. Standard analysis shows that λ^* lies in the point spectrum of W_∞^+. The desired result follows from the first part of the proof and the identity $||(\lambda I - W_b)^{-1}||_{[0,b]} = ||(\lambda^* I - W_b^+)^{-1}||_{[0,b]}$ [**9**]. □

For upper-triangular W-H operators, the resolvent norm $||(\lambda I - W_b)^{-1}||_{[0,b]}$ grows exponentially as $b \to \infty$ for $\lambda \in \mathrm{int}(\Lambda(W_\infty))$, and this occurs because the eigenfunctions of W_∞ are exponentials. For general W-H operators, the eigenfunctions need not be exponential functions; the form of the eigenfunctions of W_∞ and the asymptotic behavior of the resolvent norm depend on the kernel.

The next result completes the proof of Theorem 5.2 by showing that $P(\{W_b\}) \subseteq \Lambda(W_\infty)$.

Lemma 5.4. *If $\lambda \notin \Lambda(W_\infty)$, then $||(\lambda I - W_b)^{-1}||_{[0,b]}$ is uniformly bounded for all sufficiently large b.*

Proof. The lemma can be proved by adapting an argument used by Baxter in his proof of an analogous result for Toeplitz matrices [**3**]. We present a sketch of the proof.

We are done if we can show that the solution of the inhomogeneous problem $\lambda u - W_b u = f$ for $\lambda \notin \Lambda(W_\infty)$ satisfies $||u||_{[0,b]}/||f||_{[0,b]} < C$ independent of b for all $f \in L^2[0,b]$ and all sufficiently large b. Choose $u \in L^2[0,b]$ and define $\tilde{u}$ for $x \in (-\infty, \infty)$ so that it has compact support in the interval $[0,b]$ and $\tilde{u}(x) = u(x)$ for $x \in [0,b]$. Extending the inhomogeneous problem to the infinite interval, we have

$$(5.7)\quad \underset{[0,b]}{(\lambda-\tilde{W})\tilde{u} = \lambda\tilde{u}-\int_0^b \kappa(x-y)\tilde{u}(y)\,dy = f(x)} \; + \underset{(-\infty,0)}{\tilde{g}_1(x)} \; + \underset{(b,\infty)}{\tilde{g}_2(x)}.$$

The second line in this last expression indicates the support of f, $\tilde{g}_1$, and $\tilde{g}_2$.

Multiplying by $(\lambda I - \tilde{W})^{-1}$ and using the factorization defined above, we have

$$(5.8)\qquad \tilde{u} = \tilde{V}_+^{-1}\tilde{V}_-^{-1}f + \tilde{V}_-^{-1}\tilde{V}_+^{-1}\tilde{g}_1 + \tilde{V}_+^{-1}\tilde{V}_-^{-1}\tilde{g}_2.$$

The operators $\tilde{V}_+^{-1}$ and $\tilde{V}_-^{-1}$ are both bounded operators. The proof is completed by showing that $||\tilde{V}_+^{-1}\tilde{g}_1||_{[0,b]}$ and $||\tilde{V}_-^{-1}\tilde{g}_2||_{[0,b]}$ are both bounded by $C_1||f||_{[0,b]}$ for some constant C_1 independent of b for all sufficiently large b. □

Theorem 5.2 shows that if $\lambda \notin \Lambda(W_\infty)$, then λ is not an eigenvalue of W_b for all sufficiently large b. However, the theorem does not directly state what happens to the eigenvalues as $b \to \infty$. It can be shown that for any $\delta > 0$,

$$(5.9)\qquad \Lambda(W_b)\subseteq\Lambda(W_\infty) + \Delta_\delta$$

for all sufficiently large b. The proof is the same as that in [**1**] and relies on Theorem 5.2 and the fact that $\Lambda(W_b)$ lies in a fixed compact set for all b.

We conclude this section by examining what can be said about the pseudospectra of W_b as $b \to \infty$. As mentioned in the Introduction, for general Toeplitz matrices $\Lambda_\varepsilon(T_N) \to \Lambda_\varepsilon(T_\infty)$ as the dimension $N \to \infty$ for all $\varepsilon > 0$ [**19, 24**]. The close relationship between Toeplitz matrices and Wiener-Hopf integral operators suggests that a similar convergence result ought to be true for W-H integral operators. We have been unable to prove such a result.

We have been able to take a first step towards a convergence result. It can be shown that

$$(5.10)\qquad \Lambda_\varepsilon(W_\infty)\subseteq \lim_{b\to\infty} \Lambda_\varepsilon(W_b), \quad \forall\,\varepsilon > 0,$$

and the proof is similar to the first part of the proof of Lemma 5.3. For normal W-H operators we can obtain a convergence result by proving that the opposite inclusion holds. For normal operators, $\Lambda_\varepsilon(W_b) = \Lambda(W_b) + \Delta_\varepsilon$ for all $\varepsilon \geq 0$. This last formula and (5.9) imply that for any given $\delta > 0$,

$$\Lambda_\varepsilon(W_b) \subseteq \Lambda(W_\infty) + \Delta_\varepsilon + \Delta_\delta = \Lambda_\varepsilon(W_\infty) + \Delta_\delta \tag{5.11}$$

for all sufficiently large b. Letting $b \to \infty$, which implies that we can take δ to be arbitrarily small in (5.11), it follows that $\lim_{b\to\infty} \Lambda_\varepsilon(W_b) \subseteq \Lambda_\varepsilon(W_\infty)$.

6. Pseudospectra of triangular constant-coefficient differential operators. In this section we extend the results of Section 3 to upper-triangular constant-coefficient differential operators. These operators are defined by the boundary conditions

$$u^{(i)}(b) = 0, \quad i = 0, 1, \dots, n-1, \tag{6.1}$$

where n is the order of the differential operator. The domain of A_b is

$$\mathcal{D}(A_b) = \{u \in Q_n[0,b] : u^{(i)}(b) = 0,\ i = 0, 1, \dots, n-1\}, \tag{6.2}$$

where $Q_n[0,b]$ is the maximal space of n-times differential functions introduced in Section 2. We consider A_b to be an operator from the space $L^2[0,b]$ to itself. Recall that the symbol is

$$\hat{a}(\omega) = a_n(-i\omega)^n + a_{n-1}(-i\omega)^{n-1} + \dots + a_1(-i\omega) + a_0. \tag{6.3}$$

In this and the next section we will let A denote the differential operator A_b without any boundary conditions, particularly when we consider the inhomogeneous problem

$$Au - \lambda u = f. \tag{6.4}$$

If s is a root of $\hat{a}(\omega) - \lambda$, then $u = e^{-isx}$ is a solution of the homogeneous part of (6.4).

The operator A_∞ does not have explicit boundary conditions at $x = \infty$; the condition $\mathcal{D}(A_\infty) = Q_n[0,\infty)$ ensures that (6.1) is satisfied for $b = \infty$. The pseudospectra of A_∞ can be characterized in a simple manner. The following theorem and its proof are similar to the result for upper-triangular W-H operators.

Theorem 6.1. *Let A_∞ be an upper-triangular constant-coefficient differential operator defined on $[0, \infty)$ with symbol $\hat{a}$. The pseudospectra of A_∞ are given by*

$$\Lambda_\varepsilon(A_\infty) = \hat{a}(\mathbf{C}^-) + \Delta_\varepsilon \quad \forall\, \varepsilon \geq 0. \tag{6.5}$$

We now turn to the finite interval case. It is straightforward to verify that the solution of the inhomogeneous problem $A_b u - \lambda u = f$, where $f \in L^2[0, b]$, can be written

$$u(x) = \int_x^b g_\lambda(x - y) f(y)\, dy. \tag{6.6}$$

For all λ, the Green's function $g_\lambda(x)$ is continuous for $x \in [0, b]$ [**14**], and this implies that the integral operator in (6.6) is bounded. Hence, $\Lambda(A_b)$ is the empty set. In a certain sense, which will become more clear below, we can consider $\lambda = \infty$ to be the sole point in the spectrum.

The behavior of the pseudospectra of A_b is similar to that of W_b. It is straightforward to show that if $b \leq b'$, then

$$\Lambda_\varepsilon(A_b) \subseteq \Lambda_\varepsilon(A_{b'}) \quad \forall\, \varepsilon \geq 0; \tag{6.7}$$

the proof is similar to that given in Section 3. It follows that

$$\Lambda_\varepsilon(A_b) \subseteq \Lambda_\varepsilon(A_\infty) \quad \forall\, \varepsilon \geq 0. \tag{6.8}$$

We have been unable to derive a general lower bound formula for the pseudospectra analogous to (3.17). We can, however, verify that the resolvent norm of A_b satisfies an exponential growth estimate similar to condition (II). Suppose that $\gamma \in \operatorname{int}(\mathbf{C}^-)$ and $\lambda = \hat{a}(\gamma)$. Roughly speaking, (II) holds because $u = e^{-i\gamma x}$ satisfies the boundary conditions to within a factor of $O(e^{b \operatorname{Im} \gamma})$ as $b \to \infty$. We cannot use u as a pseudo-eigenfunction because $u \notin \mathcal{D}(A_b)$. Instead, we consider

$w(x) = u(x) + v(x)$, where the small correction function v is chosen so that $w \in \mathcal{D}(A_b)$. The idea is to pick v so that
(6.9)
$$||Av-\lambda v||_{[0,b]} = O(e^{b\mathrm{Im}\,\gamma}) \quad \text{and} \quad ||v||_{[0,b]} = o(||u||_{[0,b]}) \quad b \to \infty.$$

This choice implies

$$\frac{||A_b w-\lambda w||_{[0,b]}}{||w||_{[0,b]}} = \frac{||Av-\lambda v||_{[0,b]}}{||u+v||_{[0,b]}} = O(e^{b\mathrm{Im}\,\gamma}) \quad b \to \infty. \tag{6.10}$$

Here we have used the fact that $||u||_{[0,b]} = O(1)$ as $b \to \infty$.

A candidate for v is

$$v(x) = \sum_{j=1}^{n} d_i e^{\beta_j(x-b)}. \tag{6.11}$$

Here the constants $\{\beta_j\}$ satisfy $\beta_j > 0$ and $\beta_j \neq \beta_k$ if $j \neq k$. The constants d_i are chosen so that w satisfies the boundary conditions, and these constants exist because the exponentials in the sum in (6.11) form a linearly independent set. Straightforward calculations show that the conditions in (6.9) hold.

The above arguments show that $\lambda \in \Lambda_\varepsilon(A_b)$ for $\varepsilon = Ce^{b\mathrm{Im}\,\gamma}$ and $b \gg 1$. In this case C depends on λ and the constants $\{\beta_j\}$. Hence, for $\varepsilon \ll 1$, an approximate lower bound for the ε-pseudospectrum is $L_\varepsilon(A_b) \approx \hat{a}(C_t^-)$, where $t = \log(\varepsilon/C)/b$. We do not have a precise formula for C.

Let us illustrate these ideas with an example. The simplest upper-triangular C-C differential operator is the operator defined by $A_b u = du/dx$ and $u(b) = 0$, which has the symbol $\hat{a} = -i\omega$. The spectra of this and related operators are analyzed in [**15**]. Theorem 6.1 implies that the spectrum of A_∞ is the left half-plane; if $\mathrm{Re}\,\lambda < 0$, then $e^{\lambda x}$ is an eigenfunction with eigenvalue λ. For each ε, the $\Lambda_\varepsilon(A_\infty)$ is the half-plane $\{z : \mathrm{Re}\,z \leq \varepsilon\}$.

To get a lower bound for $\Lambda_\varepsilon(A_b)$ we choose a pseudo-eigenfunction of the form $w = e^{\lambda x} - e^{x\mathrm{Im}\,\lambda + b\mathrm{Re}\,\lambda} e^{x-b}$ for $\mathrm{Re}\,\lambda < 0$ and a modified version of this function for $\mathrm{Re}\,\lambda \geq 0$. This choice leads to lower bound sets $L_\varepsilon(A_b)$ that are half-planes. To get a sharp upper bound we start with the formula (6.6), where in this case $g_\lambda(x) = -e^{\lambda x}$, and this leads

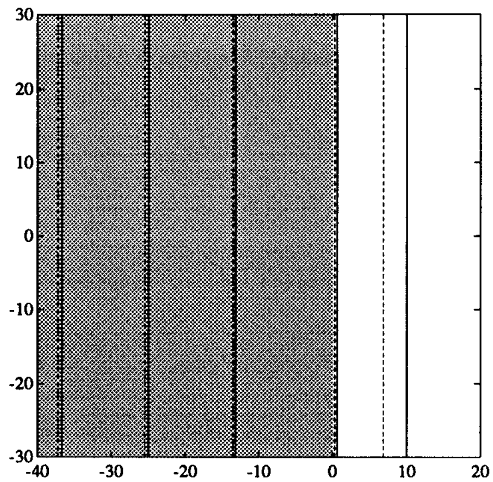

FIGURE 4. Pseudospectra of the derivative operator with boundary condition at $x = b$. The spectrum of A_∞ is the shaded region, and for each $\varepsilon > 0$, the ε-pseudospectrum of A_∞ is the shaded region plus a strip of thickness ε. The dashed and solid lines (from left to right) are lower and upper bounds for the pseudospectra of A_b for $\varepsilon = 10^{-30}, 10^{-20}, 10^{-10}, 10^0, 10^1$. Here $b = 2$.

to a resolvent bound similar to (4.2). Again, when this last method yields a poor result we use (6.8) in our calculations. It can be shown that the resulting upper bound set $U_\varepsilon(A_b)$ is a half-plane as well.

The results for the example are shown in Figure 4. The shaded region is the spectrum of A_∞. The solid and dashed lines are the boundaries of the upper and lower bound sets for A_b for $\varepsilon = 10^{-30}, 10^{-20}, 10^{-10}, 10^0, 10^1$. The sets $L_\varepsilon(A_b)$ and $U_\varepsilon(A_b)$ are the half-planes lying to the left of these vertical lines. It can be shown that the sets $\Lambda_\varepsilon(A_b)$ are half-planes as well.

Again the upper and lower bounds are reasonable sharp for $\varepsilon \ll 1$. Note the logarithmic spacing of the contours in the left half-plane. Using the pseudo-eigenfunction above, it can be shown that

$$||(\lambda I - A_b)^{-1}|| \approx e^{-b\operatorname{Re}\lambda} \quad \operatorname{Re}\lambda \to -\infty, \tag{6.12}$$

On the other hand, (6.8) implies that

$$||(\lambda I - A_b)^{-1}|| = O((\operatorname{Re}\lambda)^{-1}) \quad \operatorname{Re}\lambda \to \infty. \tag{6.13}$$

If we consider $\lambda = \infty$ as the sole point in the spectrum of A_b, then this behavior of the resolvent is similar to that for triangular W-H operators.

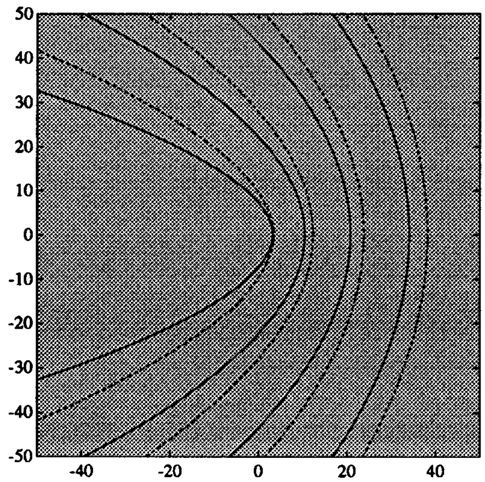

FIGURE 5. Same as Figure 4 for the operator $A_b u = u''$ with $u(b) = u'(b) = 0$ and $b = 4$. The dashed and solid lines (from right to left) are lower and upper bounds for the ε-pseudospectra for $\varepsilon = 10^{-8}, 10^{-6}, 10^{-4}, 10^{-2}$.

The behavior of the pseudospectra of triangular C-C differential operators in the limit $b \to \infty$ is similar to that of W-H operators:

Theorem 6.2. *Let A_b be an upper-triangular constant-coefficient differential operator defined on the interval $[0, b]$ with symbol $\hat{a}$. For $b \leq b' \leq \infty$, we have*

$$\Lambda_\varepsilon(A_b) \subseteq \Lambda_\varepsilon(A_{b'}) \quad \forall\, \varepsilon \geq 0. \tag{6.14}$$

In addition, for each $\varepsilon > 0$ the pseudospectra satisfy

$$\lim_{b \to \infty} \Lambda_\varepsilon(A_b) \to \Lambda_\varepsilon(A_\infty). \tag{6.15}$$

We conclude this section with a second example. Let $A_b = d^2/dx^2$ and $u(b) = u'(b) = 0$. The symbol in this case is $\hat{a}(\omega) = -\omega^2$. The symbol maps $\mathbf{C}^-$ to the entire complex plane. Hence, $\Lambda_\varepsilon(A_\infty) = \mathbf{C}$ for all $\varepsilon \geq 0$. The pseudospectra of A_b can be computed using the techniques described above, and the results are shown in Figure 5. The pseudospectra are shaped like giant mouths. As $b \to \infty$, the mouths close.

Finally, we note that the results of this section can be extended to lower-triangular C-C operators, defined with n boundary conditions at $x = 0$. The adjoint of a lower-triangular C-C operator with symbol $\hat{a}$ is an upper-triangular C-C operator with symbol $\hat{a}^+ = \hat{a}^*$ [**14**]. Again, we can compute the pseudospectra using the formula $\Lambda_\varepsilon(A_b) = \Lambda_\varepsilon^*(A_b^+)$.

7. Results for general constant-coefficient differential operators. In this section we extend the results on the spectrum of the family in Section 5 to constant-coefficient differential operators with boundary conditions at both endpoints.

Let A_b be an n^{th} order C-C differential operator with n boundary conditions of the form (1.5). The domain of A_b is

(7.1)
$$\begin{aligned}\mathcal{D}(A_b) = \{u \in Q_n[0,b] : u^{(j)}(0) = 0,\ j = 0, 1, \ldots, s_0 - 1,\\ u^{(j)}(b) = 0,\ j = 0, 1, \ldots, s_b - 1\}.\end{aligned}$$

The operator A_∞ is defined with s_0 boundary conditions at $x = 0$ and no explicit boundary conditions at $x = \infty$.

As in the case of W-H operators, it is convenient to introduce a winding number. The appropriate function is

$$J(\hat{a}(\mathbf{R}), \lambda) = \frac{n}{2} - s_0 - \frac{1}{2\pi} \arg[\lambda - \hat{a}(\omega)]|_{-\infty}^{\infty} \quad \lambda \notin \hat{a}(\mathbf{R}). \tag{7.2}$$

The winding number is not defined for $\lambda \in \hat{a}(\mathbf{R})$. Applying the principle of the argument, it can be shown that $J(\hat{a}(\mathbf{R}), \lambda) + s_0$ is equal to the number of roots of $\hat{a}(\omega) - \lambda$ lying in the lower half-plane [**13**]. This quantity is equal to the number of exponentially decaying solutions of the homogeneous equation

$$Au - \lambda u = 0. \tag{7.3}$$

(Recall that A denotes the operator A_b without any boundary conditions.) The curve $\hat{a}(\mathbf{R})$ divides the complex plane into a number of open components, and $J(\hat{a}(\mathbf{R}), \lambda)$ is constant in each component.

The following theorem is similar to Theorem 5.1.

Theorem 7.1. *Let A_∞ be an n^{th} order constant-coefficient differential operator defined on $[0, \infty)$ with symbol $\hat{a}$ and winding number $J(\hat{a}(\mathbf{R}), \lambda)$. We have*

$$\Lambda(A_\infty) = \{z \notin \hat{a}(\mathbf{R}) : J(\hat{a}(\mathbf{R}), z) \neq 0\} \cup \hat{a}(\mathbf{R}). \tag{7.4}$$

Proof. Choose $\lambda \in \mathbf{C}$ and let us first suppose that $\lambda \notin \hat{a}(\mathbf{R})$. As the above discussion indicates, if $J(\hat{a}(\mathbf{R}), \lambda) > 0$, the equation (6.4) has more than s_0 linearly independent decaying solutions. These solutions can be linearly superposed to create a function that satisfies the s_0 boundary conditions. Hence, λ is an eigenvalue of A_∞.

Second, let us suppose that $J(\hat{a}(\mathbf{R}), \lambda,) < 0$. We show that $\lambda \in \Lambda(A_\infty)$ by considering the adjoint A_∞^+. A straightforward calculation shows that A_∞^+ is the C-C differential operator defined by the symbol $\hat{a}^*(\omega)$ with $s_0^+ = n - s_0$ boundary conditions at $x = 0$ [**14**]. It can be shown that the equation $A^+ u = \lambda^* u$ has more that s_0^+ linearly independent decaying solutions. Hence, $\lambda^* \in \Lambda(A_\infty^+)$, which implies that $\lambda \in \Lambda(A_\infty)$.

If $\lambda \in \hat{a}(\mathbf{R})$, then (7.3) has at least one purely sinusoidal solution, which does not grow or decay. If (7.3) has more than s_0 linearly independent decaying solutions, then λ is an eigenvalue. If there are more than $n - s_0$ growing solutions, then λ lies in the residual spectrum. If there are exactly s_0 decaying solutions or $n - s_0$ growing solutions, then it can be shown that λ lies in the continuous spectrum. We will omit the details. Hence, we have $\{z \notin \hat{a}(\mathbf{R}) : J(\hat{a}(\mathbf{R}), \lambda) \neq 0\} \cup \hat{a}(\mathbf{R}) \subseteq \Lambda(A_b)$.

Now, let us suppose that $\lambda \notin \hat{a}(\mathbf{R})$ and $J(\hat{a}(\mathbf{R}), \lambda) = 0$. We show that $\lambda \notin \Lambda(A_\infty)$ by showing that $(\lambda I - A_\infty)^{-1}$ is bounded. Consider the inhomogeneous problem $Au - \lambda u = f$ with s_0 homogeneous boundary conditions at $x = 0$. Let us assume that $||f||_{[0,\infty]} = 1$. The homogeneous part of this equation has exactly s_0 linearly independent decaying solutions, which we will denote by $\{\xi_j\}$. A bounded solution of the inhomogeneous problem can be written in the form

$$u(x) = \tilde{u}(x) + \sum_{j=1}^{s_0} \alpha_j \xi_j(x). \tag{7.5}$$

Here $\tilde{u}$ is the solution of the infinite interval problem $\tilde{A}\tilde{u} - \lambda\tilde{u} = \tilde{f}$, where $\tilde{f}(x) = f(x)$ for $x \in [0, \infty)$ and $\tilde{f}(x) = 0$ for $x \in (-\infty, 0)$. The function

u is a solution to the full inhomogeneous problem if the constants $\{\alpha_j\}$ can be chosen so that u satisfies the boundary conditions. The results in Section 2 show that $||\tilde{u}||_{[0,\infty)}$ is bounded independent of f. Since each of the functions ξ_j satisfies $||\xi_j||_{[0,\infty)} < \infty$, we are done if we can show that each of the constants in (7.5) is bounded uniformly for all f.

The constants are the solution of the linear system $F_d(0)\alpha = -v_0$, where

$$F_d(x) = \begin{bmatrix} \xi_1^{(0)}(x) & \xi_2^{(0)}(x) & \cdots & \xi_{s_0}^{(0)}(x) \\ \xi_1^{(1)}(x) & \xi_2^{(1)}(x) & \cdots & \xi_{s_0}^{(1)}(x) \\ \vdots & \vdots & \vdots & \\ \xi_1^{(s_0-1)}(x) & \xi_2^{(s_0-1)}(x) & \cdots & \xi_{s_0}^{(s_0-1)}(x) \end{bmatrix} \tag{7.6}$$

is the Wronskian of the functions $\{\xi_j\}$, $v_0 = (\tilde{u}(0), \tilde{u}^{(1)}(0), \dots, \tilde{u}^{(s_0-1)}(0))^T$, and α is the vector of the constants. Since the functions $\{\xi_j\}$ are linearly independent, $F_d(0)$ is invertible. As mentioned in Section 2, $|\tilde{u}^{(j)}(0)|$ is bounded independent of f for $j = 0, 1, \dots s_0 - 1$. This and the invertibility of $F_d(0)$ imply that each of the constants satisfy $|\alpha_j| < C$ for some C, independent of f [**6**]. □

As in the case of general W-H operators, there does not appear to be a simple formula for the pseudospectra of A_∞, except for normal and other special classes of C-C differential. For normal operators, $\Lambda_\varepsilon(A_\infty) = \Lambda(A_\infty) + \Delta_\varepsilon$ for all $\varepsilon \geq 0$. This formula is also valid for the convection-diffusion operators defined by $A_\infty = u'' + u'$ with $s_0 = 1$ [**18**].

For finite b, the spectrum of A_b consists of a countable number of discrete points, and these eigenvalues have no finite accumulation point [**14**]. Upper and lower bounds for the pseudospectra can be computed using the techniques described in the previous section; see [**16**] for results on the convection-diffusion operator.

Our main focus is on the spectrum of the family of the operators $\{A_b\}$. The following theorem is similar to Theorem 5.2.

Theorem 7.2. *The spectrum of the family of the operators* $\{A_b\}$ *satisfies*

$$P(\{A_b\}) = \Lambda(A_\infty). \tag{7.7}$$

We again split the proof of the theorem into two parts. The following lemma is similar to Lemma 5.3.

Theorem 7.3. *If* $\lambda \in \Lambda(A_\infty)$, *then* $||(\lambda I - A_b)^{-1}||_{[0,b]} \to \infty$ *as* $b \to \infty$.

Proof. The proof is similar in spirit to the proof of Lemma 5.3. As in that previous proof, we need only consider the point spectrum and continuous spectrum.

Suppose that λ lies in the point spectrum or the continuous spectrum of A_∞. Let $\delta > 0$ be given. Then there exists $\phi \in \mathcal{D}(A_\infty)$ such that $||A\phi - \lambda\phi||_{[0,b]}/||\phi||_{[0,b]} < \delta$ for all sufficiently large b. The function ϕ cannot be used as a pseudo-eigenfunction because it need not satisfy the boundary conditions at $x = b$. Let us instead consider a pseudo-eigenfunction of the form

$$w(x) = \phi(x) + v(x) = \phi(x) + \sum_{j=1}^{s_0} d_j \xi_j(x) + \sum_{j=1}^{s_b} e_j \chi_j(x - b), \tag{7.8}$$

where $\{\xi_j\}$ and $\{\chi_j\}$ are arbitrary sets of linearly independent exponentially decaying and growing functions, respectively. This formula for v in (7.8) is similar to (6.11) except now decaying functions $\{\chi_i\}$ are required to handle the boundary conditions at $x = 0$. The constants $\{d_j\}$ and $\{e_j\}$ are chosen so that $w \in \mathcal{D}(A_b)$. Estimating $||A_b w - \lambda w||_{[0,b]}/||w||_{[0,b]}$, we obtain

$$\frac{||A_b w - \lambda w||_{[0,b]}}{||w||_{[0,b]}} \le \frac{||A\phi - \lambda\phi||_{[0,b]}}{||\phi + v||_{[0,b]}} + \frac{||Av - \lambda v||_{[0,b]}}{||\phi + v||_{[0,b]}} \tag{7.9}$$

We must show that the two terms on the right-hand side decay to 0 as $b \to \infty$. We do this by showing that $||Av - \lambda v||_{[0,b]} \to 0$ and $||v||_{[0,b]} \to 0$ as $b \to \infty$. Since $||\xi_j(x)||_{[0,b]}$ and $||\chi_j(x-b)||_{[0,b]}$ are uniformly bounded as $b \to \infty$ for all j, we are done if we can show that the constants $\{d_j\}$ and $\{e_j\}$ decrease to 0 as $b \to \infty$.

The constants in (7.8) can be determined from the linear system

$$\begin{bmatrix} F_d(0) & \tilde{F}_g(b) \\ \tilde{F}_d(b) & F_g(b) \end{bmatrix} \begin{bmatrix} d \\ e \end{bmatrix} = \begin{bmatrix} \phi_0 \\ \phi_b \end{bmatrix}, \tag{7.10}$$

which is obtained from the boundary conditions. Let us explain the notation. Here d and e are the s_0- and s_v-vectors of the constants. The vector on the right-hand side consists of ϕ_0, the zero-vector of length s_0, and $\phi_b = -(\phi(b), \phi^{(1)}(b), \ldots, \phi^{(s_b-1)}(b))^T$. Let $\mathbf{F}$ denote the full 2×2 block matrix. The upper left block $F_d(0)$ is given in (7.6). The block $F_g(x)$ is the Wronskian of the functions $\{\chi_j(x-b)\}$, and $F_g(b)$ has the same form as $F_d(0)$ with ξ_j replaced by χ_j. The elements of $\tilde{F}_g(b)$ and $\tilde{F}_d(b)$ are of the form $\chi_j^{(i)}(-b)$ and $\xi_j^{(i)}(b)$, respectively. These elements become exponentially small as $b \to \infty$.

For each j, the constants satisfy the bound

$$(7.11) \qquad |d_j| \le ||\mathbf{F}^{-1}||_2 ||\phi_b||_2, \qquad |e_j| \le ||\mathbf{F}^{-1}||_2 ||\phi_b||_2,$$

where the 2 denotes the Euclidian norm [**6**]. Since $\phi \in \mathcal{D}(A_\infty)$, we have

$$(7.12) \qquad ||\phi_b||_2 \to 0, \quad b \to \infty.$$

The determinant of $\mathbf{F}$ satisfies $\det(\mathbf{F}) = \det(F_d(0))\det(F_g(b)) + \tilde{F}$, where the last term is the contribution from $\tilde{F}_g(b)$ and $\tilde{F}_d(b)$. Since the functions $\{\chi_j\}$ and $\{\xi_j\}$ are linearly independent, $\det(F_d(0)) \ne 0$ and $\det(F_g(b)) \ne 0$. The term $\tilde{F}$ becomes exponentially small as $b \to \infty$. Hence, $|\det(\mathbf{F})| > C_1$ for all sufficiently large b for some constant $C_1 > 0$. It follows that $\mathbf{F}$ is invertible. Each element of $\mathbf{F}$ is uniformly bounded as $b \to \infty$. This last fact and the bound on the determinant imply that $||\mathbf{F}^{-1}||_2$ is uniformly bounded for sufficiently large b. Combined with (7.11) and (7.12), this last result implies that the constants satisfy $|d_j| \to 0$ and $|e_j| \to 0$ as $b \to \infty$. $\square$

Unlike the W-H case, we can estimate the behavior of the resolvent norm in Lemma 7.3. If $J(\hat{a}(\mathbf{R}), \lambda) > 0$, then we can choose ϕ to be an eigenfunction of A_∞. This function is exponentially decaying. If we analyze the proof of Lemma 7.3, then it can be shown that $||(\lambda I - A_b)^{-1}||_{[0,b]} \to \infty$ exponentially as $b \to \infty$. A similar result holds if $J(\hat{a}(\mathbf{R}), \lambda) < 0$.

The following result completes the proof of Theorem 7.2.

Lemma 7.4. *If $\lambda \notin \Lambda(A_\infty)$, then $||(\lambda I - A_b)^{-1}||_{[0,b]}$ is uniformly bounded for all sufficiently large b.*

Proof. Suppose that $\lambda \notin \Lambda(A_\infty)$ and consider the inhomogeneous problem $Au - \lambda u = f$ with the n boundary conditions (1.5). Let us assume that $||f||_{[0,b]} = 1$. The result is proved if we can show that $||u||_{[0,b]}$ is uniformly bounded as $b \to \infty$, independent of $f \in L^2[0,b]$.

If $\lambda \notin \Lambda(A_\infty)$, then the homogeneous part of the equation has s_0 linearly independent decaying solutions, $\{\xi_j\}$ and s_b linearly independent growing solutions, $\{\chi_j\}$. We can write the solution to the full problem in the form

$$u(x) = \tilde{u}(x) + \sum_{j=1}^{s_0} d_j \xi_j(x) + \sum_{j=1}^{s_b} e_j \chi_j(x-b). \tag{7.13}$$

Here $\tilde{u}$ is the solution of the inhomogeneous problem $\tilde{A}\tilde{u} - \lambda\tilde{u} = \tilde{f}$ on the infinite interval, where $\tilde{f}(x) = f(x)$ for $x \in [0,b]$ and $\tilde{f}(x) = 0$ for $x \notin [0,b]$. The results in Section 2 show that $||\tilde{u}||_{[0,b]}$ is bounded independent of f and b. To complete the proof we need to show that the constants $\{d_j\}$ and $\{e_j\}$ are bounded independent of f and b, for sufficiently large b. The proof is similar to the proof of Lemma 7.3. $\square$

The precise behavior of the sets $\Lambda_\varepsilon(A_b)$ in the limit $b \to \infty$ is not known for general C-C operators. Again, we can show that the $\Lambda_\varepsilon(A_\infty) \subseteq \lim_{b\to\infty} \Lambda_\varepsilon(A_b)$ for all $\varepsilon > 0$, and the proof is similar to the proof of Lemma 7.3.

Finally, we note that we have examined the pseudospectra of the convection-diffusion operator defined by $A_b u = u'' + u'$ in detail, and the results are reported in [**18**]. For this operator we show that $\Lambda_\varepsilon(A_\infty) = \Lambda(A_\infty) + \Delta_\varepsilon$ for all $\varepsilon \geq 0$ and that $\lim_{b\to\infty} \Lambda_\varepsilon(A_b) \to \Lambda_\varepsilon(A_\infty)$ for all $\varepsilon > 0$. In addition, we derive asymptotic formulas for large b for the resolvent norm $||(\lambda I - A_b)^{-1}||_{[0,b]}$.

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