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MATHEMATICAL ASPECTS OF VARIATIONAL BOUNDARY INTEGRAL EQUATIONS FOR TIME DEPENDENT WAVE PROPAGATION

PATRICK JOLY AND JERÓNIMO RODRÍGUEZ

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ABSTRACT. In this work, we provide a review of recent results on the mathematical analysis of space-time variational bilinear forms associated to transient boundary integral operators for the wave equation. Most of the results will be proven directly in the time domain and compared to similar results (most of them obtained in the Laplace domain) that can be found in the literature.

1. Introduction. Time domain boundary integral equations for wave propagation problems (TDBIEs), also called retarded potential methods in 3D, have been initially used for the numerical approximation of time domain scattering problems. Rather recently, they have also been used, in association to volumic methods, for the design of transparent boundary conditions: this is, in fact, the target application we had in mind when writing this paper. The recent progress of rapid algorithms for the inversion of matrices originated from the discretization of boundary integral equations (see, for instance, **[4, 8, 9, 31, 36, 44]**), such methods have now become in a number of cases a credible alternative to more standard methods such as local absorbing boundary conditions **[28, 29, 35]** or perfectly matched layers **[14, 50]**. Moreover, with respect to the above-mentioned methods, the boundary integral equations present the clear advantage of authorizing non convex computational domains. For instance, in **[1]**, a method has

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been proposed based on the coupling between a discontinuous Galerkin approach to the volumic part of the problem and a space-time variational approach to boundary integral equations. On the other hand, in [**11**], the authors looked at the coupling between a finite element method for the discretization of the volumic equations and a convolution quadrature approach to boundary integral equations.

As originally used by engineers [**43, 47**], their discretization was based on collocation methods that present some advantages in terms of the simplicity of their implementation. However, such methods are known to face some reliability and stability problems [**24, 25**]. Space-time variational approaches, as well as associated Galerkin approximations, were designed in the 1980's [**5, 6, 13, 33, 34**] as an alternative to collocation methods [**24, 25, 43, 44, 47**]. Although rather difficult to implement [**49**] (as far as the effective computation of the matrices involved in the discrete problem), those methods benefit from unconditional stability properties which make them quite attractive. More recently, convolution quadrature methods have gained interest in the community of applied mathematicians for their application to wave propagation problems [**7, 48**]. Such methods also benefit in general from unconditional stability, and their numerical analysis (convergence analysis and error estimates) is more or less completely understood [**10, 19, 37**]. As a matter of fact, in such methods, the time discretization of integral equations can be seen as exact boundary conditions (or transparent boundary conditions in the case of the coupling with an interior problem) once the exterior problem has been semi-discretized in time using an unconditional implicit numerical scheme (a multi-step or Runge-Kutta method). As a consequence, the analysis of convolution quadrature methods benefits from well-known properties of these schemes.

On the other hand, the space-time variational approach benefits from some nice properties that are not shared by the convolution quadrature approach. For instance, the method does not introduce any numerical dissipation (nor dispersion) since the exact Green's kernel of the wave equation is used. Moreover, it naturally takes into account finite propagation velocity or Huygens principle (in 3D). However, compared to the convolution quadrature approach, numerical analysis of space-time boundary element methods is less advanced. For instance, one can hardly find articles reporting on the error analysis of such methods, even though some (unpublished) results in this direction can be found in the habilitation thesis of Ha-Duong **[32]**. As a consequence, the theoretical analysis in **[1]** is less advanced than that in **[11]**.

In 2003, in a review article devoted to time domain integral equations **[23]**, Costabel wrote that "the theory of error estimates for the BEM (Boundary Element Method) in the hyperbolic case is still incomplete." To our knowledge, this assertion is still valid in 2016. One explanation is that the analysis of variational methods naturally relies on the mathematical properties of involved space-time bilinear forms (think typically to the application, or adaptation, of Cea's or Strang's type lemmas **[27]**). Since the pioneering results of Bamberger and Ha-Duong **[5, 6]**, known results on the coercivity/continuity of these bilinear forms rely on Laplace transform (in time) methods, Plancherel's theorem and causality properties (in order to derive finite time coercivity and continuity estimates). However, the use of such properties for the analysis of Galerkin methods faces difficulties related to the following facts:

(1) The coercivity result is obtained for a weighted (with a decreasing exponential in time) version of the bilinear forms. As a consequence, a too naive adaptation of "classical" methods would provide a result for a space-time method that does not coincide with the methods which are implemented in practice, which do not involve any weight. On the other hand, the interest of developing (theory driven) numerical methods with a weighted bilinear form are far from obvious:
   - the choice of the decay rate of the exponential weight is somewhat arbitrary. Looking at finite time estimates, it seems that it should be chosen as a function of the final time T, which would represent practical drawbacks.
   - After time discretization, due to the presence of the weight, the matrix to be inverted at each time step would depend upon the time step considered, which would greatly affect the practical efficiency of the resulting algorithm.

(2) Even if one looks at weighted bilinear forms, in the results obtained by the Laplace method, there is a gap between the (space-time Sobolev) coercivity norm (i.e., the norm for which the coercivity holds) and the continuity norm (the norm of the space in which the bilinear form is proven to be continuous) which is strictly stronger.

As a consequence, as far as the error analysis is concerned, this must be paid (through the use of inverse inequalities **[20, 27]**) in terms of the powers of h (the space step) or Δt (the time step) and in terms of the required regularity of the solution: this is at least what appears in the unpublished results of Ha-Duong **[32]**.

The main goal of this paper is to provide a state of the art of recent results concerning the mathematical analysis of space time variational bilinear forms associated to space-time integral operators. We shall put the emphasis on a direct time domain approach to these results (by opposition to Fourier-Laplace methods) in the spirit of what has been done in **[26]**. Such approaches should be, we think, of interest for the numerical analysis of Galerkin methods. In particular, we shall address the following questions related to the above points (1) and (2):

- As far as the coercivity results mentioned in point (1) are concerned, it is not clear at first glance whether or not the presence of a weight is needed for ensuring coercivity (it could be a priori due to the technique); we shall see that this is nevertheless the case. Moreover, we shall see that coercivity results can be derived in an elementary way for a large class of weights including the decreasing exponentials.
- As far as the coercivity results and continuity results (mentioned in point (2)) are concerned, it is not clear at all that the results obtained by the Laplace method are sharp. First, in Section 3, we shall see that direct time domain approaches bring a small improvement to the above results. Then we shall study in more detail two particular situations, similar to those previously considered in **[2]**, which are quite academic but nevertheless very instructive:
  – In the case of a flat surface in dimension $d \geq 2$, we shall see that the use of a more "micro-local" (space-time) approach analysis leads to an improvement of coercivity and continuity estimates. In particular, it would justify the use of discontinuous finite elements in time, an important feature in the method in **[1]**.
  – In the case of the 1D wave equation, we shall provide optimal coercivity and continuity results. In this case, there is no longer any gap between the coercivity and continuity norms, which opens the door to an optimal and

complete numerical analysis of the method in [**1**]. (This will be the subject of future work).

2. Motivation: Solving the wave equation on unbounded domains. Let us consider the transient wave propagation problem

$$\left|\begin{array}{lll} \dfrac{1}{\rho c^2}\dfrac{\partial p}{\partial t} + \operatorname{div}\mathbf{v} & = f & \text{in } \mathbb{R}^d \times [0,T], \\ \rho\dfrac{\partial \mathbf{v}}{\partial t} + \nabla p & = \mathbf{0} & \text{in } \mathbb{R}^d \times [0,T], \\ p(\mathbf{x},0) & = p_0(\mathbf{x}) & \text{in } \mathbb{R}^d, \\ \mathbf{v}(\mathbf{x},0) & = \mathbf{v}_0(\mathbf{x}), & \text{in } \mathbb{R}^d. \end{array}\right. \tag{2.1}$$

In these equations $\rho(\cdot)$ and $c(\cdot)$ are, respectively, the density of the fluid and sound velocity that are assumed to satisfy

$$0 < \rho_{\min} \le \rho(\mathbf{x}) \le \rho_{\max} < \infty$$

and

$$0 < c_{\min} \le c(\mathbf{x}) \le c_{\max} < \infty.$$

We suppose that there exists a smooth open, connected (for simplicity) and bounded set $\Omega_i \subset \mathbb{R}^d$, that will be called the *interior domain*, outside of which (i.e., in the exterior domain $\Omega_e = \mathbb{R}^d \setminus \overline{\Omega_i}$) the fluid is homogeneous and does not support any source term or initial condition, i.e., (see Figure 1 on the left) for all $(\mathbf{x}, t) \in \Omega_e \times [0,T]$,

$$\begin{array}{ll} \rho(\mathbf{x}) = \rho_0, & c(\mathbf{x}) = c_0, \\ \mathbf{v}_0(\mathbf{x}) = \mathbf{0}, & p_0(\mathbf{x}) = f(\mathbf{x},t) = 0. \end{array} \tag{2.2}$$

Let us denote the interface between both domains that will be assumed to be smooth (of class C^∞) by $\Gamma = \partial\Omega_i \equiv \partial\Omega_e$. Our goal is to artificially bound the computational domain, restricting the computations to the domain Ω_i and the boundary Γ. To this end, several techniques can be used, such as, for example, perfectly matched layers [**14, 50**], absorbing boundary conditions [**28, 29, 35**] or infinite elements [**3, 15**]. In [**1**], we proposed to use transparent boundary conditions based on boundary integral equations that we recall below.

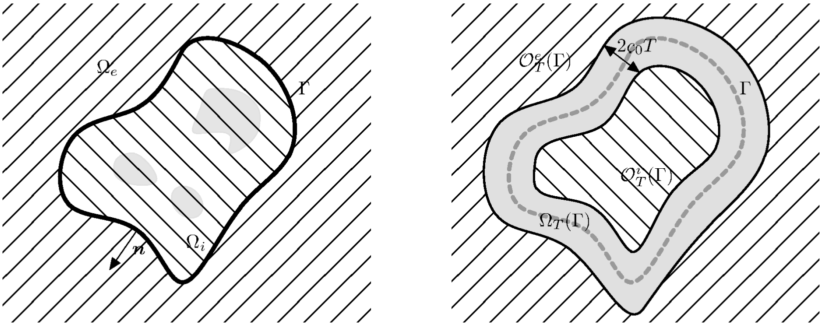

FIGURE 1. Left: geometry of the problem (in gray, the support of the heterogeneities, initial conditions and sources). Right: definitions of the sets $\Omega_T(\Gamma)$ and $\mathcal{O}^l_T(\Gamma)$, $l \in \{i, e\}$.

First, one formulates an artificial transmission problem between Ω_i and Ω_e equivalent to equation (2.1). Denoting $(p_i, \mathbf{v}_i)$ (respectively, $(p_e, \mathbf{v}_e)$) the restriction of $(p, \mathbf{v})$ to $\Omega_i \times [0, T]$ (respectively, $\Omega_e \times [0, T]$) it is well known that $((p_i, \mathbf{v}_i), (p_e, \mathbf{v}_e))$ is solution of the transmission problem:

$$
(2.3) \qquad \left|\begin{array}{lll}
\dfrac{1}{\rho c^2}\dfrac{\partial p_i}{\partial t} + \operatorname{div} \mathbf{v}_i & = f & \text{in } \Omega_i \times [0, T], \\
\rho \dfrac{\partial \mathbf{v}_i}{\partial t} + \nabla p_i & = \mathbf{0} & \text{in } \Omega_i \times [0, T], \\
p_i(\mathbf{x}, 0) & = p_0(\mathbf{x}) & \text{in } \Omega_i, \\
\mathbf{v}_i(\mathbf{x}, 0) & = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega_i,
\end{array}\right.
$$

$$
(2.4) \qquad \left|\begin{array}{lll}
\dfrac{1}{\rho_0 c_0^2}\dfrac{\partial p_e}{\partial t} + \operatorname{div} \mathbf{v}_e & = 0 & \text{in } \Omega_e \times [0, T], \\
\rho_0 \dfrac{\partial \mathbf{v}_e}{\partial t} + \nabla p_e & = \mathbf{0} & \text{in } \Omega_e \times [0, T], \\
p_i(\mathbf{x}, 0) & = 0 & \text{in } \Omega_e, \\
\mathbf{v}_i(\mathbf{x}, 0) & = \mathbf{0} & \text{in } \Omega_e,
\end{array}\right.
$$

equations that are coupled through the transmission conditions

$$\left| \begin{array}{lll} \mathbf{v}_e \cdot \mathbf{n} & = \mathbf{v}_i \cdot \mathbf{n} & \text{on } \Gamma, \\ p_e & = p_i & \text{on } \Gamma, \end{array} \right. \tag{2.5}$$

where $\mathbf{n}$ is the unit normal vector to Γ, outgoing with respect to Ω_i. Reciprocally, if $((p_i, \mathbf{v}_i), (p_e, \mathbf{v}_e))$ is a solution of the above transmission problem, the field $(p, \mathbf{v})$ obtained by concatenation of $(p_i, \mathbf{v}_i)$ and $(p_e, \mathbf{v}_e)$ is the solution of equation (2.1).

The next step is to replace equation (2.4) by a set of boundary integral equations defined on Γ whose unknowns are the traces of the exterior solution, i.e., the functions $(\psi, \varphi) := (p_{e|\Gamma}, \mathbf{v}_{e|\Gamma} \cdot \mathbf{n})$. This formulation will be obtained in subsection 2.3. The integral operators and associated bilinear forms needed on this process are introduced in the following sections. For simplicity, we suppose that time and space units are chosen in such a way that

$$\rho_0 = 1 \quad \text{and} \quad c_0 = 1. \tag{2.6}$$

2.1. Integral operators and associated bilinear forms: Abstract definitions. Before introducing the boundary integral operators let us introduce some notation concerning traces, jumps, mean values and Green's formula that will be useful for the sequel. For any function defined in $\mathbb{R}^d$, we set

$$f_e := f_{|\Omega_e}, \qquad f_i := f_{|\Omega_i}.$$

Next, for any function $q \in L^2(\mathbb{R}^d) \cap H^1(\Omega_i \cup \Omega_e)$, we introduce the jumps and the mean values on Γ by

$$\begin{aligned} [\![q]\!]_\Gamma &:= q_{e|\Gamma} - q_{i|\Gamma} \in H^{1/2}(\Gamma), \\ \{\!\{q\}\!\}_\Gamma &:= \frac{1}{2}\left(q_{e|\Gamma} + q_{i|\Gamma}\right) \in H^{1/2}(\Gamma). \end{aligned} \tag{2.7}$$

Analogously, for $\mathbf{w} \in [L^2(\mathbb{R}^d)]^n \cap H(\mathrm{div}; \Omega_i \cup \Omega_e)$, we introduce the normal jumps and mean values

$$\begin{aligned} [\![\mathbf{w} \cdot \mathbf{n}]\!]_\Gamma &:= \mathbf{w}_e \cdot \mathbf{n}_{|\Gamma} - \mathbf{w}_i \cdot \mathbf{n}_{|\Gamma} \in H^{-1/2}(\Gamma), \\ \{\!\{\mathbf{w} \cdot \mathbf{n}\}\!\}_\Gamma &:= \frac{1}{2}\left(\mathbf{w}_i \cdot \mathbf{n}_{|\Gamma} + \mathbf{w}_e \cdot \mathbf{n}_{|\Gamma}\right) \in H^{-1/2}(\Gamma). \end{aligned} \tag{2.8}$$

We now recall the following Green's formula.

Lemma 2.1. *For any* $q \in H^1(\Omega_i \cup \Omega_e)$ *and* $\mathbf{w} \in H(\mathrm{div}; \Omega_i \cup \Omega_e)$*, one has*

$$(2.9)\qquad -\int_{\mathbb{R}^d \setminus \Gamma} \big(\mathrm{div}\, \mathbf{w}\, q + \mathbf{w} \cdot \nabla q\big)\, \mathrm{d}\mathbf{x} = \int_\Gamma \big([\![q]\!]_\Gamma \{\!\{\mathbf{w} \cdot \mathbf{n}\}\!\}_\Gamma + \{\!\{q\}\!\}_\Gamma [\![\mathbf{w} \cdot \mathbf{n}]\!]_\Gamma\big)\, \mathrm{d}\gamma.$$

Proof. Since $\mathbf{n}$ is exterior to Ω_i we have

$$-\int_{\Omega_i} \mathrm{div}\, \mathbf{w}\, q\, \mathrm{d}\mathbf{x} = \int_{\Omega_i} \boldsymbol{w} \cdot \nabla q\, \mathrm{d}\mathbf{x} - \int_\Gamma \mathbf{w}_i \cdot \mathbf{n}\, q_i\, \mathrm{d}\gamma.$$

In the same way, $-\mathbf{n}$ being exterior to Ω_e,

$$-\int_{\Omega_e} \mathrm{div}\, \mathbf{w}\, q\, \mathrm{d}\mathbf{x} = \int_{\Omega_e} \mathbf{w} \cdot \nabla q\, \mathrm{d}\mathbf{x} + \int_\Gamma \mathbf{w}_e \cdot \mathbf{n}\, q_e\, \mathrm{d}\gamma.$$

Adding the last two equalities leads to equation (2.9) while observing that

$$\mathbf{w}_e \cdot \mathbf{n}\, q_e - \mathbf{w}_i \cdot \mathbf{n}\, q_i = [\![q]\!]_\Gamma \{\!\{\mathbf{w} \cdot \mathbf{n}\}\!\}_\Gamma + \{\!\{q\}\!\}_\Gamma [\![\mathbf{w} \cdot \mathbf{n}]\!]_\Gamma.$$

This concludes the proof. □

In what follows, for any $D \subset \mathbb{R}^d$ and $I \subset \mathbb{R}$ (for instance $D = \Gamma$, $\overline{\Omega}_i$ or $\overline{\Omega}_e$ and $I = \mathbb{R}$ or $[0, T]$), we shall set (the index c stands for "causal")

$$(2.10)\qquad C_c^\infty(D \times I) = \{g \in C^\infty(D \times I) / g(x, t) = 0 \text{ for } t \leq 0\}.$$

Assuming that (ψ, φ) belongs to $C_c^\infty(\Gamma \times \mathbb{R})^2$, we introduce $(p_{\psi,\varphi}, \mathbf{v}_{\psi,\varphi})$ as the unique solution of the transmission problem

$$(2.11)\qquad \left|\begin{array}{llll} \dfrac{\partial p_{\psi,\varphi}}{\partial t} + \mathrm{div}\, \mathbf{v}_{\psi,\varphi} & = 0 & \text{in } \mathbb{R}^d \setminus \Gamma \times \mathbb{R}^+, & \text{(a)} \\ \dfrac{\partial \mathbf{v}_{\psi,\varphi}}{\partial t} + \nabla p_{\psi,\varphi} & = \mathbf{0} & \text{in } \mathbb{R}^d \setminus \Gamma \times \mathbb{R}^+, & \text{(b)} \\ [\![p_{\psi,\varphi}]\!]_\Gamma & = \psi & \text{on } \Gamma \times \mathbb{R}^+, & \text{(c)} \\ [\![\mathbf{v}_{\psi,\varphi} \cdot \mathbf{n}]\!]_\Gamma & = \varphi & \text{on } \Gamma \times \mathbb{R}^+, & \text{(d)} \\ p_{\psi,\varphi}(\mathbf{x}, 0) & = 0 & \text{in } \mathbb{R}^d \setminus \Gamma, & \text{(e)} \\ \mathbf{v}_{\psi,\varphi}(\mathbf{x}, 0) & = \mathbf{0} & \text{in } \mathbb{R}^d \setminus \Gamma, & \text{(f)} \end{array}\right.$$

that has the regularity (with obvious notation, the letters e and i are set here as superscripts for convenience):

$$(p^i_{\psi,\varphi}, \mathbf{v}^i_{\psi,\varphi}) \in C^\infty_c(\overline{\Omega}_i \times \mathbb{R}) \times \left(C^\infty_c(\overline{\Omega}_i \times \mathbb{R})\right)^d,$$
$$(p^e_{\psi,\varphi}, \mathbf{v}^e_{\psi,\varphi}) \in C^\infty_c(\overline{\Omega}_e \times \mathbb{R}) \times \left(C^\infty_c(\overline{\Omega}_e \times \mathbb{R})\right)^d. \tag{2.12}$$

This allows us to define a boundary operator that associates to the jumps of $(p_{\psi,\varphi}, \mathbf{v}_{\psi,\varphi})$, namely, (ψ,φ), the corresponding mean values of the traces of $(p_{\psi,\varphi}, \mathbf{v}_{\psi,\varphi})$:

$$M_\Gamma \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \equiv M_\Gamma \begin{pmatrix} [\![p_{\psi,\varphi}]\!]_\Gamma \\ [\![\mathbf{v}_{\psi,\varphi} \cdot \mathbf{n}]\!]_\Gamma \end{pmatrix} := \begin{pmatrix} \{\!\{p_{\psi,\varphi}\}\!\}_\Gamma \\ \{\!\{\mathbf{v}_{\psi,\varphi} \cdot \mathbf{n}\}\!\}_\Gamma \end{pmatrix}.$$

In order to exhibit a block decomposition of M_Γ we use linearity of $(\psi,\varphi) \mapsto (p_{\psi,\varphi}, \mathbf{v}_{\psi,\varphi})$ to write

$$\mathbf{v}_{\psi,\varphi} = \mathbf{v}_{\psi,0} + \mathbf{v}_{0,\varphi} \quad \text{and} \quad p_{\psi,\varphi} = p_{\psi,0} + p_{0,\varphi},$$

so that we can rewrite

$$M_\Gamma \Phi = \begin{pmatrix} \mathcal{W}_\Gamma & \mathcal{Z}_\Gamma \\ \mathcal{Y}_\Gamma & \mathcal{W}^*_\Gamma \end{pmatrix} \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \quad \text{where} \quad \Phi = \begin{pmatrix} \psi \\ \varphi \end{pmatrix}, \tag{2.13}$$

where the operators are defined by

$$\left| \begin{array}{llll} \mathcal{Z}_\Gamma \varphi := \{\!\{p_{0,\varphi}\}\!\}_\Gamma & \text{(a)}, & \mathcal{W}_\Gamma \psi := \{\!\{p_{\psi,0}\}\!\}_\Gamma & \text{(b)}, \\ \mathcal{W}^*_\Gamma \varphi := \{\!\{\mathbf{v}_{0,\varphi} \cdot \mathbf{n}\}\!\}_\Gamma & \text{(c)}, & \mathcal{Y}_\Gamma \psi := \{\!\{\mathbf{v}_{\psi,0} \cdot \mathbf{n}\}\!\}_\Gamma & \text{(d)}. \end{array} \right. \tag{2.14}$$

For the purpose of writing the coupled DG-BEM formulation of the problem (2.3)–(2.5), it is also useful to introduce the operator B_Γ defined by

$$B_\Gamma := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} M_\Gamma \equiv \begin{pmatrix} \mathcal{Y}_\Gamma & \mathcal{W}^*_\Gamma \\ \mathcal{W}_\Gamma & \mathcal{Z}_\Gamma \end{pmatrix} \tag{2.15}$$

so that

$$B_\Gamma \begin{pmatrix} [\![p_{\psi,\varphi}]\!]_\Gamma \\ [\![\mathbf{v}_{\psi,\varphi} \cdot \mathbf{n}]\!]_\Gamma \end{pmatrix} = \begin{pmatrix} \{\!\{\mathbf{v}_{\psi,\varphi} \cdot \mathbf{n}\}\!\}_\Gamma \\ \{\!\{p_{\psi,\varphi}\}\!\}_\Gamma \end{pmatrix}.$$

Given a finite interval $[0,T]$, we introduce the associated bilinear form (setting $\Phi = (\psi,\varphi)$, $\widetilde{\Phi} = (\widetilde{\psi}, \widetilde{\varphi})$)

$$b_T(\Phi, \widetilde{\Phi}) := \int_0^T \int_\Gamma (B_\Gamma \Phi, \widetilde{\Phi}) \, \mathrm{d}\gamma \, \mathrm{d}t. \tag{2.16}$$

Using the block decomposition for B_Γ, we obtain

$$b_T((\psi,\varphi),(\widetilde{\psi},\widetilde{\varphi})) = b_T^{\mathcal{Y}}(\psi,\widetilde{\psi})+b_T^{\mathcal{W}^*}(\varphi,\widetilde{\psi})+b_T^{\mathcal{W}}(\psi,\widetilde{\varphi})+b_T^{\mathcal{Z}}(\varphi,\widetilde{\varphi}), \tag{2.17}$$

where

$$\left| \begin{array}{ll} b_T^{\mathcal{Y}}(\psi,\widetilde{\psi}) = \displaystyle\int_0^T\!\!\int_\Gamma \mathcal{Y}_\Gamma\psi\,\widetilde{\psi}\,\mathrm{d}\gamma\,\mathrm{d}t, & b_T^{\mathcal{W}^*}(\varphi,\widetilde{\psi}) = \displaystyle\int_0^T\!\!\int_\Gamma \mathcal{W}_\Gamma^*\varphi\,\widetilde{\psi}\,\mathrm{d}\gamma\,\mathrm{d}t, \\ b_T^{\mathcal{W}}(\psi,\widetilde{\varphi}) = \displaystyle\int_0^T\!\!\int_\Gamma \mathcal{W}_\Gamma\psi\,\widetilde{\varphi}\,\mathrm{d}\gamma\,\mathrm{d}t, & b_T^{\mathcal{Z}}(\varphi,\widetilde{\varphi}) = \displaystyle\int_0^T\!\!\int_\Gamma \mathcal{Z}_\Gamma\varphi\,\widetilde{\varphi}\,\mathrm{d}\gamma\,\mathrm{d}t. \end{array}\right. \tag{2.18}$$

We also see from equations (2.11) and (2.14) that

$$\begin{aligned} &b_T\big((\psi,\varphi),(\widetilde{\psi},\widetilde{\varphi})\big) \\ &\quad= \int_0^T\int_\Gamma \big([\![p_{\tilde{\psi},\tilde{\varphi}}]\!]_\Gamma\{\!\{\mathbf{v}_{\psi,\varphi}\cdot\mathbf{n}\}\!\}_\Gamma + \{\!\{p_{\psi,\varphi}\}\!\}_\Gamma[\![\mathbf{v}_{\tilde{\psi},\tilde{\varphi}}\cdot\mathbf{n}]\!]_\Gamma\big)\,\mathrm{d}\gamma\,\mathrm{d}t. \end{aligned} \tag{2.19}$$

This last formula clearly emphasizes the link between the bilinear form $b_T(\cdot,\cdot)$ and Green's formula (2.9) and explains the introduction of the operator B_Γ.

2.2. Integral operators and associated bilinear forms: Explicit expressions. Since the system in equation (2.11) has constant coefficients, it can be solved using the fundamental solution of the wave equation. As a consequence, for any $(\mathbf{x},t)\in\mathbb{R}^d\backslash\Gamma\times\mathbb{R}^+$, one can obtain the expressions of $p_{\psi,\varphi}(\mathbf{x},t)$ and $\mathbf{v}_{\psi,\varphi}(\mathbf{x},t)$ as integrals on $\Gamma\times[0,T]$ involving φ and ψ. One can then obtain expressions for $\mathcal{Z}_\Gamma\varphi$, $\mathcal{W}_\Gamma\psi$, $\mathcal{W}_\Gamma^*\varphi$ and $\mathcal{Y}_\Gamma\psi$ by computing the interior and exterior traces of $p_{\psi,\varphi}(\mathbf{x},t)$ and $\mathbf{v}_{\psi,\varphi}(\mathbf{x},t)$ and using equation (2.14). These expressions depend upon d, the space dimension.

Below we give the analytic expressions of the bilinear forms in (2.18) on the 3D case, which is the physically relevant one, and for the 1D case, for the purpose of the 1D analysis that will be presented in subsection 4.1.

2.2.1. The 3D case. Let us state the corresponding formulas in the following proposition (we refer the reader to **[32]** for the proof).

Proposition 2.2. *For any $T \geq 0$ and smooth enough functions $\varphi, \widetilde{\varphi}, \psi, \widetilde{\psi} : \Gamma \times \mathbb{R}^+ \to \mathbb{R}$, one has the formulas*

(2.20)

$$
\left|\begin{aligned}
b_T^{\mathcal{Z}}(\varphi,\widetilde{\varphi}) &= \int_0^T \int_\Gamma \int_\Gamma \frac{\dot{\varphi}(\mathbf{y},\tau)}{4\pi|\mathbf{x}-\mathbf{y}|}\,\widetilde{\varphi}(\mathbf{x},t)\,\mathrm{d}\gamma_{\mathbf{y}}\,\mathrm{d}\gamma_{\mathbf{x}}\,\mathrm{d}t,\\
b_T^{\mathcal{W}}(\psi,\widetilde{\varphi}) &= \int_0^T \int_\Gamma \int_\Gamma \frac{\mathbf{n}_{\mathbf{y}}\cdot(\mathbf{x}-\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\left(\frac{\psi(\mathbf{y},\tau)}{|\mathbf{x}-\mathbf{y}|^2}\right.\\
&\qquad\left.+\frac{\dot{\psi}(\mathbf{y},\tau)}{c_0|\mathbf{x}-\mathbf{y}|}\right)\widetilde{\varphi}(\mathbf{x},t)\,\mathrm{d}\gamma_{\mathbf{y}}\,\mathrm{d}\gamma_{\mathbf{x}}\,\mathrm{d}t,\\
b_T^{\mathcal{W}^*}(\varphi,\widetilde{\psi}) &= \int_0^T \int_\Gamma \int_\Gamma \frac{\mathbf{n}_{\mathbf{x}}\cdot(\mathbf{x}-\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\left(\frac{\varphi(\mathbf{y},\tau)}{|\mathbf{x}-\mathbf{y}|^2}\right.\\
&\qquad\left.+\frac{\dot{\varphi}(\mathbf{y},\tau)}{c_0|\mathbf{x}-\mathbf{y}|}\right)\widetilde{\psi}(\mathbf{x},t)\,\mathrm{d}\gamma_{\mathbf{y}}\,\mathrm{d}\gamma_{\mathbf{x}}\,\mathrm{d}t,\\
b_T^{\mathcal{Y}}(\psi,\widetilde{\psi}) &= -\int_0^T \int_\Gamma \int_\Gamma \frac{\psi(\mathbf{y},\tau)\,\dot{\widetilde{\psi}}(\mathbf{x},t)}{4\pi|\mathbf{x}-\mathbf{y}|}\,\mathbf{n}_{\mathbf{x}}\cdot\mathbf{n}_{\mathbf{y}}\,\mathrm{d}\gamma_{\mathbf{x}}\,\mathrm{d}\gamma_{\mathbf{y}}\,\mathrm{d}t\\
&\quad -\int_0^T \int_\Gamma \int_\Gamma \frac{\int_0^t \mathbf{rot}_\Gamma\psi(\mathbf{y},\sigma)\,\mathrm{d}s\cdot\mathbf{rot}_\Gamma\widetilde{\psi}(\mathbf{x},t)}{4\pi|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\gamma_{\mathbf{x}}\,\mathrm{d}\gamma_{\mathbf{y}}\,\mathrm{d}t,
\end{aligned}\right.
$$

where $\tau = t - |\mathbf{x}-\mathbf{y}|$ and $\sigma = s - |\mathbf{x}-\mathbf{y}|$ are retarded times, $\mathbf{rot}_\Gamma$ is the tangential curl operator (see **[33, 45, 46]***) and $\dot{\varphi}$ holds for the time derivative of φ.*

2.2.2. The 1D case. This case is very degenerate from the geometrical point of view since we have for some $a < b$,

$$\Omega_i = (a,b) \quad \text{and} \quad \Omega_e = \mathbb{R} \setminus [a,b],$$

the common boundary to both sub-domains being reduced to two points, $\Gamma = \{a\} \cup \{b\}$. For any function μ defined on Γ, we introduce the notation $\mu(c) = \mu_c$, $c \in \{a,b\}$. The solution to the auxiliary problem (2.11) is obtain through the method of characteristics and is given by

(2.21)

$$
\begin{aligned}
p_{\psi,\varphi}(x,t) = -\frac{\operatorname{sign}(x-a)}{2}\,\psi_a(t-|x-a|) + \frac{\operatorname{sign}(x-b)}{2}\,\psi_b(t-|x-b|)\\
+\frac{1}{2}\,\varphi_a(t-|x-a|) + \frac{1}{2}\,\varphi_b(t-|x-b|),
\end{aligned}
$$

(2.22)

$$v_{\psi,\varphi}(x,t) = \frac{1}{2}\,\psi_b(t-|x-b|) - \frac{1}{2}\,\psi_a(t-|x-a|) + \frac{\operatorname{sign}(x-a)}{2}\,\varphi_a(t-|x-a|) + \frac{\operatorname{sign}(x-b)}{2}\,\varphi_b(t-|x-b|).$$

Thus, using equation (2.14), one gets the expressions:

(2.23)

$$\begin{aligned}
\big(\mathcal{Z}_\Gamma\varphi\big)_a(t) &= \frac{1}{2}\varphi_a(t) + \frac{1}{2}\varphi_b(t-|b-a|),\\
\big(\mathcal{Z}_\Gamma\varphi\big)_b(t) &= \frac{1}{2}\varphi_a(t-|b-a|) + \frac{1}{2}\varphi_b(t),\\
\big(\mathcal{W}_\Gamma\psi\big)_a(t) &= -\frac{1}{2}\psi_b(t-|b-a|),\\
\big(\mathcal{W}_\Gamma\psi\big)_b(t) &= -\frac{1}{2}\psi_a(t-|b-a|),\\
\big(\mathcal{W}^*_\Gamma\varphi\big)_a(t) &= \frac{1}{2}\varphi_b(t-|b-a|),\\
\big(\mathcal{W}^*_\Gamma\varphi\big)_b(t) &= \frac{1}{2}\varphi_a(t-|b-a|),\\
\big(\mathcal{Y}_\Gamma\psi\big)_a(t) &= \frac{1}{2}\psi_a(t) - \frac{1}{2}\psi_b(t-|b-a|),\\
\big(\mathcal{Y}_\Gamma\psi\big)_b(t) &= -\frac{1}{2}\psi_a(t-|b-a|) + \frac{1}{2}\psi_b(t).
\end{aligned}$$

The corresponding bilinear forms are given by

(2.24)

$$\left|\begin{aligned}
b_T^{\mathcal{Z}}(\varphi,\widetilde{\varphi}) &= \frac{1}{2}\int_0^T \big(\varphi_a(s)\widetilde{\varphi}_a(s) \\
&\qquad + \varphi_b(s)\widetilde{\varphi}_b(s) + \varphi_b(s-|b-a|)\widetilde{\varphi}_a(s) + \varphi_a(s-|b-a|)\widetilde{\varphi}_b(s)\big)\,\mathrm{d}s,\\
b_T^{\mathcal{W}}(\psi,\tilde{\varphi}) &= -\frac{1}{2}\int_0^T \big(\psi_b(s-|b-a|)\widetilde{\varphi}_a(s) + \psi_a(s-|b-a|)\widetilde{\varphi}_b(s)\big)\,\mathrm{d}s,\\
b_T^{\mathcal{W}^*}(\varphi,\widetilde{\psi}) &= \frac{1}{2}\int_0^T \big(\varphi_b(s-|b-a|)\widetilde{\psi}_a(s) + \varphi_a(s-|b-a|)\widetilde{\psi}_b(s)\big)\,\mathrm{d}s,\\
b_T^{\mathcal{Y}}(\psi,\widetilde{\psi}) &= \frac{1}{2}\int_0^T \big(\psi_a(s)\widetilde{\psi}_a(s) + \psi_b(s)\widetilde{\psi}_b(s) \\
&\qquad - \psi_b(s-|b-a|)\widetilde{\psi}_a(s) - \psi_a(s-|b-a|)\widetilde{\psi}_b(s)\big)\,\mathrm{d}s.
\end{aligned}\right.$$

2.3. Applications to BEM/DGFEM couplings. As will be recalled in subsection 3.2, the equations on Ω_e, see equation (2.4), can be replaced by the system of boundary integral equations:

$$\left| \begin{aligned} \frac{1}{2}\psi_e &= \mathcal{W}_\Gamma \psi_e + \mathcal{Z}_\Gamma \varphi_e \quad \text{on } \Gamma, \\ \frac{1}{2}\varphi_e &= \mathcal{Y}_\Gamma \psi_e + \mathcal{W}^*_\Gamma \varphi_e \quad \text{on } \Gamma, \end{aligned} \right. \tag{2.25}$$

where the unknowns $(\psi_e, \varphi_e) \equiv (p_{e|\Gamma}, (\mathbf{v}_e \cdot \mathbf{n})_{|\Gamma})$ designate the traces of the exterior solution $(p_e, \mathbf{v}_e)$. Using this notation, the transmission conditions (2.5) can be reformulated as:

$$\left| \begin{aligned} \varphi_e &= \mathbf{v}_i \cdot \mathbf{n}, \quad \text{on } \Gamma, \\ \psi_e &= p_i, \quad \text{on } \Gamma. \end{aligned} \right. \tag{2.26}$$

As a consequence, the coupled PDE system (2.3)–(2.5) is equivalent to the coupled integro-differential system given by (2.3), (2.25) and (2.26). Notice that the latter is set on a bounded domain.

In what follows, we recall the weak formulation of this problem on which the BEM-FEM coupling in **[1]** was based. For simplicity, this formulation is presented in a nonrigorous way without paying attention to the correct functional spaces.

2.3.1. Space variational formulation in Ω_i. This weak formulation is obtained in the spirit of that used for the derivation of discontinuous Galerkin methods for first order PDE systems using central fluxes. To do so, we multiply the first two equations in (2.3) by the corresponding test functions $(\widetilde{p}_i, \widetilde{\mathbf{v}}_i)$ (assumed to be sufficiently smooth functions of the space variables) and integrate in Ω_i to obtain

$$\int_{\Omega_i} \left(\frac{1}{\rho c^2} \frac{\partial p_i}{\partial t} + \operatorname{div} \mathbf{v}_i \right) \widetilde{p}_i \, \mathrm{d}\mathbf{x} = \int_{\Omega_i} (f \widetilde{p}_i \, \mathrm{d}\mathbf{x}),$$
$$\int_{\Omega_i} \left(\rho \frac{\partial \mathbf{v}_i}{\partial t} + \nabla p_i \right) \cdot \widetilde{\mathbf{v}}_i \, \mathrm{d}\mathbf{x} = 0.$$

Splitting the terms involving differential operators in space into two identical parts and integration by parts on one of them, we obtain the following formulation for the interior problem:

$$(2.27)\qquad \left|\begin{array}{l} \displaystyle\int_{\Omega_i}\left(\frac{1}{\rho c^2}\frac{\partial p_i}{\partial t}\widetilde{p}_i+\frac{1}{2}\mathrm{div}\,\mathbf{v}_i\,\widetilde{p}_i-\frac{1}{2}\nabla\widetilde{p}_i\cdot\mathbf{v}_i\right)\mathrm{d}\mathbf{x} \\ \displaystyle\qquad +\frac{1}{2}\int_\Gamma \mathbf{v}_i\cdot\mathbf{n}\,\widetilde{p}_i\,\mathrm{d}\gamma=\int_{\Omega_i} f\,\widetilde{p}_i\,\mathrm{d}\mathbf{x}, \\ \displaystyle\int_{\Omega_i}\left(\rho\frac{\partial \mathbf{v}_i}{\partial t}\cdot\widetilde{\mathbf{v}}_i+\frac{1}{2}\nabla p_i\cdot\widetilde{\mathbf{v}}_i-\frac{1}{2}\mathrm{div}\,\widetilde{v}_i\,p_i\right)\mathrm{d}\mathbf{x} \\ \displaystyle\qquad +\frac{1}{2}\int_\Gamma \widetilde{\mathbf{v}}_i\cdot\mathbf{n}\,p_i\,\mathrm{d}\gamma=0. \end{array}\right.$$

2.3.2. Space-time variational formulation on Γ. In order to obtain a space-time variational formulation of (2.25), these equations are multiplied by test functions $(\widetilde{\varphi}_e,\widetilde{\psi}_e)$ and integrated in space and time as

$$\left|\begin{array}{l} \displaystyle\frac{1}{2}\int_0^T\int_\Gamma \psi_e\,\widetilde{\varphi}_e\,\mathrm{d}\gamma\,\mathrm{d}t=\int_0^T\int_\Gamma\Big(\mathcal{W}_\Gamma\psi_e+\mathcal{Z}_\Gamma\varphi_e\Big)\,\widetilde{\varphi}_e\,\mathrm{d}\gamma\,\mathrm{d}t, \\ \displaystyle\frac{1}{2}\int_0^T\int_\Gamma \varphi_e\,\widetilde{\psi}_e\,\mathrm{d}\gamma\,\mathrm{d}t=\int_0^T\int_\Gamma\Big(\mathcal{Y}_\Gamma\psi_e+\mathcal{W}_\Gamma^*\varphi_e\Big)\widetilde{\psi}_e\,\mathrm{d}\gamma\,\mathrm{d}t. \end{array}\right.$$

Adding both equations, one gets

$$(2.28)\qquad \frac{1}{2}\int_0^T\int_\Gamma\Big(\psi_e\,\widetilde{\varphi}_e\,\mathrm{d}\gamma+\varphi_e\widetilde{\psi}_e\Big)\,\mathrm{d}\gamma\,\mathrm{d}t=b_T((\psi_e,\varphi_e),(\widetilde{\psi}_e,\widetilde{\varphi}_e)).$$

2.3.3. Coupled formulation. It remains to couple equations (2.27) and (2.28) using the transmission conditions in (2.26). In **[1]**, we first proposed replacing ψ_e (respectively φ_e) by p_i (respectively $\mathbf{v}_i\cdot\mathbf{n}$) on the left hand side in equation (2.28). Next, in the boundary terms of (2.27), we proposed replacing p_i (respectively $\mathbf{v}_i\cdot\mathbf{n}$) by ψ_e (respectively φ_e). This led to:

(2.29)

$$\left|\begin{array}{l} \displaystyle\int_{\Omega_i}\left(\frac{1}{\rho c^2}\frac{\partial p_i}{\partial t}\widetilde{p}_i+\frac{1}{2}\mathrm{div}\,\mathbf{v}_i\,\widetilde{p}_i-\frac{1}{2}\nabla\widetilde{p}_i\cdot\mathbf{v}_i\right)\mathrm{d}\mathbf{x}=-\frac{1}{2}\int_\Gamma\varphi_e\,\widetilde{p}_i\,\mathrm{d}\gamma \\ \displaystyle\qquad\qquad\qquad\qquad\qquad\qquad +\int_{\Omega_i} f\,\widetilde{p}_i\,\mathrm{d}\mathbf{x}, \\ \displaystyle\int_{\Omega_i}\left(\rho\frac{\partial \mathbf{v}_i}{\partial t}\cdot\widetilde{\mathbf{v}}_i+\frac{1}{2}\nabla p_i\cdot\widetilde{\mathbf{v}}_i-\frac{1}{2}\mathrm{div}\,\widetilde{v}_i\,p_i\right)\mathrm{d}\mathbf{x}=-\frac{1}{2}\int_\Gamma\widetilde{\mathbf{v}}_i\cdot\mathbf{n}\,\psi_e\,\mathrm{d}\gamma, \\ \displaystyle b_T((\psi_e,\varphi_e),(\widetilde{\psi}_e,\widetilde{\varphi}_e))=\frac{1}{2}\int_0^T\int_\Gamma\Big(p_i\,\widetilde{\varphi}_e\,\mathrm{d}\gamma\;+\;\mathbf{v}_i\cdot\mathbf{n}\,\widetilde{\psi}_e\Big)\,\mathrm{d}\gamma\,\mathrm{d}t. \end{array}\right.$$

2.3.4. Space-time discretization. In [**1**], we proposed a space-time discretization method based on

- space discontinuous Galerkin approximation in space and a explicit centered finite difference method in time for the interior unknowns,
- space-time finite elements (discontinuous in time) for the boundary unknowns ψ_e and φ_e.

One of the key points of the method lay in the way the coupling integral terms in Γ were taken into account at the discrete level. This was done in order to ensure the conservation of some discrete energy (refer to [**1**] for the details). The well posedness of the discrete problem as well as the stability were established using some of the properties of the bilinear for $b_T(\cdot,\cdot)$ that will be presented in subsection 3.3. However, the complete convergence and error analysis remains an open question.

3. General properties of the integral operators. We recall or establish in this section the main mathematical results that play a role in the justification and the analysis of the coupled problem, as well as in the numerical analysis of the method of subsection 2.3. The results of subsections 3.1 and 3.2 are not really original but are recalled here, with elementary proofs, for completeness. They are time domain counterparts of well-known results in the time harmonic case or can be found in the literature (most often for the second order wave equation rather than for the first order system version).

The results of subsections 3.3, 3.4 and 3.5 are new for most of them, either by their own content or by the method used for obtaining them, directly in the time domain, which make them easier to extend to discretized problems.

3.1. Calderon projector properties.

Proposition 3.1. *The operator M_Γ satisfies $M_\Gamma^2 = (1/4)I$, which is equivalent to*

$$\mathcal{W}_\Gamma^2 + \mathcal{Z}_\Gamma \mathcal{Y}_\Gamma = \frac{1}{4} I, \qquad \mathcal{W}_\Gamma \mathcal{Z}_\Gamma + \mathcal{Z}_\Gamma \mathcal{W}_\Gamma^* = 0,$$
$$\mathcal{Y}_\Gamma \mathcal{W}_\Gamma + \mathcal{W}_\Gamma^* \mathcal{Y}_\Gamma = 0, \qquad \mathcal{Y}_\Gamma \mathcal{Z}_\Gamma + \mathcal{W}_\Gamma^{*2} = \frac{1}{4} I. \tag{3.1}$$

As a consequence, the operators $(1/2)I \pm M_\Gamma$ *are projectors.*

Proof. Let $(\psi,\varphi) \in C_c^\infty(\Gamma\times\mathbb{R})^2$, by definition of M_Γ, we have

$$(\psi_1,\varphi_1) := M_\Gamma(\psi,\varphi) = \big(\{\!\{p_{\psi,\varphi}\}\!\}_\Gamma, \{\!\{\mathbf{v}_{\psi,\varphi}\cdot\mathbf{n}\}\!\}_\Gamma\big),$$
$$(\psi_2,\varphi_2) := M_\Gamma^2(\psi,\varphi) = M_\Gamma(\psi_1,\varphi_1) = \big(\{\!\{p_{\psi_1,\varphi_1}\}\!\}_\Gamma, \{\!\{\mathbf{v}_{\psi_1,\varphi_1}\cdot\mathbf{n}\}\!\}_\Gamma\big).$$

We have to prove that $(\psi_2,\varphi_2) = (\psi/4,\varphi/4)$. Let us define $(p^a_{\psi,\varphi},\mathbf{v}^a_{\psi,\varphi})$ in $\mathbb{R}^d\setminus\Gamma\times\mathbb{R}^+$ by

$$(p^a_{\psi,\varphi},\mathbf{v}^a_{\psi,\varphi})|_{\Omega_e} = (p_{\psi,\varphi},\mathbf{v}_{\psi,\varphi})|_{\Omega_e},$$
$$(p^a_{\psi,\varphi},\mathbf{v}^a_{\psi,\varphi})|_{\Omega_i} = -(p_{\psi,\varphi},\mathbf{v}_{\psi,\varphi})|_{\Omega_i}.$$

By linearity, $(p^a_{\psi,\varphi},\mathbf{v}^a_{\psi,\varphi})$ obviously solves the wave equation (2.11) (a), (b) with zero initial conditions. Moreover, by definition of the mean and jump operators (2.7), we have:

$$[\![p^a_{\psi,\varphi}]\!]_\Gamma = 2\,\{\!\{p_{\psi,\varphi}\}\!\}_\Gamma = 2\,\psi_1,$$
$$[\![\mathbf{v}^a_{\psi,\varphi}\cdot\mathbf{n}]\!]_\Gamma = 2\,\{\!\{\mathbf{v}_{\psi,\varphi}\cdot\mathbf{n}\}\!\}_\Gamma = 2\,\varphi_1.$$

By well-posedness of the transmission problem (2.11), we thus deduce that $(p^a_{\psi,\varphi},\mathbf{v}^a_{\psi,\varphi}) = 2\,(p_{\psi_1,\varphi_1},\mathbf{v}_{\psi_1,\varphi_1})$. Therefore, again using the mean and jump operators, we get

$$\psi_2 = \{\!\{p_{\psi_1,\varphi_1}\}\!\}_\Gamma = \frac{1}{2}\{\!\{p^a_{\psi,\varphi}\}\!\}_\Gamma = \frac{1}{4}[\![p_{\psi,\varphi}]\!]_\Gamma = \frac{\psi}{4},$$
$$\varphi_2 = \{\!\{v_{\psi_1,\varphi_1}\cdot\mathbf{n}\}\!\}_\Gamma = \frac{1}{2}\{\!\{v^a_{\psi,\varphi}\cdot\mathbf{n}\}\!\}_\Gamma = \frac{1}{4}[\![v_{\psi,\varphi}\cdot\mathbf{n}]\!]_\Gamma = \frac{\varphi}{4}.$$

This shows the first part of the theorem. Finally, we remark that

$$\left(\frac{1}{2}I \pm M_\Gamma\right)^2 = \frac{1}{4}I + M_\Gamma^2 \pm M_\Gamma = \frac{1}{2}I \pm M_\Gamma,$$

as claimed. □

3.2. Transparent conditions. We wish to characterize the subset of $C_c^\infty(\Gamma\times\mathbb{R})^2$ of functions (ψ_e,φ_e) which are traces on Γ of a solution $(p_e,\mathbf{v}_e)$ of the wave equation in the exterior domain Ω_e with zero initial conditions, that is, solution of (2.4).

Let us denote $(p,\mathbf{v})$ as the extension of $(p_e,\mathbf{v}_e)$ by 0 inside Ω_i. By construction, $[\![p]\!]_\Gamma = \psi_e$, $[\![\mathbf{v}\cdot\mathbf{n}]\!]_\Gamma = \varphi_e$, while $\{\!\{p\}\!\}_\Gamma = \psi_e/2$, $\{\!\{\mathbf{v}\cdot\mathbf{n}\}\!\}_\Gamma =$

$\varphi_e/2$. Since $(p, \mathbf{v})$ obviously satisfies equation (2.11) (a), (b), (e) and (f), we have

$$M_\Gamma \begin{pmatrix} \psi_e \\ \varphi_e \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \psi_e \\ \varphi_e \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \psi_e \\ \varphi_e \end{pmatrix} \in \operatorname{Ker} \left(\frac{1}{2} I - M_\Gamma \right), \tag{3.2}$$

which means that (ψ_e, φ_e) are related by the linear equations

$$\text{(a)} \quad \frac{1}{2}\, \psi_e = \mathcal{W}_\Gamma \psi_e + \mathcal{Z}_\Gamma \varphi_e, \qquad \text{(b)} \quad \frac{1}{2}\, \varphi_e = \mathcal{Y}_\Gamma \psi_e + \mathcal{W}_\Gamma^* \varphi_e. \tag{3.3}$$

Since each of conditions (3.3) (a) and (3.3) (b) are satisfied by any couple of traces of the solution of a homogeneous wave equation, we call them *transparent conditions.* In fact, only one of them is sufficient to characterize the kernel of $I/2 - M_\Gamma$. This is a consequence of Proposition 3.1 and the injectivity of the operators $\mathcal{Z}_\Gamma$ and $\mathcal{Y}_\Gamma$ (that will be demonstrated in subsection 3.4, see Remark 3.12).

Proposition 3.2. *The two transparent conditions* (3.3) (a) *and* (3.3) (b) *are equivalent. Moreover, each is equivalent to saying that* (ψ_e, φ_e) *are traces on* Γ *of a solution* $(p_e, \mathbf{v}_e)$ *of the wave equation in the exterior domain* Ω_e *with zero initial conditions, that is, equation* (2.4)*.*

Proof. We show that equation (3.3) (a) implies equation (3.3) (b), the reverse statement being proved in the same way.

Let us remark that the first two identities of (3.1) can be rewritten:

$$\begin{aligned} \frac{1}{2} I - \mathcal{W}_\Gamma &= \left(\frac{1}{2} I - \mathcal{W}_\Gamma \right)^2 + \mathcal{Z}_\Gamma \mathcal{Y}_\Gamma, \\ \mathcal{Z}_\Gamma &= \left(\frac{1}{2} I - \mathcal{W}_\Gamma \right) \mathcal{Z}_\Gamma + \mathcal{Z}_\Gamma \left(\frac{1}{2} I - \mathcal{W}_\Gamma^* \right). \end{aligned} \tag{3.4}$$

Then, we observe that

$$\begin{aligned} \text{(3.3) (a)} \Longleftrightarrow & \left(\frac{1}{2} I - \mathcal{W}_\Gamma \right) \psi_e - \mathcal{Z}_\Gamma\, \varphi_e = 0 \\ \Longrightarrow & \left(\frac{1}{2} I - \mathcal{W}_\Gamma \right)^2 \psi_e + \mathcal{Z}_\Gamma \mathcal{Y}_\Gamma\, \psi_e - \left(\frac{1}{2} I - \mathcal{W}_\Gamma \right) \mathcal{Z}_\Gamma\, \varphi_e \\ & - \mathcal{Z}_\Gamma \left(\frac{1}{2} I - \mathcal{W}_\Gamma^* \right) \varphi_e = 0 \quad \text{(by (3.4))} \end{aligned}$$

$$\begin{aligned}\iff \left(\frac{1}{2}I - \mathcal{W}_\Gamma\right)\left(\left(\frac{1}{2}I - \mathcal{W}_\Gamma\right)\psi_e - \mathcal{Z}_\Gamma\,\varphi_e\right)\\ + \mathcal{Z}_\Gamma\left(\mathcal{Y}_\Gamma\,\psi_e - \left(\frac{1}{2}I - \mathcal{W}_\Gamma^*\right)\varphi_e\right) = 0\\ \implies \mathcal{Z}_\Gamma\left(\mathcal{Y}_\Gamma\,\psi_e - \left(\frac{1}{2}I - \mathcal{W}_\Gamma^*\right)\varphi_e\right) = 0\end{aligned}$$

(by (3.3) (a)) $\implies$ (3.3) (b) (by injectivity of $\mathcal{Z}_\Gamma$).

Next, assume that (ψ_e, φ_e) satisfies (3.3) (a) and introduce $(p_e, \mathbf{v}_e)$ as the solution of the wave equation in the exterior domain Ω_e with zero initial conditions (that is, (2.4)) and Dirichlet condition $p_e|_\Gamma = \psi_e$. Letting $\varphi_e' := \mathbf{v}_e \cdot \mathbf{n}|_\Gamma$, we know (see first part of this section) that $\psi_e/2 = \mathcal{W}_\Gamma\psi_e + \mathcal{Z}_\Gamma\varphi_e'$. Taking the difference of this last equation with (3.3) (a), we get $\mathcal{Z}_\Gamma(\varphi_e' - \varphi_e) = 0$. Therefore, by injectivity of $\mathcal{Z}_\Gamma$, $\varphi_e = \varphi_e' = \mathbf{v}_e \cdot \mathbf{n}|_\Gamma$. □

3.3. Basic properties of $b(\cdot,\cdot)$. Our first result concerns the positivity of the bilinear form $b_T(\cdot,\cdot)$, which appears to be a straightforward consequence of the energy identity for the wave equation.

Proposition 3.3. *For any $T > 0$, the bilinear form $b_T(\cdot,\cdot)$ is nonnegative: for any $(\psi,\varphi) : \Gamma \times \mathbb{R}^+ \mapsto \mathbb{R} \times \mathbb{R}$,*

$$b_T\big((\psi,\varphi),(\psi,\varphi)\big) = \frac{1}{2}\int_{\mathbb{R}^d\setminus\Gamma} \left(\,|p_{\psi,\varphi}(\mathbf{x},T)|^2 + |\mathbf{v}_{\psi,\varphi}(\mathbf{x},T)|^2\,\right) \mathrm{d}\mathbf{x}, \tag{3.5}$$

where $(p_{\psi,\varphi}, \mathbf{v}_{\psi,\varphi})$ is the solution of (2.11).

Proof. Take the inner product in $\mathbb{R}^3$ of (2.11) (b) with $\mathbf{v}_{\psi,\varphi}$, multiply (2.11) (a) by $p_{\psi,\varphi}$, add the two equalities and integrate the result over $\Omega_i \cup \Omega_e$. One obtains

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d\setminus\Gamma} \left(\,|p_{\psi,\varphi}|^2 + |\mathbf{v}_{\psi,\varphi}|^2\,\right)\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d\setminus\Gamma} \big(\mathrm{div}\,\mathbf{v}_{\psi,\varphi}\,p_{\psi,\varphi} + \mathbf{v}_{\psi,\varphi}\cdot\nabla p_{\psi,\varphi}\big)\,\mathrm{d}\mathbf{x} = 0. \tag{3.6}$$

Then, using Green's formula (2.9) yields

$$\begin{aligned}(3.7)\quad \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d\setminus\Gamma} \left(\left|p_{\psi,\varphi}\right|^2 + \left|\mathbf{v}_{\psi,\varphi}\right|^2\right) \mathrm{d}\mathbf{x} \\ = \int_\Gamma \left([\![p_{\psi,\varphi}]\!]_\Gamma \{\!\{\mathbf{v}_{\psi,\varphi}\cdot\mathbf{n}\}\!\}_\Gamma + \{\!\{p_{\psi,\varphi}\}\!\}_\Gamma [\![\mathbf{v}_{\psi,\varphi}\cdot\mathbf{n}]\!]_\Gamma\right) \mathrm{d}\gamma.\end{aligned}$$

Substituting (2.11) (c), (d) and (2.15) into (3.7) then gives

$$\begin{aligned}(3.8)\quad \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d\setminus\Gamma} \left(\left|p_{\psi,\varphi}\right|^2 + \left|\mathbf{v}_{\psi,\varphi}\right|^2\right) \mathrm{d}\mathbf{x} \\ = \int_\Gamma \left(\mathcal{Z}_\Gamma\varphi\,\varphi + \mathcal{W}_\Gamma\psi\,\varphi + \mathcal{W}_\Gamma^*\varphi\,\psi + \mathcal{Y}_\Gamma\psi\,\psi\right) \mathrm{d}\gamma.\end{aligned}$$

Finally, integrating (3.8) in time between 0 and T leads to (3.5), taking into account (2.17), (2.19) and the initial conditions (2.11) (e), (f). □

A natural subsequent question to Proposition 3.3 is to know whether $b_T((\psi,\varphi),(\varphi,\psi))$ is positive definite, in other words, whether it defines the square of a norm.

Curiously, addressing this question is far from obvious and one has two different results depending on the fact whether the time T is small or large. Let us begin with a positive result when T is small enough. For this, we need to introduce some notation. For each $T>0$, we introduce the set of points in $\mathbb{R}^d$ whose distance to Γ is less than T, as well as its complement in $\mathbb{R}^d$ (see Figure 1 on the right):

$$(3.9)\quad \begin{aligned}\Omega_T(\Gamma) &:= \left\{\mathbf{x}\in\mathbb{R}^d\setminus\Gamma / d(\mathbf{x},\Gamma) < T\right\},\\ \mathcal{O}_T(\Gamma) &:= \mathbb{R}^d \setminus \overline{\Omega_T(\Gamma)},\end{aligned}$$

which obviously satisfy

(3.10)

$$\begin{aligned}\Omega_T(\Gamma) &= \Omega_T^e(\Gamma)\cup\Omega_T^i(\Gamma), \qquad \Omega_T^\ell(\Gamma) = \Omega_T(\Gamma)\cap\Omega_\ell, \quad \ell\in\{i,e\},\\ \mathcal{O}_T(\Gamma) &= \mathcal{O}_T^e(\Gamma)\cup\mathcal{O}_T^i(\Gamma), \qquad \mathcal{O}_T^\ell(\Gamma) = \mathcal{O}_T(\Gamma)\cap\Omega_\ell, \quad \ell\in\{i,e\}.\end{aligned}$$

By construction, the sets $\mathcal{O}_T^e(\Gamma)$ and $\mathcal{O}_T^i(\Gamma)$ decrease when T increases. Since Ω_e is unbounded, $\mathcal{O}_T^e(\Gamma)$ itself remains unbounded for all $T>0$, and thus non-empty. For $\mathcal{O}_T^i(\Gamma)$ this is only true for T small enough, which is why we introduce:

$$T^*(\Omega_i) = \sup\big\{T > 0 / \mathcal{O}_T^i(\Gamma) \neq \emptyset\big\}, \tag{3.11}$$

that satisfies $0 < T^*(\Omega_i) \leq \operatorname{diam}(\Omega_i)$, (note that $T^*(\Omega_i) = \operatorname{diam}(\Omega_i)$ when Ω_i is a ball).

Proposition 3.4. *For any $0 < T < 2\,T^*(\Omega_i)$, the bilinear form $b_T(\cdot,\cdot)$ is positive definite:*

$$b_T\big((\psi,\varphi),(\psi,\varphi)\big) = 0 \Longrightarrow \varphi(\mathbf{x},t) = \psi(\mathbf{x},t) = 0, \qquad \mathbf{x} \in \Gamma, \quad t \in [0,T]. \tag{3.12}$$

Proof. The idea of the following proof has been suggested to us by Lebeau **[38]**. According to equation (3.5), we have to show that

$$\begin{aligned} p_{\psi,\varphi}(\mathbf{x},T) = 0, \qquad \mathbf{v}_{\psi,\varphi}(\mathbf{x},T) = \mathbf{0}, \quad \text{for all } \mathbf{x} \in \mathbb{R}^d \setminus \Gamma, \\ \Longrightarrow \varphi(\mathbf{x},t) = \psi(\mathbf{x},t) = 0, \quad \mathbf{x} \in \Gamma,\ t \in [0,T]. \end{aligned}$$

We first note that, due to the finite propagation velocity of the wave equation

$$\operatorname{supp} p_{\psi,\varphi}(\cdot,t) \cup \operatorname{supp} \mathbf{v}_{\psi,\varphi}(\cdot,t) \subset \Omega_t(\Gamma), \quad \text{for all } t > 0.$$

Similarly, by time reversibility, changing t in $T-t$ and using $p_{\psi,\varphi}(\mathbf{x},T) = 0$, $\mathbf{v}_{\psi,\varphi}(\mathbf{x},T) = \mathbf{0}$, for all $\mathbf{x} \in \mathbb{R}^3 \setminus \Gamma$, we have

$$\operatorname{supp} p_{\psi,\varphi}(\cdot,t) \cup \operatorname{supp} \mathbf{v}_{\psi,\varphi}(\cdot,t) \subset \Omega_{T-t}(\Gamma), \quad \text{for all } t \in\,]0,T[\,.$$

Therefore, for all $t \in\,]0,T[$,

$$\operatorname{supp} p_{\psi,\varphi}(\cdot,t) \cup \operatorname{supp} \mathbf{v}_{\psi,\varphi}(\cdot,t) \subset \Omega_t(\Gamma) \cap \Omega_{T-t}(\Gamma),$$

and in particular, for all $t \in\,]0,T[$,

$$\operatorname{supp} p_{\psi,\varphi}(\cdot,t) \cup \operatorname{supp} \mathbf{v}_{\psi,\varphi}(\cdot,t) \subset \Omega_{T/2}(\Gamma). \tag{3.13}$$

Let us consider $\widetilde{p}_{\psi,\varphi}(\mathbf{x},t) : \mathbb{R}^3 \setminus \Gamma \times \mathbb{R} \to \mathbb{R}$ and $\widetilde{\mathbf{v}}_{\psi,\varphi}(\mathbf{x},t) : \mathbb{R}^3 \setminus \Gamma \times \mathbb{R} \to \mathbb{R}^3$ defined for all times by extension by 0:

$$\begin{cases} \widetilde{p}_{\psi,\varphi}(\mathbf{x},t) = p_{\psi,\varphi}(\mathbf{x},t) & \text{if } t \in\,]0,T[, \\ \widetilde{p}_{\psi,\varphi}(\mathbf{x},t) = 0 & \text{if } t \in\,]-\infty,0[\, \cup\,]T,+\infty[, \\ \widetilde{\mathbf{v}}_{\psi,\varphi}(\mathbf{x},t) = \mathbf{v}_{\psi,\varphi}(\mathbf{x},t) & \text{if } t \in\,]0,T[, \\ \widetilde{\mathbf{v}}_{\psi,\varphi}(\mathbf{x},t) = 0 & \text{if } t \in\,]-\infty,0[\, \cup\,]T,+\infty[. \end{cases}$$

Since ψ and φ are traces of $\widetilde{p}_{\psi,\varphi}$ and $\widetilde{\mathbf{v}}_{\psi,\varphi}$, the proof will be achieved if we show that $\widetilde{p}_{\psi,\varphi}$ and $\widetilde{\mathbf{v}}_{\psi,\varphi}$ vanish identically.

Because of the zero initial ($t=0$) and final ($t=T$) conditions satisfied by $(p_{\psi,\varphi}, \mathbf{v}_{\psi,\varphi})$, $(\widetilde{p}_{\psi,\varphi}, \widetilde{\mathbf{v}}_{\psi,\varphi})$ satisfy equations (2.11) (a), (b) in $\mathbb{R}^d \setminus \Gamma \times \mathbb{R}$. Thus, the Fourier transforms in time of $\widetilde{p}_{\psi,\varphi}$ and $\widetilde{\mathbf{v}}_{\psi,\varphi}$, namely,

$$\widehat{p}_{\psi,\varphi}(\mathbf{x},\tau) = \int_{-\infty}^{+\infty} \widetilde{p}_{\psi,\varphi}(\mathbf{x},t) e^{-i\tau t}\,\mathrm{d}t,$$
$$\widehat{\mathbf{v}}_{\psi,\varphi}(\mathbf{x},\tau) = \int_{-\infty}^{+\infty} \widetilde{\mathbf{v}}_{\psi,\varphi}(\mathbf{x},t) e^{-i\tau t}\,\mathrm{d}t,$$

satisfy, for each $\tau \in \mathbb{R}$,
(3.14)
$$\begin{cases} \operatorname{div} \widehat{\mathbf{v}}_{\psi,\varphi} + i\tau \widehat{p}_{\psi,\varphi} = 0 & \text{in } \mathbb{R}^3 \setminus \Gamma, \\ \nabla \widehat{p}_{\psi,\varphi} + i\tau \widehat{\mathbf{v}}_{\psi,\varphi} = 0 & \text{in } \mathbb{R}^3 \setminus \Gamma. \end{cases} \Longrightarrow \Delta\, \widehat{p}_{\psi,\varphi} + \tau^2 \widehat{p}_{\psi,\varphi} = 0, \text{ in } \mathbb{R}^3 \setminus \Gamma.$$

On the other hand, we deduce from equation (3.13) that for all $\tau \in \mathbb{R}$,

$$\operatorname{supp} \widehat{p}_{\psi,\varphi}(\cdot,\tau) \subset \Omega_{T/2}(\Gamma).$$

In particular, $\widehat{p}_{\psi,\varphi}$ vanishes in $\mathcal{O}^e_{T/2}(\Gamma)$ and $\mathcal{O}^i_{T/2}(\Gamma)$, which are both nonempty open sets since $T < 2T^*(\Omega_i)$. As Ω_i and Ω_e are connected, we can use a unique continuation argument (Holmgren's theorem **[21, 22]**) to assert that:

$$\widehat{p}_{\psi,\varphi} = 0, \quad \text{in } \mathbb{R}^3 \setminus \Gamma, \quad \text{which implies (cf., (3.14)) } \widehat{\mathbf{v}}_{\psi,\varphi} = 0, \text{ in } \mathbb{R}^3 \setminus \Gamma.$$

This concludes the proof. □

Remark 3.5. Proposition 3.4 plays an important role in the numerical analysis of the method developed in **[1]** and that we briefly recalled in subsection 2.3.4. Indeed, the well posedness of the discrete problem relies on the positive definiteness of $b_{\Delta t}(\cdot,\cdot)$, where Δt is the time step. Thus, Proposition 3.4 gives an upper bound for the time step (whatever the space discretization) to be satisfied in order to ensure the well posedness of the method (also see **[34]** for analogous discussions).

Remark 3.6. Once we know that, for $T < 2T^*(\Omega_i)$, $(\psi,\varphi) \mapsto b_T((\psi,\varphi),(\psi,\varphi))$ defines the square of a norm, it is natural to wonder whether this norm is equivalent to a more standard norm, of Sobolev type for instance. Except in the case of the dimension 1, as we shall see in subsection 4.1, the answer is negative. It can be shown, at least in the case of a flat boundary Γ, that this is a very weak norm in the sense that it cannot be bounded from below by any (even arbitrarily negative) Sobolev norm (see **[2]** for instance).

As stated earlier, the fact that T must be small enough to ensure that b_T is positive definite is not only a technical convenience. This is also a necessary condition. Let us set:

$$\mathcal{P} = \big\{T > 0 / b_T(\cdot,\cdot) \text{ is positive definite}\big\}. \tag{3.15}$$

By definition of $\mathcal{P}$,

$$T \in \mathbb{R}^+ \setminus \mathcal{P} \Longleftrightarrow \mathcal{N}(b_T) := \big\{(\psi,\varphi)|_{[0,T]} / b_T\big((\psi,\varphi),(\psi,\varphi)\big) = 0\big\}$$

(kernel of b_T) is different from $\{0\}$.

According to the proof of Proposition 3.4, we know that the kernel $\mathcal{N}(b_T)$ is also defined by

$$\mathcal{N}(b_T) = \big\{(\psi,\varphi)|_{[0,T]} / p_{\psi,\varphi}(\mathbf{x},T) = 0,\ \mathbf{v}_{\psi,\varphi}(\mathbf{x},T) = \mathbf{0} \text{ for all } \mathbf{x} \in \mathbb{R}^3\backslash\Gamma\big\}. \tag{3.16}$$

It is clear that, for $T_2 > T_1$, $\mathcal{N}(b_{T_2}) \supset \mathcal{N}(b_{T_1})$ (note that, if $(\psi,\varphi) \in \mathcal{N}(b_{T_1})$, then (ψ_*,φ_*), the extension of (ψ,φ) by 0 in $]T_1,T_2]$, belongs to $\mathcal{N}(b_{T_2})$). Thus, there exists $T_{\max}(\Omega_i) > 0$ such that

$$\mathcal{P} = \,]0, T_{\max}(\Omega_i)[\,. \tag{3.17}$$

From Proposition 3.4, we know that

$$T_{\max}(\Omega_i) \geq 2\,T^*(\Omega_i). \tag{3.18}$$

The question of obtaining a good upper bound for $T_{\max}(\Omega_i) > 0$ is clearly related to the boundary controllability theory for the wave equation inside Ω_i. Let us recall that the wave equation in Ω_i is controllable from the boundary with Dirichlet data in time T if and only if, for any initial data $(p_0,\mathbf{v}_0)$, there exists a boundary data ψ defined on $\Gamma \times [0,T]$ such that $(p_\psi,\mathbf{v}_\psi)$, the solution of the following problem

$$\left|\begin{array}{ll} \dfrac{\partial p_\psi}{\partial t} + \text{div}\mathbf{v}_\psi = 0 & \text{in } \Omega_i \times [0,T], \\ \dfrac{\partial \mathbf{v}_\psi}{\partial t} + \nabla p_\psi = \mathbf{0} & \text{in } \Omega_i \times [0,T], \\ p_\psi = -\psi & \text{on } \Gamma \times [0,T], \\ p_\psi(\mathbf{x},0) = p_0 & \text{in } \Omega_i, \\ \mathbf{v}_\psi(\mathbf{x},0) = \mathbf{v}_0 & \text{in } \Omega_i, \end{array}\right.$$

satisfies $(p_\psi(T), \mathbf{v}_\psi(T))) = (0, \mathbf{0})$. We can then introduce

(3.19)

$$T_c(\Omega_i) = \inf\big\{T > 0/\text{the wave equation in } \Omega_i \text{ is controllable in time } T \text{ from Dirichlet data on } \Gamma\big\}.$$

It is well known that $T_c(\Omega_i)$ is linked to the geometry of Ω_i and that, in particular, $T_c(\Omega_i) \geq \text{diam}\,(\Omega_i)$, cf., **[39, 12]**.

Proposition 3.7. *For any $T > T_c(\Omega_i)$, the kernel $\mathcal{N}(b_T)$ of the bilinear form $b_T(\cdot,\cdot)$ is non trivial, namely, $T_{\max}(\Omega_i) \leq T_c(\Omega_i)$.*

Proof. Indeed, let $T_0 > 0$ and $\psi_0 : \Gamma \times [0, T_0] \to \mathbb{R}$ with $\psi_0 \neq 0$. Let $(p_{\psi_0}, \mathbf{v}_{\psi_0}) : \Omega_i \times [0, T_0] \to \mathbb{R} \times \mathbb{R}^d$ be the solution of the wave equation (2.11) (a), (b) inside $\Omega_i \times [0, T_0]$ with zero initial data and non homogeneous Dirichlet condition $p_{\psi_0} = -\psi_0$ on $\Gamma \times [0, T_0]$. Let us denote the solution at the final time $t = T_0$ as

$$p_0(\mathbf{x}) := p_{\psi_0}(\mathbf{x}, T_0), \qquad \mathbf{v}_0(\mathbf{x}) := \mathbf{v}_{\psi_0}(\mathbf{x}, T_0), \quad \text{for all } \mathbf{x} \in \Omega_i.$$

By definition of $T_c(\Omega_i)$, one can find a Dirichlet boundary control $\psi_0^c : \Gamma \times [T_0, T_0 + T_c(\Omega_i)] \to \mathbb{R}$ such that $(p_0^c, \mathbf{v}_0^c) : \Omega_i \times [T_0, T_0 + T_c(\Omega_i)] \to \mathbb{R} \times \mathbb{R}^d$, defined as the solution of wave equation (2.11) (a), (b) in $\Omega_i \times [T_0, T_0 + T_c(\Omega_i)]$ with initial data $(p_0, \mathbf{v}_0)$ at $t = T_0$ and Dirichlet condition $p_0^c = -\psi_0^c$ on $\Gamma \times [T_0, T_0 + T_c(\Omega_i)]$, vanishes at the final time $T_0 + T_c(\Omega_i)$:

(3.20)

$$p_0^c\big(\mathbf{x}, T_0 + T_c(\Omega_i)\big) = 0, \qquad \mathbf{v}_0^c\big(\mathbf{x}, T_0 + T_c(\Omega_i)\big) = \mathbf{0}, \quad \text{for all } \mathbf{x} \in \Omega_i.$$

Let $(\psi, \varphi) : \Gamma \times [0, T_0 + T_c(\Omega_i)]$ be defined (note that $\psi_0 \neq 0 \Rightarrow (\psi, \varphi) \neq 0$) by

$$\psi|_{[0,T_0]} = \psi_0, \qquad \psi|_{[T_0, T_0 + T_c(\Omega_i)]} = \psi_0^c,$$
$$\varphi|_{[0,T_0]} = -\mathbf{v}_{\psi_0} \cdot \mathbf{n}|_\Gamma, \qquad \varphi|_{[T_0, T_0 + T_c(\Omega_i)]} = -\mathbf{v}_0^c \cdot \mathbf{n}|_\Gamma,$$

and let $(p, \mathbf{v}) : \mathbb{R}^d \times [0, T_0 + T_c(\Omega_i)]$ be defined by

$$\begin{cases} p|_{\Omega_i \times [0,T_0]} = p_{\psi_0}, & \mathbf{v}|_{\Omega_i \times [0,T_0]} = \mathbf{v}_{\psi_0}, \\ p|_{\Omega_i \times [T_0, T_0+T_c(\Omega_i)]} = p_0^c, & \mathbf{v}|_{\Omega_i \times [T_0,T_0+T_c(\Omega_i)]} = \mathbf{v}_0^c, \\ p|_{\Omega_e \times [0,T_0+T_c(\Omega_i)]} = 0, & \mathbf{v}|_{\Omega_e \times [0,T_0+T_c(\Omega_i)]} = 0. \end{cases}$$

By construction, (p, v) satisfies the wave equation (2.11) (a), (b) in $\mathbb{R}^d \setminus \Gamma \times [0, T_0 + T_c(\Omega_i)]$ with jump conditions

$$[\![p]\!]_\Gamma = \psi, \quad [\![\mathbf{v} \cdot \mathbf{n}]\!]_\Gamma = \varphi, \quad t \in [0, T_0 + T_c(\Omega_i)],$$

which means that $p = p_{\psi,\varphi}$ and $\mathbf{v} = \mathbf{v}_{\psi,\varphi}$. Then equation (3.20) reads $p_{\psi,\varphi}(\mathbf{x}, T_0 + T_c(\Omega_i)) = 0$, $\mathbf{v}_{\psi,\varphi}(\mathbf{x}, T_0 + T_c(\Omega_i)) = 0$, i.e., according to equation (3.16), shows that $(\psi, \varphi) \in \mathcal{N}(b_T)$ with $T = T_0 + T_c(\Omega_i)$. Thus, $T_0 + T_c(\Omega_i)$ belongs to $[T_{\max}(\Omega_i), +\infty[$. As this is true for any $T_0 > 0$, the proof is complete. □

3.4. The weighted bilinear form and its properties. Let us introduce a positive weight function

$$\omega(t) \in C^1(0, T), \quad \omega(t) > 0, \text{ for all } t \in [0, T], \ \omega(0) = 1, \tag{3.21}$$

and let us set

$$\mu(t) := -\omega'(t). \tag{3.22}$$

We then define the corresponding weighted bilinear form associated to the operator B_Γ (denoting $\Phi = (\psi, \varphi)$) by

$$b_{\omega,T}(\Phi, \widetilde{\Phi}) := \int_0^T \int_\Gamma (B_\Gamma \Phi, \widetilde{\Phi}) \omega(t) \, \mathrm{d}\gamma \, \mathrm{d}t, \tag{3.23}$$

or equivalently, (see subsection 2.1)

(3.24)

$$\begin{aligned} & b_{\omega,T}\big((\psi, \varphi), (\widetilde{\psi}, \widetilde{\varphi})\big) \\ & \quad = \int_0^T \int_\Gamma \big(\mathcal{Y}_\Gamma \psi \, \widetilde{\psi} + \mathcal{W}_\Gamma^* \varphi \, \widetilde{\psi} + \mathcal{W}_\Gamma \psi \, \widetilde{\varphi} + \mathcal{Z}_\Gamma \varphi \, \widetilde{\varphi}\big) \omega(t) \, \mathrm{d}\gamma \, \mathrm{d}t \\ & \quad = \int_0^T \int_\Gamma \big([\![p_{\tilde{\psi},\tilde{\varphi}}]\!]_\Gamma \{\!\{\mathbf{v}_{\psi,\varphi} \cdot \mathbf{n}\}\!\}_\Gamma + \{\!\{p_{\psi,\varphi}\}\!\}_\Gamma [\![\mathbf{v}_{\tilde{\psi},\tilde{\varphi}} \cdot \mathbf{n}]\!]_\Gamma\big) \omega(t) \, \mathrm{d}\gamma \, \mathrm{d}t. \end{aligned}$$

Of course, when $\omega(t) \equiv 1$, one obtains $b_{\omega,T} \equiv b_T$. We now state an extension of Proposition 3.3:

Proposition 3.8. *For any* $(\psi,\varphi) : \Gamma \times \mathbb{R}^+ \mapsto \mathbb{R}\times\mathbb{R}$,

$$b_{\omega,T}\big((\psi,\varphi),(\psi,\varphi)\big) = \frac{1}{2}\omega(T)\int_{\mathbb{R}^d\setminus\Gamma}\big(\,|p_{\psi,\varphi}(\mathbf{x},T)|^2 + |\mathbf{v}_{\psi,\varphi}(\mathbf{x},T)|^2\,\big)\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_0^T\int_{\mathbb{R}^d\setminus\Gamma}\big(|p_{\psi,\varphi}(\mathbf{x},t)|^2+|\mathbf{v}_{\psi,\varphi}(\mathbf{x},t)|^2\big)\mu(t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t, \tag{3.25}$$

where $(p_{\psi,\varphi},\mathbf{v}_{\psi,\varphi})$ *is the solution of equation* (2.11)*.*

Proof. Take the inner product in $\mathbb{R}^3$ of equation (2.11) (b) with $\mathbf{v}_{\psi,\varphi}\,\omega(t)$, multiply equation (2.11) (a) by $p_{\psi,\varphi}\omega(t)$, add the two equalities and integrate the result over $\Omega_i\cup\Omega_e$. One obtains

$$\frac{1}{2}\int_{\mathbb{R}^d\setminus\Gamma}\frac{\partial}{\partial t}\big(\,|p_{\psi,\varphi}|^2+|\mathbf{v}_{\psi,\varphi}|^2\,\big)\omega(t)\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d\setminus\Gamma}\big(\operatorname{div}\mathbf{v}_{\psi,\varphi}p_{\psi,\varphi} + \mathbf{v}_{\psi,\varphi}\cdot\nabla p_{\psi,\varphi}\big)\omega(t)\,\mathrm{d}\mathbf{x} = 0, \tag{3.26}$$

which gives, using Green's formula (2.9),

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d\setminus\Gamma}\big(\,|p_{\psi,\varphi}|^2+|\mathbf{v}_{\psi,\varphi}|^2\,\big)\omega(t)\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^d\setminus\Gamma}\big(\,|p_{\psi,\varphi}|^2+|\mathbf{v}_{\psi,\varphi}|^2\,\big)\mu(t)\,\mathrm{d}\mathbf{x} = \int_\Gamma\big([\![p_{\psi,\varphi}]\!]_\Gamma\{\!\{\mathbf{v}_{\psi,\varphi}\cdot\mathbf{n}\}\!\}_\Gamma + \{\!\{p_{\psi,\varphi}\}\!\}_\Gamma[\![\mathbf{v}_{\psi,\varphi}\cdot\mathbf{n}]\!]_\Gamma\big)\omega(t)\,\mathrm{d}\gamma, \tag{3.27}$$

that is to say, using equations (2.11) (c), (d) and (2.15),

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d\setminus\Gamma}\big(\,|p_{\psi,\varphi}|^2+|\mathbf{v}_{\psi,\varphi}|^2\,\big)\omega(t)\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^d\setminus\Gamma}\big(\,|p_{\psi,\varphi}|^2+|\mathbf{v}_{\psi,\varphi}|^2\,\big)\mu(t)\,\mathrm{d}\mathbf{x} = \int_\Gamma\big(\mathcal{Z}_\Gamma\varphi\varphi + \mathcal{W}_\Gamma\psi\,\varphi + \mathcal{W}_\Gamma^*\varphi\,\psi + \mathcal{Y}_\Gamma\psi\,\psi\big)\omega(t)\,\mathrm{d}\gamma. \tag{3.28}$$

Integrating equation (3.28) in time between 0 and T leads to equation (3.25), thanks to equation (3.24). □

Examples. Let us consider two particular examples of weight functions:

(1) Linearly decaying function: $\omega(t) = 1 - t/T$. In that case $\mu(t) = 1/T$, $\omega(T) = 0$ and equation (3.25) gives

$$b_{\omega,T}\big((\psi,\varphi),(\psi,\varphi)\big) = \frac{1}{2T}\int_0^T \int_{\mathbb{R}^d\setminus\Gamma} \big(\, |p_{\psi,\varphi}(\mathbf{x},t)|^2 + |\mathbf{v}_{\psi,\varphi}(\mathbf{x},t)|^2 \,\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{3.29}$$

The reader may easily verify that equation (3.29) is easily recovered from equation (3.5) after having noticed that

$$b_{\omega,T}(\cdot,\cdot) \equiv \frac{1}{T}\int_0^T b_t(\cdot,\cdot)\ \mathrm{d}t.$$

(2) Exponentially decaying weight: $\omega(t) = e^{-2\eta t}$, $\eta > 0$. In that case $\mu(t) = 2\eta e^{-2\eta t}$ and equation (3.25) give

$$\begin{aligned} b_{\omega,T}&\big((\psi,\varphi),(\psi,\varphi)\big) \\ &= \frac{1}{2}e^{-2\eta T}\int_{\mathbb{R}^d\setminus\Gamma} \big(\, |p_{\psi,\varphi}(\mathbf{x},T)|^2 + |\mathbf{v}_{\psi,\varphi}(\mathbf{x},T)|^2 \,\big)\,\mathrm{d}\mathbf{x} \\ &\quad + \eta\int_0^T\int_{\mathbb{R}^d\setminus\Gamma}\big(\, |p_{\psi,\varphi}(\mathbf{x},t)|^2 + |\mathbf{v}_{\psi,\varphi}(\mathbf{x},t)|^2\big)e^{-2\eta t}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \end{aligned} \tag{3.30}$$

This choice for $\omega(t)$ is the one used to recover the results obtained from the analysis based on the use of the Laplace transform in time. The parameter η is somewhat arbitrary. Its determination will be discussed later, cf., equation (3.48).

An immediate consequence of the previous result is the next corollary.

Corollary 3.9. *Assume that the function $\omega(t)$ is decreasing, namely, that $\mu(t) \geq 0$, for all $t \in [0,T]$. Then the quadratic form associated to $b_{\omega,T}(\cdot,\cdot)$ is positive for all (ψ,φ):*

$$\Gamma \times \mathbb{R}^+ \longmapsto \mathbb{R}\times\mathbb{R}, \qquad b_{\omega,T}\big((\psi,\varphi),(\psi,\varphi)\big) \geq 0. \tag{3.31}$$

If $\omega(t)$ is strictly decreasing, namely, $\mu(t) > 0$, for all $t \in [0,T]$, this quadratic form is positive definite:

$$b_{\omega,T}\big((\psi,\varphi),(\psi,\varphi)\big) = 0 \Longrightarrow (\psi,\varphi) = 0. \tag{3.32}$$

We end this section with a coercivity result. What follows really makes sense when $d \geq 2$. The special case $d = 1$ will be discussed in more detail in subsection 4.1.

Notation and functional spaces. X denoting any Banach space, $H_c^r(0,T;X), r \in \mathbb{R}$ is the subspace of distributions $\varphi \in H^r(0,T;X)$ whose extension by 0 for $t < 0$, denoted $\widetilde{\varphi}$, belongs to $H^r(-\infty,T;X)$ (the subscript c refers to causal functions). It is equipped with a Hilbert space norm given by

$$\|\varphi\|_{H_c^r(0,T;X)} := \|\widetilde{\varphi}\|_{H^r(-\infty,T;X)}. \tag{3.33}$$

It is well known [**40**] that $H_c^r(0,T;X)$ differs from $H^r(0,T;X)$ only for $r \geq 1/2$. Moreover, introducing the space

$$C_c^\infty(0,T;X) = \{\varphi = \varphi^*|_{[0,T]}, \varphi^* \in C_c^\infty(\mathbb{R};X)\},$$

where $C_c^\infty(\mathbb{R};X)$ is the set of causal and indefinitely differentiable functions of the time variable with values in the Banach space X, it is dense in $H_c^r(0,T;X)$.

Let us introduce the sets

$$\Omega_T := \Omega \times]0,T[\quad \text{and} \quad \Sigma_T := \Gamma \times]0,T[\,, \tag{3.34}$$

where $\Omega = \mathbb{R}^d \backslash \Gamma$. We now introduce some adequate anisotropic Sobolev spaces [**41**] of causal functions of the form $H_c^{r,s}(\Sigma_T), r, s \in \mathbb{R}$, where the first index r refers to time regularity and the second index s to space regularity. More precisely, let us set

$$H_c^{r,s}(\Sigma_T) = H_c^r\big(0,T;L^2(\Gamma)\big) \cap L^2\big(0,T;H^s(\Gamma)\big),$$

naturally equipped with a Hilbert space structure with the norm

$$\|\varphi\|_{r,s,\Sigma_T}^2 := \|\varphi\|_{H_c^r(0,T;L^2(\Gamma))}^2 + \|\varphi\|_{L^2(0,T;H^s(\Gamma))}^2. \tag{3.35}$$

We shall use the (obvious) fact that $H_c^{1/2,1/2}(\Sigma_T) \subset H^{1/2}(\Sigma_T)$ (the usual isotropic Sobolev space) and, therefore, that

$$H^{-1/2}(\Sigma_T) \equiv H^{-1/2,-1/2}(\Sigma_T) \subset H_c^{1/2,1/2}(\Sigma_T)'. \tag{3.36}$$

Note also that, due to Poincaré inequality, the trace map $u \mapsto u|_{\Sigma_T}$ satisfies the following continuity property for all

$$
\begin{gathered}
u_j \in L^2\big(0,T;H^1(\Omega_j)\big) \cap H^1_c\big(0,T;L^2(\Omega_j)\big),\\
u_j \in H_c^{1/2,1/2}(\Sigma_T) \quad \text{and} \quad \|u_j\|_{1/2,1/2,\Sigma_T} \leq C_T |u|_{H^1(\Omega_T)},\\
j = i, e,
\end{gathered}
\tag{3.37}
$$

where, by definition

$$
|u|^2_{H^1(\Omega_T)} := \int_0^T \int_{\Omega_i \cup \Omega_e} \big(|\partial_t u|^2 + |\nabla u|^2\big)\,\mathrm{d}x\,\mathrm{d}t.
$$

We shall also use the following trace inequality that does not seem to be so standard and whose proof is postponed to the Appendix (see Section A).

Lemma 3.10. *Assume that $d \geq 2$. There exists a constant $\mathbf{C}_T > 0$ such that, for $j = i, e$ and any $\mathbf{w}_j \in C_c^\infty(\overline{\Omega}_j \times [0,T])$ which has compact support,*

$$
\|\mathbf{w}_j \cdot \mathbf{n}\|^2_{H^{1/2}(0,T;H^{-1/2}(\Gamma))} \leq \mathbf{C}_T |\mathbf{w}_j|^2_{1,\mathrm{div},\Omega_j,T},
\tag{3.38}
$$

where we have defined

$$
|\mathbf{w}_j|^2_{1,\mathrm{div},\Omega_j,T} := \Big(\|\partial_t \mathbf{w}_j\|^2_{L^2(0,T;L^2(\Omega_j))} + \|\mathrm{div}\,\mathbf{w}_j\|^2_{L^2(0,T;L^2(\Omega_j))}\Big).
\tag{3.39}
$$

A coercivity result. Let us adopt, for any $\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^+;X)$, the notation

$$
\partial_t^{-1}\varphi(t) := \int_0^t \varphi(s)\,\mathrm{d}s.
\tag{3.40}
$$

Proposition 3.11. *Assume that the function $\omega(t)$ is strictly decreasing and that $\mu(t) = -\omega'(t)$ satisfies*

$$
\mu_T := \inf_{t\in[0,T]} \mu(t) > 0.
\tag{3.41}
$$

Then the quadratic form associated to $b_{\omega,T}(\cdot,\cdot)$ *satisfies the following coercivity inequality, for some* $\alpha_T > 0$*: for all* $(\psi,\varphi) \in C_c^\infty(\Gamma\times[0,T])$*,*

(3.42)
$$b_{\omega,T}\big((\psi,\varphi),(\psi,\varphi)\big) \geq \alpha_T\big(\|\partial_t^{-1}\psi\|^2_{1/2,1/2,\Sigma_T}+\|\partial_t^{-1}\varphi\|^2_{H^{1/2}(0,T;H^{-1/2}(\Gamma))}\big).$$

Proof. The proof relies on Proposition 3.8 (identity (3.25)) combined with trace inequalities (3.37) and (3.38).

Indeed, let us introduce $P_{\psi,\varphi}(x,t) = \partial_t^{-1}p_{\psi,\varphi}(x,s)$ and $\mathbf{V}_{\psi,\varphi}(x,t) = \partial_t^{-1}\mathbf{v}_{\psi,\varphi}$, which satisfy

(3.43)
$$\begin{cases} \partial_t P_{\psi,\varphi} + \mathrm{div}\mathbf{V}_{\psi,\varphi} = 0, & \\ \quad \partial_t\mathbf{V}_{\psi,\varphi} + \nabla P_{\psi,\varphi} = 0 & \text{in } \mathbb{R}^d\setminus\Gamma\times[0,T] \quad \text{(i)}, \\ [\![P_{\psi,\varphi}]\!]_\Gamma(\cdot,t) = \partial_t^{-1}\psi(\cdot,t), & \\ \quad [\![\mathbf{V}_{\psi,\varphi}\cdot\mathbf{n}]\!]_\Gamma(\cdot,t) = \partial_t^{-1}\varphi(\cdot,t) & \text{on } \Gamma\times[0,T] \quad \text{(ii)}. \end{cases}$$

From inequality (3.37) applied to $u_i = P^i_{\psi,\varphi}$ and $u_e = P^e_{\psi,\varphi}$, we have, since $\partial_t P_{\psi,\varphi} = p_{\psi,\varphi}$ and $\nabla P_{\psi,\varphi} = -\partial_t\mathbf{V}_{\psi,\varphi} = -\mathbf{v}_{\psi,\varphi}$,

(3.44)
$$\begin{aligned}\|\partial_t^{-1}\psi\|^2_{1/2,1/2,\Sigma_T} &\leq C_T\big(\|\partial_t P_{\psi,\varphi}\|^2_{L^2(\Omega_T)} + \|\nabla P_{\psi,\varphi}\|^2_{L^2(\Omega_T)}\big)\\ &= C_T\big(\|p_{\psi,\varphi}\|^2_{L^2(\Omega_T)} + \|\mathbf{v}_{\psi,\varphi}\|^2_{L^2(\Omega_T)}\big).\end{aligned}$$

Applying equation (3.38) to $\mathbf{w}_i = \mathbf{V}^i_{\psi,\varphi}$ and $\mathbf{w}_e = \mathbf{V}^e_{\psi,\varphi}$, we deduce, since $\partial_t\mathbf{V}_{\psi,\varphi} = \mathbf{v}_{\psi,\varphi}$ and $\mathrm{div}\,\mathbf{V}_{\psi,\varphi} = -\partial_t P_{\psi,\varphi} = -\,p_{\psi,\varphi}$,

(3.45)
$$\begin{aligned}\|\partial_t^{-1}\varphi\|^2_{H^{1/2}(0,T;H^{-1/2}(\Gamma))} &\leq 2\mathbf{C}_T\big(\|\partial_t\mathbf{V}_{\psi,\varphi}\|^2_{L^2(\Omega_T)} + \|\mathrm{div}\,\mathbf{V}_{\psi,\varphi}\|^2_{L^2(\Omega_T)}\big)\\ &\leq 2\mathbf{C}_T\big(\|p_{\psi,\varphi}\|^2_{L^2(\Omega_T)} + \|\mathbf{v}_{\psi,\varphi}\|^2_{L^2(\Omega_T)}\big).\end{aligned}$$

Since, according to equations (3.25) and (3.41),

(3.46)
$$b_{\omega,T}\big((\psi,\varphi),(\psi,\varphi)\big) \geq \frac{\mu_T}{2}\int_0^T\int_{\mathbb{R}^d\setminus\Gamma}\big(\,|p_{\psi,\varphi}(\mathbf{x},t)|^2+|\mathbf{v}_{\psi,\varphi}(\mathbf{x},t)|^2\,\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,$$

it suffices to combine equations (3.44) and (3.45) with (3.46) to conclude with $\alpha_T = (\mu_T/8)\min(C_T^{-1},\mathbf{C}_T^{-1})$. □

The reader will observe that the coercivity result (3.42) fits quite well with the one that can be obtained with the Laplace transform method, see for instance, [**11,** Lemma 4.1], with, however, a slight improvement that can be interpreted as a gain of 1/2 order of regularity in time. Indeed, equation (3.42) may be rewritten as

$$\begin{aligned} b_{\omega,T}\big((\psi,\varphi),(\psi,\varphi)\big) \\ \geq \alpha_T\big(\|\partial_t^{-1}\psi\|^2_{L^2(0,T;H^{1/2}(\Gamma))} + \|\partial_t^{-1}\psi\|^2_{H^{1/2}_c(0,T;L^2(\Gamma))} \\ + \|\partial_t^{-1}\varphi\|^2_{H^{1/2}(0,T;H^{-1/2}(\Gamma))}\big)\end{aligned}$$

while, in the particular case $\omega(t) = \exp(-2\eta t)$, [**11,** Lemma 4.1] reads as

$$b_{\omega,T}\big((\psi,\varphi),(\psi,\varphi)\big) \geq \alpha_T\big(\|\partial_t^{-1}\psi\|^2_{L^2(0,T;H^{1/2}(\Gamma))}+\|\partial_t^{-1}\varphi\|^2_{L^2(0,T;H^{-1/2}(\Gamma))}\big).$$

However, we shall see in the two particular examples in Section 4 that, even the coercivity result (3.42) is not optimal.

Returning to the examples.

(1) Linearly decaying function: $\omega(t) = 1 - t/T$. Inequality (3.46) becomes ($\mu_T = 1/T$)

(3.47)
$$b_{\omega,T}\big((\psi,\varphi),(\psi,\varphi)\big) \geq \frac{1}{2T}\int_0^T\int_{\mathbb{R}^d\setminus\Gamma}\big(|p_{\psi,\varphi}(\mathbf{x},t)|^2+|\mathbf{v}_{\psi,\varphi}(\mathbf{x},t)|^2\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.$$

(2) Exponentially decaying weight: $\omega(t) = e^{-2\eta t}$, $\eta > 0$. In this case $\mu_T = 2\eta e^{-2\eta T}$. This quantity can be maximized by choosing $\eta = 1/(2T)$, leading to

(3.48)
$$b_{\omega,T}\big((\psi,\varphi),(\psi,\varphi)\big) \geq \frac{1}{2eT}\int_0^T\int_{\mathbb{R}^d\setminus\Gamma}\big(|p_{\psi,\varphi}(\mathbf{x},t)|^2+|\mathbf{v}_{\psi,\varphi}(\mathbf{x},t)|^2\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.$$

Remark 3.12. If one denotes the operator $(\psi,\varphi)\mapsto\omega(t)(\psi,\varphi)$ by $\mathbf{T}_\omega$, one has $b_{\omega,T}((\psi,\varphi),(\psi,\varphi)) = b_T(\mathbf{T}_\omega(\psi,\varphi),(\psi,\varphi))$. As a consequence, the result of Proposition 3.11, for instance, can be reformulated as the $\mathbf{T}_\omega$-coercivity of $b(\cdot,\cdot)$ in the sense defined as in [**16, 17**], which also corresponds to an inf-sup condition. A particular consequence is the injectivity of operator B_Γ and thus the injectivity of the two operators $\mathcal{Z}_\Gamma$ and $\mathcal{Y}_\Gamma$.

3.5. Continuity properties of the operator B_Γ. We now state continuity results for the four operators $\mathcal{Z}_\Gamma$, $\mathcal{W}_\Gamma^*$, $\mathcal{Y}_\Gamma$ and $\mathcal{W}_\Gamma$ in anisotropic space time Sobolev spaces. We shall restrict ourselves to establish continuity estimates for smooth (ψ,φ), but the reader should understand that, even if it is not explicitly mentioned, the statements of the theorem also mean that these operators can be extended by continuity and density to the adequate functional spaces. Throughout this section, C_T will represent a non negative constant which depends only upon T but whose value may vary from one line to the other.

Proposition 3.13. *We have the continuity estimates*:
(3.49)
$$\begin{cases} \text{(i) } \|\mathcal{Z}_\Gamma\varphi\|_{1/2,1/2,\Sigma_T} \le C_T\Big(\|\partial_t\varphi\|_{-1/2,-1/2,\Sigma_T} + \|\partial_t^2\varphi\|_{-1/2,-1/2,\Sigma_T}\Big), \\ \text{(ii) } \|\mathcal{W}_\Gamma^*\varphi\|_{H_c^{1/2}(0,T;H^{-1/2}(\Gamma))} \\ \qquad\qquad \le C_T\Big(\|\partial_t\varphi\|_{-1/2,-1/2,\Sigma_T} + \|\partial_t^2\varphi\|_{-1/2,-1/2,\Sigma_T}\Big), \\ \text{(iii) } \|\mathcal{Y}_\Gamma\psi\|_{H_c^{1/2}(0,T;H^{-1/2}(\Gamma))} \\ \qquad\qquad \le C_T\Big(\|\partial_t\psi\|_{H^{-1/2}(0,T;H^{1/2}(\Gamma))} + \|\partial_t^2\psi\|_{H^{-1/2}(0,T;H^{1/2}(\Gamma))}\Big), \\ \text{(iv) } \|\mathcal{W}_\Gamma\psi\|_{1/2,1/2,\Sigma_T} \\ \qquad\qquad \le C_T\Big(\|\partial_t\psi\|_{H^{-1/2}(0,T;H^{1/2}(\Gamma))} + \|\partial_t^2\psi\|_{H^{-1/2}(0,T;H^{1/2}(\Gamma))}\Big). \end{cases}$$

Proof of (i). Let $p = p_{0,\varphi}$; we have, with $\Omega = \mathbb{R}^d \setminus \Gamma$,

$$\partial_t^2 p - \Delta p = 0 \quad \text{in } \Omega\times\mathbb{R}^+, \quad [\![\partial_n p]\!]_\Gamma = -\partial_t\varphi, \quad [\![p]\!]_\Gamma = 0,$$

and $\mathcal{Z}_\Gamma\varphi = \{\!\{p\}\!\}_\Gamma$. Thus, we have the standard energy identity

(3.50)
$$\frac{1}{2}\int_{\mathbb{R}^d\setminus\Gamma}\big(|\partial_t p|^2 + |\nabla p|^2\big)\,\mathrm{d}x = \int_0^t\int_\Gamma \partial_t\varphi\,\partial_t p\,\mathrm{d}\sigma\,\mathrm{d}s = -\int_0^t\int_\Gamma \partial_t^2\varphi\,p\,\mathrm{d}\sigma\,\mathrm{d}s + \int_\Gamma\big(\partial_t\varphi\,p\big)\,\mathrm{d}\sigma.$$

Integrating equality (3.50) between 0 and T gives
(3.51)
$$\frac{1}{2}|p|^2_{H^1(\Omega\times[0,T])} = -\int_0^T\int_\Gamma (T-t)\partial_t^2\varphi\,p\,\mathrm{d}\sigma\,\mathrm{d}s + \int_0^T\int_\Gamma \partial_t\varphi\,p\,\mathrm{d}\sigma\,\mathrm{d}t.$$

Next, we observe that, first using the duality between $H^{-1/2,-1/2}(\Sigma_T)$ and $H^{1/2,1/2}(\Sigma_T)$, see equation (3.36), then the trace inequality (3.37),

$$\left|\int_0^T \int_\Gamma (T-t)\partial_t\varphi \, p \,\mathrm{d}\sigma\,\mathrm{d}s\right| \le \|\partial_t^2\varphi\|_{-1/2,-1/2,\Sigma_T}\|(T-t)p\|_{1/2,1/2,\Sigma_T} \le C_T\|\partial_t^2\varphi\|_{-1/2,-1/2,\Sigma_T}|p|_{H^1(\Omega\times[0,T])}. \tag{3.52}$$

In the same way,

$$\left|\int_0^T \int_\Gamma \partial_t\varphi \, p \,\mathrm{d}\sigma\,\mathrm{d}t\right| \le C_T\|\partial_t\varphi\|_{-1/2,-1/2,\Sigma_T}|p|_{H^1(\Omega\times[0,T])}. \tag{3.53}$$

Substituting equations (3.52) and (3.53) into equation (3.51), we obtain

$$|p|_{H^1(\Omega\times[0,T])} \le C_T\Big(\|\partial_t\varphi\|_{-1/2,-1/2,\Sigma_T} + \|\partial_t^2\varphi\|_{-1/2,-1/2,\Sigma_T}\Big). \tag{3.54}$$

Again using inequality (3.37):

$$\begin{aligned}\|\mathcal{Z}_\Gamma\varphi\|_{1/2,1/2,\Sigma_T} &\equiv \|\{\!\{p\}\!\}_\Gamma\|_{1/2,1/2,\Sigma_T}\\ &\le C_T\Big(\|\partial_t\varphi\|_{-1/2,-1/2,\Sigma_T} + \|\partial_t^2\varphi\|_{-1/2,-1/2,\Sigma_T}\Big).\end{aligned}$$

□

Proof of (ii). Let $\mathbf{v} = \mathbf{v}_{0,\varphi}$, so that $\mathcal{W}_\Gamma^*\varphi = \{\!\{\mathbf{v}\cdot\mathbf{n}\}\!\}_\Gamma$. We have $\partial_t\mathbf{v} = -\nabla p, \operatorname{div}\mathbf{v} = -\partial_t p$, so that

$$|p|^2_{H^1(\Omega\times[0,T])} = |\mathbf{v}|^2_{1,\mathrm{div},\Omega,T} := \int_0^T\int_\Omega \big(|\partial_t\mathbf{v}|^2 + |\operatorname{div}\mathbf{v}|^2\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.$$

By the trace Lemma 3.10,

$$\begin{aligned}\|\mathcal{W}_\Gamma^*\varphi\|^2_{H^{1/2}(0,T;H^{-1/2}(\Gamma))} = \|\{\!\{\mathbf{v}\cdot\mathbf{n}\}\!\}_\Gamma\|^2_{H^{1/2}(0,T;H^{-1/2}\Gamma)} &\le C_T|\mathbf{v}|^2_{1,\mathrm{div},\Omega,T}\\ &\equiv C_T|p|^2_{H^1(\Omega\times[0,T])}.\end{aligned}$$

We then conclude thanks to (3.54). □

Proof of (iii). Let $\mathbf{v} = \mathbf{v}_{\psi,0}$. We have, with $\Omega = \mathbb{R}^d\setminus\Gamma$,

$$\begin{aligned}&\partial_t^2\mathbf{v} - \nabla\big(\operatorname{div}\mathbf{v}\big) = 0 \quad \text{in } \Omega\times\mathbb{R}^+,\\ &[\![\operatorname{div}\mathbf{v}]\!]_\Gamma = -\partial_t\psi, \quad [\![\mathbf{v}\cdot\mathbf{n}]\!]_\Gamma = 0,\end{aligned}$$

and $\mathcal{Y}_\Gamma\psi = \{\!\{\mathbf{v}\cdot\mathbf{n}\}\!\}_\Gamma$. Again, we have the energy identity

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^d}\big(|\partial_t\mathbf{v}|^2 + |\operatorname{div}\mathbf{v}|^2\big)(\cdot,t)\,\mathrm{d}\mathbf{x}\\
&= \int_0^t\int_\Gamma \partial_t\psi\,\partial_t(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}\sigma\,\mathrm{d}s \qquad (3.55)\\
&= -\int_0^t\int_\Gamma \partial_t^2\psi\,(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}\sigma\,\mathrm{d}s + \int_\Gamma\big(\partial_t\psi\,(\mathbf{v}\cdot\mathbf{n})\big)(\cdot,t)\,\mathrm{d}\sigma.
\end{aligned}$$

Integrating equality (3.55) between 0 and T gives

$$\begin{aligned}
|\mathbf{v}|^2_{1,\mathrm{div},\Omega,T} &= -\int_0^T\int_\Gamma (T-t)\,\partial_t^2\psi\,(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}\sigma\,\mathrm{d}s \qquad (3.56)\\
&\quad + \int_0^T\int_\Gamma\big(\partial_t\psi\,(\mathbf{v}\cdot\mathbf{n})\big)(\cdot,t)\,\mathrm{d}\sigma.
\end{aligned}$$

Using trace Lemma 3.10 and the inclusion $H^{-1/2}(0,T;H^{1/2}(\Gamma)) \subset H_c^{1/2}(0,T;H^{-1/2}(\Gamma))'$, we obtain as in the proof of point (i) (we omit the details)

(3.57)
$$|\mathbf{v}|_{1,\mathrm{div},\Omega,T} \le C_T\Big(\|\partial_t\psi\|_{H^{-1/2}(0,T;H^{1/2}(\Gamma))} + \|\partial_t^2\psi\|_{H^{-1/2}(0,T;H^{1/2}(\Gamma))}\Big).$$

We conclude by applying Lemma 3.10 again. □

Proof of (iv). Let $p = p_{\psi,0}$, so that $\mathcal{W}_\Gamma\psi = \{\!\{p\}\!\}_\Gamma$. We have $\partial_t p = -\operatorname{div}\mathbf{v}$, $\nabla p = -\partial_t\mathbf{v}$ so that $|p|_{H^1(\Omega\times[0,T])} = |\mathbf{v}|_{1,\mathrm{div},\Omega,T}$. We conclude with trace inequality (3.37). □

It is interesting to compare the above continuity results to the “standard” results obtained by the Fourier-Laplace method, as they are recalled for instance in [**11,** Section 3], see also [**37**]. With respect to the operators V, K^T, K, W introduced in [**11**], it is easy to check the correspondence

$$\mathcal{Z}_\Gamma \equiv \partial_t V, \qquad \mathcal{W}_\Gamma^* \equiv K^T, \qquad \mathcal{W}_\Gamma \equiv K, \qquad \partial_t\mathcal{Y}_\Gamma \equiv W.$$

It appears that both results fit quite well with some slight differences, however. For instance, for $\mathcal{Z}_\Gamma$, the standard result can be rewritten as

$$\|\mathcal{Z}_\Gamma\varphi\|_{L^2(0,T;H^{1/2}(\Gamma))} \le C_T\|\varphi\|_{H^2(0,T;H^{-1/2}(\Gamma))}, \qquad (3.58)$$

while our inequality (3.49) (i) can be rephrased as

$$\text{(3.59)}\quad \begin{aligned}&\|\mathcal{Z}_\Gamma\varphi\|_{L^2(0,T;H^{1/2}(\Gamma))} + \|\mathcal{Z}_\Gamma\varphi\|_{H^{1/2}(0,T;L^2(\Gamma))}\\ &\qquad \le C_T\big(\|\varphi\|_{H^2(0,T;H^{-1/2}(\Gamma))} + \|\varphi\|_{H^{3/2}(0,T;L^2(\Gamma))}\big).\end{aligned}$$

In inequality (3.59), we get more but we also ask for more. However, as we shall see in particular examples in Section 4, none of these results is sharp. In particular, they are quite demanding concerning the time regularity of the functions (ψ,φ) and thus do not explain why, in practice, discontinuous finite elements in time can be used for the numerical approximation as can be demonstrated in a very pedestrian way, as in **[1]** (for instance, also see **[30, 34]**).

Remark 3.14. Concerning the other three operators, the "classical" results can be reformulated as

$$\begin{aligned}\|\mathcal{W}_\Gamma^*\varphi\|_{H^{1/2}(0,T;H^{-1/2}(\Gamma))} &\le C_T\|\varphi\|_{H^2(0,T;H^{-1/2}(\Gamma))},\\ \|\mathcal{Y}_\Gamma\psi\|_{H^{1/2}(0,T;H^{-1/2}(\Gamma))} &\le C_T\|\psi\|_{H^{3/2}(0,T;H^{1/2}(\Gamma))},\\ \|\mathcal{W}_\Gamma\psi\|_{L^2(0,T;H^{1/2}(\Gamma))} &\le C_T\|\psi\|_{H^{3/2}(0,T;H^{1/2}(\Gamma))},\end{aligned}$$

while those obtained in Proposition 3.13 imply

$$\begin{aligned}\|\mathcal{W}_\Gamma^*\varphi\|_{H^{1/2}(0,T;H^{-1/2}(\Gamma))} &\le C_T\big(\|\varphi\|_{H^2(0,T;H^{-1/2}(\Gamma))}\\ &\qquad +\|\varphi\|_{H^3/2(0,T;L^2(\Gamma))}\big),\\ \|\mathcal{Y}_\Gamma\psi\|_{H^{1/2}(0,T;H^{-1/2}(\Gamma))} &\le C_T\|\psi\|_{H^{3/2}(0,T;H^{1/2}(\Gamma))},\\ \|\mathcal{W}_\Gamma\psi\|_{L^2(0,T;H^{1/2}(\Gamma))} + \|\mathcal{W}_\Gamma\psi\|_{H^{1/2}(0,T;L^2(\Gamma))} &\le C_T\|\psi\|_{H^{3/2}(0,T;H^{1/2}(\Gamma))}.\end{aligned}$$

4. Study of two particular cases.

4.1. The 1D case. For any smooth enough function ψ defined on $\Sigma_T = \Gamma\times[0,T] = \{a,b\}\times[0,T]$, we introduce

$$\|\psi\|^2_{L^2(\Sigma_T)} := \|\psi_a\|^2_{L^2([0,T])} + \|\psi_b\|^2_{L^2([0,T])}.$$

The next positivity property and $L^2(\Sigma_T)$-coercivity results follow for T small enough.

Proposition 4.1. *Let (ψ,φ) be a pair of smooth enough causal functions defined on $\Gamma\times\mathbb{R}^+$. Then*

$$b_T((\psi,\varphi),(\psi,\varphi)) \geq 0. \tag{4.1}$$

Moreover, if $0<T<|b-a|$, there exists a constant $\mathcal{C}>0$ such that

$$b_T((\psi,\varphi),(\psi,\varphi)) \geq \mathcal{C}\Big(\|\psi\|^2_{L^2(\Sigma_T)} + \|\varphi\|^2_{L^2(\Sigma_T)}\Big). \tag{4.2}$$

Proof. Positivity of the bilinear form is given by the general result in equation (3.3). Nevertheless, it is interesting to provide another proof of the result that directly uses the expressions for the bilinear form. From equations (2.17) and (2.24) with $(\widetilde{\psi},\widetilde{\varphi})=(\psi,\varphi)$ one can see that

$$b_T((\psi,\varphi),(\psi,\varphi)) = I(\psi,\varphi)+J(\psi,\varphi),$$

with

$$I(\psi,\varphi) = \frac{1}{2}\int_0^T \big(\psi_a^2(s)+\psi_b^2(s)+\varphi_a^2(s)+\varphi_b^2(s)\big)\,\mathrm{d}s, \tag{4.3}$$

and

(4.4)

$$\begin{aligned}
J(\psi,\varphi) = {}& \frac{1}{2}\int_0^T \big(\varphi_b(s-|b-a|)\varphi_a(s)+\varphi_a(s-|b-a|)\varphi_b(s)\big)\,\mathrm{d}s\\
&-\frac{1}{2}\int_0^T \big(\psi_b(s-|b-a|)\psi_a(s)+\psi_a(s-|b-a|)\psi_b(s)\big)\,\mathrm{d}s\\
&-\frac{1}{2}\int_0^T \big(\psi_b(s-|b-a|)\varphi_a(s)+\psi_a(s-|b-a|)\varphi_b(s)\big)\,\mathrm{d}s\\
&+\frac{1}{2}\int_0^T \big(\varphi_b(s-|b-a|)\psi_a(s)+\varphi_a(s-|b-a|)\psi_b(s)\big)\,\mathrm{d}s.
\end{aligned}$$

If $t<|b-a|$, then $J(\psi,\varphi)=0$ since all the integrals involve terms with delay that vanish due to causality. In consequence,

$$b_T((\psi,\varphi),(\psi,\varphi)) = I(\psi,\varphi),$$

and one obtains the $L^2(\Sigma_T)$-coercivity result with $\mathcal{C}=1/2$.

In order to obtain the positivity result for an arbitrary time, $J(\psi,\varphi)$ is rewritten as

$$J(\psi,\varphi)=\frac{1}{2}\int_0^T(\mu_a(s)+\lambda_a(s))\,(\mu_b(s-\tau)-\lambda_b(s-\tau))\\ +(\mu_b(s)+\lambda_b(s))\,(\mu_a(s-\tau)-\lambda_a(s-\tau))\;\mathrm{d}s,$$

where $\tau=|b-a|$. Using the identity $2xy=(x+y)^2-x^2-y^2$, the last expression for $J(\mu,\lambda)$ can be transformed into

$$\begin{aligned}J(\psi,\varphi)=&\frac{1}{4}\int_0^T\Big[\big(\varphi_a(s)+\varphi_b(s-\tau)\big)+\big(\psi_a(s)-\psi_b(s-\tau)\big)\Big]^2\mathrm{d}s\\&-\frac{1}{4}\int_0^T\big(\varphi_a(s)+\psi_a(s)\big)^2\mathrm{d}s\\&-\frac{1}{4}\int_0^{T-\tau}\big(\varphi_b(s)-\psi_b(s)\big)^2\mathrm{d}s\\&+\frac{1}{4}\int_0^T\Big[\big(\varphi_b(s)+\varphi_a(s-\tau)\big)+\big(\psi_b(s)-\psi_a(s-\tau)\big)\Big]^2\mathrm{d}s\\&-\frac{1}{4}\int_0^T\big(\varphi_b(s)+\psi_b(s)\big)^2\mathrm{d}s\\&-\frac{1}{4}\int_0^{T-\tau}\big(\varphi_a(s)-\psi_a(s)\big)^2\mathrm{d}s.\end{aligned}$$

Next we use the identity $2(x^2+y^2)=(x+y)^2+(x-y)^2$ to transform $I(\psi,\varphi)$ into

$$\begin{aligned}I(\varphi,\psi)=&\frac{1}{4}\int_0^T\big(\varphi_a(s)+\psi_a(s)\big)^2\mathrm{d}s+\frac{1}{4}\int_0^T\big(\varphi_a(s)-\psi_a(s)\big)^2\mathrm{d}s\\&+\frac{1}{4}\int_0^T\big(\varphi_b(s)+\psi_b(s)\big)^2\mathrm{d}s+\frac{1}{4}\int_0^T\big(\varphi_b(s)-\psi_b(s)\big)^2\mathrm{d}s.\end{aligned}$$

Using both results, one gets

$$\begin{aligned}&b_T((\psi,\varphi),(\psi,\varphi))\\&\quad=\frac{1}{4}\int_{T-\tau}^T\Big[\big(\varphi_a(s)-\psi_a(s)\big)^2+\big(\varphi_b(s)-\psi_b(s)\big)^2\Big]\mathrm{d}s\\&\quad+\frac{1}{4}\int_0^T\Big[\big(\varphi_a(s)+\varphi_b(s-\tau)\big)+\big(\psi_a(s)-\psi_b(s-\tau)\big)\Big]^2\mathrm{d}s\end{aligned}\tag{4.5}$$

$$+\frac{1}{4}\int_0^T \Big[\big(\varphi_b(s)+\varphi_a(s-\tau)\big)+\big(\psi_b(s)-\psi_a(s-\tau)\big)\Big]^2\,\mathrm{d}s,$$

leading to the positivity. $\square$

For T large enough, Proposition 3.7 shows that the bilinear form $b_T(\cdot,\cdot)$ is no longer $L^2(\Sigma_T)$-coercive. Let us prove this result for the one-dimensional case explicitly using the expressions of the bilinear form. We have the next result (the proof is inspired from [**2,** Theorem 2.1]):

Proposition 4.2. *If $T > |b-a|$, there exist non vanishing $(\psi,\varphi)\in (L^2(\Sigma_T))^2$ such that*

$$b_T((\psi,\varphi),(\psi,\varphi)) = 0. \tag{4.6}$$

More precisely, the quadratic form

$$(\psi,\varphi)\in \big(L^2([0,T])\big)^4 \longrightarrow b_T((\psi,\varphi),(\psi,\varphi))\in\mathbb{R}^+, \tag{4.7}$$

has a infinite-dimensional kernel.

Proof. As a first step, let us assume that $T = N\,L$ for some integer $N>0$ (where we have introduced the notation $L=|b-a|$). From equation (4.5), one obtains

$$\begin{aligned}
&b_T((\psi,\varphi),(\psi,\varphi))\\
&\quad=\frac{1}{4}\int_{(N-1)L}^{N\,L}\big(\varphi_a(s)-\psi_a(s)\big)^2+\big(\varphi_b(s)-\psi_b(s)\big)^2\,\mathrm{d}s\\
&\qquad+\frac{1}{4}\sum_{k=0}^{N-1}\int_{k\,L}^{(k+1)L}\big(\varphi_a(s)+\varphi_b(s-\tau)+\psi_a(s)-\psi_b(s-\tau)\big)^2\,\mathrm{d}s\\
&\qquad+\frac{1}{4}\sum_{k=0}^{N-1}\int_{k\,L}^{(k+1)L}\big(\varphi_a(s-\tau)+\varphi_b(s)+\psi_b(s)-\psi_a(s-\tau)\big)^2\,\mathrm{d}s.
\end{aligned}$$

Next, we introduce the following scalar functions indexed by the integer $k\in\{-1,0,\ldots,N-1\}$

$$\varphi^k(c,s) := \varphi_c(s+kL),\qquad \psi^k(c,s) := \psi_c(s+kL),\qquad (c,s)\in\{a,b\}\times[0,L]. \tag{4.8}$$

Notice that, due to causality, $\varphi^{-1}(c,s) = \psi^{-1}(c,s) = 0$. It is also useful to introduce the vector-valued functions consisting of the concatenation of φ^k and ψ^k, $k \in \{0, \dots, N-1\}$,

$$\begin{aligned} \varphi(c,s) &:= \big(\varphi^0(c,s), \dots, \varphi^{N-1}(c,s)\big), \\ \psi(c,s) &:= l\big(\psi^0(c,s), \dots, \psi^{N-1}(c,s)\big). \end{aligned} \tag{4.9}$$

A change of variable (consisting of a shift) allows us to write all the integrals in the interval $[0, L]$ as follows:

$$\begin{aligned} &b_T((\psi,\varphi),(\psi,\varphi)) \\ &\quad = \frac{1}{4}\int_0^L \Big[\varphi^{N-1}(a,s) - \psi^{N-1}(a,s)\Big]^2 + \Big[\varphi^{N-1}(b,s) - \psi^{N-1}(b,s)\Big]^2 \,\mathrm{d}s \\ &\qquad + \frac{1}{4}\int_0^L \sum_{k=0}^{N-1} \Big[\varphi^k(a,s) + \varphi^{k-1}(b,s) + \psi^k(a,s) - \psi^{k-1}(b,s)\Big]^2 \,\mathrm{d}s \\ &\qquad + \frac{1}{4}\int_0^L \sum_{k=0}^{N-1} \Big[\varphi^{k-1}(a,s) + \varphi^k(b,s) + \psi^k(b,s) - \psi^{k-1}(a,s)\Big]^2 \,\mathrm{d}s. \end{aligned}$$

Next, we introduce the quadratic form $Q : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, defined by

$$Q(\varphi,\overline{\varphi},\psi,\overline{\psi}) := \frac{1}{4}\sum_{k=0}^{N} \big[\varphi^k + \overline{\varphi}^{k-1} + \psi^k - \overline{\psi}^{k-1}\big]^2 + \big[\varphi^{k-1} + \overline{\varphi}^k + \overline{\psi}^k - \psi^{k-1}\big]^2,$$

where, by convention, $\varphi^N = \overline{\varphi}^N = \psi^N = \overline{\psi}^N = 0$. We clearly have that

$$b_T((\psi,\varphi),(\psi,\varphi)) = \int_0^L Q(\varphi(a,s), \varphi(b,s), \psi(a,s), \psi(b,s)) \,\mathrm{d}s.$$

The next step is to compute the kernel of Q. Introducing the notation

$$\begin{aligned} u^k &= \varphi^k + \psi^k, \qquad & x^k &= \overline{\varphi}^k + \overline{\psi}^k, \\ v^k &= \overline{\psi}^k - \overline{\varphi}^k, \qquad & y^k &= \psi^k - \varphi^k, \end{aligned}$$

the quadratic form can be written as:

$$Q(\varphi,\overline{\varphi},\psi,\overline{\psi}) := \frac{1}{4}\sum_{k=0}^{N} [u^k - v^{k-1}]^2 + [x^k - y^{k-1}]^2.$$

In consequence, $Q(\varphi, \overline{\varphi}, \psi, \overline{\psi}) = 0$ if and only if

$$v^{k-1} = u^k, \qquad y^{k-1} = x^k, \quad \text{for all } k \in \{0, \dots, N-1\}.$$

Setting $v^N = 0$ and $y^N = 0$ and considering v^k and y^k, $k \in \{0, \dots, N-1\}$ as parameters, the following expressions for φ^k, $\overline{\varphi}^k$, ψ_k and $\overline{\psi}_k$ are found

$$(4.10) \qquad \left| \begin{array}{ll} \varphi^k = \frac{1}{2}(v^{k-1} + y^k), & \overline{\varphi}^k = \frac{1}{2}(y^{k-1} + v^k), \\ \psi_k = \frac{1}{2}(v^{k-1} - y^k), & \overline{\psi}_k = \frac{1}{2}(y^{k-1} - v^k). \end{array} \right.$$

Finally, for any $v^k \in L^2([0,L])$ and $y^k \in L^2([0,L])$, $k \in \{0, \dots, N-1\}$, using equations (4.10) and (4.8), one gets

$$(4.11) \qquad \left| \begin{array}{l} \varphi_a(s + kL) = \frac{1}{2}(v^{k-1}(s) + y^k(s)), \\ \varphi_b(s + kL) = \frac{1}{2}(y^{k-1}(s) + v^k(s)), \\ \psi_a(s + kL) = \frac{1}{2}(v^{k-1}(s) - y^k(s)), \\ \psi_b(s + kL) = \frac{1}{2}(y^{k-1}(s) - v^k(s)), \end{array} \right.$$

for $s \in [0, L]$. Notice that, if $\varphi_a(\cdot)$, $\varphi_b(\cdot)$, $\psi_a(\cdot)$ and $\psi_b(\cdot)$ are given, then the corresponding functions $v_k(\cdot)$ and $y_k(\cdot)$ can be built from the last equalities. In consequence, one concludes that the kernel of the bilinear form $b_t(\cdot,\cdot)$ is isomorphic to $L^2([0,L])^{2(N-1)}$.

When one has an arbitrary time $T > |b - a| = L$, we consider $T^* = [(b-a)/L]L$ (where $[\cdot]$ represents the integer part operator), and we split the time interval $[0,T]$ into $[0,T^*] \cup [T^*, T]$. Next one builds functions $\psi_a^*(\cdot)$, $\psi_b^*(\cdot)$, $\varphi_a^*(\cdot)$ and $\varphi_b^*(\cdot)$ for $s \in [0, T^*]$ such that $b_{T^*}((\psi^*, \varphi^*), (\psi^*, \varphi^*)) = 0$ using the fact that $T^* = NL$, $N \in \mathbb{N}$, and extends them by 0 to obtain $\psi_a(\cdot)$, $\psi_b(\cdot)$, $\varphi_a(\cdot)$ and $\varphi_b(\cdot)$ defined for $s \in [0,T]$. Clearly, $b_T((\psi, \lambda), (\psi, \lambda)) = 0$. $\square$

Let us study the situation when an arbitrary weight is considered. In this case, similar computations to those presented in the proof of Proposition 4.1 lead to the expression

(4.12)
$$\begin{aligned} b_{\omega,T}&((\psi,\varphi),(\psi,\varphi)) \\ &= \frac{1}{4}\int_{T-\tau}^{T}\Big[\big(\varphi_a(s)-\psi_a(s)\big)^2+\big(\varphi_b(s)-\psi_b(s)\big)^2\Big]\chi(s)\,\mathrm{d}s \\ &\quad+\frac{1}{4}\int_0^T\Big[\big(\varphi_a(s)+\varphi_b(s-\tau)\big)+\big(\psi_a(s)-\psi_b(s-\tau)\big)\Big]^2\omega(s)\,\mathrm{d}s \\ &\quad+\frac{1}{4}\int_0^T\Big[\big(\varphi_b(s)+\varphi_a(s-\tau)\big)+\big(\psi_b(s)-\psi_a(s-\tau)\big)\Big]^2\omega(s)\,\mathrm{d}s, \end{aligned}$$

where

$$\chi(s)=\begin{cases}\omega(s) & s\in[t-\tau,t],\\ \omega(s)-\omega(s+\tau) & s\in[0,t-\tau].\end{cases} \tag{4.13}$$

This allows us to establish the next proposition.

Proposition 4.3. *Let (ψ,φ) be a pair of smooth enough causal functions defined on $\Gamma\times\mathbb{R}^+$, and assume that*

- $\omega(s)\geq\omega_*>0,\ s\in[0,T]$,
- $\omega(s)-\omega(s+\tau)\geq\alpha>0, s\in[0,T]$.

Then there exists a constant $\mathcal{C}>0$ such that

$$b_{\omega,T}((\psi,\varphi),(\psi,\varphi))\geq\mathcal{C}\min(\omega_*,\alpha)\Big(\|\psi\|^2_{L^2(\Sigma_T)}+\|\varphi\|^2_{L^2(\Sigma_T)}\|\Big). \tag{4.14}$$

Proof. First of all, we introduce the continuous operator:

$$\begin{aligned}\mathcal{A}:\big(L^2([0,T])\big)^4&\longrightarrow\big(L^2([0,T])\big)^4\\ (\psi,\varphi)&\longmapsto\mathcal{A}(\psi,\varphi):=(\mu,\lambda),\end{aligned}$$

where

$$\begin{aligned}\mu_a(s)&:=\varphi_a(s)-\psi_a(s),\\ \lambda_a(s)&:=\varphi_a(s)+\varphi_b(s-\tau)+\psi_a(s)-\psi_b(s-\tau),\\ \mu_b(s)&:=\varphi_b(s)-\psi_b(s),\\ \lambda_b(s)&:=\varphi_b(s)+\varphi_a(s-\tau)+\psi_b(s)-\psi_a(s-\tau).\end{aligned} \tag{4.15}$$

Clearly, equation (4.12) leads to

$$b_{\omega,T}((\psi,\varphi),(\psi,\varphi)) \geq \min\big(\alpha,\omega_*\big)\|\mathcal{A}(\psi,\varphi)\|^2_{(L^2([0,T]))^4}.$$

Since

$$\begin{aligned}
\psi_a(s) &= \frac{1}{2}\big(\lambda_a(s) - \mu_a(s) - \mu_b(s-\tau)\big),\\
\varphi_a(s) &= \frac{1}{2}\big(\lambda_a(s) + \mu_a(s) - \mu_b(s-\tau)\big),\\
\psi_b(s) &= \frac{1}{2}\big(\lambda_b(s) - \mu_b(s) - \mu_a(s-\tau)\big),\\
\varphi_b(s) &= \frac{1}{2}\big(\lambda_b(s) + \mu_b(s) - \mu_a(s-\tau)\big),
\end{aligned}$$

the operator $\mathcal{A}$ is bijective and with continuous inverse. In consequence, from equation (4.15), one obtains equation (4.14). □

It is worth mentioning that the weight $\omega(\cdot)$ can eventually be piecewise smooth (differentiability is not a priori needed).

Finally, the $L^2(\Sigma_T)$-continuity of the bilinear form $b_{\omega,T}(\cdot,\cdot)$ trivially holds for $\omega \in L^\infty(\mathbb{R}^+)$ (in particular for $\omega(\cdot)\equiv 1$). This leads to the next proposition.

Proposition 4.4. *Let (ψ,φ) and $(\widetilde{\psi},\widetilde{\varphi})$ be pairs of smooth enough causal functions defined on $\Gamma\times\mathbb{R}^+$, and assume that $\omega(\cdot)\in L^\infty(\mathbb{R}^+)$. Then there exists a constant $\mathcal{C}>0$ such that*

$$\text{(4.16)}\quad \begin{aligned} &b_{\omega,T}((\psi,\varphi),(\widetilde{\psi},\widetilde{\varphi}))\\ &\leq \mathcal{C}\|\omega\|_{L^\infty(\mathbb{R}^+)}\Big(\|\psi\|_{L^2(\Sigma_T)}+\|\varphi\|_{L^2(\Sigma_T)}\Big)\Big(\|\widetilde{\psi}\|_{L^2(\Sigma_T)}+\|\widetilde{\varphi}\|_{L^2(\Sigma_T)}\|\Big).\end{aligned}$$

4.2. The case of Γ in a hyperplane in $\mathbb{R}^d$. Again, the results of this section have been mainly motivated by the work of Aimi et al. [**2**], which seems to have received too little attention, and of which we give a slightly different presentation (concerning continuity results) and extension (concerning the coercivity result).

In this section, we set $\mathbb{R}^d = \{(\mathbf{x}',x_d),\ \mathbf{x}'\in\mathbb{R}^{d-1}, x_d\in\mathbb{R}\}$, $d\geq 2$, assume that $\Gamma = \big\{(\mathbf{x}',0),\mathbf{x}'\in\mathbb{R}^{d-1}\big\}$ and concentrate ourselves on the operator $\mathcal{Z}_\Gamma$ (which is the one more demanding in terms of time regularity) and the corresponding bilinear form $b_T^{\mathcal{Z}}(\cdot,\cdot)$ (and its weighted version with $\omega(t) = \exp(-2\eta t)$ denoted $b^{\mathcal{Z}}_{\eta,T}(\cdot,\cdot)$) to address the following questions:

(i) What is the smallest value of s, if any, for which the bilinear form $b^{\mathcal{Z}}_{T}(\cdot,\cdot)$ is continuous in $L^2(0,T;H^s((\Gamma)))$?

(ii) What is the largest value of s, if any, for which the bilinear form $b^{\mathcal{Z}}_{\eta,T}(\cdot,\cdot)$ is coercive $L^2(0,T;H^s(\Gamma)))$?

The reader will note that question (i) is equivalent to:

(i)′ What is the smallest value of s for which $\mathcal{Z}_\Gamma$ maps $L^2(0,T;H^r(\Gamma))$ continuously into $L^2(0,T;H^{r-2s}((\Gamma))$?

The proof relies on a detailed study of the Fourier-Laplace symbol

$$(\xi', s) \in \mathbb{R}^{d-1} \times \mathbb{C} \mapsto \widehat{\mathcal{Z}}_\Gamma(\xi', s)$$

of $\mathcal{Z}_\Gamma$ along the line $\mathcal{R}e\, s = \eta$, $\eta > 0$ being given. More precisely, denoting $\mathcal{F}$ as the Fourier transform in $\mathbf{x}'$ (with the dual variable ξ') and Laplace transform in time (with dual variable $s \in \mathbb{C}$), such that, for any $\varphi \in L^1(0,+\infty;L^1(\mathbb{R}^{d-1}))$, for all $\xi' \in \mathbb{R}^{d-1}$, $\mathcal{R}e\, s > 0$:

$$\mathcal{F}\varphi(\xi', s) := \frac{1}{(2\pi)^{d/2}} \int_0^{+\infty} \int_\Gamma \varphi(\mathbf{x}', t) e^{-i\xi'\cdot\mathbf{x}'} e^{-st}\, \mathrm{d}\mathbf{x}'\, \mathrm{d}t, \tag{4.17}$$

one easily computes that

$$\begin{aligned} &\mathcal{F}(\mathcal{Z}_\Gamma\varphi)(\xi', s) = \widehat{\mathcal{Z}}_\Gamma(\xi', s)\mathcal{F}\varphi(\xi', s), \\ &\widehat{\mathcal{Z}}_\Gamma(\xi', s) := \frac{s}{2}\big(|\xi'|^2 + s^2\big)^{-1/2}, \\ &(\text{where } \mathcal{R}e\, z^{1/2} \geq 0). \end{aligned} \tag{4.18}$$

Lemma 4.5. *There exist two constants* $0 < C_- < C_+$ *such that, for any* $\eta > 0$,

$$C_-(1+|\xi'|^2/\eta^2)^{1/4} \leq \sup_{\omega\in\mathbb{R}} |\widehat{\mathcal{Z}}_\Gamma(\xi', \eta+i\,\omega)| \leq C_+(1+|\xi'|^2/\eta^2)^{1/4}. \tag{4.19}$$

For any $\eta > 0$,

$$\frac{1}{2}(1+|\xi'|^2/\eta^2)^{-1/2} \leq \inf_{\omega\in\mathbb{R}} \mathcal{R}e\, \widehat{\mathcal{Z}}_\Gamma(k, \eta+i\,\omega) \leq (1+|\xi'|^2/\eta^2)^{-1/2}. \tag{4.20}$$

The proof of these estimates is purely computational and postponed until the Appendix (see Section B). Then, the answer to question (i), respectively, question (ii), is obtained by looking at the behavior as $|\xi'|$ goes to $+\infty$ for the bounds in inequality (4.19), respectively, (4.20). More precisely, using standard tools such as Plancherel's theorem and causality arguments, one easily shows the following.

Corollary 4.6. *The smallest real number s for which the bilinear form $b_T^{\mathcal{Z}}(\cdot,\cdot)$ is continuous in $L^2(0,T;H^s((\Gamma)))$ is $s = 1/4$. The largest value of s for which the weighted bilinear form $b_{\eta,T}^{\mathcal{Z}}(\cdot,\cdot)$ is coercive $L^2(0,T;H^s(\Gamma))$ is $s = -1/2$.*

Remark 4.7. The continuity result of $b_T^{\mathcal{Z}}(\cdot,\cdot)$ in $L^2(0,T;H^{\frac{1}{4}}(\Gamma))$ is due to **[2]**. The (quite tricky) proof they gave did not use the Laplace-transform in time and lemma 4.5, but instead the Fourier transform in time. The proof that we give here is easier and in our sense more natural. It also proves the optimality of the continuity result.

It is interesting to remark that Corollary 4.6 explains why it is possible to use piecewise polynomial functions, in both space and time variables, for the finite element approximation of the operator $\mathcal{Z}_\Gamma$ since such functions belong to $L^2(0,T;H^{1/4}(\Gamma))$.

Corollary 4.6 implies, in particular, that, for $\eta > 0$ and some positive constants α_T and $M_T \geq \alpha_T$, for all $\varphi \in L^2(0,T;H^{1/4}(\Gamma))$:

$$\alpha_T \|\varphi\|^2_{L^2(0,T;H^{-1/2}(\Gamma))} \leq b_{\eta,T}^{\mathcal{Z}}(\varphi,\varphi) \leq M_T \|\varphi\|^2_{L^2(0,T;H^{1/4}(\Gamma))}. \tag{4.21}$$

In this situation, we say that we have a coercivity-regularity gap which is 0 in time and 3/4 in space. Let us close this section by mentioning some open related questions.

• It would be interesting to address the same questions as in (i) and (ii) after inverting the role of space and time regularity (i.e., replacing $L^2(0,T;H^s(\Gamma))$ by $H^s(0,T;L^2(\Gamma))$. There is no difficulty a priori; we simply did not address this question for the moment.

• The weakness of the results of this section is in that they are restricted to flat surfaces. It is of course very much tempting to conjecture that such results should be extendable to smooth (C^∞) surfaces, by adapting, for instance, the technique of **[42]**. However, the hyperbolic nature of the wave equation encourages caution.

APPENDIX

A. Proof of trace Lemma 3.10. We restrict ourselves to giving the proof the trace estimate in the case where Ω_j is the halfspace

$$\mathbb{R}^d_+ := \{\mathbf{x} = (\mathbf{x}', x_d), \mathbf{x}' \in \mathbb{R}^{d-1},\ x_d > 0\}$$

and when $T = +\infty$. The case of a general domain with a smooth boundary is then obtained as usual (cf., **[18]**) using local charts for the parametrization and a partition of unity for passing from local to global estimates. Obtaining the results for $T < +\infty$ relies an a standard time localization process. These last two steps will not be detailed in what follows.

Let $\mathbf{w}$ be a vector field in $C^\infty(\mathbb{R}^d_+ \times \mathbb{R})$ with compact support in $\overline{\mathbb{R}^d_+} \times \mathbb{R}^+$. We have

$$\mathbf{w} \cdot \mathbf{n} = -w_d \quad \text{on } \Gamma := \partial\mathbb{R}^d_+ \equiv \{\mathbf{x} = (\mathbf{x}', x_d), \mathbf{x}' \in \mathbb{R}^{d-1}, x_d = 0\}.$$

Let $\widehat{\mathbf{w}}(\xi', x_d, \tau)$ be the partial Fourier transform of $\mathbf{w}$ in the variables $\mathbf{x}' \in \mathbb{R}^{d-1}$ (with dual variable ξ') and t (with dual variable τ). By definition of the norm in $H^{1/2}(0, +\infty; H^{-1/2}(\Gamma))$

$$\|w_d\|^2_{H^{1/2}(0,+\infty;H^{-1/2}(\Gamma))} := \int_{-\infty}^{+\infty} \int_{\mathbb{R}^{d-1}} \big(1 + |\tau|^2\big)^{1/2} \big(1 + |\xi'|^2\big)^{-1/2} |\widehat{w}_d(\xi', 0, \tau)|^2 \, \mathrm{d}\xi' \, \mathrm{d}\tau.$$

We have, for any $(\xi', \tau) \in \mathbb{R}^{d-1} \times \mathbb{R}$,

$$\begin{aligned} |\widehat{w}_d(\xi', 0, \tau)|^2 &= -\int_0^{+\infty} \partial_{x_d}\big(|\widehat{w}_d(\xi', x_d, \tau)|^2\big) \, \mathrm{d}x_d \\ &\leq 2 \int_0^{+\infty} \big|\partial_{x_d}\widehat{w}_d(\xi', x_d, \tau)\big| \big|\widehat{w}_d(\xi', x_d, \tau)\big| \, \mathrm{d}x_d. \end{aligned}$$

Setting $\widehat{\mathbf{w}}' = (\widehat{w}_1, \ldots, \widehat{w}_{d-1})$, we thus have (we use $\partial_{x_d}\widehat{w}_d = (\partial_{x_d}\widehat{w}_d + i\xi' \cdot \widehat{\mathbf{w}}') - i\xi' \cdot \widehat{\mathbf{w}}'$)

$$\begin{aligned} |\widehat{\mathbf{w}}_d(\xi', 0, \tau)|^2 \leq{} & 2 \int_0^{+\infty} \big|(\partial_{x_d}\widehat{w}_d + i\xi' \cdot \widehat{\mathbf{w}}')(\xi', x_d, \tau)\big| \big|\widehat{w}_d(\xi', x_d, \tau)\big| \, \mathrm{d}x_d \\ & + 2 \int_0^{+\infty} |\xi'| \big|\widehat{\mathbf{w}}'(\xi', x_d, \tau)\big| \big|\widehat{w}_d(\xi', x_d, \tau)\big| \, \mathrm{d}x_d. \end{aligned}$$

As a consequence, multiplying by $(1+|\tau|^2)^{1/2}$ and using Young's inequality, we get

$$\begin{aligned}&\big(1+|\tau|^2\big)^{1/2}|\widehat{w}_d(\xi',0,\tau)|^2\\&\quad\le\int_0^{+\infty}\Big(\big|\widehat{\operatorname{div}\mathbf{w}}(\xi',x_d,\tau)\big|^2+\big(1+|\tau|^2\big)\big|\widehat{w}_d(\xi',x_d,\tau)\big|^2\Big)\,\mathrm{d}x_d\\&\qquad+2\int_0^{+\infty}|\xi'|\big(1+|\tau|^2\big)^{1/2}\big|\widehat{\mathbf{w}}'(\xi',x_d,\tau)\big|\big|\widehat{w}_d(\xi',x_d,\tau)\big|\,\mathrm{d}x_d.\end{aligned}$$

We then multiply by $(1+|\xi'|^2)^{-1/2}$ and use $|\xi'|(1+|\xi'|^2)^{-1/2}\le 1$, $(1+|\xi'|^2)^{-1/2}\le 1$ and Young's inequality again to obtain

$$\begin{aligned}&\big(1+|\tau|^2\big)^{1/2}\big(1+|\xi'|^2\big)^{-1/2}|\widehat{w}_d(\xi',0,\tau)|^2\\&\quad\le\int_0^{+\infty}\Big(\big|\widehat{\operatorname{div}\mathbf{w}}(\xi',x_d,\tau)\big|^2+\big(1+|\tau|^2\big)\big|\widehat{w}_d(\xi',x_d,\tau)\big|^2\Big)\,\mathrm{d}x_d\\&\quad+\int_0^{+\infty}\Big(\big|\widehat{\mathbf{w}}'(\xi',x_d,\tau)\big|^2+\big(1+|\tau|^2\big)\big|\widehat{w}_d(\xi',x_d,\tau)\big|^2\Big)\,\mathrm{d}x_d.\end{aligned}$$

After integration of the above inequality over $(\xi',\tau)\in\mathbb{R}^{d-1}\times\mathbb{R}$, we obtain, due to Plancherel's theorem

$$\begin{aligned}&\int_{-\infty}^{+\infty}\int_{\mathbb{R}^{d-1}}\big(1+|\tau|^2\big)^{1/2}\big(1+|\xi'|^2\big)^{-1/2}|\widehat{w}_d(\xi',0,\tau)|^2\,\mathrm{d}\xi'\,\mathrm{d}\tau\\&\quad\le\int_0^{+\infty}\int_\Omega\big(|\mathbf{w}|^2+|\partial_t w_d|^2+|\operatorname{div}\mathbf{w}|^2\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\&\quad+\int_0^{+\infty}\int_\Omega\big(|\mathbf{w}|^2+|\partial_t w_d|^2\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,\end{aligned}$$

which leads finally to:

$$\begin{aligned}&\|\mathbf{w}\cdot\mathbf{n}\|^2_{H^{1/2}(0,+\infty;H^{-1/2}(\Gamma))}\\&\qquad\qquad\le 2\big\{\|\mathbf{w}\|^2_{H^1(0,+\infty;L^2(\Omega))}+\|\mathbf{w}\|^2_{L^2(0,+\infty;H(\mathrm{div};\Omega))}\big\}.\end{aligned}$$

Through a localization process, one can thus obtain the result for a finite time interval $[0,T]$. Finally, when considering causal vector fields $\mathbf{w}\in C_c^\infty(\mathbb{R}^+\times[0,T])$, one easily gets equation (3.38) using a Poincaré inequality.

B. Proof of Lemma 4.5.

Proof of (4.19). One first computes from equation (4.18) that

$$2\,|\widehat{\mathcal{Z}}_\Gamma(\xi',\eta+i\,\omega)|^4 = \frac{|\eta+i\,\omega|^4}{\big||\xi'|^2+(\eta+i\omega)^2\big|^2} = F\big(|\xi'|^2/\eta^2,\ \omega^2/\eta^2\big),$$

$$F(x,y) = \frac{(1+y)^2}{\big(y-x\big)^2+2y+2x+1} > 0.$$

Let us study the (strictly positive) function

$$F^*(x) = \sup_{y\geq 0} F(x,y), \quad \text{for } x\geq 0.$$

We first observe that $F(x,y)\to 1$ when $y\to+\infty$ and that $F(x,0) = (1+x)^{-2}\leq 1$. Looking at possible local extrema y^* of $y\mapsto F(x,y)$ leads to the equation $\partial_y F(x,y^*)=0$ which gives, after some computation,

$$x\big(x-y^*+3\big)=0 \quad \text{that is} \quad x=0 \text{ or } y^*(x) \;=\; x+3.$$

Since $F(0,y)=1$, we find that

$$F^*(x) = \max\big(F(x,y^*(x)),1\big) = 1+\frac{x}{4}.$$

One deduces that, for $x\geq 0$, the function $x\to F^*(x)/(1+x)$ never vanishes and tends to $1/4$ at infinity. It is then bounded from above and below by two positive constants, which achieves the proof of (4.19).

Proof of (4.20). First observe that (note that the meaning of the variables x and y is not the same as in the first part of the proof)

$$2\widehat{\mathcal{Z}}_\Gamma(\xi',\eta+i\,\omega) = \frac{\eta+i\,\omega}{\big(|\xi'|^2+(\eta+i\omega)^2\big)^{1/2}} = K\left(\frac{|\xi'|}{\eta},\frac{\omega}{\eta}\right)$$

with

$$K(x,y) = \frac{1+iy}{\big(x^2+(1+iy)^2\big)^{1/2}}.$$

Proving equation (4.20) amounts to obtaining upper and lower bounds for $\inf_{y\in\mathbb{R}}\mathcal{R}e\,K(x,y)$. Since $K(x,0)=(1+x^2)^{-1/2}$, we immediately have

$$\inf_{y\in\mathbb{R}}\mathcal{R}e\,K(x,y)\leq (1+x^2)^{-1/2}. \tag{B.1}$$

We now look for a lower bound. Writing $(x^2 + (1 + iy)^2)^{1/2} = \alpha + i\,\beta, (\alpha, \beta) \in \mathbb{R}$, we have

$$K(x, y) = \frac{(1 + iy)(\alpha - i\beta)}{(\alpha^2 + \beta^2)},$$

where $\alpha = \alpha(x, y) \in \mathbb{R}$ and $\beta = \beta(x, y) \in \mathbb{R}$ are entirely determined by:

$$\text{(B.2)} \qquad \alpha^2 - \beta^2 = 1 + x^2 - y^2, \quad \alpha\beta = y, \ \alpha > 0.$$

In particular,

$$\text{(B.3)} \qquad \mathcal{R}e\, K(x, y) = \frac{\alpha + y\beta}{(\alpha^2 + \beta^2)} = \frac{\alpha^2 + y^2}{\alpha(\alpha^2 + \beta^2)} \quad \text{(since } \alpha\beta = y\text{)}.$$

From the first equation of (B.2), we deduce that

$$2(\alpha^2 + y^2) = \alpha^2 + \beta^2 + (1 + x^2 + y^2) \geq \alpha^2 + \beta^2.$$

Therefore, equation (B.3) yields

$$\text{(B.4)} \qquad \mathcal{R}e\, K(x, y) \geq 1/(2\alpha).$$

A lower bound for $y \mapsto \mathcal{R}e\, K(x, y)$ will thus be obtained from an upper bound for $y \mapsto \alpha(x, y)$.

Let us look at what happens when $y \to \pm\infty$. Since

$$\alpha^2 + \beta^2 = \left(\left(x^2 + 1 - y^2\right)^2 + 4y^2\right)^{1/2},$$

one gets

$$2\alpha^2 = \left(\left(x^2 + 1 - y^2\right)^2 + 4y^2\right)^{1/2} + \left(x^2 - y^2 + 1\right).$$

Using $((x^2 - y^2 + 1)^2 + 4y^2)^{1/2} = y^2 - x^2 + 1 + O(y^{-2})$ in this last equality, we deduce that

$$\lim_{y \to \pm\infty} \alpha(x, y) = 1.$$

Looking at the local extrema of $y \mapsto \alpha(x, y)$ leads to the investigation of the following system, obtained from differentiation of equation (B.2),

$$\begin{cases} \alpha \partial_y \alpha - \beta \partial_y \beta = -y \\ \alpha \partial_y \beta + \beta \partial_y \alpha = 1, \end{cases}$$

which implies that $(\alpha^2+\beta^2)\partial_y\alpha = \beta - \alpha y$. Thus, $\partial_y\alpha = 0 \Rightarrow \beta = \alpha y$. Since, $\alpha\beta = y$, we see that

$$\partial_y\alpha = 0 \Longrightarrow (1-\alpha^2)y = 0 \Longrightarrow \alpha = 1 \quad \text{or} \quad y = 0.$$

At $y = 0$, we have $\alpha(x,0) = (1+x^2)^{1/2} \geq 1$. Thus, since $\lim_{y\to\pm\infty}\alpha(x,y) = 1$, we have proven that $\sup_{y\in\mathbb{R}}\alpha(x,y) = (1+x^2)^{1/2}$. Finally, equation (B.4) yields

$$\inf_{y\in\mathbb{R}} \mathcal{R}e\, K(x,y) \geq \frac{1}{2}(1+x^2)^{1/2}. \tag{B.5}$$

which achieves the proof. □

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POEMS, INRIA-CNRS: UMR7231-ENSTA ParisTech, 828 Boulevard des Maréchaux, 91762 Palaiseau, France
Email address: **patrick.joly@inria.fr**

Departamento de Matemática Aplicada, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, A Coruña, Spain
Email address: **jeronimo.rodriguez@usc.es**