JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 30, Number 1, Spring 2018

SOLUTION ESTIMATES FOR A SYSTEM OF NONLINEAR INTEGRAL EQUATIONS ARISING IN OPTOMETRY

WOJCIECH OKRASIŃSKI AND ŁUKASZ PŁOCINICZAK

Communicated by Colleen Kirk

ABSTRACT. In this paper, we investigate a system of nonlinear integral equations that has previously been proposed in modelling of the human cornea. The main result of our work is a construction of lower and upper estimates that bound the components of the exact solution to the system being considered. These results generalize some of the recent work by other authors. We conclude the paper with a numerical verification of our analytical estimates.

1. Introduction. Mathematical biology is a prominent and important field in applied mathematics. In addition to addressing some crucial problems concerning living beings, it also raises a plethora of interesting mathematical questions. This paper deals with a system of nonlinear integral equations that has been used in the description of corneal geometry. Our model was initiated in [**14**] and later generalized in [**15, 21**]. It created a number of interesting inverse problems investigated in [**18, 19**]. Other authors contributed to the model by providing some efficient numerical algorithms [**8, 9, 20**] or positively solving an open problem concerning unconditional existence and uniqueness [**4**]. Further, those results were generalized, and their connection to the nth dimensional prescribed mean-curvature equation was established in [**5, 6**]. Some additional results concerning a few related problems were given by other authors in [**7, 16**]. We would also like to mention a very interesting study of an experimental nature [**10**], in which a real-world problem of wind loading a permeable screen door was modelled by our equation.

2010 AMS *Mathematics subject classification.* Primary 45G15.

Keywords and phrases. System of nonlinear integral equations, estimates of solution, ophthalmology.

Received by the editors on November 13, 2016, and in revised form on January 31, 2017.

DOI:10.1216/JIE-2018-30-1-167

The main motivation behind the above cited results is a model of an important constituent of the human eye, the cornea. It is a shell-like structure that plays both optical and mechanical roles in the process of vision. Since it is a surface of the first contact with external factors, the cornea must be strong and durable. Moreover, due to optical reasons, it must be as transparent as possible in order to refract light in an effective way (the optical power of a typical cornea is equal to 2/3 of the entire eye). Therefore, corneal topography and curvature are important objects in optometrical mathematical modelling. A thorough exposition of the eye's anatomy may be found in [**12**], while a more physical description of the process of vision is in [**2**].

The very first models of corneal topography, based on conical sections, such as ellipsoids and paraboloids, were proposed by Helmholtz in the early 20th century [**11**]. This is an accurate way of describing the cornea with many uses and generalizations (see [**3, 13**]). On the other hand, more complex models constructed with the use of the structural mechanics are invaluable in material and strain analysis of the cornea [**1, 17**]. Due to complexity, their properties are very difficult to infer with analytical tools, and numerical analysis is obligatory. The main motivation for this work is to derive an intermediate-complexity model that can capture some of the important material properties of the cornea and yet still be feasible to be investigated by analytical means.

Since the main equation of our model is nonlinear, we cannot hope to obtain a closed-form solution. However, some estimates have been acquired in the works cited above. In what follows, we present an inductive method for finding accurate estimates that are constructed using elementary functions. Not only are they easier to work with than the numerical solution, but they also describe some quantitative features of the exact solution. Having the approximate form of the meridional profile of the corneal topography can help to compute the curvature, and hence, the eye power.

2. Estimates. Consider the following boundary-value problem that has arisen in corneal topography modelling

(2.1)
$$-\left(\frac{h'}{\sqrt{1+h'^2}}\right)' + ah = \frac{b}{\sqrt{1+h'^2}}, \quad h = h(t),\ 0 \le t \le 1,\ a, b \in \mathbb{R}_+,$$

supplied with boundary conditions

$$h'(0) = 0, \quad h(1) = 0. \tag{2.2}$$

In the above formulas, all variables are nondimensional and represent quantities associated with corneal geometry. If R is the typical linear dimension of the cornea (such as its diameter), then $Rh(t)$ represents its height at Rt. We assume axial symmetry, whence t is the nondimensional distance from the axis of symmetry. Moreover, the parameters a and b are associated with corneal tension T, elasticity coefficient k and intra-ocular pressure P. Specifically, $a = kR^2/T$ and $b = PR/T$. More details concerning the model may be found in the paper **[14]**.

Existence, uniqueness and stability of solutions for (2.1) have been proved in **[4]** with the use of elegant and elementary techniques. In what follows, we will use an integral equations approach to find some estimates on the solution of (2.1).

In order to transform (2.1) into a system of integral equations, it is only necessary to use the boundary condition at $t = 0$. Integrate from 0 to t to obtain

$$h'(t) = \sqrt{1 + h'(t)^2} \int_0^t \left(ah(s) - \frac{b}{\sqrt{1 + h'(s)^2}} \right) ds. \tag{2.3}$$

If we define

$$x(t) := h'(t), \qquad y(t) := h(t), \tag{2.4}$$

then (2.3) is a part of a fixed-point system

$$\begin{cases} x(t) = \sqrt{1 + x(t)^2} \int_0^t \left(ay(s) - \frac{b}{\sqrt{1+x(s)^2}} \right) ds, \\ y(t) = -\int_t^1 x(s)\, ds, \end{cases} \tag{2.5}$$

where we have again used (2.2). Now, both boundary conditions have been incorporated into the system of integral equations (2.5). If we define the *nonlinear* operator $\mathbf{F}$ by the formula

(2.6)

$$\mathbf{F}(\mathbf{x}) := \left(\sqrt{1 + x(t)^2} \int_0^t \left(ay(s) - \frac{b}{\sqrt{1 + x(s)^2}} \right) ds, -\int_t^1 x(s)\, ds \right),$$
$$\mathbf{x}(t) = (x(t), y(t)),$$

we can see that (2.1)–(2.2) imply the fixed-point problem

$$\mathbf{x} = \mathbf{F}(\mathbf{x}), \quad \mathbf{x}(t) = (x(t), y(t)). \tag{2.7}$$

The application of Schauder's fixed-point theorem immediately leads to the existence result; however, we will not pursue the details here. Instead, we refer the reader to the work by Coelho, Corsato and Omari [**4**] where, apart from existence and uniqueness results, the following useful theorem was proven.

Theorem 2.1 ([**4**])**.** *Let $(x(t), y(t))$ be a solution of (2.5). Then, y is positive, decreasing and concave for $t \in (0,1)$. Moreover, we have*

$$-\sqrt{\exp\left(2b^2/a\right) - 1} < x(t) < 0$$

and $0 < y(t) < b/a$.

In order to start our reasoning, we first prove an auxiliary lemma concerning sequences of real numbers that will be utilized below in defining the approximative sequence of functions.

Lemma 2.2. *Assume that $0 < b < 1$ and $0 < a \le b^2\sqrt{1-b^2}$. Define the sequences c_n and d_n by the recursive formulas*

$$c_1 = b, \qquad c_{n+1} = b - \frac{a}{d_n}\left(1 - \sqrt{1-d_n^2}\right), \qquad d_n = b\sqrt{1-c_n^2}, \quad n \ge 1. \tag{2.8}$$

Then, both of these sequences are convergent and their respective limits, which we denote by c and d, satisfy

$$\begin{cases} c = b - (a/d)\left(1 - \sqrt{1-d^2}\right), \\ d = b\sqrt{1-c^2}. \end{cases} \tag{2.9}$$

Moreover, we have the following chain of inequalities

$$0 < \frac{a}{b} \le d_n, \qquad c_n \le b < 1, \tag{2.10}$$

with c_n decreasing and d_n increasing.

Proof. We begin by showing that, for all $n \ge 1$, we have $0 < c_n$, $d_n < 1$, which implies that both of these sequences are bounded and well defined. We proceed by induction. First, note that

$$0 < c_1 = b < 1, \qquad 0 < d_1 = b\sqrt{1-b^2} < 1, \tag{2.11}$$

and assume that, for a fixed n, we have $0 < c_n < 1$. We will show that this holds for the next term. From (2.8), we have

$$0 < d_n = b\sqrt{1-c_n^2} < 1, \tag{2.12}$$

and further,

$$c_{n+1} = b - \frac{a}{d_n}\left(1 - \sqrt{1-d_n^2}\right) < 1, \tag{2.13}$$

since a and d_n are both positive. From an elementary inequality $\sqrt{1-d_n^2} \geq 1 - d_n^2$ and the assumption $a \leq b^2\sqrt{1-b^2} < b$, we can estimate

$$c_{n+1} \geq b - \frac{a}{d_n}\left(1 - (1-d_n^2)\right) = b - ad_n > b - a \geq b - b^2\sqrt{1-b^2} > 0, \tag{2.14}$$

for $0 < b < 1$. This completes the induction.

Our next step is the monotonicity. Define

$$\varphi(t) := b - \frac{a}{t}\left(1 - \sqrt{1-t^2}\right), \qquad \psi(t) := b\sqrt{1-t^2}, \tag{2.15}$$

and notice that both of these functions are decreasing. Again, the argument will be conducted by mathematical induction. The first terms of the c_n sequence are

$$\begin{aligned} c_1 &= b, \\ c_2 &= b - \frac{a}{b\sqrt{1-b^2}}\left(1 - \sqrt{1-b^2(1-b^2)}\right) < b = c_1, \end{aligned} \tag{2.16}$$

hence $c_2 < c_1$. Now, assume that $c_n < c_{n-1}$. Since ψ is decreasing, we have $d_n = \psi(c_n) > \psi(c_{n-1}) = d_{n-1}$. Using this result and the same reasoning as above, we obtain $c_{n+1} = \varphi(d_n) < \varphi(d_{n-1}) = c_n$. Therefore, c_n is decreasing while d_n is increasing.

Since both sequences considered above are monotone and bounded, they have a limit that satisfies the system of equations (2.9).

Finally, we must show the inequality $c_n, d_n \geq a/b$. Note that $d_1 = b\sqrt{1-b^2} \geq a/b$ since $a \leq b^2\sqrt{1-b^2}$. By monotonicity, we then have $d_n \geq a/b$ for all $n \in \mathbb{N}$. In order to show that $c_{n+1} \geq a/b$, we begin with an obvious inequality $(1-\sqrt{1-b^2})^2 \geq 0$ and observe that it is equivalent to

$$\sqrt{1-b^2} \leq \frac{1}{2-\sqrt{1-b^2}}. \tag{2.17}$$

By the assumption, $a \leq b^2\sqrt{1-b^2}$, and it follows that

$$\frac{a}{b^2} \leq \frac{1}{2-\sqrt{1-b^2}} \longrightarrow a \leq \frac{b^2}{2-\sqrt{1-b^2}}, \tag{2.18}$$

which, after transformation is equivalent to $a/b + (a/b)\left(1-\sqrt{1-b^2}\right) \leq b$. Hence,

$$\varphi(\psi(0)) = b - \frac{a}{b}\left(1-\sqrt{1-b^2}\right) \geq \frac{a}{b}. \tag{2.19}$$

Now, from the definition of the sequences c_n and d_n, we have

$$c_{n+1} = \varphi(d_n) = \varphi(\psi(c_n)). \tag{2.20}$$

Further, observe that, since φ and ψ are both decreasing, their composition is increasing, and hence, $\varphi(\psi(c_n)) \geq \varphi(\psi(0))$. Finally, combining (2.20) with (2.19) yields the desired result

$$c_{n+1} \geq \varphi(\psi(0)) \geq \frac{a}{b}. \tag{2.21}$$

This concludes the proof of Lemma 2.2. □

A simple calculation shows that the system (2.9) is equivalent to solving a quadric polynomial which, in principle, can always be done. Yet, formulas for its roots are very involved, and thus, not very useful in calculations. However, we can give an expression for an approximate solution of (2.9) valid for small c and d. The idea is to use the Taylor series

$$\begin{aligned} b - \frac{a}{t}\left(1-\sqrt{1-t^2}\right) &= b - \frac{a}{2}t + O(t^2), \\ b\sqrt{1-t^2} &= b\left(1-\frac{1}{2}t^2 + O(t^4)\right). \end{aligned} \tag{2.22}$$

Then, we have an approximate system of equations defining c and d

$$\begin{cases} c = b - \frac{a}{2}d, \\ d = b(1-c^2), \end{cases} \longrightarrow \begin{cases} c = \dfrac{1-\sqrt{1-2ab^2+a^2b^2}}{ab} \\ d = \dfrac{2}{a}\left(\dfrac{\sqrt{a^2b^2-2ab^2+1}}{ab} - \dfrac{1}{ab} + b\right). \end{cases} \tag{2.23}$$

We are ready to prove the main result concerning estimates of the solution to (2.5). The proof is based on an inductive construction giving bounds from below and above. Those estimates use the results from Lemma 2.2.

Theorem 2.3. *Let* $(x(t), y(t))$ *be a solution of* (2.5) *for* $0 < b < 1$ *and* $a \leq b^2\sqrt{1-b^2}$. *If we denote*

$$H_u(t) := \frac{\sqrt{1-(ut)^2} - \sqrt{1-u^2}}{u}, \qquad H'_u(t) = -\frac{ut}{\sqrt{1-(ut)^2}}, \tag{2.24}$$

then the following estimates hold

$$H'_c(t) \leq x(t) \leq H'_d(t), \tag{2.25}$$

and

$$H_d(t) \leq y(t) \leq H_c(t), \tag{2.26}$$

where c *and* d *are defined in* (2.9).

Proof. Let us turn to the system (2.5) and use the estimates found in Theorem 2.1, i.e., $x(t) \leq 0$ and $y(t) > 0$, to obtain

$$x(t) \geq -b\sqrt{1+x(t)^2}\,t, \tag{2.27}$$

which, after squaring and manipulation, gives

$$x(t)^2 \leq \frac{(bt)^2}{1-(bt)^2}. \tag{2.28}$$

Taking the root, choosing the negative one and minding the sign change yields

$$x(t) \geq -\frac{bt}{\sqrt{1-(bt)^2}} = H'_b(t) = H'_{c_1}(t), \tag{2.29}$$

where we used the assumption $0 < b \leq 1$. Here, c_1 is the first term of the c_n sequence defined in Lemma 2.2. Now, integrate the above equation from t to 1 and use the definition of y, i.e., (2.5), to arrive at

$$y(1) - y(t) \geq -b \int_t^1 \frac{sds}{\sqrt{1-(bs)^2}} = -\frac{\sqrt{1-(bt)^2} - \sqrt{1-b^2}}{b}, \tag{2.30}$$

or, since $y(1) = 0$,

$$y(t) \leq H_b(t) = H_{c_1}(t). \tag{2.31}$$

In order to proceed further, set

$$f_u(s) := \frac{a}{u}(\sqrt{1-(us)^2} - \sqrt{1-u^2}) - b\sqrt{1-(us)^2}$$

for $s \in [0,1]$ and $u \in [0,1]$. Note that, by a simple derivative test, we can show that, for $1 \geq u \geq a/b$, the function $f_u(s)$ is negative and s-increasing. Hence, for $u \geq a/b$, we have

$$\begin{aligned} \min_{s\in[0,1]} f_u(s) &= f_u(0) = \frac{a}{u}\left(1 - \sqrt{1-u^2}\right) - b, \\ \max_{s\in[0,1]} f_u(s) &= f_u(1) = -b\sqrt{1-u^2}. \end{aligned} \tag{2.32}$$

Now, observe that the definition of $x(t)$ with (2.29) and (2.31) implies

$$\begin{aligned} x(t) &\leq \sqrt{1+x(t)^2} \int_0^1 \left(\frac{a}{b}\left(\sqrt{1-(bt)^2} - \sqrt{1-b^2}\right) - b\,\frac{\sqrt{1-(bs)^2}}{\sqrt{1-(bs)^2+(bs)^2}}\right) ds \\ &= \sqrt{1+x(t)^2} \int_0^t f_b(s)\, ds \\ &\leq -b\sqrt{1-b^2}\sqrt{1+x(t)^2}\, t, \end{aligned} \tag{2.33}$$

where we have used (2.32). Put $d_1 := b\sqrt{1-b^2}$, and transform the above equation to obtain

$$x(t) \leq H'_{d_1}(t), \tag{2.34}$$

which forces

$$y(t) \geq H_{d_1}(t). \tag{2.35}$$

We have thus shown that both x and y can be bounded from below and above with functions belonging to the family H_u defined in (2.24).

We claim that this procedure can be continued inductively. In order to show this explicitly, assume that

$$x(t) \geq H'_{c_n}(t), \qquad y(t) \leq H_{c_n}(t), \tag{2.36}$$

where c_n is defined in (2.8). We will show that formula (2.36) implies further bounds on the order of $n+1$. Using (2.36) and our main equation (2.3), we have

$$\frac{x(t)}{\sqrt{1+x(t)^2}} \leq \int_0^t f_{c_n}(s)\,ds. \tag{2.37}$$

Now, since by Lemma 2.2, the sequence c_n is greater than or equal to a/b, we can use (2.32) to make a further estimate by using the definition of $x(t)$ (as in (2.5)) and taking the maximal value of f (similarly as in (2.33))

$$\frac{x(t)}{\sqrt{1+x(t)^2}} \leq -b\sqrt{1-c_n^2}\,t = -d_n t < 0. \tag{2.38}$$

Transforming this expression, gives us

$$x(t) \leq H'_{d_n}(t), \tag{2.39}$$

which, after integration from t to 1 and using boundary condition (2.2), becomes

$$y(t) \geq H_{d_n}(t). \tag{2.40}$$

We can repeat above steps to obtain another bound, this time involving c_{n+1}. To this end, use (2.3) with (2.39) and (2.40) to obtain

$$\frac{x(t)}{\sqrt{1+x(t)^2}} \geq \int_0^t f_{d_n}(s)\,ds \geq \left(\frac{a}{d_n}\left(1-\sqrt{1-d_n^2}\right)-b\right)t = -c_{n+1}t, \tag{2.41}$$

where Lemma 2.2 was used in concluding that $d_n \geq (a/b)$ so that we could use (2.32). After transformation, the above equation becomes

$$x(t) \geq H'_{c_{n+1}}(t), \tag{2.42}$$

which, after integration, gives

$$y(t) \leq H_{c_{n+1}}(t). \tag{2.43}$$

This is the assertion which had to be shown in order to conclude the induction.

Finally, invoking Lemma 2.2 helps us to conclude that c_n and d_n are convergent. This establishes the estimates given in (2.25)–(2.26). The proof is complete. □

Numerical simulations show that estimates (2.39)–(2.40) have increasing accuracy with a and b closer to 0. This is expected since, for $a = b = 0$ by (2.9), we have $c = d = 0$, while

$$\lim_{u \to 0} H_u(t) = 0, \tag{2.44}$$

which is identical to the exact solution of (2.1). Plots of our approximations to y are presented in Figure 1. Note that, for $b = 0.5$ and $a = b^2\sqrt{1-b^2}$, curves representing limits of the approximative sequences are almost indistinguishable.

As a final comment, we can state a quick estimate on the difference between the unknown exact solution of (2.5) and its bounds. In applications, the y-component of the solution is more important since it gives us the shape of the cornea. We can quickly give a quantitative estimate on the difference between y and its approximations H_c and H_d. We have

$$y(t) - H_d(t) \le H_c(t) - H_d(t), \quad H_c(t) - y(t) \le H_c(t) - H_d(t), \tag{2.45}$$

and thus, it is sufficient to find a bound on the difference $H_c(t) - H_d(t)$.

The next remark gives the precise statement. It can be used as a quick gauge for the magnitude of the corneal apex elevation. This, in turn, is one of the fundamental parameters measured in optometry.

Remark 2.4. Let H_u be as defined in (2.24) while c and d are solutions of (2.9). Then,

$$H_c(t) - H_d(t) \le \frac{1}{d^2}\left(\frac{1}{\sqrt{1-d^2}} - \frac{1}{\sqrt{1-(dt)^2}}\right)(c-d), \tag{2.46}$$

and

$$H_d'(t) - H_c'(t) \le \frac{t}{(1-(dt)^2)^{3/2}}(c-d). \tag{2.47}$$

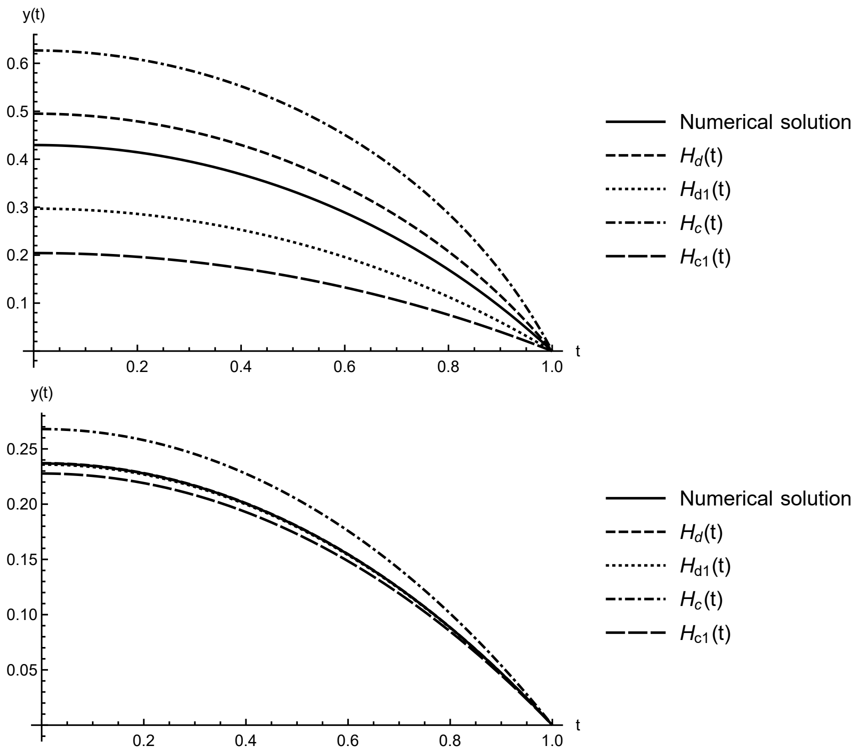

FIGURE 1. Plots of y-component of numerical solution to (2.5) along with its estimates found in Theorem 2.3. Inset legends describe various curves. The choice of parameters is the following: $a = b^2\sqrt{1-b^2} = 0.353$, $b = 0.9$ (top) and $a = b^2\sqrt{1-b^2} = 0.216$, $b = 0.5$ (bottom).

In particular, for the corneal apex, we have

$$\max\{y(0) - H_d(0), H_c(0) - y(0)\} \leq \frac{c-d}{d^2}\left(\frac{1}{\sqrt{1-d^2}} - 1\right). \tag{2.48}$$

Proof. Let us start with finding bounds on the derivatives. To this end, use Lagrange's theorem, and write

$$H_d'(t) - H_c'(t) = \frac{\partial}{\partial u} H_u'(t)\Big|_{u=u_*}(c-d), \tag{2.49}$$

for some $d \leq u_* \leq c$. Now, simple calculation yields

$$\frac{\partial}{\partial u} H_u'(t)\Big|_{u=u_*} = \frac{t}{(1-(u_* t)^2)^{3/2}} \leq \frac{t}{(1-(dt)^2)^{3/2}}. \tag{2.50}$$

Hence, the bound on $H_d'(t) - H_c'(t)$. Finally, integrating (2.47) from t to 1 yields (2.46). This completes the proof. □

3. Conclusion. As numerical analysis has verified that our estimates can be very accurate, especially for small values of b. This, in turn, is equivalent to saying that the intra-ocular pressure has low values. This situation leads to a condition known as *hypotony*, which has a number of negative consequences for the eye. We hope that our analysis can provide some insights about this disease as well as the ophthalmological aspects of mathematical biology.

REFERENCES

1. Kevin Anderson, Ahmed El-Sheikh and Timothy Newson, *Application of structural analysis to the mechanical behaviour of the cornea*, J. Roy. Soc. Interface **1** (2004).

2. David A. Atchison and George Smith, *Optics of the human eye*, Butterworth-Heinemann, Oxford, 2000.

3. Henry Burek and W.A. Douthwaite, *Mathematical models of the general corneal surface*, Ophthalmic Physiolog. Optics **13** (1993), 68–72.

4. Isabel Coelho, Chiara Corsato and Pierpaolo Omari, *A one-dimensional prescribed curvature equation modeling the corneal shape*, Bound. Value Prob. **2014** (2014).

5. Chiara Corsato, Colette De Coster and Pierpaolo Omari, *Radially symmetric solutions of an anisotropic mean curvature equation modeling the corneal shape*, Discr. Contin. Dynam. Syst. (2015).

6. ______, *The Dirichlet problem for a prescribed anisotropic mean curvature equation*: *Existence, uniqueness and regularity of solutions*, J. Diff. Eqs. **260** (2016), 4572–4618.

7. Meiqiang Feng and Xuemei Zhang, *Time-map analysis to establish the exact number of positive solutions of one-dimensional prescribed mean curvature equations*, Bound. Value Prob. **2014** (2014), 1–16.

8. G.W. Griffiths and W.E. Schiesser, et al., *Analysis of cornea curvature using radial basis functions, Part* I: *Methodology*, Computers Biol. Med. **77** (2016), 274–284.

9. ______, *Analysis of cornea curvature using radial basis functions, Part* II: *Fitting to data-set*, Computers Biol. Med. **77** (2016), 285–296.

10. Cameron Hall, Matthew Mason, et al., *Structural modelling of deformable screens for large door openings*, Austral. & New Zealand Indust. Appl. Math. J. **57** (2016), 55–114.

11. Hermann von Helmholtz, *Helmholtz's treatise on physiological optics*, Optical Society of America **3** (1925).

12. Jack J. Kanski and Brad Bowling, *Clinical ophthalmology: A systematic approach*, Elsevier Health Sciences, Elsevier Saunders, Ltd., London, 2011.

13. Richard Lindsay, George Smith and David Atchison, *Descriptors of corneal shape*, Optometry Vision Sci. **75** (1998), 156–158.

14. Wojciech Okrasiński and Łukasz Płociniczak, *A nonlinear mathematical model of the corneal shape*, Nonlin. Anal. RWA **13** (2012), 1498–1505.

15. ______, *Bessel function model of corneal topography*, Appl. Math. Comp. **223** (2013), 436–443.

16. Hongjing Pan and Ruixiang Xing, *Applications of total positivity theory to* 1D *prescribed curvature problems*, J. Math. Anal. Appl. **428** (2015), 113–144.

17. A. Pandolfi and F. Manganiello, *A model for the human cornea: Constitutive formulation and numerical analysis*, Biomech. Model. Mechanobiol. **5** (2006), 237–246.

18. L. Płociniczak and W. Okrasiński, *Nonlinear parameter identification in a corneal geometry model*, Inv. Prob. Sci. Eng. **23** (2015),443–456.

19. ______, *Regularization of an ill-posed problem in corneal topography*, Inv. Prob. Sci. Eng. **21** (2013), 1090–1097.

20. Łukasz Płociniczak, Graham W. Griffiths and William E. Schiesser, ODE/PDE *analysis of corneal curvature*, Comp. Biol. Med. **53** (2014), 30–41.

21. Łukasz Płociniczak, Wojciech Okrasiński, et al., *On a nonlinear boundary value problem modeling corneal shape*, J. Math. Anal. Appl. **414** (2014), 461–471.

Wrocław University of Science and Technology, Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland
Email address: wojciech.okrasinski@pwr.edu.pl

Wrocław University of Science and Technology, Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland
Email address: lukasz.plociniczak@pwr.edu.pl