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EXPONENTIAL DECAY ESTIMATES OF THE EIGENVALUES FOR THE NEUMANN-POINCARÉ OPERATOR ON ANALYTIC BOUNDARIES IN TWO DIMENSIONS

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ABSTRACT. We show that the eigenvalues of the Neumann-Poincaré operator on analytic boundaries of simply connected bounded planar domains tend to zero exponentially fast, and the exponential convergence rate is determined by the maximal Grauert radius of the boundary. We present a few examples of boundaries to show that the estimate is optimal.

1. Introduction. The Neumann-Poincaré (NP) operator is an integral operator defined on the boundary of a bounded domain. It naturally arises when solving the Dirichlet and Neumann boundary value problems for the Laplacian in terms of layer potentials. As the name suggests, its study goes back to Neumann [**19**] and Poincaré [**22**]. It was a central object in the Fredholm theory of integral equations and the theory of singular integral operators. The notion of the 'double layer operator' is also commonly used for the NP operator, see [**8, 13**].

In this paper, we consider spectral properties of the NP operator. There was some work on spectral properties of the NP operator in the 1950s (see, for example, [**23**] and the references therein). More recently, we have seen rapidly growing interest in spectral properties of the NP operator, due to its connection to plasmon resonance and cloaking by anomalous localized resonance, see e.g., [**15, 16, 17**] and the references therein. In fact, in the quasi-static limit, the plasmon resonance takes place at the eigenvalues of the NP operator [**3, 5**], and the anomalous

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localized resonance takes place at the accumulation point of eigenvalues **[1, 4]**.

Recently, there has been considerable progress in the spectral theory of the NP operator. In **[12]**, Poincaré's variational problem was revisited with the modern language of mathematics. Among other findings of the paper is that the NP operator can be symmetrized by introducing a new, but equivalent, inner product to $H^{-1/2}$ space, the Sobolev $-1/2$ space. This is a quite important discovery for the spectral theory of the NP operator. The NP operator, as a self-adjoint operator, has only two types of spectra: the continuous spectrum and the discrete spectrum (see, for example, **[26]**). If a given domain has a smooth boundary, then the NP operator is compact and has only eigenvalues accumulating to 0. If the boundary has a corner, then the NP operator is a singular integral operator and may have a continuous spectrum. We refer to **[7, 10, 20, 21]** for recent developments on NP spectral theory on planar domains with corners. We also mention that a spectral radius of the NP operator was obtained in **[24]**.

As mentioned above, if the domain has a smooth, $C^{1,\alpha}$, $\alpha > 0$, to be precise, boundary, then the NP operator is compact and has eigenvalues converging to 0. Here and afterwards, the NP *spectrum* is an abbreviation of the spectrum of the NP operator. In the recent paper **[18]**, a quantitative estimate of the decay rate of NP eigenvalues was obtained: Let $\{\lambda_j\}$ be the NP eigenvalues arranged in such a way that $|\lambda_1| = |\lambda_2| \geq |\lambda_3| = |\lambda_4| \geq \cdots$. It is proven that, if the boundary of the domain is C^k, $k \geq 2$, then

$$|\lambda_n| = o(n^{\alpha}) \quad \text{as } n \to \infty, \tag{1.1}$$

for any $\alpha > -k + 3/2$. If, in particular, the boundary is C^∞ smooth, then NP eigenvalues decay faster than any algebraic order. On the other hand, the NP eigenvalues on the ellipse of the long axis a and the short axis b are known to be

$$\pm\frac{1}{2}\left(\frac{a-b}{a+b}\right)^n, \quad n = 1, 2, \ldots. \tag{1.2}$$

Therefore, it is suspected that NP eigenvalues on analytic boundaries tend to 0 exponentially fast. We prove it in this paper.

We show that, if the boundary is analytic, then NP eigenvalues converge to 0 exponentially fast, and the exponential convergence rate is determined by the modified maximal Grauert radius. See Theorem 3.1 for the precise statement of the result and subsection 2.2 for the definition of the modified maximal Grauert radius. We do not know if the convergence rate is optimal in general. However, we show that it is optimal on domains like disks, ellipses and limaçons of Pascal. It is worth emphasizing that the main theorem is proven using the Weyl-Courant min-max principle and a Paley-Wiener type lemma (Lemma 3.2).

This paper is organized as follows. In Section 2, we review symmetrization of the NP operator, define the modified maximal Grauert radius (and tube), and show that the integral kernel of the NP operator admits analytic continuation to the modified maximal Grauert tube. Section 3 presents and proves the main result of this paper. Section 4 provides some examples to show that the modified maximal Grauert radius yields the best possible bound for the convergence.

2. Preliminaries.

2.1. The NP operator and symmetrization. Throughout, we assume that Ω is a bounded planar domain whose boundary, $\partial\Omega$, is analytic. The single layer potential of a function φ on $\partial\Omega$ is defined by

$$\mathcal{S}_{\partial\Omega}[\varphi](x) = \frac{1}{2\pi}\int_{\partial\Omega} \ln|x-y|\varphi(y)\,d\sigma(y), \quad x \in \mathbb{R}^2,$$

where $d\sigma$ is the length element of $\partial\Omega$. The NP operator on $\partial\Omega$ is defined by

$$\mathcal{K}_{\partial\Omega}[\varphi](x) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{\langle y-x,\nu_y\rangle}{|x-y|^2}\varphi(y)\,d\sigma(y), \quad x \in \partial\Omega,$$

where ν_y is the outward unit normal vector at $y \in \partial\Omega$. The relation between the NP operator and the single layer potential is given by the jump relation for which we refer, for example, to **[2]**. It is well known that Plemelj's symmetrization principle (also known as Calderón's identity) holds:

$$\mathcal{S}_{\partial\Omega}\mathcal{K}^*_{\partial\Omega} = \mathcal{K}_{\partial\Omega}\mathcal{S}_{\partial\Omega}, \tag{2.1}$$

where $\mathcal{K}^*_{\partial\Omega}$ is the $L^2(\partial\Omega)$-adjoint of $\mathcal{K}_{\partial\Omega}$.

We denote by $H^s = H^s(\partial\Omega)$, $s \in \mathbb{R}$, the usual Sobolev space on $\partial\Omega$, and its norm is denoted by $\|\cdot\|_s$. The single layer potential $\mathcal{S}_{\partial\Omega}$ as an operator on $\partial\Omega$ maps $H^{-1/2}(\partial\Omega)$ into $H^{1/2}(\partial\Omega)$. It may not be invertible. In fact, there is a domain Ω in two dimensions such that

$$\mathcal{S}_{\partial\Omega} : H^{-1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega)$$

has a kernel of dimension 1, see **[25]**. However, we may redefine $\mathcal{S}_{\partial\Omega}$ on its kernel so that the redefined operator, which we still denote by $\mathcal{S}_{\partial\Omega}$, is invertible from $H^{-1/2}(\partial\Omega)$ onto $H^{1/2}(\partial\Omega)$, see **[4,** Section 2]. Define

$$\langle \varphi, \psi \rangle_{\mathcal{H}} := -\langle \varphi, \mathcal{S}_{\partial\Omega}^{-1}[\psi] \rangle \tag{2.2}$$

for $\varphi, \psi \in \mathcal{H}_0 := \{\varphi \in H^{1/2} : \langle \varphi, \mathcal{S}_{\partial\Omega}[1] \rangle = 0\}$, equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Here and afterwards, $\langle \cdot, \cdot \rangle$ denotes the $H^{1/2} - H^{-1/2}$ duality product.

It is known that $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is, in fact, an inner product on $\mathcal{H}_0$ and induces the norm equivalent to $\|\cdot\|_{1/2}$, namely, there are constants C_1 and C_2 such that

$$C_1 \|\varphi\|_{1/2} \le \|\varphi\|_{\mathcal{H}} \le C_2 \|\varphi\|_{1/2} \tag{2.3}$$

for all $\varphi \in \mathcal{H}_0$, see **[9]**. Then, $\mathcal{K}_{\partial\Omega}$ is a self-adjoint operator on $\mathcal{H}_0$. In fact, we have the following from (2.1):

$$\langle \varphi, \mathcal{K}_{\partial\Omega}[\psi] \rangle_{\mathcal{H}} = -\langle \varphi, \mathcal{S}_{\partial\Omega}^{-1} \mathcal{K}_{\partial\Omega}[\psi] \rangle = -\langle \varphi, \mathcal{K}_{\partial\Omega}^* \mathcal{S}_{\partial\Omega}^{-1}[\psi] \rangle = \langle \mathcal{K}_{\partial\Omega}[\varphi], \psi \rangle_{\mathcal{H}}.$$

Thus, as a self-adjoint compact operator on a Hilbert space, $\mathcal{K}_{\partial\Omega}$ has eigenvalues converging to 0. It is known that all eigenvalues lie in $(-1/2, 1/2)$, see **[11]**. It is worth mentioning that $1/2$ is an eigenvalue of $\mathcal{K}_{\partial\Omega}$ of multiplicity 1 if we consider $\mathcal{K}_{\partial\Omega}$ as an operator on $H^{1/2}$, not on $\mathcal{H}_0$.

2.2. Maximal Grauert radius. Let S^1 be the unit circle and $Q : S^1 \to \partial\Omega \subset \mathbb{C}$ a regular real analytic parametrization of $\partial\Omega$. Such a parametrization exists since $\partial\Omega$ is real analytic. Hereon, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Then, Q admits an extension as an analytic function from an annulus

$$A_\varepsilon := \{\tau \in \mathbb{C} : e^{-\varepsilon} < |\tau| < e^{\varepsilon}\} \tag{2.4}$$

for some $\varepsilon > 0$ onto a tubular neighborhood of $\partial\Omega$ in $\mathbb{C}$. Let

$$q(t) := Q(e^{it}), \quad t \in \mathbb{R} \times i(-\varepsilon, \varepsilon). \tag{2.5}$$

Then, q is an analytic function from $\mathbb{R} \times i(-\varepsilon, \varepsilon)$ onto a tubular neighborhood of $\partial\Omega$. Moreover, q is a 2π-periodic function, namely, $q(t+2\pi) = q(t)$. The supremum, denoted by ε_*, of the collection of such ε is called the *maximal Grauert radius* of q, and the set $\mathbb{R} \times i(-\varepsilon_*, \varepsilon_*)$ the *maximal Grauert tube.*

In this paper, we consider the numbers ε such that q satisfies an additional condition:

(G) if $q(t) = q(s)$ for $t \in [-\pi, \pi) \times i(-\varepsilon, \varepsilon)$ and $s \in [-\pi, \pi)$, then $t = s$.

It is worth emphasizing that condition (G) is weaker than univalence. It only requires that q attain values $q(s)$, $s \in [-\pi, \pi)$, only at s. Condition (G) is imposed for the integral kernel of the NP operator to be analytically continued, see (2.9). We will see that this condition yields an optimal convergence rate of the NP operator in examples in Section 4. Note that the only points $\partial\Omega$, to which the function $q : \mathbb{R} \times i(-\varepsilon, \varepsilon) \to \mathbb{C}$ maps, are those on the real line.

Since Q is one-to-one on $\partial\Omega$, the extended function is univalent in A_ε if ε is sufficiently small. Therefore, condition (G) is fulfilled if ε is small. We denote the supremum of such an ε by ε_q. We emphasize that ε_q may differ depending on the parametrization q, see Example 4.3 in Section 4. Let

$$\varepsilon_{\partial\Omega} := \sup_q \varepsilon_q, \tag{2.6}$$

where the supremum is taken over all regular real analytic parametrization q of $\partial\Omega$. We call $\varepsilon_{\partial\Omega}$ the *modified maximal Grauert radius* of $\partial\Omega$. The set $\mathbb{R} \times i(-\varepsilon_{\partial\Omega}, \varepsilon_{\partial\Omega})$ is called the *modified maximal Grauert tube*, which we denote by $G_{\partial\Omega}$.

2.3. Analytic extension of the NP operator. Let q be a regular real analytic parametrization on $[-\pi, \pi)$ of $\partial\Omega$. For $x, y \in \partial\Omega$, let $x = q(t)$ and $y = q(s)$. Then the outward unit normal vector ν_y is given by $-iq'(s)/|q'(s)|$ in the complex form. So we have

$$\langle y - x, \nu_y \rangle = \frac{1}{|q'(s)|} \Re \left[(q(s) - q(t)) \overline{(-iq'(s))} \right],$$

and hence,

$$\frac{\langle y-x,\nu_y\rangle}{|x-y|^2}=\frac{-1}{2i|q'(s)|}\left[\frac{q'(s)}{q(t)-q(s)}-\frac{\overline{q'(s)}}{\overline{q(t)}-\overline{q(s)}}\right].$$

Therefore, we have

$$\mathcal{K}_{\partial\Omega}[\varphi](q(t))=\frac{-1}{4\pi i}\int_0^{2\pi}\left[\frac{q'(s)}{q(t)-q(s)}-\frac{\overline{q'(s)}}{\overline{q(t)}-\overline{q(s)}}\right]\varphi(q(s))\,ds.$$

Define

$$K_q(t,s):=\frac{-1}{4\pi i}\left[\frac{q'(s)}{q(t)-q(s)}-\frac{\overline{q'(s)}}{\overline{q(t)}-\overline{q(s)}}\right] \tag{2.7}$$

and

$$\mathcal{K}_q[f](t):=\int_{-\pi}^{\pi}K_q(t,s)f(s)\,ds,\quad -\pi\le t\le\pi. \tag{2.8}$$

Then, we have the relation

$$\mathcal{K}_q[\varphi\circ q](t)=\mathcal{K}_{\partial\Omega}[\varphi](q(t)).$$

For each fixed $s\in\mathbb{R}$, $K_q(t,s)$ as a function of the t variable has an analytic continuation to $\mathbb{R}\times i(-\varepsilon_q,\varepsilon_q)\setminus(s+2\pi\mathbb{Z})$, which is given by

$$K_q(t,s):=\frac{-1}{4\pi i}\left[\frac{q'(s)}{q(t)-q(s)}-\frac{\overline{q'(s)}}{\overline{q(\bar{t})}-\overline{q(s)}}\right]. \tag{2.9}$$

We emphasize that $\overline{q(\bar{t})}$ is analytic by the Schwarz reflection principle. Therefore, due to condition (G), $K_q(t,s)$ is a meromorphic function in $\mathbb{R}\times i(-\varepsilon_q,\varepsilon_q)$ with the singularity on $s+2\pi\mathbb{Z}$. Moreover, it can easily be seen that

$$\lim_{t\to s}\left[\frac{q'(s)}{q(t)-q(s)}-\frac{\overline{q'(s)}}{\overline{q(\bar{t})}-\overline{q(s)}}\right]=\frac{-q'(s)\overline{q''(s)}+\overline{q'(s)}q''(s)}{2|q'(s)|^2}. \tag{2.10}$$

Note that $q'(s)\neq 0$. This means that $K_q(t,s)$ has a removable singularity at $t=s$. Thus, for each fixed $s\in\mathbb{R}$, $K_q(t,s)$ has an analytic continuation (as a function of the t variable) in $\mathbb{R}\times i(-\varepsilon_q,\varepsilon_q)$, and the extended function is 2π-periodic, namely,

$$K_q(t+2\pi,s)=K_q(t,s). \tag{2.11}$$

Define the space H_0 by

$$H_0 := \{f : f = \varphi \circ q,\ \varphi \in \mathcal{H}_0\}. \tag{2.12}$$

We emphasize that H_0 is the collection of 2π-periodic functions equipped with the inner product inherited from $\mathcal{H}_0$:

$$\langle f, g\rangle_H = \langle \varphi, \psi\rangle_{\mathcal{H}},$$

where $f = \varphi \circ q$ and $g = \psi \circ q$ for $\varphi, \psi \in \mathcal{H}_0$.

In the next section, we study the spectrum of $\mathcal{K}_q$ on the space H_0.

3. The main result. Let $\{\lambda_n\}_{n=1}^{\infty}$ be the eigenvalues of the NP operator $\mathcal{K}_{\partial\Omega}$ on $\mathcal{H}_0$, or equivalently, of $\mathcal{K}_q$ on H_0 as defined in (2.8). Since eigenvalues of the NP operator in two dimensions are symmetric with respect to the origin (see, e.g., [**6, 23**], as well as [**7**]), we may assume that eigenvalues are enumerated in the following manner:

$$\frac{1}{2} > |\lambda_1| = |\lambda_2| \geq |\lambda_3| = |\lambda_4| \geq \cdots . \tag{3.1}$$

The next theorem is the main result of this paper.

Theorem 3.1. *Let Ω be a bounded planar domain with the analytic boundary $\partial\Omega$ and $\varepsilon_{\partial\Omega}$ the modified maximal Grauert radius of $\partial\Omega$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be the eigenvalues of the NP operator $\mathcal{K}_{\partial\Omega}$ on $\mathcal{H}_0$ enumerated as* (3.1). *For any $\varepsilon < \varepsilon_{\partial\Omega}$, there is a constant C such that*

$$|\lambda_{2n-1}| = |\lambda_{2n}| \leq Ce^{-n\varepsilon} \tag{3.2}$$

for all n.

The remainder of this section is devoted to proving Theorem 3.1.

We first emphasize that the operator $\mathcal{K}_q$ is symmetric on H_0. In fact, we have

$$\langle f, \mathcal{K}_q[g]\rangle_H = \langle \varphi, \mathcal{K}_{\partial\Omega}[\psi]\rangle_{\mathcal{H}} = \langle \mathcal{K}_{\partial\Omega}[\varphi], \psi\rangle_{\mathcal{H}} = \langle \mathcal{K}_q[f], g\rangle_H,$$

where $f = \varphi \circ q$ and $g = \psi \circ q$ for $\varphi, \psi \in \mathcal{H}_0$. Since the kernel $K_q(t, s)$ of the operator $\mathcal{K}_q$ is 2π-periodic with respect to the t variable, it admits the Fourier series expansion:

$$K_q(t,s) = \sum_{k\in\mathbb{Z}} a_k^q(s)e^{ikt}, \qquad a_k^q(s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_q(t,s)e^{-ikt}dt. \tag{3.3}$$

We obtain the following lemma.

Lemma 3.2. *Suppose that* Ω *is a bounded planar domain with the analytic boundary, and let* q *be a regular real analytic parametrization on* $[-\pi,\pi)$ *of* $\partial\Omega$*. For any* $0<\varepsilon<\varepsilon_q$*, there is a constant* C *such that*

$$|a_k^q(s)| \le Ce^{-\varepsilon|k|} \tag{3.4}$$

for all integers k *and* $s\in[-\pi,\pi)$*.*

Proof. If $k>0$, then we take a rectangular contour R with the clockwise orientation in $\mathbb{R}\times i(-\varepsilon_q,\varepsilon_q)$:

$$R = R_1\cup R_2\cup R_3\cup R_4 := [-\pi,\pi]\cup[\pi,\pi-i\varepsilon]\cup[\pi-i\varepsilon,-\pi-i\varepsilon]\cup[-\pi-i\varepsilon,-\pi].$$

Since $K_q(t,s)$ is analytic in $G_{\partial\Omega}$ and 2π-periodic with respect to the t variable, we have

$$\begin{aligned} 0 = \int_R K_q(t,s)e^{-ikt}dt &= \left\{\int_{R_1}+\int_{R_2}+\int_{R_3}+\int_{R_4}\right\}K_q(t,s)e^{-ikt}dt \\ &= \left\{\int_{R_1}+\int_{R_3}\right\}K_q(t,s)e^{-ikt}dt, \end{aligned}$$

which implies that

$$\begin{aligned} 2\pi a_k^q(s) &= \int_{R_1} K_q(t,s)e^{-ikt}dt \\ &= -\int_{R_3} K_q(t,s)e^{-ikt}dt \\ &= -\int_{\pi-i\varepsilon}^{-\pi-i\varepsilon} K_q(t,s)e^{-ikt}dt. \end{aligned}$$

Since

$$|K_q(t,s)| \le C_0, \quad s\in\mathbb{R},\ t\in R_3, \tag{3.5}$$

it follows immediately

$$|a_k^q(s)| \le C_0 e^{-\varepsilon k}. \tag{3.6}$$

We emphasize that the constant $C_0 > 0$ is independent of k and $s \in \mathbb{R}$.

If $k < 0$, we can prove (3.4) by taking the rectangular contour

$$R = [-\pi, \pi] \cup [\pi, \pi + i\varepsilon] \cup [\pi + i\varepsilon, -\pi + i\varepsilon] \cup [-\pi + i\varepsilon, -\pi].$$

The estimate (3.4) for $k = 0$ is obvious. Thus, the lemma follows. □

We now recall the Weyl-Courant min-max principle (see, for example, **[14]** for a proof).

Theorem 3.3 (the Weyl-Courant min-max principle)**.** *Let $\mathcal{T}$ be a compact symmetric operator on a Hilbert space, whose eigenvalues $\{\kappa_n\}_{n=1}^{\infty}$ are arranged as*

$$|\kappa_1| \ge |\kappa_2| \ge \cdots \ge |\kappa_n| \ge \cdots .$$

If $\mathcal{S}$ is an operator of rank less than or equal to n, then

$$\|\mathcal{T} - \mathcal{S}\| \ge |\kappa_{n+1}|.$$

Proof of Theorem 3.1. Suppose that $\varepsilon < \varepsilon_{\partial\Omega}$, and let q be a regular real analytic parametrization of $\partial\Omega$ such that $\varepsilon < \varepsilon_q \le \varepsilon_{\partial\Omega}$. Using the Fourier expansion of $K_q(t,s)$ given in (3.3), we define

$$S_n(t,s) = \sum_{|k| \le n-1} a_k^q(s) e^{ikt}$$

and

$$\mathcal{S}_n[f](t) = \int_{-\pi}^{\pi} S_n(t,s) f(s) ds.$$

Then, $\mathcal{S}_n$ is of rank at most $2n-1$ on H_0. Thus, it follows from the Weyl-Courant min-max principle that

$$\|\mathcal{K}_q - \mathcal{S}_n\| \ge |\lambda_{2n}|. \tag{3.7}$$

Let $f \in H_0$. It holds that

$$\|f\|_{1/2}^2 \approx \sum_k (1 + |k|) |\widehat{f}(k)|^2, \tag{3.8}$$

where $\widehat{f}(k)$ is the kth Fourier coefficient of f, namely,

$$\widehat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-iks} ds.$$

Note that

$$(\mathcal{K}_q - \mathcal{S}_n)[f](t) = \sum_{|k| \geq n} e^{ikt} \int_{-\pi}^{\pi} a_k^q(s) f(s)\, ds.$$

Since

$$\left| \int_{-\pi}^{\pi} a_k^q(s) f(s)\, ds \right| \leq \|f\|_{1/2} \|a_k^q\|_{-1/2},$$

it follows from the Cauchy-Schwarz inequality that

$$\|(\mathcal{K}_q - \mathcal{S}_n)[f]\|_{1/2}^2 \leq C \|f\|_{1/2}^2 \sum_{|k| \geq n} (1 + |k|) \|a_k^q\|_{-1/2}^2$$

for some constant C. Note that

$$\|a_k^q\|_{-1/2} \leq \|a_k^q\|_{L^2} \leq C e^{-\varepsilon |k|} \tag{3.9}$$

for all $0 < \varepsilon < \varepsilon_q$, which is a consequence of (3.4). It follows that, if $0 < \varepsilon < \varepsilon' < \varepsilon_q$, then

$$\begin{aligned} \sum_{|k| \geq n} (1 + |k|) \|a_k^q\|_{-1/2}^2 &\leq C_1 \sum_{|k| \geq n} (1 + |k|) e^{-2\varepsilon' |k|} \\ &\leq C_2 \sum_{|k| \geq n} e^{-2\varepsilon |k|} \leq C_3 e^{-2\varepsilon n}, \end{aligned}$$

and hence,

$$\|(\mathcal{K}_q - \mathcal{S}_n)[f]\|_{1/2} \leq C e^{-\varepsilon n} \|f\|_{1/2}. \tag{3.10}$$

We then obtain (3.2) from (2.3), (3.7) and (3.10). This completes the proof. □

It is worth mentioning that the exponential decay of the eigenvalues for $\mathcal{K}_q$ can also be shown by using the Chebyshev expansion of $K_q(t, s)$. The Chebyshev expansion has been used in **[14]** to study eigenvalues of operators with real analytic symmetric kernels. Using this method, it can be shown that (3.2) holds for all ε such that

$$\varepsilon < \varepsilon_0 := \log\left(\frac{1}{\pi}\left(\varepsilon_{\partial\Omega} + \sqrt{\pi^2 + \varepsilon_{\partial\Omega}^2}\right)\right). \tag{3.11}$$

This result is weaker than Theorem 3.1 since $\varepsilon_0 < \varepsilon_{\partial\Omega}$. Thus, we omit the details.

4. Examples. Theorem 3.1 shows that (3.2) holds for all $\varepsilon < \varepsilon_{\partial\Omega}$. In this section we present a few examples of domains to show that this result is optimal in the sense that $\varepsilon_{\partial\Omega}$ is the smallest number with such a property.

Example 4.1 (Circles)**.** Suppose that $\partial\Omega$ is a circle. Then, it can easily be seen that $K_q(t,s) \equiv 1$, and hence, $\varepsilon_{\partial\Omega} = +\infty$. Thus, (3.2) shows that, for any number $\beta > 0$, there is a constant C such that

$$|\lambda_{2n}| \leq C\beta^n$$

for all n. Indeed, it is known that 0 is the only eigenvalue of the NP operator on $\mathcal{H}_0$.

Example 4.2 (Ellipses)**.** Suppose that $\partial\Omega$ is the ellipse given by

$$\partial\Omega : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0.$$

A parametrization of $\partial\Omega$ is given by

$$q(t) = a\cos t + ib\sin t = \frac{a+b}{2}e^{it} + \frac{a-b}{2}e^{-it}, \quad t \in [-\pi, \pi).$$

Note that q admits analytic continuation to the entire complex plane, and hence, the maximal Grauert radius is ∞.

In order to compute the modified maximal Grauert radius $\varepsilon_{\partial\Omega}$, suppose that $q(t) = q(s)$, where $t \in [-\pi, \pi) \times i\mathbb{R}$ and $s \in [-\pi, \pi)$. Nontrivial solutions of this equation are given by

$$e^{it} = \frac{a-b}{a+b}e^{-is},$$

which implies that

$$e^{-\Im t} = \frac{a-b}{a+b},$$

and hence,

$$\varepsilon_q = \log \frac{a+b}{a-b}. \tag{4.1}$$

Therefore, from Theorem 3.1, we have the exponential decay estimate

$$|\lambda_{2n-1}| = |\lambda_{2n}| \leq C\beta^n \quad \text{for any } \beta > \frac{a-b}{a+b}. \tag{4.2}$$

In view of (1.2), we see that the number $a-b/a+b$ in (4.1) is optimal. This means, in particular, that condition (G) is necessary for the definition of the modified maximal Grauert radius in this paper.

Example 4.3 (Pascal's limaçons)**.** Let A be a number such that $0 < A < 1/2$. The limaçon of Pascal $\partial\Omega_A$ is defined by

$$\partial\Omega_A : w = z + Az^2, \quad z = e^{it},\ t \in [-\pi, \pi). \tag{4.3}$$

See Figure 1 for the limaçon with $A = 0.4$.

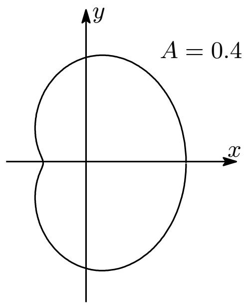

FIGURE 1.

Let us first compute eigenvalues of the NP operator on $\partial\Omega_A$. In order to do so, we recall that the polar equation of an ellipse with one focus at the origin is, up to similarity,

$$r = \frac{1}{1 + e\cos\theta}$$

where e is the eccentricity. We denote the ellipse by ∂E_e. In complex notation, ∂E_e is given by

$$w = f(z) := \frac{z}{1 + e(z + z^{-1})/2} = \frac{2}{e + 2z^{-1} + ez^{-2}}, \quad |z| = 1.$$

Let h be the Möbius transformation defined by

$$h(w) := \frac{-ew+2}{2w}. \tag{4.4}$$

Then, we have

$$h(f(z)) = z^{-1} + \frac{e}{2}z^{-2}. \tag{4.5}$$

This is the limaçon with $A = e/2$. In short, we have

$$h(\partial E_{2A}) = \partial\Omega_A. \tag{4.6}$$

According to [**23,** page 1195], eigenvalues of the NP operator are invariant under the Möbius transformations, and hence, NP operators on ∂E_{2A} and $\partial\Omega_A$ have identical eigenvalues. In view of (1.2), we see that eigenvalues of the NP operator on $\partial\Omega_A$ are

$$\pm\frac{1}{2}\left(\frac{1-\sqrt{1-4A^2}}{1+\sqrt{1-4A^2}}\right)^n. \tag{4.7}$$

A straightforward parametrization of the limaçon $\partial\Omega_A$ is given by

$$q(t) := e^{it} + Ae^{2it}, \quad t \in [-\pi,\pi). \tag{4.8}$$

This shows that q can be analytically extended to the entire complex plane. To find ε_q we suppose $q(t) = q(s)$ for some $t \in [-\pi,\pi) \times i\mathbb{R}$ and $s \in [-\pi,\pi)$. Then, non-trivial solutions are $e^{it} = -e^{is} - 1/A$, and hence, $e^{-\Im t} = |e^{is} + 1/A|$. Therefore, we have

$$\varepsilon_q = \inf_s \log\left|e^{is} + \frac{1}{A}\right| = \log\left(\frac{1}{A} - 1\right).$$

Thus, we infer from Theorem 3.1 that

$$|\lambda_{2n-1}| = |\lambda_{2n}| \le C\beta^n \quad \text{for any } \beta > \frac{A}{1-A}.$$

It can be seen from (4.7) that this estimate is not optimal since

$$\frac{1-\sqrt{1-4A^2}}{1+\sqrt{1-4A^2}} < \frac{A}{1-A}.$$

However, we may use another parametrization of $\partial\Omega_A$ to obtain an optimal estimate. In fact, let $e = 2A$, and

$$a := \frac{1}{1-e^2}, \qquad b := a\sqrt{1-e^2}$$

so that

$$g(z) = \frac{a+b}{2}z + \frac{a-b}{2}z^{-1} + ae, \quad |z| = 1,$$

is a complex parametrization of ∂E_e. Using the Möbius transformation h in (4.4), define

$$q_1(t) := h(g(e^{it})). \tag{4.9}$$

Then, (4.6) shows that $q_1(t)$, $t \in [-\pi, \pi)$, is a parametrization of $\partial\Omega_A$.

If $q_1(t) = q_1(s)$, then $g(e^{it}) = g(e^{is})$. Therefore, as shown in Example 4.2, we have

$$\varepsilon_{q_1} = \log\frac{a+b}{a-b} = \log\frac{1+\sqrt{1-4A^2}}{1-\sqrt{1-4A^2}},$$

which yields an optimal estimate.

It is worth mentioning that all three above examples show that (3.2) holds even for $\varepsilon = \varepsilon_{\partial\Omega}$, while Theorem 3.1 only shows that it holds for $\varepsilon < \varepsilon_{\partial\Omega}$. It is interesting to show that (3.2) holds even for $\varepsilon = \varepsilon_{\partial\Omega}$ in general.

We now present one more example of a curve on which the NP eigenvalues are unknown.

Example 4.4 (Transcendental curves)**.** We consider the transcendental curve

$$\partial\Omega : w = e^{Az}, \quad |z| = 1,\ 0 < |A| < \pi.$$

(See Figure 2.)

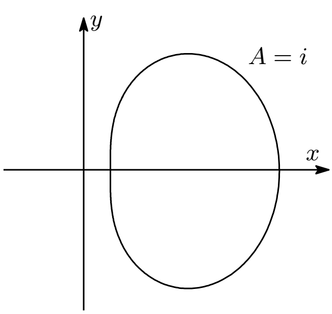

FIGURE 2.

An obvious parametrization of $\partial\Omega$ is given by

$$q(t) := \exp(Ae^{it}), \quad t \in [-\pi, \pi).$$

If $q(t) = q(s)$ for some $t \in [-\pi, \pi) \times i\mathbb{R}$ and $s \in [-\pi, \pi)$, then non-trivial solutions are given by

$$Ae^{it} = Ae^{is} + i2\pi n, \quad n \in \mathbb{Z}\ (n \neq 0).$$

It then follows that

$$\varepsilon_q = \inf_{n,s} \log \left| e^{is} + \frac{i2\pi n}{A} \right| = \log \left(\frac{2\pi}{|A|} - 1 \right).$$

Thus, we have

$$|\lambda_{2n-1}| = |\lambda_{2n}| \leq C\beta^n \quad \text{for any } \beta > \frac{|A|}{2\pi - |A|}. \tag{4.10}$$

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