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EXISTENCE OF A MILD SOLUTION FOR A NEUTRAL STOCHASTIC FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSION WITH A NONLOCAL CONDITION

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ABSTRACT. This paper mainly concerns the existence of a mild solution for a neutral stochastic fractional integro-differential inclusion of order $1 < \beta < 2$ with a nonlocal condition in a separable Hilbert space. Utilizing the fixed point theorem for multi-valued operators due to O' Regan **[29]**, we establish an existence result involving a β-resolvent operator. An illustrative example is provided to show the effectiveness of the established results.

1. Introduction. In the past few decades, the theory of fractional calculus has become a most interesting area for researchers due to its wide applicability in sciences and engineering in such areas as material sciences, mechanics, seepage flow in porous media, fluid dynamic traffic models, population dynamics, economics, chemical technology, medicine and many others. One of the major applications of fractional calculus is the hypothesis of fractional evolution equations. The fractional derivatives give a phenomenal instrument for describing the memory and the process of genetic properties of different materials is a major advantage of fractional calculus. For more details regarding fractional calculus and fractional differential equations, the interested reader is referred to the monographs **[21, 32]**, and the references cited therein. Moreover, the investigation of the abstract nonlocal Cauchy problem was introduced in **[8]**. It has been observed that differential equations with nonlocal conditions are more realistic for describing many phenomena and have better effects in applications than the prob-

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lem without nonlocal conditions. Many researchers have investigated the differential equations with nonlocal condition, and certain results have been obtained, see [**7**]–[**39**], and the references cited therein.

Fractional differential inclusions play a significant role in the investigation of different dynamical processes and phenomena represented by a discontinuous or multivalued equation, arising, specifically, in the investigation of the dynamics of economics, dynamic Coulomb friction problems and biological macrosystems. In addition, they are extremely valuable in demonstrating existence theorems in control theory and differential variational inequalities. Fractional differential inclusions in infinite-dimensional spaces have not been considered, in particular, fractional differential inclusions with a nonconvex multivalued term. For more details on differential inclusions, the reader is referred to [**19, 34**] as well as [**4, 5, 10, 17, 24, 25, 30, 35, 37, 38, 39**]. In addition, stochastic differential equations have gained much attention due to the large number of problems in real life situations to which mathematical models are applicable, and which are fundamentally stochastic instead of deterministic. Stochastic differential equations have an extraordinary application in different fields of science and engineering, for example, finance, numerical analysis, physics, biology, medical, control theory and social sciences, see [**26, 28**]. The theory of stochastic differential equations has been very rapidly developed, and there are numerous, fascinating results on the qualitative properties of the solution, such as existence, uniqueness and stability of solutions of different stochastic differential equations and integro-differential equations, see [**12, 33, 36**] and the references given therein.

An existence result for neutral delay fractional integro-differential equations with nonlocal condition in a separable Banach space is studied in [**22**], utilizing the theory of the measures of noncompactness and condensing maps. Ezzinbi et al. [**16**] extended the results of [**15**] and discussed the existence and regularity of solutions for some nonlocal neutral partial differential equations using the strongly continuous semigroup. Using the fixed point theorem for multi-valued operators, due to Dhage [**14**], and fractional power of operators, Lin and Hu [**24**] derived sufficient conditions for the existence of the mild solution to neutral stochastic functional integro-differential inclusions involving nonlocal and impulsive conditions. Yan and Zhang [**36**] obtained the existence of the mild solution to nonlocal stochastic integro-differential equations with the aid of Schaefer's fixed point theorem and the analytic resolvent operator. Sufficient conditions proving the existence of the mild solution of fractional stochastic differential inclusions with state-dependent delays were obtained in **[17]** via the nonlinear alternative of Leray-Schauder type for multivalued maps due to O'Regan. Li and Liu **[23]** considered neutral stochastic differential inclusions with infinite delay and proved existence results utilizing a fixed point theorem for condensing maps due to Martelli **[27]**. The existence of mild solutions in the mean square moment for impulsive neutral stochastic integro-differential inclusions of the fractional order with state-dependent delay was studied in **[37]** with the aid of the nonlinear alternative of Leray-Schauder type for multivalued maps, due to O'Regan **[29]** and the solution operator. Balasubramaniam, et al. **[3]**, discussed the existence results in the pth moment for stochastic delay evolution inclusions by utilizing the compact semigroup and fixed point theorem for condensing maps, due to Martelli **[27]**. Liu and Liu **[25]** considered the existence of the mild solution for fractional semi-linear differential inclusions involving a nonconvex set-valued function. With the aid of a fixed point theorem for the condensing multivalued map and an analytic resolvent operator, sufficient conditions providing existence results was derived in Chang and Nieto **[11]**. After reviewing the previous work, we find that there is very little research, to the best of our knowledge, which has been done in regards to the mild solution for nonlocal neutral stochastic integro-differential inclusions involving the fractional derivative in a pth moment utilizing resolvent operators. This fact is the inspiration of our present work.

In this paper, we consider the following, neutral stochastic integro-differential inclusion with nonlocal conditions in a separable Hilbert space $(\mathbb{U}; \|\cdot\|_{\mathbb{U}})$ with inner product $(\cdot,\cdot)_{\mathbb{U}}$,

$$\begin{aligned}(1.1)\quad {}^cD_t^\beta[y(t) - F(t, y(h_1(t)), \int_0^t a_1(t,s,y(h_2(s)))\,ds)] \in Ay(t)\\ + \int_0^t f(t-s)y(s)\,ds + G(t, y(h_3(t)), \int_0^t a_2(t,s,y(h_4(s)))\,ds)\,\frac{dw(t)}{dt},\\ t \in J = [0,T],\end{aligned}$$

$$(1.2)\qquad y(0) = y_0 + h(y) \in \mathbb{U}, \quad y'(0) = 0,$$

where ${}^cD_t^\beta$ means the Caputo fractional derivative of order $1 < \beta < 2$, $0 < T < \infty$, A and $f(t)$, $t \geq 0$, are closed, densely linear operators defined on a common domain in a Hilbert space $\mathbb{U}$. The functions F, G, h_1, h_2, h_3, h are appropriate continuous functions to be specified later and $h_j \in C(J, J)$, $j = 1, 2, 3, 4$. We assume that $\{w(t) : t \geq 0\}$ is a given $\mathbb{V}$-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $\mathcal{Q} \geq 0$ to be defined later; here, $\mathbb{V}$ means another separable Hilbert space with norm $\|\cdot\|_{\mathbb{V}}$ and inner product $(\cdot,\cdot)_{\mathbb{V}}$.

The purpose of this work is to study the existence of the mild solution for the nonlocal fractional order system (1.1)–(1.2) utilizing fixed point theory for multivalued maps, a generalization of previous results. As a motivation example for this class of equations, we consider the following boundary value problem with nonlocal condition

$$\begin{aligned} \frac{\partial}{\partial t}\left[z(t,x) - F\left(t, z(\cos t, x), \frac{\partial z}{\partial t}(\cos t, x)\right)\right] \\ = \frac{\partial^2 z(t,x)}{\partial x^2} + G\left(t, z(\cos t, x), \frac{\partial z}{\partial x}(\cos t, x)\right)\frac{\partial w(t)}{\partial t}, \end{aligned} \tag{1.3}$$

$$z(t,0) = z(t,\pi) = 0, \tag{1.4}$$

$$z(0,x) = z_0(x) + h(z(t,x)), \quad x \in [0,\pi],\ t \in [0,1]. \tag{1.5}$$

This nonlocal fractional stochastic system can also be incorporated into an abstract neutral equation, as mentioned above. Since F and G involve the spatial partial derivative, the results obtained by other authors cannot be applied to our system even if $h(\cdot) = 0$. This is the main motivation of this paper. In addition, this work proposes a framework for studying the neutral stochastic fractional integro-differential equation with nonlocal conditions, the main contribution of the work. The rest of the paper is organized as follows. Section 2 discusses some basic definitions, lemmas and theorems, useful in proving our results. Section 3 focuses on the existence of a mild solution to system (1.1)–(1.2) with the aid of a fixed point theorem of multi-valued mapping and a resolvent operator. Section 4 provides an example.

2. Preliminaries. In this section, we discuss some basic definitions, notation, theorems, lemmas and some basic facts regarding analytic resolvent operators. Throughout, the notation $(\mathbb{U}, \|\cdot\|_{\mathbb{U}}, (\cdot,\cdot)_{\mathbb{U}})$ and $(\mathbb{V}, \|\cdot\|_{\mathbb{V}}, (\cdot,\cdot)_{\mathbb{V}})$ stand for the separable Hilbert spaces. The notation $C(J,\mathbb{U})$ stands for the Banach space of continuous functions from J to $\mathbb{U}$ with supremum norm, i.e., $\|y\|_J = \sup_{t\in J}\|y(t)\|_{\mathbb{U}}$, for all $y \in C(J,\mathbb{U})$, and $L^1(J,\mathbb{U})$ denotes the Banach space of functions $y : J \to \mathbb{U}$ which are Bochner integrable, normed by

$$\|y\|_{L^1} = \int_0^T \|y(t)\|_{\mathbb{U}}\, dt,$$

for all $y \in L^1(J,\mathbb{U})$. A measurable function $y : J \to \mathbb{U}$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable. For more details concerning the Bochner integral, the reader is referred to **[38]**. The notation $\mathbb{B}(\mathbb{U})$ stands for the Banach space of all linear bounded operators from $\mathbb{U}$ onto itself with norm

$$\|g\|_{\mathbb{B}(\mathbb{U})} = \sup\{\|g(y)\| : \|y\| \le 1\}, \quad \text{for all } g \in \mathbb{B}(\mathbb{U}). \tag{2.1}$$

Herein, we assume that A, $f(t)$, $t \ge 0$, are closed linear operators densely defined on a common domain $D(A)$ on the Hilbert space $\mathbb{U}$. Let $[D(A)]$ denote the domain of A with the graph norm. For $0 < \eta \le 1$, the notation $(-A)^\eta$ represents the fractional power of the operator $-A$ with dense domain $D((-A)^\eta)$ in $\mathbb{U}$. It is easy to verify that $D((-A)^\eta)$ is a Banach space with the norm

$$\|y\|_\eta = \|(-A)^\eta y\|, \quad \text{for all } y \in D((-A)^\eta). \tag{2.2}$$

Hence, we signify the space $D((-A)^\eta)$ by $\mathbb{U}_\eta$ endowed with the η-norm $(\|\cdot\|_\eta)$ and this norm is equivalent to the graph norm of $(-A)^\eta$, that is, $\|y\|_\eta = (\|y\|^2 + \|A^\eta y\|^2)^{1/2}$. Also, we have that $\mathbb{U}_\kappa \hookrightarrow \mathbb{U}_\eta$ for $0 < \eta < \kappa$, and therefore, the embedding is continuous. Then, we define $\mathbb{U}_{-\eta} = (\mathbb{U}_\eta)^*$ for each $\eta > 0$. The space $\mathbb{U}_{-\eta}$ stands for a Banach space with the norm $\|z\|_{-\eta} = \|A^{-\eta}z\|$, $z \in \mathbb{U}_{-\eta}$, known as the dual space of $\mathbb{U}_\eta$. For more details on the fractional powers of closed linear operator, the reader is referred to **[31]**.

Let $(\Omega, \mathcal{F}, \mathbb{P}; \mathbf{F})$ $(\mathbf{F} = \{\mathcal{F}_t\}_{t\ge 0})$ be a complete filtered probability space satisfying the condition that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets, where Ω is a space, $\mathcal{F}$ is a σ-algebra of subsets of Ω and $\mathbb{P}$ is a countably additive, non-negative measure on $(\Omega, \mathcal{F})$ with total mass $\mathbb{P}(\Omega) = 1$. A filtration $\mathbf{F}$ is a sequence of σ-algebra $\{\mathcal{F}_t\}_{t\geq 0}$ with $\mathcal{F}_t \subset \mathcal{F}$ for each t and $t_1 \leq t_2 \Rightarrow \mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$. A $\mathbb{U}$-valued random variable is an $\mathcal{F}_t$-measurable function

$$u(t) : \Omega \longrightarrow \mathbb{U},$$

and the space

$$S = \{u(t,\omega) : \Omega \longrightarrow \mathbb{U} : t \in [0,T]\}$$

which contains all random variables is called a *stochastic process*. In addition, we use the notation $u(t)$ instead of $u(t,w)$ and $u(t) : J \to \mathbb{U} \in S$.

Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of K. We assume that $\{w(t) : t \geq 0\}$ is a cylindrical $\mathbb{V}$-valued Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P}; \mathbf{F})$ with a finite trace nuclear covariance operator $\mathcal{Q} \geq 0$, i.e.,

$$\mathrm{Tr}(\mathcal{Q}) := \sum_{i=1}^{\infty} \lambda_i = \lambda < \infty$$

such that $\mathcal{Q}e_i = \lambda_i e_i$. Then, we obtain $w(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} w_i(t) e_i$; here, $\{w_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Brownian motions. At this point, the above $\mathbb{V}$-valued stochastic process $w(t)$ is called a $\mathcal{Q}$-*Wiener process*. The symbol $L(\mathbb{V}, \mathbb{U})$ stands for the space of all bounded linear operators from $\mathbb{V}$ into $\mathbb{U}$ with the usual norms $\|\cdot\|_{L(\mathbb{V},\mathbb{U})}$ and $L(\mathbb{U})$ when $\mathbb{V} = \mathbb{U}$. Suppose that $\mathcal{F}_t = \sigma\{w(s) : 0 \leq s \leq t\}$ is the σ-algebra generated by w and $\mathcal{F}_T = \mathcal{F}$. For $\psi \in L(\mathbb{V}, \mathbb{U})$, we define

$$\|\psi\|_{\mathcal{Q}}^2 = \mathrm{Tr}(\psi \mathcal{Q} \psi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \psi e_n\|^2. \tag{2.3}$$

The operator ψ is a $\mathcal{Q}$-Hilbert-Schmidt operator if $\|\psi\|_{\mathcal{Q}}^2 < \infty$. The symbol $L_{\mathcal{Q}}(\mathbb{V}, \mathbb{U})$ stands for the space containing all $\mathcal{Q}$-Hilbert-Schmidt operators $\psi : \mathbb{V} \to \mathbb{U}$. The completion $L_{\mathcal{Q}}(\mathbb{V}, \mathbb{U})$ of $L(\mathbb{V}, \mathbb{U})$ with the topology induced by the norm $\|\cdot\|_{\mathcal{Q}}$ is a Hilbert space with the same norm topology; here, $\|\psi\|_{\mathcal{Q}}^2 = (\psi, \psi)$. For a basic study on stochastic differential equations, the reader is referred to **[12]**.

In order to set the structure for our primary existence results, we provide the following definitions.

Definition 2.1. The Riemann-Liouville fractional integral operator $\mathbb{J}$ of order $\beta > 0$ is defined by

$$ {}^{RL}\mathbb{J}_t^\beta F(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F(s)\, ds, \tag{2.4}$$

where $F \in L^1((0,T), \mathbb{U})$.

Definition 2.2. The Riemann-Liouville fractional derivative is given as

$$ {}^{RL}D_t^\beta F(t) = D_t^m \mathbb{J}_t^{m-\beta} F(t), \quad m-1 < \beta < m,\ m \in \mathbb{N}, \tag{2.5}$$

where $D_t^m = d^m/dt^m$, $F \in L^1((0,T),\mathbb{U})$ and ${}^{RL}\mathbb{J}_t^{m-\beta} F \in W^{m,1}((0,T), \mathbb{U})$. Here, the notation $W^{m,1}((0,T),\mathbb{U})$ stands for the Sobolev space defined by:

$$ \begin{aligned} W^{m,1}&((0,T),\mathbb{U}) \\ &= \Big\{ y \in \mathbb{U} : \text{ there exists a } z \in L^1((0,T),\mathbb{U}) : \\ &\qquad y(t) = \sum_{k=0}^{m-1} d_k \frac{t^k}{k!} + \frac{t^{m-1}}{(m-1)!} * z(t),\ t \in (0,T) \Big\}. \end{aligned} \tag{2.6}$$

Note that $z(t) = y^m(t)$ and $d_k = y^k(0)$.

Definition 2.3. The Caputo fractional derivative is given as

$$ {}^cD_t^\beta F(t) = \frac{1}{\Gamma(m-\beta)} \int_0^t (t-s)^{m-\beta-1} F^m(s)\, ds, \quad m-1 < \beta < m, \tag{2.7}$$

where $F \in C^{m-1}((0,T),\mathbb{U}) \cap L^1((0,T),\mathbb{U})$.

Let $L_p(\Omega, \mathcal{F}_t, \mathbb{U})$ be the Banach space of all $\mathcal{F}_t$-measurable pth power integrable random variables with values in $\mathbb{U}$. The notation $L_p^{\mathcal{F}}([0,T],\mathbb{U})$ stands for the Banach space of all pth power integrable and $\mathcal{F}_t$-measurable processes with the values in $\mathbb{U}$. Let $C([0,T], L_p(\mathcal{F},\mathbb{U}))$ be the Banach space of all continuous mappings from $[0,T]$ into $L_p(\mathcal{F},\mathbb{U})$ with $\sup_{t\in[0,T]} E\|y(t)\|_{\mathbb{U}}^p < \infty$. In particular, the notation $\mathcal{C}$ denotes the Banach space $C([0,T], L_p(\Omega,\mathcal{F},\mathbb{U}))$, the family of all $\mathcal{F}_t$-measurable $\mathbb{U}$-valued stochastic processes with the norm

$$\|y\|_{\mathcal{C}} = \big(\sup_{t\in[0,T]} E\|y(t)\|_{\mathbb{U}}^{p}\big)^{1/p}.$$

Here, E denotes the expectation with respect to a probability $\mathbb{P}$, i.e., $Ey = \int_{\Omega} y d\mathbb{P}$.

Let $L_p^0(\Omega, \mathcal{C})$ be the family of all $\mathcal{F}_0$-measurable, $\mathcal{C}$-valued random variables $y(0)$. We use the symbol $\mathcal{P}(\mathbb{U})$ for the family of all subsets of $\mathbb{U}$, and denote:

$$\begin{aligned}
&\mathcal{P}_{cl}(\mathbb{U}) = \{Z \in \mathcal{P}(\mathbb{U}) : Z \text{ is closed}\},\\
&\mathcal{P}_{bd}(\mathbb{U}) = \{Z \in \mathcal{P}(\mathbb{U}) : Z \text{ is bounded}\},\\
&\mathcal{P}_{cv}(\mathbb{U}) = \{Z \in \mathcal{P}(\mathbb{U}) : Z \text{ is convex}\},\\
&\mathcal{P}_{cp}(\mathbb{U}) = \{Z \in \mathcal{P}(\mathbb{U}) : Z \text{ is compact}\}.
\end{aligned} \tag{2.8}$$

Now, we present a few facts on multi-valued analysis. The multi-valued $\Upsilon : \mathbb{U} \to \mathcal{P}(\mathbb{U})$ is called *convex* (*closed*) *valued* if $\Upsilon(x)$ is convex (closed) for every $x \in \mathbb{U}$. If $\Upsilon(B) = \cup_{u\in B}\Upsilon(u)$ is bounded in $\mathbb{U}$ for all $B \in \mathcal{P}_{bd}(\mathbb{U})$ i.e., $\sup_{u\in B}\{\sup\{\|z\| : z \in \Upsilon(u)\}\} < \infty$, then map Υ is bounded on bounded sets.

A multi-valued map $\Upsilon : \mathbb{U} \to \mathcal{P}(\mathbb{U})$ is called *upper semicontinuous* (usc) if, for any $u \in \mathbb{U}$, the set $\Upsilon(u)$ is a nonempty closed subset of $\mathbb{U}$ and if, for each open set $\mathcal{G}$ of $\mathbb{U}$ which is contained in $\Upsilon(u)$, there exists an open neighborhood $\mathcal{N}$ of u such that $\Upsilon(\mathcal{N}) \subset \mathcal{G}$. The map Υ is called *completely continuous* if $\Upsilon(\mathcal{G})$ is relatively compact for every bounded subset of $\mathcal{G} \subseteq \mathbb{U}$. If the multi-valued map Υ is completely continuous with nonempty compact values, then Υ is usc if and only if Υ has a closed graph, i.e., $u_n \to u$, $v_n \to v$, $v_n \in \Upsilon(u_n) \Rightarrow v \in \Upsilon(u)$. For $y \in \mathbb{U}$ and $\mathcal{N}$, $\mathcal{G} \in \mathcal{P}_{bd,cl}$, we denote by

$$\mathbb{D}(y, \mathcal{N}) = \inf\{\|y - z\|_{\mathbb{U}} : \ z \in \mathcal{N}\}$$

and

$$\widetilde{\rho}(\mathcal{N}, \mathcal{G}) = \sup_{u\in\mathcal{N}} \mathbb{D}(u, \mathcal{N}),$$

and the Hausdorff metric $\mathbb{U}_d : \mathcal{P}_{bd,cl}(\mathbb{U}) \to \mathcal{P}_{bd,cl}(\mathbb{U}) \to \mathbb{R}^+$ by

$$\mathbb{U}_d(\widetilde{B}, \widetilde{C}) = \max\{\widetilde{\rho}(\widetilde{B}, \widetilde{C}), \widetilde{\rho}(\widetilde{C}, \widetilde{B})\}.$$

The map Υ has a fixed point if there exists a $y \in \mathbb{U}$ with $y \in \Upsilon(y)$. A multi-valued map $\Upsilon : J \to \mathcal{P}_{bd,cl,cv}(\mathbb{U})$ is called *measurable* if, for each $y \in \mathbb{U}$, the function $t \mapsto \mathbb{D}(y, \Upsilon(t))$ is a measurable function for each $t \in [0,T]$. For more details on multi-valued maps, the reader is referred to **[13, 20]**.

Definition 2.4. Let $\Upsilon : \mathbb{U} \to \mathcal{P}_{bd,cl}(\mathbb{U})$ be a multivalued mapping. Then, Υ is called a *multivalued contraction* if there exists a constant $\mu \in (0,1)$ such that

$$\mathbb{U}_d(\Upsilon(x) - \Upsilon(y)) \leq \mu \|x - y\|_{\mathbb{U}}, \tag{2.9}$$

for each $x, y \in \mathbb{U}$. The constant μ is called a *contraction constant* of Υ.

Definition 2.5. The multi-valued map $G : [0,T] \times \mathbb{U} \times \mathbb{U} \to \mathcal{P}_{bd,cl,cv}(L(\mathbb{V}, \mathbb{U}))$ is called L^p-*Carathéodory* if

(a) the map $t \mapsto G(t,y,z)$ is measurable for each $(y,z) \in \mathbb{U} \times \mathbb{U}$;
(b) the map $(y,z) \mapsto G(t,y,z)$ is usc for almost all $t \in J$;
(c) there exist a continuous function $W_g \in L^1([0,T];\mathbb{R}_+)$ and a continuous increasing function $\Theta_g : [0,\infty) \to (0,\infty)$ such that

$$\|G(t,y,z)\|_{\mathbb{U}}^p = \sup_{g \in G(t,y,z)} E\|g\|_{\mathbb{U}}^p \leq W_g(t)\Theta_g(E\|y\|_{\mathbb{U}}^p + E\|z\|_{\mathbb{U}}^p), \tag{2.10}$$

for all $(y,z) \in \mathbb{U} \times \mathbb{U}$ and for almost every $t \in J$.

Thus, we have the following result stated as

Lemma 2.6 (**[13]**)**.** *Let $\mathbb{U}$ be a Hilbert space and I a compact interval. If G is an L^p-Carathéodory multi-valued map with $\mathcal{N}_{G,u} \neq 0$, and Υ is a linear continuous mapping from $L^p(I,\mathbb{U})$ to $C(I,\mathbb{U})$, then the map*

$$\begin{aligned} &\Upsilon \circ \mathcal{N}_G : C(I,\mathbb{U}) \longrightarrow \mathcal{P}_{cp,cv}(\mathbb{U}), \\ &u \longmapsto (\Upsilon \circ \mathcal{N}_G)(u) = \Upsilon(\mathcal{N}_{G,u}), \end{aligned} \tag{2.11}$$

is a closed graph operator in $C(I,\mathbb{U}) \times C(I,\mathbb{U})$, where $\mathcal{N}_{G,u}$ denotes the selectors set from G, defined as

$$\begin{aligned} &g \in \mathcal{N}_{G,u} \\ &= \Big\{ g \in L^p(I, L(\mathbb{V},\mathbb{U})) : g(t) \in G\Big(t, u(h_3(t)), \int_0^t a_2(t,s,u(h_4(t)))\, ds\Big) \\ &\qquad\qquad \textit{for almost every } t \in [0,T] \Big\}. \end{aligned} \tag{2.12}$$

Now, we present an α-resolvent operator, which appeared in **[1]**.

Definition 2.7 ([**1**]). A one-parameter family of bounded linear operators $S_\beta(t)$, $t \geq 0$, on $\mathbb{U}$ is said to be a β-resolvent operator for

$$ {}^cD_t^\beta y(t) = Ay(t) + \int_0^t f(t-s)y(s)\,ds, \tag{2.13} $$

$$ y(0) = x, \quad y'(0) = 0, \tag{2.14} $$

if

(i) the function $S_\beta(\cdot) : [0,\infty) \to L(\mathbb{U})$ is strongly continuous;
(ii) $S_\beta(0)x = x$, for all $x \in \mathbb{U}$ and $\alpha \in (1,2)$;
(iii) for $x \in D(A)$, $S_\beta(\cdot)x \in C([0,\infty),[D(A)]) \cap C^1((0,\infty),\mathbb{U})$; and

$$ \begin{aligned} {}^cD_t^\beta S_\beta(t)x &= AS_\beta(t)x + \int_0^t f(t-s)S_\beta(s)x\,ds, \\ &= S_\beta(t)Ax + \int_0^t S_\beta(t-s)f(s)x\,ds, \quad t \geq 0. \end{aligned} \tag{2.15} $$

In what follows, we consider the following assumptions:

(P1) The operator $A : D(A) \subset \mathbb{U} \to \mathbb{U}$ is a closed, densely linear operator. Let $\beta \in (1,2)$. For some $\phi_0 \in (0,\pi/2]$ for every $\phi < \phi_0$, there exists a constant $C_0 = C_0(\phi) > 0$ such that $\lambda \in \rho(A)$ for each

$$ \lambda \in \sum_{0,\beta\eta} = \{\lambda \in \mathbb{C},\ \lambda \neq 0,\ |\arg(\lambda)| < \beta\eta\}; \tag{2.16} $$

here, $\eta = \phi + \pi/2$ and $\|R(\lambda,A)\| \leq C_0/|\lambda|$ for all $\lambda \in \sum_{0,\beta\eta}$.

(P2) $f(t) : D(f(t)) \subseteq \mathbb{U} \to \mathbb{U}$ for $t \geq 0$ is a closed linear operator with $D(A) \subseteq D(f(t))$, and $f(\cdot)x$ is strongly measurable on $(0,\infty)$ for every $x \in D(A)$. For $t > 0$ and $x \in D(A)$, there exists a $d(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ such that $\widehat{d}(\lambda)$ (Laplace of $d(\cdot)$) exists for $\mathrm{Re}(\lambda) > 0$ and $\|f(t)x\| \leq d(t)\|x\|_1$. Furthermore, the operator-valued function

$$ \widehat{f} : \sum_{0,\pi/2} \longrightarrow \mathcal{L}([D(A)],\mathbb{U}) $$

has an analytical extension, denoted by $\widehat{f}$ to $\sum_{0,\eta}$ such that

$$ \|\widehat{f}(\lambda)y\| \leq \|\widehat{f}(\lambda)\| \cdot \|y\|_1 $$

for each $x \in D(A)$ and

$$\|\widehat{f}(\lambda)\| = O\bigg(\frac{1}{|\lambda|}\bigg), \quad \lambda \to \infty.$$

(P3) There exist positive constants C_i, $i = 1, 2$, and a subspace $\widehat{\mathcal{D}} \subseteq D(A)$, dense in $[D(A)]$, such that

$$A(\widehat{\mathcal{D}}) \subseteq D(A),$$
$$\widehat{f}(\lambda)(\widehat{\mathcal{D}}) \subseteq D(A)$$

and

$$\|A\widehat{f}(\lambda)y\| \le C_1\|y\|$$

for all $y \in \widehat{\mathcal{D}}$ and $\lambda \in \sum_{0,\eta}$.

In the continuation, we have that, for $\theta \in (\pi/2, \eta)$ and $r > 0$,

$$\sum_{r,\theta} = \{\lambda \in \mathbb{C} : \lambda \ne 0,\ r < |\lambda|,\ |\arg(\lambda)| < \theta\},$$

and, for $\Gamma_{r,\theta}$,

$$\begin{aligned} \Gamma^1_{r,\theta} &= \{te^{i\theta} :\ t \ge r\}, \\ \Gamma^2_{r,\theta} &= \{re^{i\zeta} :\ -\theta \le \zeta \le \theta\}, \\ \Gamma^3_{r,\theta} &= \{te^{-i\theta} :\ t \ge r\}, \end{aligned} \tag{2.17}$$

where $\Gamma^i_{r,\theta}$, $i = 1, 2, 3$, are the paths such that

$$\Gamma_{r,\theta} = \bigcup_{i=1}^{3} \Gamma^i_{r,\theta}$$

is oriented counterclockwise. Let $G_\beta(\lambda) = \lambda^{\beta-1}(\lambda^\beta I - A - A\widehat{f}(\lambda))^{-1}$, and define the set $\rho_\beta(G_\beta)$ as

$$\rho_\beta(G_\beta) = \{\lambda \in \mathbb{C} : G_\beta(\lambda) \in L(\mathbb{U})\}. \tag{2.18}$$

Now, we define the operator family $S_\beta(t)$, $t \ge 0$, by

$$S_\beta(t) = \begin{cases} 1/2\pi i \int_{\Gamma_{r,\theta}} e^{\lambda t} G_\beta(\lambda)\, d\lambda & t > 0, \\ I & t = 0. \end{cases} \tag{2.19}$$

Definition 2.8 ([**2**])**.** Let $\beta \in (1,2)$. Then, $R_\beta(t)$, $t \geq 0$, is defined by

$$R_\beta(t)x = \int_0^t g_{\beta-1}(t-s)S_\beta(s)\,ds, \quad t \geq 0, \tag{2.20}$$

where $g_{\beta-1}(t) = (t^{\beta-2})/(\Gamma(\beta-1))$, $t > 0$, $\beta - 1 \geq 0$.

For more details, see [**1**].

Lemma 2.9 ([**2**])**.** *There exists a positive number r_1 such that*

$$\sum_{r_1,\eta} \subseteq \rho_\beta(G_\beta)$$

and the map

$$G_\beta : \sum_{r_1,\eta} \longrightarrow L(\mathbb{U})$$

is analytic. Furthermore, we have

$$G_\beta(\lambda) = \lambda^{\beta-1}R(\lambda^\beta, A)[I - \widehat{f}(\lambda)R(\lambda^\beta, A)]^{-1}, \tag{2.21}$$

and there are constants $\widetilde{M_i}$ for $i = 1, 2$, such that

$$\begin{aligned} \|G_\beta(\lambda)\| &\leq \frac{\widetilde{M_1}}{|\lambda|}, \\ \|AG_\beta(\lambda)y\| &\leq \frac{\widetilde{M_2}}{|\lambda|}\|y\|_1, \quad y \in D(A), \\ \|AG_\beta(\lambda)\| &\leq \frac{\widetilde{M_2}}{|\lambda|^{1-\beta}} \end{aligned} \tag{2.22}$$

for each $\lambda \in \sum_{r_1,\eta}$.

Lemma 2.10 ([**1**])**.** *We assume that conditions* (P1)–(P3) *are satisfied. Then, there exists a unique β-resolvent operator for the system* (2.13)–(2.14).

Lemma 2.11 ([**1**])**.** *The function $S_\beta : [0,\infty) \to L(\mathbb{U})$ is strongly continuous and $S_\beta : (0,\infty) \to L(\mathbb{U})$ is uniformly continuous.*

Lemma 2.12 ([**1**])**.** *If the function* $S_\beta(\cdot)$ *is exponentially bounded in* $L([D(A)])$*, then* $R_\beta(\cdot)$ *is exponentially bounded in* $L([D(A)])$*.*

Lemma 2.13 ([**1**])**.** *The operator families* $S_\beta(t)$ *and* $R_\beta(t)$ *are compact for all* $t \geq 0$ *if* $R(\lambda_0^\beta, A)$ *is compact for some* $\lambda_0^\beta \in \rho(A)$*.*

Lemma 2.14 ([**12**])**.** *For any* $p \geq 1$ *and for arbitrary* $L_2^0(\mathbb{V}, \mathbb{U})$*-valued predictable process* $\varphi(\cdot)$ *such that*

$$\sup_{s\in[0,t]} E\|\int_0^s \varphi(\tau)\, dw(\tau)\|_{\mathbb{U}}^{2p} \leq C_p\bigg(\int_0^t (E\|\varphi(s)\|_{L_2^0}^{2p})^{1/p} ds\bigg)^p, \tag{2.23}$$

for each $t \in [0, \infty)$*, where* $C_p = (p(2p-1))^p$*.*

Next, we present the fixed point theorem, which is the main tool for establishing the result.

Theorem 2.15 (Nonlinear alternative of Leray-Schauder type for multivalued maps [**29**])**.** *Let* $\mathbb{U}$ *be a Hilbert space,* B *an open, convex subset of* $\mathbb{U}$ *and* $y \in \mathbb{U}$*. Assume that*:

(i) $\Lambda : \overline{B} \to \mathcal{P}_{cd}(\mathbb{U})$ *has a closed graph*; *and*
(ii) $\Lambda : \overline{B} \to \mathcal{P}_{cd}(\mathbb{U})$ *is a condensing map such that* $\Lambda(\overline{B})$*, a subset of a bounded set in* $\mathbb{U}$*, holds. Then either*:
  (a) *there exists a fixed point of the mapping* Λ *in* $\overline{B}$; *or*
  (b) *there exist* $y \in \partial B$ *and* $\lambda \in (0,1)$ *such that* $y \in \lambda\Lambda(y) + (1-\lambda)\{y_0\}$*.*

3. Main result. Before expressing and demonstrating the main result, we present the definition of the mild solution to problem (1.1)–(1.2).

Definition 3.1. An $\mathcal{F}_t$-adapted stochastic process $y \in \mathcal{C}$ is called a *mild solution* of the problem (1.1)–(1.2) if:

(i) $y_0, h \in L_p^0(\Omega, \mathcal{C})$;
(ii) $y(0) = y_0 + h(y)$, $y'(0) = 0$;

(iii) $y(t) \in \mathbb{U}$ has càdlàg paths on $t \in [0,T]$ almost surely, and there is a function $g \in \mathcal{N}_{G,y(h_3(t)),\int_0^t a_2(t,s,y(h_4(s)))\,ds}$ such that:

(3.1)
$$\begin{aligned}
y(t) \in S_\beta(t)&[y_0 + h(y) - F(0, y(h_1(0)), 0)] \\
&+ F\left(t, y(h_1(t)), \int_0^t a_1(t,s,y(h_2(s)))\,ds\right) \\
&+ \int_0^t AR_\beta(t-s)F\left(s, y(h_1(s)), \int_0^s a_1(s,\tau,y(h_2(\tau)))\,d\tau\right) ds \\
&+ \int_0^t \int_0^s R_\beta(t-s)f(s-\xi) \\
&\qquad F\left(\xi, y(h_1(\xi)), \int_0^\xi a_1(\xi,\tau,y(h_2(\tau)))\,d\tau\right) d\xi\,ds \\
&+ \int_0^t R_\beta(t-s)g(s)\,dw(s), \quad t \in [0,T],
\end{aligned}$$

Now, we make the following assumptions to establish the required result.

(B1) The operator families $S_\beta(t)$, $t > 0$, and $R_\beta(t)$, $t > 0$ are compact, and there exist constants M_1 and $\delta > 0$ such that $\|S_\beta(t)\|_{L(\mathbb{U})} \le M_1 e^{-\delta t}$ and $\|R_\beta(t)\|_{L(\mathbb{U})} \le M_1 e^{-\delta t}$ for each $t > 0$ and

$$\|(-A)^\eta R_\beta(t)\|_{\mathbb{U}} \le M_2 t^{\beta(1-\eta)-1}, \quad t \in (0,T].$$

(B2) For each $z \in [D((-A)^{1-\eta})]$, $f(\cdot)z \in C([0,T],\mathbb{U})$, and there is a positive function $\mathcal{W}(\cdot) \in L^1([0,T],\mathbb{R}^+)$ and a constant $M_3 > 0$ such that

$$\|f(s)R_\beta(t)\|_{L([D((-A)^\eta)],\mathbb{U})} \le M_3 \mathcal{W}(s) t^{\beta\eta-1}, \quad 0 \le s < t \le T.$$

(B3) The function $F : [0,T] \times \mathbb{U} \times \mathbb{U} \to [D((-A)^\vartheta)]$ is a Lipschitz continuous, and there is a constant $L_F > 0$ such that

$$\begin{aligned}
&\|(-A)^\vartheta F(t_1, y_1, z_1) - (-A)^\vartheta F(t_2, y_2, z_2)\|_{\mathbb{U}}^p \\
&\qquad\qquad \le L_F[|t_1 - t_2| + \|y_1 - y_2\|_{\mathbb{U}}^p + \|z_1 - z_2\|_{\mathbb{U}}^p],
\end{aligned}$$

for each (t, y, z), (t_1, y_1, z_1), $(t_2, y_2, z_2) \in [0, T] \times \mathbb{U} \times \mathbb{U}$ and $0 < \vartheta < 1$ with

$$\mathcal{L}_F^2 = \sup_{t \in J} \|(-A)^{\vartheta} F(t, 0, 0)\|^p;$$

here, $\mathcal{L}_F^2$ is a positive constant.

(B4) The map $a_1 : D_1 \times \mathbb{U} \to \mathbb{U}$ is continuous, and there exists a positive constant L_{a_1} such that

$$\left\| \int_0^t [a_1(t, s, z_1) - a_1(t, s, z_2)]\, ds \right\|_{\mathbb{U}}^p \leq L_{a_1} \|z_1 - z_2\|_{\mathbb{U}}^p,$$

for all $z_1, z_2 \in \mathbb{U}$ and $t \in [0, T]$ with $\mathcal{L}_{a_1}^1 = T \sup_{(t,s) \in D_1} \|a_1(t, s, 0)\|_{\mathbb{U}}^p$, where $D_1 = \{(t, s) \in [0, T] \times [0, T] : t \geq s\}$.

(B5) The multi-valued map $G : J \times \mathbb{U} \times \mathbb{U} \to \mathcal{P}_{bd,cl,cv}(L(\mathbb{V}, \mathbb{U}))$ is an L^p-Carathéodory function such that:

(i) the map $G(t, \cdot, \cdot) : \mathbb{U} \times \mathbb{U} \to \mathcal{P}_{bd,cl,cv}(L(\mathbb{V}, \mathbb{U}))$ is usc for each $t \in [0, T]$ and $G(\cdot, y, z)$ is measurable for each $(y, z) \in \mathbb{U} \times \mathbb{U}$. Then, the set

$$\mathcal{N}_{G,z} = \Bigg\{ \varrho \in L^p([0, T], L(\mathbb{V}, \mathbb{U})) : \varrho(t) \in G\Bigg(t, z(h_3(t)), \int_0^t a_2(t, s, z(h_4(s)))\, ds\Bigg) \text{ for almost every } t \in [0, T] \Bigg\}$$

for fixed $z \in \mathcal{C}$, is nonempty;

(ii) there exist a continuous function $m_g : [0, T] \to [0, \infty)$ and a continuous increasing function $\Theta_g : [0, \infty) \to (0, \infty)$ such that

$$\|G(t, y, z)\|_{\mathbb{U}}^p = \sup\{E\|g\|_{\mathbb{U}}^p : g \in G(t, y, z)\} \leq m_g(t)\Theta_g(E\|y\|_{\mathbb{U}}^p + E\|z\|_{\mathbb{U}}^p),$$

for almost every $t \in [0, T]$ and each $(y, z) \in \mathbb{U} \times \mathbb{U}$ with

$$\int_1^{\infty} \frac{ds}{s + \Theta_g(s) + \Theta_{a_2}(s)} = \infty.$$

(B6)

(i) The function $a_2(t,s,\cdot):\mathbb{U}\to\mathbb{U}$ is continuous for each $(t,s)\in D_1$, and the function $a_2(\cdot,\cdot,y):D_1\to\mathbb{U}$ is strongly measurable for each $y\in\mathbb{U}$.

(ii) There exist a continuous function $m_{a_2}:D_1\to[0,\infty)$ and an increasing function $\Theta_{a_2}:[0,\infty)\to(0,\infty)$ such that

$$E\|a_2(t,s,y)\|_{\mathbb{U}}^p\le m_{a_2}(t,s)\Theta_{a_2}(E\|y\|_{\mathbb{U}}^p),$$

for almost every $t,s\in[0,T]$ and $y\in\mathbb{U}$.

(B7) $h:\mathcal{C}\to\mathbb{U}$ is completely continuous, and there exist positive constants $\mathcal{C}_1$ and $\mathcal{C}_2$ such that

$$\|h(u)\|_{\mathbb{U}}^p\le\mathcal{C}_1\|u\|_{\mathbb{U}}^p+\mathcal{C}_2.$$

Now, we present the following theorem, which is our main result.

Theorem 3.2. *Let* $y_0\in L_p^0(\Omega,\mathcal{C})$. *If conditions* (B1)–(B7) *are fulfilled, and*

$$\begin{aligned}\mathcal{K}_*=4^{p-1}\Big[&M_1^p\|(-A)^{\vartheta}\|^pL_F+\|(-A)^{\vartheta}\|^pL_F(1+L_{a_1})\\&+(M_2^pT^{p-1}+M_3^pT^{2(p-1)}\|\mathcal{W}^p\|_{L^1})\times L_F(1+L_{a_1})\\&\times\frac{T^{p(\beta\vartheta-1)+1}}{p(\beta\vartheta-1)+1}\Big]<1,\end{aligned}\tag{3.2}$$

$$\mathbb{K}^*=15^{p-1}M_1^pe^{-\delta pt}(\mathcal{C}_1+2^{p-1}L_F)\tag{3.3}$$

$$\begin{aligned}&+10^{p-1}\Big\{\|(-A)^{-\vartheta}\|^p+(M_2^pT^{p-1}+M_3^pT^{2(p-1)}\|\mathcal{W}^p\|_{L^1})\\&\times\frac{T^{p(\beta\vartheta-1)+1}}{p(\beta\vartheta-1)+1}\Big\}\times L_F(1+2^{p-1}L_{a_1})<1,\end{aligned}\tag{3.4}$$

then system (1.1)–(1.2) *admits at least one mild solution on* $[0,T]$.

Proof. In order to demonstrate the theorem, we firstly define the operator $\Phi:\mathcal{C}\to\mathcal{P}(\mathcal{C})$ by Φy, the set of $\varrho\in\mathcal{C}$ such that:

(3.5)
$$\begin{aligned}\varrho(t) = {} & S_\beta(t)[y_0 + h(y) - F(0, y(h_1(0)), 0)] \\ & + F\left(t, y(h_1(t)), \int_0^t a_1(t,s,y(h_2(s)))ds\right) + \int_0^t AR_\beta(t-s) \\ & \times F\left(s, y(h_1(s)), \int_0^s a_1(s,\tau,y(h_2(\tau)))\,d\tau\right) ds + \int_0^t \int_0^s R_\beta(t-s) \\ & \times f(s-\xi) F\left(\xi, y(h_1(\xi)), \int_0^\xi a_1(\xi,\tau,y(h_2(\tau)))\,d\tau\right) d\xi\, ds \\ & + \int_0^t R_\alpha(t-s) g(s)\, dw(s), \quad t \in [0,T];\end{aligned}$$

here,

$$\begin{aligned} g \in \mathcal{N}_{G,y} = \{ & g \in L^p(L(\mathbb{V}, \mathbb{U})) : \\ & g(t) \in G\left(t, y(h_3(t)), \int_0^t a_2(t,s,y(h_4(s)))\,ds\right)\end{aligned}$$

almost everywhere $t \in [0,T]\}$. Clearly, the map Φ is well defined from $\mathcal{C}$ into $\mathcal{P}(\mathcal{C})$ by using the facts that F, g and h are continuous functions. In order to show that there exists a mild solution for the problem (1.1)–(1.2), it is sufficient to prove that Φ has a fixed point.

Now, we will prove the result in several steps.

Step 1. We show that there is an open set $B \subset \mathcal{C}$ such that $y \in \lambda(\Phi y)$ for each $\lambda \in (0,1)$ and $y \notin \partial B$. Let us consider $y \in \mathcal{C}$. Then, we have that there exists a $g \in \mathcal{N}_{G,y}$ such that, for each $\lambda \in (0,1)$,

(3.6)
$$\begin{aligned} y(t) = {} & \lambda S_\beta(t)[y_0 + h(y) - F(0, y(h_1(0)), 0)] \\ & + \lambda F\left(t, y(h_1(t)), \int_0^t a_1(t,s,y(h_2(s)))\,ds\right) \\ & + \lambda \int_0^t AR_\beta(t-s) F\left(s, y(h_1(s)), \int_0^s a_1(s,\tau,y(h_2(\tau)))\,d\tau\right) ds \\ & + \lambda \int_0^t \int_0^s R_\beta(t-s) f(s-\xi)\end{aligned}$$

$$F\bigg(\xi, y(h_1(\xi)), \int_0^\xi a_1(\xi,\tau,y(h_2(\tau)))\,d\tau\bigg)\,d\xi\,ds$$
$$+\lambda\int_0^t R_\alpha(t-s)g(s)\,dw(s), \quad t\in[0,T].$$

This yields the following:

$$\begin{aligned}
&E\|y(t)\|_{\mathbb{U}}^p\\
&\le 5^{p-1}E\|S_\beta(t)[y_0+h(y)-F(0,y(h_1(0)),0)]\|_{\mathbb{U}}^p\\
&\quad+5^{p-1}E\bigg\|F\bigg(t,y(h_1(t)),\int_0^t a_1(t,s,y(h_2(s)))\,ds\bigg)\bigg\|_{\mathbb{U}}^p\\
&\quad+5^{p-1}\bigg\|\int_0^t AR_\beta(t-s)F\bigg(s,y(h_1(s)),\int_0^s a_1(s,\tau,y(h_2(\tau)))\,d\tau\bigg)\,ds\bigg\|_{\mathbb{U}}^p\\
&\quad+5^{p-1}\bigg\|\int_0^t\int_0^s R_\beta(t-s)f(s-\xi)\\
&\qquad\qquad F\bigg(\xi,y(h_1(\xi)),\int_0^\xi a_1(\xi,tau,y(h_2(\tau)))\,d\tau\bigg)\,d\xi\,ds\bigg\|_{\mathbb{U}}^p\\
&\quad+5^{p-1}\bigg\|\int_0^t R_\beta(t-s)g(s)\,dw(s)\bigg\|_{\mathbb{U}}^p\\
&\le 15^{p-1}M_1^pe^{-\delta pt}[\|y_0\|^p+\mathcal{C}_1\|y\|^p+\mathcal{C}_2\\
&\quad+2^{p-1}\|(-A)^{-\vartheta}\|^p(L_F\|y(h_1(0))\|_{\mathbb{U}}^p+\mathcal{L}_F^2)]\\
&\quad+5^{p-1}\|(-A)^{-\vartheta}\|^p[2^{p-1}L_F(\|y(t)\|_{\mathbb{U}}^p\\
&\quad+2^{p-1}L_{a_1}\|y(t)\|+2^{p-1}\mathcal{L}_{a_1}^1)+2^{p-1}\mathcal{L}_F^2]\\
&\quad+5^{p-1}M_2^pT^{p-1}\int_0^t(t-s)^{p(\beta\vartheta-1)}[2^{p-1}L_F(\|y(t)\|_{\mathbb{U}}^p+2^{p-1}L_{a_1}\|y(t)\|\\
&\qquad\qquad\qquad\qquad+2^{p-1}\mathcal{L}_{a_1}^1)+2^{p-1}\mathcal{L}_F^2]\,ds\\
&\quad+5^{p-1}M_3^pT^{2(p-1)}\int_0^t\int_0^s\mathcal{W}^p(t-\xi)(t-s)^{p(\beta\vartheta-1)}\\
&\times[2^{p-1}L_F(\|y(t)\|_{\mathbb{U}}^p+2^{p-1}L_{a_1}\|y(t)\|+2^{p-1}\mathcal{L}_{a_1}^1)+2^{p-1}\mathcal{L}_F^2]\,d\xi\,ds\\
&\quad+5^{p-1}M_1^pC_pT^{p/2-1}e^{-\delta pt}\int_0^t e^{p\delta s}m_g(s)\Theta_g\bigg(E\|y\|_{\mathbb{U}}^p+\int_0^s m_{a_2}(s,\tau)\Theta_{a_2}\\
&\qquad\qquad\qquad\qquad(E\|y\|_{\mathbb{U}}^p\,d\tau)\bigg)\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leq 15^{p-1}M_1^p e^{-\delta pt}[\|y_0\|^p + \mathcal{C}_1\|y\|_{\mathbb{U}}^p + \mathcal{C}_2 + 2^{p-1}(L_F\|y(h_1(0))\|_{\mathbb{U}}^p + \mathcal{L}_F^2)] \\
&\quad + 10^{p-1}\|(-A)^{-\vartheta}\|^p[L_F(\|y(t)\|_{\mathbb{U}}^p + 2^{p-1}L_{a_1}\|y(t)\| + 2^{p-1}\mathcal{L}_{a_1}^1) + \mathcal{L}_F^2] \\
&\quad + 10^{p-1}M_2^pT^{p-1}[L_F(\|y(t)\|_{\mathbb{U}}^p + 2^{p-1}L_{a_1}\|y(t)\| \\
&\qquad\qquad + 2^{p-1}\mathcal{L}_{a_1}^1) + \mathcal{L}_F^2]\frac{T^{p(\beta\vartheta-1)+1}}{p(\beta\vartheta-1)+1} \\
&\quad + 10^{p-1}M_3^pT^{2(p-1)}\|\mathcal{W}^p\|_{L^1}[L_F(\|y(t)\|_{\mathbb{U}}^p + 2^{p-1}L_{a_1}\|y(t)\| \\
&\qquad\qquad + 2^{p-1}\mathcal{L}_{a_1}^1) + \mathcal{L}_F^2]\frac{T^{p(\beta\vartheta-1)+1}}{p(\beta\vartheta-1)+1} + 5^{p-1}M_1^pC_pT^{p/2-1}e^{-\delta pt} \\
&\times \int_0^t e^{p\delta s}m_g(s)\Theta_g\left(E\|y\|_{\mathbb{U}}^p + \int_0^s m_{a_2}(s,\tau)\Theta_{a_2}(E\|y\|_{\mathbb{U}}^p)\,d\tau\right)ds, \\
&\leq 15^{p-1}M_1^pe^{-\delta pt}(\|y_0\|^p + \mathcal{C}_2 + 2^{p-1}\mathcal{L}_F^2) + 10^{p-1}\Bigg[\|(-A)^{-\vartheta}\|^p \\
&\quad + (M_2^pT^{p-1} + M_3^pT^{2(p-1)}\|\mathcal{W}^p\|_{L^1})\frac{T^{p(\beta\vartheta-1)+1}}{p(\beta\vartheta-1)+1}\Bigg]\times(2^{p-1}L_F\mathcal{L}_{a_1}^1 + \mathcal{L}_F^2) \\
&\quad + \Bigg[15^{p-1}M_1^pe^{-\delta pt}(\mathcal{C}_1 + 2^{p-1}L_F) + 10^{p-1}\Bigg\{\|(-A)^{-\vartheta}\|^p \\
&\qquad\qquad + (M_2^pT^{p-1} + M_3^pT^{2(p-1)}\|\mathcal{W}^p\|_{L^1})\frac{T^{p(\beta\vartheta-1)+1}}{p(\beta\vartheta-1)+1}\Bigg\} \\
&\qquad\qquad \times L_F(1 + 2^{p-1}L_{a_1})\Bigg]\sup_{t\in[0,T]}E\|y(t)\|^p + 5^{p-1}M_1^pC_pT^{p/2-1}e^{-\delta pt} \\
&\times \int_0^t e^{p\delta s}m_g(s)\Theta_g\left(E\|y\|_{\mathbb{U}}^p + \int_0^s m_{a_2}(s,\tau)\Theta_{a_2}(E\|y\|_{\mathbb{U}}^p)\,d\tau\right)ds.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\sup_{t\in[0,T]}E\|y(t)\|_{\mathbb{U}}^p \\
&\leq \frac{1}{1-\mathbb{K}^*}\Bigg[15^{p-1}M_1^pe^{-\delta pt}(\|y_0\|^p + \mathcal{C}_2 + 2^{p-1}\mathcal{L}_F^2) \\
&\quad + 10^{p-1}\Bigg[\|(-A)^{-\vartheta}\|^p + (M_2^pT^{p-1} + M_3^pT^{2(p-1)}\|\mathcal{W}^p\|_{L^1})\frac{T^{p(\beta\vartheta-1)+1}}{p(\beta\vartheta-1)+1}\Bigg] \\
&\qquad\qquad \times(2^{p-1}L_F\mathcal{L}_{a_1}^1 + \mathcal{L}_F^2)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{5^{p-1} M_1^p C_p T^{p/2-1} e^{-\delta pt}}{1-\mathbb{K}^*} \\
&\times \int_0^t e^{p\delta s} m_g(s) \Theta_g\left(E\|y\|_{\mathbb{U}}^p + \int_0^s m_{a_2}(s,\tau)\Theta_{a_2}(E\|y\|_{\mathbb{U}}^p)\, d\tau \right) ds,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{K}^* &= 15^{p-1} M_1^p e^{-\delta pt}(\mathcal{C}_1 + 2^{p-1} L_F) \\
&\quad + 10^{p-1} \Big\{ \|(-A)^{-\vartheta}\|^p + (M_2^p T^{p-1} + M_3^p T^{2(p-1)} \|\mathcal{W}^p\|_{L^1}) \\
&\qquad \times \frac{T^{p(\beta\vartheta-1)+1}}{p(\beta\vartheta-1)+1} \Big\} L_F (1 + 2^{p-1} L_{a_1}) < 1.
\end{aligned}
$$

Let $\zeta(t) = \sup_{t\in[0,T]} E\|y(t)\|_{\mathbb{U}}^p$ and $\mathcal{N}^* = M_1 \max\{1, e^{-\delta T}\}$. Since $\mathbb{K}^* < 1$, we obtain

$$
\begin{aligned}
e^{\delta pt}\zeta(t) \le \frac{1}{1-\mathbb{K}^*} \Big[& 15^{p-1} M_1^p (\|y_0\|^p + \mathcal{C}_2 + 2^{p-1}\mathcal{L}_F^2) \\
& + 10^{p-1} \mathcal{N}^* \Big[\|(-A)^{-\vartheta}\|^p \\
& + (M_2^p T^{p-1} + M_3^p T^{2(p-1)} \|\mathcal{W}^p\|_{L^1}) \times \frac{T^{p(\beta\vartheta-1)+1}}{p(\beta\vartheta-1)+1} \Big] \\
& \times (2^{p-1} L_F \mathcal{L}_{a_1}^1 + \mathcal{L}_F^2) \Big] \\
& + \frac{5^{p-1} M_1^p C_p T^{p/2-1}}{1-\mathbb{K}^*} \int_0^t e^{p\delta s} m_g(s) \\
& \times \Theta_g\left(\zeta(s) + \int_0^s m_{a_2}(s,\tau)\Theta_{a_2}(\zeta(\tau))\, d\tau \right) ds.
\end{aligned}
$$

We denote by χ the right-hand side of the above inequality and obtain

$$
\zeta(t) \le e^{-\delta pt}\chi(t) \quad \text{for all } t \in [0,T], \tag{3.7}
$$

with

$$
\begin{aligned}
\chi(0) = \frac{1}{1-\mathbb{K}^*} \Big[& 15^{p-1} M_1^p (\|y_0\|^p + \mathcal{C}_2 + 2^{p-1}\mathcal{L}_F^2) \\
& + 10^{p-1} \mathcal{N}^* \Big[\|(-A)^{-\vartheta}\|^p + (M_2^p T^{p-1} + M_3^p T^{2(p-1)} \|\mathcal{W}^p\|_{L^1}) \\
& \times \frac{T^{p(\beta\vartheta-1)+1}}{p(\beta\vartheta-1)+1} \Big] \times (2^{p-1} L_F \mathcal{L}_{a_1}^1 + \mathcal{L}_F^2) \Big],
\end{aligned}
$$

and

$$\begin{aligned}\chi'(t) &= \frac{5^{p-1}M_1^pC_pT^{p/2-1}}{1-\mathbb{K}^*}e^{p\delta t}m_g(t)\\ &\quad\times\Theta_g\bigg(\zeta(t)+\int_0^t m_{a_2}(t,s)\Theta_{a_2}(\zeta(s))\,ds\bigg),\\ &\le \frac{5^{p-1}M_1^pC_pT^{p/2-1}}{1-\mathbb{K}^*}\,e^{p\delta t}m_g(t)\\ &\quad\times\Theta_g\bigg(e^{-\delta pt}\chi(t)+\int_0^t m_{a_2}(t,s)\Theta_{a_2}(e^{-p\delta s}\chi(s))\,ds\bigg),\quad t\in[0,T].\end{aligned}$$

Now, we take $\Psi = e^{-\delta pt}\chi(t) + \int_0^t m_{a_2}(t,s)\Theta_{a_2}(e^{-p\delta s}\chi(s))\,ds$. Then, $\Psi(0)=\chi(0)$, $e^{-p\delta t}\chi(t)\le\Psi(t)$ and, for each $t\in[0,T]$, we get

$$\begin{aligned}\Psi'(t) &= -p\delta e^{-p\delta t}\chi(t)+e^{-\delta pt}\chi'(t)+m_{a_2}(t,t)\Theta_{a_2}(e^{-p\delta t}\chi(t)),\\ &\le -p\delta\Psi(t)+\frac{5^{p-1}M_1^pC_pT^{p/2-1}}{1-\mathbb{K}^*}m_g(t)\Theta_g(\Psi(t))+m_{a_2}(t,t)\Theta_{a_2}(\Psi),\\ &\le \max\bigg\{(-p\delta),\frac{5^{p-1}M_1^pC_pT^{p/2-1}}{1-\mathbb{K}^*}m_g(t),m_{a_2}(t,t)\bigg\}\\ &\quad\times[\Psi(t)+\Theta_g(\Psi(t))+\Theta_{a_2}(\Psi)],\quad t\in[0,T].\end{aligned}$$

Thus, this yields that

$$\begin{aligned}&\int_{\Psi(0)}^{\Psi(t)}\frac{d\varsigma}{\varsigma+\Theta_g(\varsigma)+\Theta_{a_2}(\varsigma)}\\ &\quad\le\int_0^T\max\bigg\{(-p\delta),\frac{5^{p-1}M_1^pC_pT^{p/2-1}}{1-\mathbb{K}^*}m_g(t),m_{a_2}(t,t)\bigg\}\,dt<\infty.\end{aligned}$$

Hence, from the above inequality, we deduce that there exists a constant $\widetilde{Q}$ such that $\Psi(t)\le\widetilde{Q}$, for each $t\in[0,T]$, and thus, we conclude that $\|y(t)\|_{\mathbb{U}}^p\le\zeta(t)\le e^{-\delta pt}\chi(t)\le\Psi(t)\le\widetilde{Q}$, where $\widetilde{Q}$ depends only upon M_1, δ, p, C_p, T and on functions $m_g(\cdot)$, $m_{a_2}(\cdot,\cdot)$, $\Theta_g(\cdot)$ and $\Theta_{a_2}(\cdot)$. Therefore, there exists an $r>0$ such that $\|y\|_{\mathcal{C}}^p\ne r$. We set

$$B=\{y\in\mathcal{C}:\ \|y\|_{\mathcal{C}}^p<r\}.$$

From the choice of B, there is no $y\in\partial B$ such that $y\in\lambda\Phi y$ for some $0<\lambda<1$.

Step 2. Φ has a closed graph. We consider $y^q \to y^*$, $\varrho^q \in \Phi y^q$, $y^q \in \overline{B}$ and $\varrho^q \to \varrho^*$. We show that $\varrho^* \in \Phi y^*$. Now, for $\varrho^q \in \Phi y^q$, it implies that there exists a $g_q \in \mathcal{N}_{G,y^q)}$ such that

(3.8)

$$\begin{aligned}\varrho^q(t) = S_\beta(t)&[y_0 + h(y^q) - F(0, y^q(h_1(0)), 0)]\\ &+ F\left(t, y^q(h_1(t)), \int_0^t a_1(t,s,y^q(h_2(s)))\,ds\right)\\ &+ \int_0^t AR_\beta(t-s)F\left(s, y^q(h_1(s)), \int_0^s a_1(s,\tau,y^q(h_2(\tau)))\,d\tau\right) ds\\ &+ \int_0^t \int_0^s R_\beta(t-s)f(s-\xi)F\biggl(\xi, y^q(h_1(\xi)),\\ &\qquad\qquad \int_0^\xi a_1(\xi,\tau,y^q(h_2(\tau)))\,d\tau\biggr)\,d\xi\,ds\\ &+ \int_0^t R_\beta(t-s)g^q(s)\,dw(s), \quad t \in [0,T].\end{aligned}$$

We must prove that there exists a $g^* \in \mathcal{N}_{G,y^*)}$ such that

(3.9)

$$\begin{aligned}\varrho^*(t) = S_\beta(t)&[y_0 + h(y^*) - F(0, y^*(h_1(0)), 0)]\\ &+ F\left(t, y^*(h_1(t)), \int_0^t a_1(t,s,y^*(h_2(s)))\,ds\right)\\ &+ \int_0^t AR_\beta(t-s)F\left(s, y^*(h_1(s)), \int_0^s a_1(s,\tau,y^*(h_2(\tau)))\,d\tau\right) ds\\ &+ \int_0^t \int_0^s R_\beta(t-s)f(s-\xi)F(\xi, y^*(h_1(\xi)),\\ &\qquad\qquad \int_0^\xi a_1(\xi,\tau,y^*(h_2(\tau)))d\tau)\,d\xi\,ds\\ &+ \int_0^t R_\beta(t-s)g^*(s)\,dw(s), \quad t \in [0,T].\end{aligned}$$

Now, for each $t \in [0,T]$, we obtain

(3.10)

$$\begin{aligned}
\bigg\|\bigg[&\varrho^q(t) - S_\beta(t)[y_0 + h(y^q) - F(0, y^q(h_1(0)), 0)] \\
&- F\bigg(t, y^q(h_1(t)), \int_0^t a_1(t, s, y^q(h_2(s)))\, ds\bigg) \\
&- \int_0^t AR_\beta(t-s) F\bigg(s, y^q(h_1(s)), \int_0^s a_1(s, \tau, y^q(h_2(\tau)))\, d\tau\bigg)\, ds \\
&- \int_0^t \int_0^s R_\beta(t-s) f(s-\xi) F\bigg(\xi, y^q(h_1(\xi)), \\
&\qquad\qquad \int_0^\xi a_1(\xi, \tau, y^q(h_2(\tau)))\, d\tau\bigg)\, d\xi\, ds\bigg] \\
&- \bigg[\varrho^*(t) - S_\beta(t)[y_0 + h(y^*) - F(0, y^*(h_1(0)), 0)] \\
&\qquad - F\bigg(t, y^*(h_1(t)), \int_0^t a_1(t, s, y^*(h_2(s)))\, ds\bigg) \\
&- \int_0^t AR_\beta(t-s) F\bigg(s, y^*(h_1(s)), \int_0^s a_1(s, \tau, y^*(h_2(\tau)))\, d\tau\bigg)\, ds \\
&- \int_0^t \int_0^s R_\beta(t-s) f(s-\xi) F\bigg(\xi, y^*(h_1(\xi)), \\
&\qquad\qquad \int_0^\xi a_1(\xi, \tau, y^*(h_2(\tau)))\, d\tau\bigg)\, d\xi\, ds\bigg]\bigg\|_{\mathcal{C}}^p \\
&\qquad\qquad \longrightarrow 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

We consider the following, continuous linear operator

$$\Xi : L^p([0,T], \mathbb{U}) \to C([0,T], \mathbb{U}),$$

defined by

$$(\Xi g)(t) = \int_0^t R_\beta(t-s) g(s)\, dw(s), \quad t \in [0,T]. \tag{3.11}$$

Thus, Lemma 2.6 provides that $\Xi \circ \mathcal{N}_G$ is a closed graph mapping. By the definition of Ξ, we also have

$$\begin{aligned}
&\varrho^q(t) - S_\beta(t)[y_0 + h(y^q) - F(0, y^q(h_1(0)), 0)] \\
&\quad - F\left(t, y^q(h_1(t)), \int_0^t a_1(t, s, y^q(h_2(s)))\, ds\right) \\
&\quad - \int_0^t AR_\beta(t-s) F\left(s, y^q(h_1(s)), \int_0^s a_1(s, \tau, y^q(h_2(\tau)))\, d\tau\right) ds \\
&\quad - \int_0^t \int_0^s R_\beta(t-s) f(s-\xi) F\Bigg(\xi, y^q(h_1(\xi)), \\
&\qquad\qquad \int_0^\xi a_1(\xi, \tau, y^q(h_2(\tau)))\, d\tau\Bigg)\, d\xi\, ds \in \Xi(\mathcal{N}_{G,y^q}),
\end{aligned}$$

for each $t \in [0, T]$. Since $y^q \to y^*$ for some $g^* \in \mathcal{N}_{G,y^*}$, we obtain that, for every $t \in [0, T]$,

$$\begin{aligned}
&\varrho^*(t) - S_\beta(t)[y_0 + h(y^*) - F(0, y^*(h_1(0)), 0)] \\
&\quad - F\left(t, y^*(h_1(t)), \int_0^t a_1(t, s, y^*(h_2(s)))\, ds\right) \\
&\quad - \int_0^t AR_\beta(t-s) F\left(s, y^*(h_1(s)), \int_0^s a_1(s, \tau, y^*(h_2(\tau)))\, d\tau\right) ds \\
&\quad - \int_0^t \int_0^s R_\beta(t-s) f(s-\xi) F\Bigg(\xi, y^*(h_1(\xi)), \\
&\qquad\qquad \int_0^\xi a_1(\xi, \tau, y^*(h_2(\tau)))\, d\tau\Bigg)\, d\xi\, ds \\
&= \int_0^t R_\beta(t-s) g^*(s)\, dw(s).
\end{aligned}$$

Therefore, Φ has a closed graph.

Next, we show that the mapping Φ is a condensing operator. We introduce the following decomposition of the map Φ into Φ_1 and Φ_2, where the mapping

$$\Phi_1 : \overline{B} \longrightarrow \mathcal{C}$$

is given by $\Phi_1 y$, the set $\varrho_1 \in \mathcal{C}$ such that

$$\begin{aligned}
\varrho_1(t) &= -S_\beta(t) F(0, y(h_1(0)), 0) + F\left(t, y(h_1(t)), \int_0^t a_1(t, s, y(h_2(s)))\, ds\right) \\
&\quad + \int_0^t AR_\beta(t-s) F\left(s, y(h_1(s)), \int_0^s a_1(s, \tau, y(h_2(\tau)))\, d\tau\right) ds
\end{aligned}$$

$$+\int_0^t\int_0^s R_\beta(t-s)f(s-\xi)F\bigg(\xi, y(h_1(\xi)),\\ \int_0^\xi a_1(\xi,\tau,y(h_2(\tau)))\,d\tau\bigg)\,d\xi\,ds,$$

and the mapping

$$\Phi_2 : \overline{B} \longrightarrow \mathcal{C}$$

is given by $\Phi_2 y$, the set $\varrho_2 \in \mathcal{C}$, such that

$$\varrho_2(t) = S_\beta(t)[y_0 + h(y)] + \int_0^t R_\beta(t-s)g(s)\,dw(s),$$

for each $t \in [0,T]$. In order to prove the result, we first prove that Φ_1 is a contraction while Φ_2 is a completely continuous operator.

Step 3. Φ_1 is a contraction mapping in $\mathcal{C}$. Let $y^*, y^{**} \in \mathcal{C}$ and $t \in [0,T]$. Thus, we obtain:

$$\begin{aligned}
&E\|(\Phi_1 y^*)(t) - (\Phi_1 y^{**})(t)\|_{\mathbb{U}}^p\\
&\quad\le 4^{p-1}\|S_\beta(t)[F(0,y^*(h_1(0)),0) - F(0,y^{**}(h_1(0)),0)]\|_{\mathbb{U}}^p\\
&\qquad + 4^{p-1}\bigg\|F\bigg(t, y^*(h_1(t)), \int_0^t a_1(t,s,y^*(h_2(s)))\,ds\bigg)\\
&\qquad\qquad - F\bigg(t, y^{**}(h_1(t)), \int_0^t a_1(t,s,y^{**}(h_2(s)))\,ds\bigg)\bigg\|_{\mathbb{U}}^p\\
&\qquad + 4^{p-1}\bigg\|\int_0^t AR_\beta(t-s)\bigg[F\bigg(s, y^*(h_1(s)), \int_0^s a_1(s,\tau,y^*(h_2(\tau)))\,d\tau\bigg)\\
&\qquad\qquad - F\bigg(s, y^{**}(h_1(s)), \int_0^s a_1(s,\tau,y^{**}(h_2(\tau)))\,d\tau\bigg)\bigg]\,ds\bigg\|_{\mathbb{U}}^p\\
&\qquad + 4^{p-1}\bigg\|\int_0^t\int_0^s R_\beta(t-s)f(s-\xi)\\
&\qquad\qquad \times\bigg[F\bigg(\xi, y^*(h_1(\xi)), \int_0^\xi a_1(\xi,\tau,y^*(h_2(\tau)))\,d\tau\bigg)\\
&\qquad\qquad - F\bigg(\xi, y^{**}(h_1(\xi)), \int_0^\xi a_1(\xi,\tau,y^{**}(h_2(\tau)))\,d\tau\bigg)\bigg]\,d\xi\,ds\bigg\|_{\mathbb{U}}^p\\
&\quad\le 4^{p-1}M_1^p\|(-A)^\vartheta\|^p L_F \sup_{t\in[0,T]} E\|y^*(t) - y^{**}(t)\|_{\mathbb{U}}^p + 4^{p-1}\|(-A)^\vartheta\|^p L_F(1+L_{a_1})
\end{aligned}$$

$$
\begin{aligned}
&\times \sup_{t\in[0,T]} E\|y^*(t)-y^{**}(t)\|_{\mathbb{U}}^p + 4^{p-1}M_2^pT^{p-1}L_F(1+L_{a_1})\\
&\times \int_0^t (t-s)^{p(\beta\vartheta-1)} \sup_{s\in[0,t]} E\|y^*(s)-y^{**}(s)\|_{\mathbb{U}}^p\,ds\\
&\quad + 4^{p-1}M_3^pT^{2(p-1)}L_F(1+L_{a_1})\\
&\times \int_0^t\int_0^s \mathcal{W}^p(s-\xi)(t-s)^{p(\beta\vartheta-1)} \sup_{\xi\in[0,s]} E\|y^*(\xi)-y^{**}(\xi)\|_{\mathbb{U}}^p\,d\xi\,ds\\
&\le 4^{p-1}\bigg[M_1^p\|(-A)^\vartheta\|^pL_F + \|(-A)^\vartheta\|^pL_F(1+L_{a_1})\\
&\qquad\qquad + (M_2^pT^{p-1}+M_3^pT^{2(p-1)}\|\mathcal{W}^p\|_{L^1})L_F(1+L_{a_1})\\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \frac{T^{p(\beta\vartheta-1)+1}}{p(\beta\vartheta-1)+1}\bigg]\|y^*-y^{**}\|_{\mathcal{C}}^p,\\
&= \mathcal{K}_*\|y^*-y^{**}\|_{\mathcal{C}}^p,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{K}_* = 4^{p-1}\bigg[&M_1^p\|(-A)^\vartheta\|^pL_F + \|(-A)^\vartheta\|^pL_F(1+L_{a_1})\\
&\qquad + (M_2^pT^{p-1}+M_3^pT^{2(p-1)}\|\mathcal{W}^p\|_{L^1})\\
&\qquad\qquad\qquad \times L_F(1+L_{a_1})\frac{T^{p(\beta\vartheta-1)+1}}{p(\beta\vartheta-1)+1}\bigg] < 1.
\end{aligned}
$$

Thus, we deduce that, for each $t\in[0,T]$,

$$
\sup_{t\in[0,T]} E\|(\Phi_1y^*)(t)-(\Phi_1y^{**})(t)\|_{\mathbb{U}}^p \le \mathcal{K}_*\|y^*-y^{**}\|_{\mathcal{C}}^p, \tag{3.12}
$$

with $\mathcal{K}_*<1$. Hence, from (3.12) and inequality (3.2), we deduce that Φ_1 is a contraction on $\mathcal{C}$.

Step 4. The map Φ_2u is convex for every $y\in\overline{B}$. Indeed, if $\widetilde{u}_1,\widetilde{u}_2\in\Phi_2y$, then there exist $g_1,g_2\in\mathcal{N}_{G,y}$ such that

(3.13)

$$
\widetilde{u}_j(t) = S_\beta(t)[y_0+h(y)] + \int_0^t R_\beta(t-s)g_j(s)\,dw(s), \quad j=1,2,\ t\in[0,T].
$$

Let $\widetilde{\lambda} \in [0,1]$. Thus, for each $t \in [0,T]$, we get

(3.14)
$$(\widetilde{\lambda}\widetilde{u}_1(t) + (1-\widetilde{\lambda})\widetilde{u}_2(t)) = S_\beta(t)[y_0 + h(y)] + \int_0^t R_\beta(t-s)(\widetilde{\lambda}g_1(s) + (1-\widetilde{\lambda})g_2(s))\, dw(s).$$

Since we have $\mathcal{N}_{G,y}$ is convex since G has convex value, therefore, we have $\widetilde{\lambda}\widetilde{u}_1(t) + (1-\widetilde{\lambda})\widetilde{u}_2(t) \in \Phi_2 y$.

Step 5. Φ_2 maps bounded sets into bounded sets in $\overline{B}$. In fact, it is sufficient to show that there exists a constant $\mathfrak{L} > 0$ such that, for each $\widetilde{u} \in \Phi_2(y)$, $y \in \overline{B}$, it can easily be seen that $E\|\widetilde{u}(t)\|_{\mathbb{U}}^p \leq \mathfrak{L}$.

If $\widetilde{u} \in \Phi_2(y)$, then there exists a $g \in \mathcal{N}_{G,y}$ such that

$$\widetilde{u}(t) = S_\beta(t)[y_0 + h(y)] + \int_0^t R_\beta(t-s)g(s)\, dw(s), \tag{3.15}$$

for each $t \in [0,T]$. Thus, by the assumptions and for each $t \in [0,T]$, we have

(3.16)
$$\begin{aligned}
E\|\widetilde{u}(t)\|_{\mathbb{U}}^p &\leq 2^{p-1}E\|S_\beta(t)[y_0 + h(y)]\|_{\mathbb{U}}^p \\
&\quad + 2^{p-1}E\left\|\int_0^t R_\beta(t-s)g(s)\, dw(s)\right\|_{\mathbb{U}}^p \\
&\leq 2^{p-1}M_1^p e^{-p\delta t}[\|y_0\|_{\mathbb{U}}^p + \mathcal{C}_1 E\|y\|_{\mathbb{U}}^p + \mathcal{C}_2] \\
&\quad + 2^{p-1}C_p M_1^p \left[\int_0^t [e^{-p\delta(t-s)}E\|g(s)\|_{\mathbb{U}}^p]^{2/p} ds\right]^{p/2} \\
&\leq 2^{p-1}M_1^p e^{-p\delta t}[\|y_0\|_{\mathbb{U}}^p + \mathcal{C}_1 E\|y\|_{\mathbb{U}}^p + \mathcal{C}_2] + 2^{p-1}C_p M_1^p e^{-p\delta t} \\
&\quad \times \int_0^t e^{p\delta s} m_g(s)\Theta_g\bigg(E\|y(s)\|_{\mathbb{U}}^p \\
&\qquad\qquad + \int_0^s m_{a_2}(s,\tau)\Theta_{a_2}(E\|y(\tau)\|_{\mathbb{U}}^p)\, d\tau\bigg)\, ds \\
&\leq 2^{p-1}M_*^p[\|y_0\|_{\mathbb{U}}^p + \mathcal{C}_1 r + \mathcal{C}_2] \\
&\quad + 2^{p-1}C_p M_*^p \Theta_g(r_{**}) \int_0^T e^{p\delta s} m_g(s)\, ds := \mathfrak{L},
\end{aligned}$$

where $r_{**} = r + \Theta_{a_2}(r)\int_0^T m_{a_2}(\mu,\mu)\,d\mu$, $M_* = M_1\max\{1, e^{-\delta T}\}$. Therefore, for each $\widetilde{u} \in \Phi_2 y$, we obtain $E\|\widetilde{u}\|_{\mathcal{C}}^p \leq \mathfrak{L}$.

Step 6. The operator Φ_2 maps bounded sets into equicontinuous sets of $\overline{B}$. Let $d_1, d_2 \in [0,T]$ with $d_1 < d_2$. Thus, for each $y \in \overline{B}$ and $\varrho_2 \in \Phi_2 y$, we have that there exists a $g \in \mathcal{N}_{G,y}$ such that

$$\varrho_2(t) = S_\beta(t)[y_0 + h(y)] + \int_0^t R_\beta(t-s)g(s)\,dw(s), \quad t \in [0,T]. \tag{3.17}$$

Then,

(3.18)

$$\begin{aligned}
&E\|\varrho_2(d_2) - \varrho_2(d_1)\|_{\mathbb{U}}^p \\
&\leq 4^{p-1}E\|(S_\beta(d_2) - S_\beta(d_1))[y_0 + h(y)]\|_{\mathbb{U}}^p \\
&\quad + 4^{p-1}E\bigg\|\int_0^{d_1-\epsilon}[R_\beta(d_2-s) - R_\beta(d_1-s)]g(s)\,dw(s)\bigg\|_{\mathbb{U}}^p \\
&\quad + 4^{p-1}E\bigg\|\int_{d_1-\epsilon}^{d_1}[R_\beta(d_2-s) - R_\beta(d_1-s)]g(s)\,dw(s)\bigg\|_{\mathbb{U}}^p \\
&\quad + 4^{p-1}E\bigg\|\int_{d_1}^{d_2}R_\beta(d_2-s)g(s)\,dw(s)\bigg\|_{\mathbb{U}}^p \\
&\leq 4^{p-1}E\|(S_\beta(d_2) - S_\beta(d_1))(y_0 + h(y))\|_{\mathbb{U}}^p \\
&\quad + 4^{p-1}C_p\bigg[\int_0^{d_1-\epsilon}[\|R_\beta(d_2-s) - R_\beta(d_1-s)\|_{\mathbb{U}}^p E\|g(s)\|_{\mathbb{U}}^p]^{2/p}ds\bigg]^{p/2} \\
&\quad + 4^{p-1}C_p\bigg[\int_{d_1-\epsilon}^{d_1}[\|R_\beta(d_2-s) - R_\beta(d_1-s)\|_{\mathbb{U}}^p E\|g(s)\|_{\mathbb{U}}^p]^{2/p}ds\bigg]^{p/2} \\
&\quad + 4^{p-1}C_p\bigg[\int_{d_1}^{d_2}[\|R_\beta(d_2-s)\|_{\mathbb{U}}^p E\|g(s)\|_{\mathbb{U}}^p]^{2/p}ds\bigg]^{p/2} \\
&\leq 4^{p-1}E\|(S_\beta(d_2) - S_\beta(d_1))(y_0 + h(y))\|_{\mathbb{U}}^p \\
&\quad + 4^{p-1}C_p\bigg[\int_0^{d_1-\epsilon}\bigg[\|R_\beta(d_2-s) - R_\beta(d_1-s)\|_{\mathbb{U}}^p \times m_g(s)\Theta_g(E\|y(s)\|^p \\
&\qquad\qquad + \int_0^s m_{a_2}(s,\tau)\Theta_{a_2}(E\|y(\tau)\|^p)\,d\tau\bigg]^{2/p} ds\bigg]^{p/2} + 8^{p-1}C_pM_1^p
\end{aligned}$$

$$\times \Bigg[\int_{d_1-\epsilon}^{d_1}\Bigg[e^{-p\delta(d_1-s)} \times m_g(s)\Theta_g\Bigg(E\|y(s)\|^p + \int_0^s m_{a_2}(s,\tau)\Theta_{a_2}(E\|y(\tau)\|^p)\,d\tau\Bigg)\Bigg]^{2/p} ds\Bigg]^{p/2}$$

$$+ 4^{p-1}C_pM_1^p\Bigg[\int_{d_1}^{d_2}\Bigg[e^{-p\delta(d_2-s)} \times m_g(s) \times \Theta_g\Bigg(E\|y(s)\|^p + \int_0^s m_{a_2}(s,\tau)\Theta_{a_2}(E\|y(\tau)\|^p)\,d\tau\Bigg)\Bigg]^{2/p} ds\Bigg]^{p/2}$$

$$\le 4^{p-1}E\|(S_\beta(d_2) - S_\beta(d_1))(y_0 + h(y))\|_{\mathbb{U}}^p$$

$$+ 4^{p-1}C_p\Theta_g(r_{**})\Bigg[\int_0^{d_1-\epsilon}[\|R_\beta(d_2-s) - R_\beta(d_1-s)\|_{\mathbb{U}}^p m_g(s)]^{2/p}ds\Bigg]^{p/2}$$

$$+ 8^{p-1}C_pM_1^p\Theta_g(r_{**})\Bigg[\int_{d_1-\epsilon}^{d_1} e^{-[(2(p-1))/(p-2)]\delta(d_1-s)}ds\Bigg]^{p/2-1} \times \int_{d_1-\epsilon}^{d_1} e^{-\delta(d_1-s)}m_g(s)\,ds$$

$$+ 4^{p-1}C_pM_1^p\Theta_f(r_{**})\Bigg[\int_{d_1}^{d_2} e^{-[(2(p-1))/(p-2)]\delta(d_2-s)}ds\Bigg]^{p/2-1} \times \int_{d_1}^{d_2} e^{-\delta(d_2-s)}m_f(s)\,ds$$

$$\le 4^{p-1}E\|(S_\beta(d_2) - S_\beta(d_1))(y_0 + h(y))\|_{\mathbb{U}}^p$$

$$+ 4^{p-1}C_p\Theta_g(r_{**})\Bigg[\int_0^{d_1-\epsilon}[\|R_\beta(d_2-s) - R_\beta(d_1-s)\|_{\mathbb{U}}^p m_g(s)]^{2/p}ds\Bigg]^{p/2}$$

$$+ 8^{p-1}C_pM_1^p\Theta_g(r_{**})\left[\frac{2\delta(p-1)}{p-2}\right]^{1-p/2}\int_{d_1-\epsilon}^{d_1} e^{-\delta(d_1-s)}m_g(s)\,ds$$

$$+ 4^{p-1}C_pM_1^p\Theta_g(r_{**})\left[\frac{2\delta(p-1)}{p-2}\right]^{1-p/2}\int_{d_1}^{d_2} e^{-\delta(d_2-s)}m_g(s)\,ds,$$

where

$$r_{**} = r + \Theta_{a_2}(r)\int_0^T m_{a_2}(\mu,\mu)\,d\mu.$$

By inequality (3.18), we can see that the right-hand side of $E\|y(d_2) - y(d_1)\|_{\mathbb{U}}^p$, which is independent of $y \in \overline{B}$, tends to zero as $d_2 \to d_1$

with ϵ sufficiently small. Since $S_\beta(t)$ and $R_\beta(t)$ are strongly continuous compact operators, the compactness of $S_\beta(t)$, $R_\beta(t)$ for $t > 0$ imply the continuity in the uniform operator topology. Hence, the set $\{\Phi_2(y) : y \in \overline{B}\}$ is equicontinuous. For $d_1 < d_2 \leq 0$, or $d_1 \leq 0 \leq d_2 \leq T$, the proof of equicontinuity is simple in these cases.

Step 7. The operator Φ_2 maps $\overline{B}$ into a precompact set. Let $0 < t \leq s \leq T$ be fixed, and let ϵ be a real number that satisfies $0 < \epsilon < t$. For each $y \in \overline{B}$, we consider the operator ϱ_2^ϵ given by

$$\varrho_2^\epsilon(t) = S_\beta(t)[y_0 + h(y)] + \int_0^{t-\epsilon} R_\beta(t-s)g(s)\,dw(s), \tag{3.19}$$

for each $g \in \mathcal{N}_{G,y}$ and $t \in [0,T]$. Since $S_\beta(t)$, $R_\beta(t)$, $t \geq 0$, are compact operators and h is a completely continuous function, we deduce that the set $V_\epsilon(t) = \{\varrho_2^\epsilon(t) : y \in \overline{B}\}$ is precompact in $\mathbb{U}$ for all ϵ, $\epsilon \in (0,t)$. Furthermore, we have that, for each $y \in \overline{B}$,

$$\begin{aligned} E\|\varrho_2(t) - \varrho_2^\epsilon(t)\|_{\mathbb{U}}^p &= E\left\| \int_{t-\epsilon}^t R_\beta(t-s)g(s)\,dw(s)\right\|_{\mathbb{U}}^p, \\ &\leq C_p M_1^p \Theta_g(r_{**}) \left[\frac{2\delta(p-1)}{p-2}\right]^{1-p/2} \\ &\qquad \int_{t-\epsilon}^t e^{-\delta(t-s)} m_g(s)\,ds. \end{aligned} \tag{3.20}$$

Thus, we observe that the right hand side of the above inequality tends to zero as $\epsilon \to 0$ since there are relatively compact sets which arbitrarily close to the set $V(t) = \{\varrho_2(t) : y \in \overline{B}\}$. Hence, we deduce that $V(t)$ is relatively compact, and thus, Φ_2 maps $\overline{B}$ into a precompact set. Therefore, we conclude that $\Phi_2 : \overline{B} \to \mathcal{P}(\mathcal{C})$ is completely continuous by the Arzelá-Ascoli theorem.

As a result of Steps 1–7, we deduce that $\Phi = \Phi_1 + \Phi_2$ is a condensing operator. Thus, by Theorem 2.15, we conclude that Φ has a fixed point $y \in \mathcal{C}$ which is a mild solution of system (1.1)–(1.2). The proof of the theorem is finished. $\square$

4. Applications. Let us consider the following, neutral stochastic fractional integro-differential inclusions illustrated by

$$
\begin{aligned}
(4.1)\qquad & D_t^{\beta}\left[y(t,x) - \int_0^{\pi} a_1(t,x,\theta) y(\sin t,\theta)\,d\theta\right] \in \frac{\partial^2}{\partial x^2} y(t,x) \\
& + \int_0^t (t-s)^{\xi} e^{-\zeta(t-s)} \frac{\partial^2}{\partial x^2} y(s,x)\,ds \\
& + \mathcal{G}\left(t, \frac{\partial}{\partial x} y(\sin(t),x), \int_0^t a_2\left(t,s,\frac{\partial y(\sin(s),x)}{\partial x}\right) ds\right) \frac{dw(t)}{dt}, \\
& \qquad 0 \le t \le 1, \quad x \in [0,\pi],
\end{aligned}
$$

$$
(4.2)\qquad y(t,0) = y(t,\pi) = 0, \quad 0 \le t \le 1,
$$

$$
(4.3)\qquad y(0,x) = y_0(x) + \int_0^{\pi} b(x,\theta)\,d\theta, \quad 0 \le x \le \pi,
$$

where $w(t)$ represents a one-dimensional standard Wiener process and D_t^{β} denotes the Caputo derivative of order β. The nonlinear functions $a_1 : [0,1] \times [0,\pi] \times [0,\pi] \to \mathbb{R}$, $a_2 : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$, $b : [0,\pi] \times [0,\pi] \to \mathbb{R}$ and $\mathcal{G} : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ are continuous mappings, and $y_0(\cdot)$ belongs to $L^p([0,\pi])$ and $\mathcal{F}_0$-measurable with $E\|y_0\|^p < \infty$.

We consider $\mathbb{U} = L^2([0,\pi])$ with the norm $\|\cdot\|$. We now define the operator $A : \mathbb{U} \to \mathbb{U}$ by $Au = u''$. The domain of A is given by

$$
\begin{aligned}
(4.4)\qquad D(A) = \{u \in \mathbb{U} :& \\
& u,\ u' \text{ are absolutely continuous } u'' \in \mathbb{U} \text{ with } u(0) = u(\pi) = 0\}.
\end{aligned}
$$

Then, we have

(a) $Au = \sum_{n=1}^{\infty} n^2 (u,u_n) u_n$, $u \in D(A)$, where $u_n(x) = \sqrt{(2/\pi)\sin(nx)}$, $n = 1,\dots,$ is the orthogonal set of eigenvectors of A. Thus, it is well known that the operator A is the infinitesimal generator of a strongly continuous, compact, analytic semigroup $T(t)$, which is compact, analytic and self-adjoint in $\mathbb{U}$. Thus, it is possible to define the fractional power $(-A)^{\widetilde{\beta}}$, $0 < \widetilde{\beta} \le 1$, of A as a closed linear operator over its domain $D[(-A)^{\widetilde{\beta}}]$. Moreover, A is sectorial of type, and (P1) is fulfilled. The operator $f(t) : D(A) \subset \mathbb{U} \to \mathbb{U}$, $t \ge 0$, $f(t)x = t^{\xi} e^{-\zeta t} x''$ for $x \in D(A)$. Moreover, it is not difficult to see that the hypotheses (P2) and (P3) are fulfilled with $t^{\xi} e^{-\zeta t}$ and $D(A) = C_0^{\infty}([0,\pi])$; here,

$C_0^\infty([0,\pi])$ is the space of infinitely differentiable functions that vanish at $x=0$ and $x=\pi$. We also assume that the following conditions hold.

(i) The function a_1 is measurable and

$$\sup_{t\in[0,1]} \int_0^\pi \int_0^\pi a_1^2(t,y,x)\,dy\,dx < \infty.$$

(ii) The function $\partial^2/\partial x^2$ is measurable, $a_1(t,y,0)=a_1(t,y,\pi)=0$, and

$$\sup_{t\in[0,1]} \left[\int_0^\pi \int_0^\pi \left(\frac{\partial^2}{\partial x^2} a_1(t,y,x)\right)^2 dy\,dx\right]^{1/2} < \infty.$$

Therefore, we consider

$$F(t,y)(\cdot) = \int_0^\pi a(t,\cdot,\theta)y(\theta)\,d\theta, \tag{4.5}$$

$$G(t,y,z)(\cdot) = \mathcal{G}\biggl(t, y'(\cdot), \int_0^t a_2(t,s,y'(\cdot))\,ds\biggr), \tag{4.6}$$

$$h(z)(\cdot) = \int_0^\pi b(\cdot,\theta)z(\theta)\,d\theta, \quad z\in\mathcal{C}. \tag{4.7}$$

Take $h_1(t)=h_2(t)=h_3(t)=h_4(t)=\sin(t)$. Thus, system (4.1)–(4.3) can be written as

(4.8)

$$D_t^\alpha[u(t)-F(t,u(h_1(t)))] \in Au(t) + \int_0^t f(t-s)u(s)\,ds + G(t,u(h_3(t)),$$
$$\int_0^t a_2(t,s,h_4(t))ds)\frac{dw(t)}{dt}, \quad t\in[0,T],\ u(0)=y_0+h(u).$$

Furthermore, $F : [0,T]\times\mathbb{U} \to \mathbb{U}_{1/2}$ (we choose $\vartheta = 1/2$) $G : [0,T]\times\mathbb{U}\times\mathbb{U} \to L(\mathbb{V},\mathbb{U})$. $(-A)^{1/2}F, G$ are bounded linear operators. Hence, there exists a mild solution for (4.1)–(4.3) under appropriate functions G, F, h and I_i satisfying suitable conditions which verify the assumptions on Theorem 3.1.

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