JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 23, Number 2, Summer 2011

A NYSTRÖM METHOD FOR FREDHOLM INTEGRAL EQUATIONS ON THE REAL LINE

G. MASTROIANNI AND I. NOTARANGELO

Communicated by Giovanni Monegato

ABSTRACT. The authors introduce a new procedure for the numerical treatment of Fredholm equations of the second kind on the real axis, based on a Nyström method. The convergence of the method is proved and a priori estimates of the error are given. The case of kernels containing a Hilbert transform is also considered.

1. Introduction. This paper concerns the numerical treatment of the class of integral equations defined by

$$f(y) - \mu \int_{\mathbf{R}} \Gamma(x,y) f(x) w(x)\, \mathrm{d}x = g(y), \quad y \in \mathbf{R}, \tag{1.1}$$

where

$$\Gamma(x,y) = k(x,y) + \nu \int_{\mathbf{R}} \frac{\eta(x,t)\sqrt{w(t)}}{t-y}\, \mathrm{d}t, \tag{1.2}$$

μ and ν are real parameters, w is a Freud weight, k, η and g are given continuous functions, f is an unknown function. The integral in (1.2) is understood in the Cauchy principal value sense, i.e., it defines the Hilbert transform of the function $\eta(x,\cdot)\sqrt{w(\cdot)}$.

2010 AMS *Mathematics subject classification.* Primary 65R20, 45E05, 41A05, 41A10, 65D30.
Keywords and phrases. Fredholm integral equation, Nyström method, Hilbert transform.
This work was supported by Università degli Studi della Basilicata (local funds) and by PRIN 2008 "Equazioni integrali e sistemi lineari con struttura" No. 20083KLJEZ.
Received by the editors on October 8, 2008, and in revised form on March 3, 2009.
DOI:10.1216/JIE-2011-23-2-253

In the literature a first natural approach was to consider this equation in L^p-spaces, because the Hilbert transform is bounded in such spaces (see, for instance, [**4**]). On the other hand, until now also the Fredholm equations on the real line have been studied only in L^2 (see [**11**]).

In this paper we consider equation (1.1) in a space of continuous functions equipped with a weighted uniform norm. We believe that in numerical analysis this approach is more suitable than the previous one. In fact, usually pointwise estimates of the error (and not L^p-norms of the error) are shown. On the other hand, if we express the L^p-norm of the error in terms of the uniform norm by means of embedding theorems, we obtain a convergence order less than the previous one. Obviously our approach generates further theoretical difficulties, due to the unboundedness of the Hilbert transform in spaces of continuous functions.

Our strategy consists essentially of two steps. First, assuming the functions k and g to be continuous, we solve the Fredholm equation

$$f(y) - \mu \int_{\mathbf{R}} k(x,y) f(x) w(x)\, \mathrm{d}x = g(y), \quad y \in \mathbf{R}, \tag{1.3}$$

using a Nyström method based on a Gauss-type rule. Then, coming back to equation (1.1), we assign the conditions on the functions k and η under which the kernel Γ is "smooth" (see Lemmas 4.1 and 4.2). Hence (1.1) turns out to be a Fredholm equation. Consequently, we use again the Nyström method, mutatis mutandis.

The stability and the convergence of the method are proved. Moreover, the error estimates and the well-conditioning of the linear system (associated to the Nyström method) are stated.

The exposed method can be adapted for solving Cauchy singular equations of the form

$$f(y) - \nu \int_{\mathbf{R}} \left[\frac{1}{x-y} + \tau(x,y) w(x) \sqrt{w(y)} \right] f(x)\, \mathrm{d}x = \widetilde{g}(y), \quad y \in \mathbf{R}, \tag{1.4}$$

where $\nu \in \mathbf{R}$, τ and $\widetilde{g}$ are given continuous functions. We observe that the numerical treatment of equation (1.4) by global approximation has received very little attention in the literature, unlike the Cauchy integral equations on $[-1,1]$. For the details, the reader can consult [**18, 20**] and the references therein.

Obviously we cannot apply the Nyström method directly to (1.4). Nevertheless, regularizing this equation (see Section 6), under suitable hypotheses, we get a Fredholm equation of the form (1.1) that can be solved by the proposed method.

The results are new and, in the analysis of the stability and the convergence of the method, we use as main tool the polynomial approximation theory on the real line (see [**9**, pages 180–196]).

This paper is organized as follows. In Section 2 we recall some basic facts. In Section 3 we study the Nyström method for equation (1.3). In Section 4 we consider equation (1.1). In Section 5 we suggest a numerical method for evaluating the coefficients of the linear system and the Nyström interpolant. In Section 6 we reduce equation (1.4) to the form (1.1). In Section 7 some numerical examples are shown. Finally in Section 8 we prove our results.

2. Notations and preliminary results. Let us consider the weight

$$u(x) = (1+|x|)^{\beta} \mathrm{e}^{-|x|^{\alpha}/2}, \quad \beta \geq 0,\ \alpha > 1. \tag{2.1}$$

We are going to study the integral equations in the space

$$C_u = \left\{ f \in C^0(\mathbf{R}) : \lim_{x \to \pm\infty} f(x)u(x) = 0 \right\},$$

equipped with the norm

$$\|f\|_{C_u} := \|fu\|_{\infty} = \sup_{x \in \mathbf{R}} |f(x)u(x)|.$$

We emphasize that $f \in C_u$ can increase exponentially for $x \to \pm\infty$.

Moreover, by L^p_u, $1 \leq p < \infty$, we denote the set of all measurable functions f such that

$$\|f\|_{L^p_u} := \|fu\|_p = \left(\int_{\mathbf{R}} |f(x)|^p u^p(x)\, \mathrm{d}x \right)^{1/p} < \infty.$$

For simplicity of notation, we will write L^∞_u instead of C_u. Such spaces L^p_u, $1 \leq p \leq \infty$, with the above-defined norms are Banach spaces.

Subspaces of L^p_u are the Sobolev spaces, defined by

$$W^p_r(u) = \left\{ f \in L^p_u : \; f^{(r-1)} \in AC(\mathbf{R}), \; \|f^{(r)}u\|_p < \infty \right\}, \quad r \geq 1,$$

where $AC(\mathbf{R})$ denotes the set of all the functions which are absolutely continuous on every closed subset of $\mathbf{R}$. We equip these spaces with the norm

$$\|f\|_{W^p_r(u)} = \|fu\|_p + \|f^{(r)}u\|_p.$$

For any $f \in L^p_u$, $1 \leq p \leq \infty$, we consider the following r-th ($r \in \mathbf{N}$) modulus of smoothness with a sufficiently small step δ (see [**9**, pages 182–183]):

$$\omega^r(f,\delta)_{u,p} = \Omega^r(f,\delta)_{u,p} + \sum_{k=1}^{2} \inf_{P \in \mathbf{P}_{r-1}} \|(f-P)u\|_{L^p(\mathcal{J}_k)},$$

where $\mathcal{J}_1 = (-\infty, -\mathrm{A}r\,\delta^{-1/(\alpha-1)})$, $\mathcal{J}_2 = (\mathrm{A}r\,\delta^{-1/(\alpha-1)}, +\infty)$. Its main part $\Omega^r(f,\delta)_{u,p}$ is defined by

$$\Omega^r(f,\delta)_{u,p} = \sup_{0<h\leq\delta} \|\Delta^r_h(f)u\|_{L^p(\mathcal{I}_h)}$$

with $\mathcal{I}_h = [-\mathrm{A}r\,h^{-1/(\alpha-1)}, \mathrm{A}r\,h^{-1/(\alpha-1)}]$, $\mathrm{A} > 1$ a fixed constant and the notation

$$\Delta_h(f;x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right), \quad \Delta^r_h = \Delta_h(\Delta^{r-1}_h). \tag{2.2}$$

By definition we have

$$\Omega^{r+1}(f,\delta)_{u,p} \leq 2\,\Omega^r(f,\delta)_{u,p}. \tag{2.3}$$

We will write $\omega(f,\delta)_{u,p} = \omega^1(f,\delta)_{u,p}$ and $\Omega(f,\delta)_{u,p} = \Omega^1(f,\delta)_{u,p}$.

In the sequel C will stand for a positive constant that can assume different values in each formula and we shall write $\mathrm{C} \neq \mathrm{C}(a,b,\dots)$ when C is independent of $a, b, \dots$. Furthermore $A \sim B$ will mean that if A and B are positive quantities depending on some parameters, then there exists a positive constant C independent of these parameters such that $(A/B)^{\pm 1} \le \mathrm{C}$. Finally $\lfloor a \rfloor$ will denote the largest integer smaller than or equal to $a \in \mathbf{R}^+$.

Let us denote by $\mathbf{P}_m$ the set of all algebraic polynomials of degree at most m and by $E_m(f)_{u,p} = \inf_{P \in \mathbf{P}_m} \|(f-P)u\|_p$ the error of best polynomial approximation in L^p_u. The following Jackson and Stechkin-type inequalities hold true (see [**9**, page 185]):

$$E_m(f)_{u,p} \le \mathrm{C}\, \omega^r\left(f, \frac{a_m}{m}\right)_{u,p}, \quad r < m, \tag{2.4}$$

$$\omega^r\left(f, \frac{a_m}{m}\right)_{u,p} \le \mathrm{C}\left(\frac{a_m}{m}\right)^r \sum_{k=1}^{m} \left(\frac{k}{a_k}\right)^r \frac{E_k(f)_{u,p}}{k}, \tag{2.5}$$

where $f \in L^p_u$, $1 \le p \le \infty$, $a_m \sim m^{1/\alpha}$ is the Mhaskar-Rahmanov-Saff (M-R-S for short) number related to the weight u and $\mathrm{C} \ne \mathrm{C}(f, m)$.

Hence, by (2.4) and (2.5), we have

$$f \in L^p_u \Longleftrightarrow \lim_{m \to \infty} E_m(f)_{u,p} = 0 \tag{2.6}$$

for $1 \le p \le \infty$.

An estimate of the error of best polynomial approximation, weaker than (2.4), is given by:

$$E_m(f)_{u,p} \le \mathrm{C} \int_0^{a_m/m} \frac{\Omega^r(f,\delta)_{u,p}}{\delta}\, \mathrm{d}\delta, \tag{2.7}$$

where $1 \le p \le \infty$ and $\mathrm{C} \ne \mathrm{C}(m, f)$. Obviously inequality (2.7) holds true if the integral at the right-hand side exists.

By means of the modulus of smoothness, for $1 \le p \le \infty$, we can define the Zygmund space

$$Z^p_s(u) := Z^p_{s,r}(u) = \left\{ f \in L^p_u : \sup_{\delta > 0} \frac{\omega^r(f,\delta)_{u,p}}{\delta^s} < \infty,\ r > s \right\},\ s \in \mathbf{R}^+,$$

equipped with the norm

$$\|f\|_{Z^p_{s,r}(u)} = \|f\|_{L^p_u} + \sup_{\delta > 0} \frac{\omega^r(f,\delta)_{u,p}}{\delta^s}.$$

In the sequel we will denote these subspaces briefly by $Z_s^p(u)$, without the second index r and with the assumption $r > s$.

We remark that, by (2.5) and (2.7), we deduce $\Omega^r(f,\delta)_{u,p} \sim \omega^r(f,\delta)_{u,p}$ for every $f \in Z_s^p(u)$.

Let us consider the Freud weight

$$w(x) = \mathrm{e}^{-|x|^\alpha}, \qquad \alpha > 1, \tag{2.8}$$

and the related M-R-S number a_m, given by (see for instance [**15**])

$$a_m = a_m(w) = \left[\frac{2\pi}{\alpha\, B\left((\alpha+1)/2, (1/2)\right)}\right]^{1/\alpha} m^{1/\alpha}, \tag{2.9}$$

where B is the beta function. Since $a_m(w) \sim a_m(\sqrt{w}) \sim a_m(u)$, in the sequel we will write a_m, when we will not need to make a distinction among the weights.

Let $\{p_m(w)\}_{m\in\mathbf{N}}$ be the sequence of the polynomials which are orthonormal with respect to w and having positive leading coefficients. We denote by x_k, $1 \le k \le \lfloor m/2 \rfloor$, the positive zeros of $p_m(w)$ and by $x_{-k} = -x_k$ the negative ones. If m is odd then $x_0 = 0$ is a zero of $p_m(w)$. These zeros satisfy

$$\begin{aligned} -a_m\left(1 - \frac{\mathrm{C}}{m^{2/3}}\right) &\le x_{-\lfloor m/2\rfloor} < \cdots < x_1 < x_2 < \cdots < x_{\lfloor m/2 \rfloor} \\ &\le a_m\left(1 - \frac{\mathrm{C}}{m^{2/3}}\right), \end{aligned} \tag{2.10}$$

where C is a positive constant independent of m (see [**16**, page 28]).

Our aim is to use the Nyström method for integral equations; then we need a quadrature rule. In order to introduce it, for a fixed $\theta \in (0,1)$, we define an index $j = j(m,\theta)$ such that

$$x_j = \min_{1 \le k \le \lfloor m/2 \rfloor} \left\{x_k :\ x_k \ge \theta a_m\right\}. \tag{2.11}$$

With the same θ we set

$$M = \left\lfloor \left(\frac{\theta}{\theta+1}\right)^\alpha \frac{m}{2} \right\rfloor =: \mathrm{C}m, \tag{2.12}$$

with $0 < \mathrm{C} < 1$.

Thus we consider the Gauss-type rule

$$\int_{\mathbf{R}} f(x)w(x)\,\mathrm{d}x = \sum_{|k|\le j} \lambda_k(w) f(x_k) + e_m(f), \tag{2.13}$$

where x_k are the above introduced Freud zeros, $\lambda_k(w)$ are the Christoffel numbers and $e_m(f)$ is the remainder term. The following proposition gives a simple estimate for $e_m(f)$ (see [**17**]).

Proposition 2.1. *Let* $f \in C_\sigma$, *where* $\sigma(x) = (1+|x|)^\beta \mathrm{e}^{-a|x|^\alpha}$, $0 < a \le 1$. *If* $\int_{\mathbf{R}} w(x)\sigma^{-1}(x)\,\mathrm{d}x < \infty$, *then*

$$|e_m(f)| \le \mathrm{C}\left\{E_M(f)_{\sigma,\infty} + \mathrm{e}^{-\mathrm{A}m}\|f\sigma\|_\infty\right\}, \tag{2.14}$$

where $M \sim m$ *is given by* (2.12), C *and* A *are positive constant independent of* m *and* f.

3. The numerical method. Let us consider the integral equation

$$f(y) - \mu \int_{\mathbf{R}} k(x,y) f(x) w(x)\,\mathrm{d}x = g(y), \quad y \in \mathbf{R}, \tag{3.1}$$

where μ is a real parameter, f is the unknown function, k and g are given continuous functions and $w(x) = \mathrm{e}^{-|x|^\alpha}$, $\alpha > 1$, is a Freud weight.

Denoting by K the integral operator given by

$$(Kf)(y) = \int_{\mathbf{R}} k(x,y) f(x) w(x)\,\mathrm{d}x \tag{3.2}$$

and by I the identity operator, we rewrite (3.1) in the form

$$(I - \mu K) f = g. \tag{3.3}$$

Furthermore, we will write $k_x(y) = k(x,y) = k_y(x)$.

Proposition 3.1. *Let* $u(x) = (1+|x|)^\beta \mathrm{e}^{-|x|^\alpha/2}$, *with* $\alpha > 1$ *and* $\beta > 1/2$. *If*

$$\lim_{m\to\infty} \sup_{x\in\mathbf{R}} u(x) E_m(k_x)_{u,\infty} = 0 \tag{3.4}$$

then the operator $K : C_u \to C_u$ *defined by* (3.2) *is compact and the Fredholm alternative holds true for equation* (3.1).

Notice that, by (2.6), (3.4) implies $k_x \in C_u$ for any fixed $x \in \mathbf{R}$.

We want to use a Nyström method for solving equation (3.1) under the assumptions of Proposition 3.1. Thus we approximate the integral in (3.1) applying the Gauss-type rule (2.13), and we introduce the operators K_m ($m \in \mathbf{N}$), defined by

$$(K_m f)(y) = \sum_{|l| \le j} \lambda_l(w) k(x_l, y) f(x_l), \quad y \in \mathbf{R}, \tag{3.5}$$

for any $f \in C_u$. Hence, for a sufficiently large m (say $m \ge m_0$), we will solve the equation

$$f_m(y) - \mu \sum_{|l| \le j} \lambda_l(w) \frac{k(x_l, y)}{u(x_l)} f(x_l) u(x_l) = g(y), \quad y \in \mathbf{R}, \tag{3.6}$$

where x_l ($|l| \le j$) are zeros of $p_m(w)$, j is defined by (2.11) and $\lambda_l(w)$ are the Christoffel numbers.

Let us set $\xi_l := f(x_l)u(x_l)$. Multiplying equation (3.6) by $u(y)$ and collocating at the points x_i ($|i| \le j$), we obtain the system of linear equations

$$\sum_{|l| \le j} \left[\delta_{il} - \mu \lambda_l(w) k(x_l, x_i) \frac{u(x_i)}{u(x_l)} \right] \xi_l = b_i, \quad |i| \le j, \tag{3.7}$$

where $b_i := g(x_i)u(x_i)$ and δ_{il} is the Kronecker delta.

If system (3.7) admits a unique solution $\xi = (\xi^*_{-j}, \dots, \xi^*_1, \xi^*_2, \dots, \xi^*_j)^T$, by (3.6), we can define the Nyström interpolant

$$f^*_m(y) = g(y) + \mu \sum_{|l| \le j} \lambda_l(w) \frac{k(x_l, y)}{u(x_l)} \xi^*_l, \quad y \in \mathbf{R}, \tag{3.8}$$

and we will compare it with the solution of equation (3.3).

We rewrite system (3.7) as follows

$$A_j \, \xi = b, \tag{3.9}$$

letting

$$A_j = \left(\delta_{il} - \mu \lambda_l(w) k(x_l, x_i) \frac{u(x_i)}{u(x_l)} \right)_{i,l=-j}^{j} \tag{3.10}$$

$\xi = (\xi^*_{-j}, \dots, \xi^*_1, \xi^*_2, \dots, \xi^*_j)^T$ and $b = (b^*_{-j}, \dots, b^*_1, b^*_2, \dots, b^*_j)^T$. Furthermore, we denote by

$$\operatorname{cond}(A_j) = \|A_j\|_\infty \|A_j^{-1}\|_\infty$$

the condition number of the matrix A_j in infinity norm.

In conclusion, the proposed method consists in solving the system of linear equations in (3.9) and then in computing the Nyström interpolant defined by (3.8). We emphasize that system (3.9) has dimension $2j$ (or $2j+1$ if m is odd) and not m, that is the degree of $p_m(w)$, thus we omit the evaluation of $\lfloor cm^2 \rfloor$ $(0 < c < 1)$ coefficients. Moreover the construction of system (3.9) requires only the computations of the zeros x_l and the Christoffel numbers $\lambda_l(w)$ $(|l| \le j)$. In order to compute these quantities, in the Hermite case $(\alpha = 2)$ one can use the routine "gaussq" (see [**13, 14**]) or routines "recur" and "gauss" (see [**12**]). While if $\alpha \neq 2$ one can use the Mathematica Package "OrthogonalPolynomials" (see [**2**]).

The next theorem gives the conditions for the stability and the convergence of the method.

Theorem 3.2. *Let* $\ker(I - \mu K) = \{0\}$ *in* C_u, *where* u *is the weight in* (2.1) *with* $\beta > 1/2$. *Thus equation* (3.1) *has a unique solution* f^* *for any* $g \in C_u$. *If the conditions*

$$\lim_{m\to\infty} \sup_{z\in\mathbf{R}} u(z) E_m(k_z)_{u,\infty} = 0, \quad z \in \{x, y\}, \tag{3.11}$$

and

$$\lim_{m\to\infty} E_m(g)_{u,\infty} = 0, \tag{3.12}$$

are fulfilled, then the linear system in (3.9) *has a unique solution* ξ^* *for a sufficiently large* m *(say* $m \ge m_0$*),*

$$\sup_{m\ge m_0} \operatorname{cond}(A_j) \le \sup_{m \ge m_0} \operatorname{cond}(I - \mu K_m) < \infty, \tag{3.13}$$

and, denoting by f_m^* the Nyström interpolant defined by (3.8)*, we get*

$$\|[f^* - f_m^*]\, u\|_\infty \le \mathrm{C}\, \|f^* u\|_\infty \sup_{y\in\mathbf{R}} u(y) \left\{E_n(k_y)_{u,\infty} + \mathrm{e}^{-\mathrm{A}m} \|k_y u\|_\infty\right\} + \mathrm{C}\, E_n(f^*)_{u,\infty} \sup_{y\in\mathbf{R}} u(y) \|k_y u\|_\infty, \tag{3.14}$$

where $n = \lfloor M/2 \rfloor$, $M \sim m$ *is defined by* (2.12)*, the constants* C *and* A *are independent of* m *and* f*.*

We remark that, by (2.6) and assumptions (3.11) and (3.12), all the given functions k_x, k_y, g belong to C_u. It follows that $f^* \in C_u$ and all the quantities at the right hand side of (3.14) tend to 0 for $m \to \infty$. Therefore, the Nyström interpolant f_m^* given by (3.8) converges to the solution f^* of equation (3.1). Obviously, stronger assumptions on the kernel k and the function g lead to a faster convergence, as the following corollary shows.

Corollary 3.3. *Under the hypotheses of Theorem* 3.2*, with* (3.11) *and* (3.12) *replaced by*

$$\sup_{z\in\mathbf{R}} u(z) \|k_z\|_{Z_s^\infty(u)} < \infty, \quad z \in \{x, y\}, \tag{3.15}$$

and

$$g \in Z_s^\infty(u), \quad s > 0, \tag{3.16}$$

respectively, the linear system in (3.9) *has a unique solution for* $m \ge m_0$ *and* (3.13) *holds true. Moreover* (3.14) *becomes*

$$\|[f^* - f_m^*]\, u\|_\infty \le \mathrm{C} \left(\frac{a_n}{n}\right)^s \|f^*\|_{Z_s^\infty(u)} \sup_{y\in\mathbf{R}} u(y) \|k_y\|_{Z_s^\infty(u)}, \tag{3.17}$$

where $n = \lfloor M/2 \rfloor$, $M \sim m$ *is defined by* (2.12) *and* C *is independent of* m *and* f*.*

We point out that, by (3.17) and (2.4), the Nyström interpolant f_m^* in (3.8) converges to the solution f^* of (3.1) with the same order of the polynomial of best approximation of functions in the Zygmund spaces.

Theorem 3.2 is a consequence of the following lemma. To state it, let us recall definition (3.5) of the operators K_m:

$$(K_m f)(y) = \sum_{|l| \le j} \lambda_l(w) k(x_l, y) f(x_l), \quad y \in \mathbf{R},\ f \in C_u,\ m \in \mathbf{N}.$$

Lemma 3.4. *Let* $u(x) = (1+|x|)^\beta \mathrm{e}^{-|x|^\alpha/2}$, $\alpha > 1$ *and* $\beta > 1/2$. *Let* $k_x, k_y \in C_u$ *and assume that the kernel* k *satisfies the condition* (3.11). *Then we have*:

$$\sup_{m \in \mathbf{N}} \|K_m\|_{C_u \to C_u} < \infty \tag{3.18}$$

$$\lim_{n\to\infty} \sup_{m\in\mathbf{N}} \sup_{f \in C_u} \frac{E_n(K_m f)_{u,\infty}}{\|fu\|_\infty} = 0, \tag{3.19}$$

$$\lim_{m\to\infty} \|(Kf - K_m f)\, u\|_\infty = 0, \tag{3.20}$$

for all $f \in C_u$, *and*

$$\lim_{m\to\infty} \|(K - K_m)\, K_m\|_{C_u \to C_u} = 0. \tag{3.21}$$

We remark that inequality (3.19) is equivalent to the collectively compactness of the family of operators $\{K_m\}_m$. Moreover (3.20) implies (3.18) by the uniform boundedness principle.

4. The complete equation. We now consider the equation of second kind

$$f(y) - \mu \int_{\mathbf{R}} \Gamma(x,y) f(x) w(x)\, \mathrm{d}x = g(y), \quad y \in \mathbf{R}, \tag{4.1}$$

where μ is a real parameter, f is the unknown function, g is a given continuous function, $w(x) = \mathrm{e}^{-|x|^\alpha}$, $\alpha > 1$, is a Freud weight and the kernel Γ is defined by

$$\begin{aligned}\Gamma(x,y) &= k(x,y) + \nu \int_{\mathbf{R}} \frac{\eta(x,t)\sqrt{w(t)}}{t-y}\, \mathrm{d}t \\ &=: k(x,y) + \nu \mathcal{H}\left(\eta_x \sqrt{w}, y\right),\end{aligned} \tag{4.2}$$

with k and η given continuous functions, ν a real parameter and $\mathcal{H}$ denotes the Hilbert transform.

We can rewrite equation (4.1) as

$$(I - \mu K)f = g,$$

where

$$(Kf)(y) = \int_{\mathbf{R}} \Gamma(x, y) f(x) w(x)\, \mathrm{d}x \tag{4.3}$$

and I is the identity operator.

We remark that, if the kernel Γ defined by (4.2) is "smooth," then we could use, mutatis mutandis, the numerical method proposed in Section 3. Thus we want to establish the hypotheses on the functions k and η, under which the kernel Γ is in some sense smooth. The following lemmas provide the criteria for this aim. Before stating them we introduce the weights

$$u_i(x) = (1 + |x|)^{i(\alpha-1)} u(x), \quad i \in \mathbf{Z}, \tag{4.4}$$

where

$$u(x) = (1 + |x|)^{\beta} \sqrt{w(x)} = (1 + |x|)^{\beta} \mathrm{e}^{-|x|^{\alpha}/2},$$

and the notations

$$\widehat{\eta}(x, y) := \frac{\partial}{\partial y} \eta(x, y) \quad \Psi(x, y) := \mathcal{H}\left(\eta_x \sqrt{w}, y\right).$$

Lemma 4.1. *Let* $1 < p < \infty$, *w and u with parameters* $\alpha > 1$ *and* $\beta \geq 0$. *Assume that*

$$\sup_{x \in \mathbf{R}} u(x) \|\eta_x\|_{Z^p_{s+(1/p)}(u)} < \infty, \quad s > 0,$$

$$\sup_{x \in \mathbf{R}} u(x) \sum_{i=0}^{r-1} \|\eta_x^{(i)} u_{r-i}\|_p < \infty, \quad r = \left\lfloor s + \frac{1}{p} \right\rfloor + 1. \tag{4.5}$$

Then we have

$$\sup_{x\in\mathbf{R}} u(x)\,\|\Psi_x\|_{Z_s^\infty(u)} < \infty. \tag{4.6}$$

Shortly speaking, the previous lemma asserts that if a function F belongs to a Zygmund space $Z^p_{s+1/p}(u)$, $s>0$ and $1<p<\infty$, and it fulfills the further assumption $\|F^{(i)}u_{r-i}\|_p<\infty$, $i\le\lfloor s+1/p\rfloor+1$, then the Hilbert transform $\mathcal{H}(F\sqrt{w})$ belongs to the space $Z_s^\infty(u)$. On the other hand, in [**8**] it was proved that $F\in Z^p_{s+1/p}(u)$ implies $F\in Z_s^\infty(u)$. Combining these two results we obtain that if, for some $s>0$ and $1<p<\infty$, $F\in Z^p_{s+1/p}(u)$ and $\|F^{(i)}u_{r-i}\|_p<\infty$, $i\le\lfloor s+1/p\rfloor+1$, then F and $\mathcal{H}(F\sqrt{w})$ belong to the same Zygmund space $Z_s^\infty(u)$.

In analogy with Lemma 4.1, the next lemma ensures the smoothness of Ψ as a function of x.

Lemma 4.2. *Let w and u be defined as in Lemma* 4.1. *If*

$$\sup_{y\in\mathbf{R}} u_1(y)\,\|\eta_y\|_{Z_s^\infty(u)} < \infty \tag{4.7}$$

and

$$\sup_{y\in\mathbf{R}} u(y)\,\|\widehat{\eta}_y\|_{Z_s^\infty(u)} < \infty, \tag{4.8}$$

then we have

$$\sup_{y\in\mathbf{R}} u(y)\,\|\Psi_y\|_{Z_s^\infty(u)} < \infty. \tag{4.9}$$

Therefore, if the function η fulfills the assumptions of Lemmas 4.1 and 4.2, and if the function k is such that

$$\sup_{z\in\mathbf{R}} u(z)\|k_z\|_{Z_s^\infty(u)} < \infty, \quad z\in\{x,y\},$$

then we have

$$\sup_{z\in\mathbf{R}} u(z)\,\|\Gamma_z\|_{Z_s^\infty(u)} < \infty, \quad z \in \{x, y\}.$$

Hence, by Proposition 3.1, the operator K given by (4.3) is compact and for equation (4.1) the Fredholm alternative holds true. Moreover, since the kernel Γ satisfies the condition (3.15) of Corollary 3.3, for solving equation (4.1), we can apply the numerical method described in Section 3 for equation (3.1).

In this case the system of linear equations assumes the form

$$\sum_{|l|\le j}\left[\delta_{il} - \mu\lambda_l(w)\Gamma(x_l, x_i)\frac{u(x_i)}{u(x_l)}\right]\xi_l = b_i, \quad |i| \le j, \tag{4.10}$$

where $\xi_l := f(x_l)u(x_l)$, $b_i := g(x_i)u(x_i)$ and

$$\Gamma(x_l, x_i) = k(x_l, x_i) + \nu\int_{\mathbf{R}} \frac{\eta(x_l, t)\sqrt{w(t)}}{t - x_i}\,\mathrm{d}t, \tag{4.11}$$

and the Nyström interpolant becomes

$$f_m^*(y) = g(y) + \mu\sum_{|l|\le j}\lambda_l(w)\frac{\Gamma(x_l, y)}{u(x_l)}\xi_l^*. \tag{4.12}$$

Moreover, we derive the following theorem.

Theorem 4.3. *Let* $\ker(I - \mu K) = \{0\}$ *in* C_u, *where* u *is the weight in* (2.1) *and* $\beta > 1/2$. *Denote by* f^* *the unique solution of equation* (4.1) *for a fixed* g. *If the function* η *fulfills the assumptions of Lemmas* 4.1 *and* 4.2, *the function* k *satisfies*

$$\sup_{z\in\mathbf{R}} u(z)\|k_z\|_{Z_s^\infty(u)} < \infty, \quad z \in \{x, y\},$$

and

$$g \in Z_s^\infty(u),$$

then the linear system in (4.10) *has a unique solution* ξ^* *for* $m \ge m_0$ *and its matrix* A_j *is such that*

$$\sup_{m\ge m_0} \operatorname{cond}(A_j) \le \sup_{m\ge m_0} \operatorname{cond}(I - \mu K_m) < \infty. \tag{4.13}$$

Moreover, for the Nyström interpolant f_m^* defined by (4.12)*, the following estimate holds true*

$$\|[f^* - f_m^*]\,u\|_\infty = O\left(\left(\frac{a_n}{n}\right)^s\right), \tag{4.14}$$

where $n = \lfloor M/2 \rfloor$, $M \sim m$ is defined by (2.12) *and the constants in "O" are independent of m and f.*

Remark. We have proved the stability and the convergence of the method assuming that the given functions belong to Zygmund spaces. Nevertheless, we can obtain the same results with minor effort if these functions belong to Sobolev spaces.

5. The approximation of the Hilbert transform. In many cases an explicit form of the expression of $\Gamma(x,y)$, appearing in (4.11) and (4.12), is not known. Therefore, in order to construct the matrix of system (4.10) and the Nyström interpolant in (4.12), the main effort is to compute integrals of the form

$$\mathcal{H}(F\sqrt{w}, y) = \int_{\mathbf{R}} \frac{F(t)\sqrt{w(t)}}{t-y}\,\mathrm{d}t, \quad y \in \mathbf{R}, \tag{5.1}$$

where $F \in C_u$, $u(t) = (1+|t|)^\beta \sqrt{w(t)}$.

The quadrature rules we are going to consider have been introduced and studied in [**19**], following an idea previously used in [**3, 5**].

Letting B be the beta function, we denote by

$$a_m(\sqrt{w}) = \left[\frac{4\pi}{\alpha\, B\left((\alpha+1)/2, (1/2)\right)}\right]^{1/\alpha} m^{1/\alpha} =: c_\alpha m^{1/\alpha}, \tag{5.2}$$

the M-R-S number related to the weight $\sqrt{w}$, by $t_{m,k}$ ($|k| \le \lfloor m/2 \rfloor$) the zeros of $p_m(\sqrt{w})$ and by $\lambda_{m,k}(\sqrt{w})$ the corresponding Christoffel numbers. As in Section 2 we set

$$t_{m,j_1} = \min_{|k| \le \lfloor m/2 \rfloor} \left\{ t_{m,k} : t_{m,k} \ge \theta a_m(\sqrt{w}) \right\},$$

with $\theta \in (0,1)$ the same parameter chosen in definition (2.11) for the construction of system (4.10). For a fixed $y \in [-\theta a_m(\sqrt{w}), \theta a_m(\sqrt{w})]$, we are going to define the sequence of integers $\{m^*\}$, $m^* \in \{m, m+1\}$ as follows. Let $t_{m,d}$ be a zero of $p_m(\sqrt{w})$ closest to y. Then, for $m \geq m_0$, two cases are possible

$$-t_{m,j_1} \leq t_{m,d} \leq y \leq t_{m+1,d+1}$$

or

$$t_{m+1,d+1} \leq y \leq t_{m,d+1} \leq t_{m,j_1}.$$

In the first case, if $y < (t_{m,d}+t_{m+1,d+1})/2$ we set $m^* = m+1$, otherwise $m^* = m$. We make a similar choice in the second case.

With these notations we introduce the following quadrature rule

$$\begin{aligned} H_{m^*}(F,y) &= F(y)\left[\mathcal{H}(\sqrt{w},y) - \sum_{|k|\leq j_1} \frac{\lambda_{m^*,k}(\sqrt{w})}{t_{m^*,k}-y}\right] \\ &\quad + \sum_{|k|\leq j_1} \lambda_{m^*,k}(\sqrt{w}) \frac{F(t_{m^*,k})}{t_{m^*,k}-y} \\ &=: F(y)A_{m^*}(y) + \sum_{|k|\leq j_1} \lambda_{m^*,k}(\sqrt{w}) \frac{F(t_{m^*,k})}{t_{m^*,k}-y} \end{aligned} \tag{5.3}$$

for any fixed $y \neq t_{m^*,k}$, $|k| \leq j_1$. Notice that $\mathcal{H}(\sqrt{w}, y)$ can be evaluated with the required precision and that the rule in (5.3) can be applied if m^* is such that $y \in [-t_{m^*,j_1}, t_{m^*,j_1}]$. The stability (apart from an extra $\log m$ factor) and the convergence of the rule $H_{m^*}(F, y)$ are shown by the following theorem, proved in [**19**].

Theorem 5.1. *For all $F \in C_u$, with $\beta > 1$, and for $y \in [-t_{m^*,j_1}, t_{m^*,j_1}]$ we have*

$$|H_{m^*}(F,y)| \leq \mathrm{C} \log m \, \|Fu\|_\infty. \tag{5.4}$$

Moreover, if $F \in Z_s^\infty(u)$, $s > 0$, for $m \geq m_0$ we get

$$\left|\mathcal{H}(F\sqrt{w},y) - H_{m^*}(F,y)\right| \leq \mathrm{C} \log m \left(\frac{a_m}{m}\right)^s \|F\|_{Z_s^\infty(u)}, \tag{5.5}$$

where C is a positive constant independent of m, y and F.

Coming back to the evaluation of the coefficients of system (4.10), we can approximate the integrals

$$\int_{\mathbf{R}} \frac{\eta(x_l,t)\sqrt{w(t)}}{t-x_i}\,\mathrm{d}t$$

in (4.11) by means of the rule in (5.3) and then the corresponding error is given by (5.5). In fact we have $x_i = x_{m,i}(w) \in [-x_j, x_j] \subset [-t_{m,j_1}, t_{m,j_1}]$ and, moreover, for every x_i we can choose m^* such that the zeros $t_{m^*,k}$ of $p_{m^*}(\sqrt{w})$ are sufficiently far from x_i.

Hence, setting

$$\Gamma_{m^*}(x_l,x_i) = k(x_l,x_i) + \nu H_{m^*}(\eta_{x_l}, x_i),$$

we obtain the new matrix

$$\widetilde{A}_j = \left(\delta_{il} - \mu\lambda_l(w)\Gamma_{m^*}(x_l,x_i)\frac{u(x_i)}{u(x_l)}\right)_{i,l=-j}^{j}. \tag{5.6}$$

But $\widetilde{A}_j$ is a perturbed matrix of A_j of system (4.10). In fact, under the hypotheses of Theorem 4.3, by virtue of (5.5), we get

$$\|A_j - \widetilde{A}_j\|_\infty \le \mathrm{C}\left(\frac{a_m}{m}\right)^s \log m \tag{5.7}$$

for $m \ge m_0$. Moreover, Theorem 4.3 and inequality (5.7) imply that (see, for instance, Proposition 3.3 in [**6**]) the matrix $\widetilde{A}_j$ is invertible and

$$\lim_{m\to\infty} \frac{\mathrm{cond}\,(\widetilde{A}_j)}{\mathrm{cond}\,(A_j)} = 1.$$

Namely if system (4.10) admits a unique solution ξ^*, so also does the linear system

$$\widetilde{A}_j \xi = b. \tag{5.8}$$

Concerning the evaluation of the integral in

$$\Gamma(x_l,y) = k(x_l,y) + \nu \int_{\mathbf{R}} \frac{\eta(x_l,t)\sqrt{w(t)}}{t-y}\mathrm{d}t,$$

which appears in the definition of the Nyström interpolant, we first observe that

$$|y| \leq \theta a_m(\sqrt{w}) \sim t_{m,j_1} \tag{5.9}$$

implies $m > (|y|/\theta c_\alpha)^\alpha$. For instance, if $\alpha = 2$, $\theta = 1/2$ and $y = 50$, we should take $m > 2500$; but with such an m we cannot compute the Christoffel numbers $\lambda_{m,k}(\sqrt{w})$ and the zeros $t_{m,k}$. Then in this case the rule in (5.3) cannot be used.

Therefore, since m is given by the Nyström method, if y satisfies (5.9) we will use (5.3); otherwise, we will consider the simpler rule (see [**19**])

$$\mathcal{H}\left(F\sqrt{w}, y\right) = \sum_{|k|\leq j_1} \lambda_{m,k}(\sqrt{w}) \frac{F(t_k)}{t_k - y} + \rho_m(F, y), \tag{5.10}$$

where, for every $F \in Z_s^\infty(u)$, $s > 0$ and $\beta > 1$, we have

$$|\rho_m(F, y)| = O\left(\left(\frac{a_m}{m}\right)^s\right). \tag{5.11}$$

Let f^* be the solution of equation (4.1). We denote by ξ^{**} is the solution of (5.8) and f_m^{**} is the corresponding Nyström interpolant given by

$$f_m^{**}(y) = g(y) + \mu \sum_{|l|\leq j} \lambda_l(w) \frac{\Gamma_{m^*}(x_l, y)}{u(x_l)} \xi_l^{**}, \tag{5.12}$$

where by $\Gamma_{m^*}(x_l, y)$ we denote the approximation of $\Gamma(x_l, y)$ obtained using either (5.3) or (5.10).

Under the assumptions of Theorem 4.3, by (5.5) and (5.11) we get

$$\|[f^* - f_m^{**}]u\|_\infty \leq \mathrm{C}\left(\frac{a_n}{n}\right)^s \log m, \tag{5.13}$$

where $n = \lfloor M/2 \rfloor$, $M \sim m$ and C is independent of m and f. Namely the rule $H_{m^*}(F, y)$ introduces only an extra $\log m$ factor in the estimate of the error of the Nyström method.

6. An application to Cauchy equations. Let us consider a Cauchy singular equation of the form

$$f(y) - \nu \int_{\mathbf{R}} \frac{f(x)}{x-y}\, \mathrm{d}x - \nu \int_{\mathbf{R}} \tau(x,y) w(x) \sqrt{w(y)} f(x)\, \mathrm{d}x = \widetilde{g}(y), \quad y \in \mathbf{R}, \tag{6.1}$$

where $\nu \in \mathbf{R}$ and $k(x,y) := \tau(x,y)\sqrt{w(y)}$ satisfies the condition

$$\sup_{x \in \mathbf{R}} u(x) \|k_x\|_{Z_s^\infty(u)} < \infty. \tag{6.2}$$

We assume that equation (6.1) admits a unique solution in L^p, $1 < p < \infty$. Using an argument in [**22**, page 173], applying the Hilbert transform to both sides of (6.1) and using the reciprocity theorem, we get

$$\mathcal{H}(f,y) + \nu\pi^2 f(y) - \nu \int_{\mathbf{R}} \frac{\sqrt{w(t)}}{t-y} \left[\int_{\mathbf{R}} \tau(x,t) f(x) w(x)\, \mathrm{d}x \right] \mathrm{d}t = \mathcal{H}(\widetilde{g}, y).$$

Then, by using (6.2), we have

$$\mathcal{H}(f,y) + \nu\pi^2 f(y) - \nu \int_{\mathbf{R}} f(x) w(x) \left[\int_{\mathbf{R}} \frac{\tau(x,t)\sqrt{w(t)}}{t-y}\, \mathrm{d}t \right] \mathrm{d}x = \mathcal{H}(\widetilde{g}, y). \tag{6.3}$$

Combining (6.3) and (6.1), we obtain the following Fredholm equation equivalent to (6.1)

$$f(y) - \frac{\nu}{1+\nu^2\pi^2} \int_{\mathbf{R}} \left[\tau(x,y)\sqrt{w(y)} + \nu \int_{\mathbf{R}} \frac{\tau(x,t)\sqrt{w(t)}}{t-y}\, \mathrm{d}t \right] f(x) w(x) \mathrm{d}x = \frac{1}{1+\nu^2\pi^2} \widetilde{g}(y) + \frac{\nu}{1+\nu^2\pi^2} \mathcal{H}(\widetilde{g}, y). \tag{6.4}$$

Note that equation (6.4) is of the form (4.1), setting

$$\begin{aligned} \mu &= \frac{\nu}{1+\nu^2\pi^2}, \\ \Gamma(x,y) &= \tau(x,y)\sqrt{w(y)} + \nu \mathcal{H}\left(\tau_x \sqrt{w}, y\right), \end{aligned} \tag{6.5}$$

and

$$g(y) = \frac{1}{1+\nu^2\pi^2}\left[\widetilde{g}(y) + \nu\mathcal{H}\left(\widetilde{g}, y\right)\right]. \tag{6.6}$$

Therefore the method described in Section 4 can also be used for solving equation (6.4). In this case Theorem 4.3 becomes the following proposition.

Proposition 6.1. *Assume that equation* (6.4) *admits a unique solution* $f^* \in C_u$ *for any* $g \in C_u$*, where* u *is the weight given by* (2.1) *with* $\beta > 1/2$*. If for some* $s > 0$ *the function* τ *fulfills the assumptions of Lemmas* 4.1 *and* 4.2*, the function* $k(x,y) := \tau(x,y)\sqrt{w(y)}$ *satisfies*

$$\sup_{z\in\mathbf{R}} u(z)\|k_z\|_{Z_s^\infty(u)} < \infty, \quad z \in \{x, y\},$$

and the function g *in* (6.6) *belongs to* $Z_s^\infty(u)$*, then the corresponding linear system in* (4.10) *has a unique solution* ξ^* *for* $m \geq m_0$ *and its matrix of the coefficients* A_j *is such that*

$$\sup_{m\geq m_0} \operatorname{cond}(A_j) \leq \sup_{m\geq m_0} \operatorname{cond}(I - \mu K_m) < \infty. \tag{6.7}$$

Moreover, for the Nyström interpolant f_m^* *defined by* (4.12)*, the estimate*

$$\left\|\left[f^* - f_m^*\right]u\right\|_\infty = O\left(\left(\frac{a_n}{n}\right)^s\right) \tag{6.8}$$

holds, where $n = \lfloor M/2 \rfloor$*,* $M \sim m$ *is defined by* (2.12) *and the constants in "O" are independent of* m *and* f*.*

Concerning the assumptions on the function $k(x,y) := \tau(x,y)\sqrt{w(y)}$, we remark that if the parameter α of weight w in (2.8) is an even integer or it is sufficiently large, then k belongs to the same class of τ, as a function of y (see Example 3). Otherwise, if α is not an even integer or it is "small", the class of k is very large (see Example 4).

From a numerical point of view, we note that if an explicit form of the Hilbert transforms in (6.5) and (6.6) is not known, we can use the method described in Section 5.

7. Numerical examples. In this section we show some approximations of the weighted solutions of integral equations of the forms (3.1), (4.1) and (6.1). Accordingly to Theorems 3.2 and 4.3, the integral equation which we are going to consider should admit solution in a space C_u, where the parameter β of the weight u in (2.1) is greater than $1/2$. Moreover, since in Examples 2, 3 and 4 we want to use the quadrature rule $H_{m^*}(F)$ in (5.3), we need $\beta > 1$.

For a fixed parameter $\theta \in (0,1)$ and with j defined by (2.11), in the following examples we will denote by A_j and $\widetilde{A}_j$ the matrices of the coefficients of systems (3.7) and (5.8), respectively. Moreover, f_m^* and f_m^{**} will stand for the corresponding Nyström interpolants given by (3.8) and (5.12), respectively, (here in the notation we omit the dependence on the truncation index j).

Since the exact solutions of the equations are not known, in the tables we will report only the digits which are correct according to the value of the weighted approximate solution obtained for $m = 512$ in Examples 2 and 3 and for $m = 400$ in Examples 1 and 4.

All the computations have been done in double precision arithmetic.

Example 1. Let us consider the Fredholm integral equation

$$f(y) - \frac{1}{4}\int_{\mathbf{R}} \frac{(1+x^2)f(x)\mathrm{e}^{-|x|^3}}{\sqrt{\pi}(1+x^2+y^2)}\,\mathrm{d}x = \arctan(1+y).$$

This equation admits a unique solution in the space C_u, with $u(x) = (1+|x|)\mathrm{e}^{-|x|^3/2}$. Choosing $\theta = 2/5$, as shown in Table 1, the machine precision in double arithmetic is obtained solving a linear system of order $2j = 132$. Moreover, the matrix A_j of the coefficient of system (3.7) is such that

$$\mathrm{cond}\,(A_j) < 1.6304.$$

Example 2. We consider the integral equation

$$f(y) - \frac{1}{10}\int_{\mathbf{R}} \Gamma(x,y)f(x)\mathrm{e}^{-x^2}\,\mathrm{d}x = \arctan(1+y),$$

where

$$\Gamma(x,y) = \frac{1}{(1+x^2+y^2)^5} + \int_{\mathbf{R}} \frac{\mathrm{e}^{-t^2/2}}{(1+x^2+t^2)^5(t-y)}\,\mathrm{d}t.$$

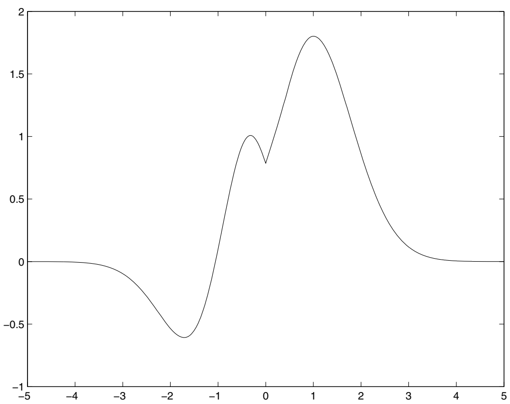

FIGURE 1.

TABLE 1.

m	$2j$	$(f_m^* u)(0.7)$	$(f_m^* u)(2)$	$(f_m^* u)(-1)$
16	6	1.7	$7.1e-2$	$1.e-1$
32	14	1.715	$7.15e-2$	$1.49e-1$
64	26	1.7157	$7.1587e-2$	$1.496e-1$
128	52	1.715725	$7.158741e-2$	$1.496905e-1$
256	106	1.7157253048945	$7.1587419268479e-2$	$1.496905171886e-1$
320	132	1.715725304894554	$7.158741926847953e-2$	$1.496905171886349e-1$

This equation is uniquely solvable in the space C_u, with $u(x) = (1+|x|)^{3/2}\mathrm{e}^{-x^2/2}$. Choosing $\theta = 1/4$, we get the machine precision for $2j = 152$, as we may notice in Table 2. Moreover, the condition number in infinity norm of the matrix $\widetilde{A}_j$ in (5.6) is such that

$$\mathrm{cond}\,(\widetilde{A}_j) < 1.3722.$$

In Figure 1 we show the graph of the weighted approximate solution $f_{256}^{**}u$, with a max absolute error of the order of 10^{-11}.

TABLE 2.

m	$2j$	$(f_m^{**}u)(0.5)$	$(f_m^{**}u)(2)$
16	6	1.42	8.6$e-1$
64	20	1.4243	8.60$e-1$
128	40	1.424344	8.606993$e-1$
256	80	1.4243444912	8.606993172$e-1$
384	122	1.4243444912311	8.6069931722905$e-1$
480	152	1.424344491231125	8.606993172290570$e-1$

Example 3. Proceeding as described in Section 6, we can show that the Cauchy singular equation

$$f(y) - \nu \int_{\mathbf{R}} \left[\frac{1}{x-y} + \frac{\arctan(y+1)\mathrm{e}^{-y^2/2}\mathrm{e}^{-x^2}}{4+x^2} \right] f(x)\,\mathrm{d}x = \frac{y}{1+y^2},$$

with $\nu = 1/10$, is equivalent to an equation of the form (6.4). To be more precise, it is equivalent to

$$f(y) - \frac{\nu}{1+\nu^2\pi^2} \int_{\mathbf{R}} \left[\tau(x,y)\mathrm{e}^{-y^2/2} + \nu \int_{\mathbf{R}} \frac{\tau(x,t)\mathrm{e}^{-y^2/2}}{t-y}\,\mathrm{d}t \right] f(x)\mathrm{e}^{-x^2}\,\mathrm{d}x = g(y), \tag{7.1}$$

where

$$\tau(x,y) = \frac{\arctan(y+1)}{4+x^2}, \qquad g(y) = \frac{y+\nu\pi}{(1+\nu^2\pi^2)(1+y^2)}$$

since the Hilbert transform of the function $\widetilde{g}(y) = y/(1+y^2)$ is known in an explicit form. Equation (7.1) has a unique solution in C_u, with $u(x) = (1+|x|)^2\mathrm{e}^{-x^2/2}$. In Table 3 we show that, choosing $\theta = 1/5$, we obtain the machine precision solving a linear system of order 98. Moreover, the matrix $\widetilde{A}_j$, given by (5.6), of the linear system is such that

$$\operatorname{cond}(\widetilde{A}_j) < 1.0749.$$

TABLE 3.

m	$2j$	$(f_m^{**}u)(0.5)$	$(f_m^{**}u)(-0.2)$
16	4	1.19	$1.50e-1$
64	16	1.19144	$1.5094e-1$
128	32	1.1914477	$1.509439e-1$
256	64	1.191447774598	$1.5094391534e-1$
384	98	1.191447774598290	$1.509439153480904e-1$

TABLE 4.

m	$2j$	$(f_m^{**}u)(-0.3)$	$(f_m^{**}u)(0.1)$	$(f_m^{**}u)(0.7)$
16	6	$2.62e-1$	$7.3e-2$	$-1.55e-1$
32	14	$2.627e-1$	$7.387e-2$	$-1.555e-1$
64	26	$2.62772e-1$	$7.387e-2$	$-1.55523e-1$
128	52	$2.627726e-1$	$7.3878e-2$	$-1.555236e-1$
256	106	$2.627726e-1$	$7.38781e-2$	$-1.555236e-1$

Example 4. Finally, we consider the Cauchy singular equation

$$f(y) - \int_{\mathbf{R}} \left[\frac{1}{x-y} + \frac{\mathrm{e}^{-|x|^3/2}}{(1+x^2+y^2)^5} \right] f(x)\,\mathrm{d}x = \frac{1}{1+y^2},$$

which is equivalent to the Fredholm equation

$$f(y) - \frac{1}{1+\pi^2} \int_{\mathbf{R}} \Gamma(x,y) f(x) \mathrm{e}^{-|x|^3}\,\mathrm{d}x = \frac{1-\pi y}{(1+\pi^2)[1+y^2]}, \tag{7.2}$$

where

$$\begin{aligned} \Gamma(x,y) &= \tau(x,y)\mathrm{e}^{-|y|^3/2} + \int_{\mathbf{R}} \frac{\tau(x,t)\mathrm{e}^{-|t|^3/2}}{t-y}\,\mathrm{d}t \\ &= \frac{\mathrm{e}^{|x|^3/2}}{(1+x^2+y^2)^5} + \int_{\mathbf{R}} \frac{\mathrm{e}^{(|x|^3+|t|^3)/2}}{(1+x^2+t^2)^5(t-y)}\,\mathrm{e}^{-|t|^3/2}\,\mathrm{d}t. \end{aligned}$$

Equation (7.2) has a unique solution in C_u, with $u(x) = (1+|x|)^{3/2}$ $\mathrm{e}^{-|x|^3/2}$. Since the functions τ and

$$k(x,y) = \tau(x,y)\sqrt{w(y)} = \frac{\mathrm{e}^{(|x|^3+|y|^3)/2}}{(1+x^2+y^2)^5}\,\mathrm{e}^{-|y|^3/2}$$

satisfy the assumptions of Proposition 6.1 with $s = 17/4$, the theoretical error converges with order $m^{-17/6}\log m$. This error is confirmed by the numerical results in Table 4, obtained for $\theta = 2/5$. In this case the matrix of the linear system is such that

$$\mathrm{cond}\,(\widetilde{A}_j) < 1.4799.$$

8. Proofs.

Proof of Proposition 3.1. We first show that K is a continuous operator from C_u into C_u. By hypotheses, for any $y \in \mathbf{R}$, we have

$$\begin{aligned}|K(f)(y)|\,u(y) &\le \|fu\|_\infty \sup_{x\in\mathbf{R}} u(x)\,\|k_x u\|_\infty \int_{\mathbf{R}} \frac{\mathrm{d}x}{(1+|x|)^{2\beta}}\\ &\le \mathrm{C}\,\|fu\|_\infty.\end{aligned}$$

Now, for any fixed $x \in \mathbf{R}$, let $P_{m,x}(y)$ be the polynomial of best approximation of $k_x(y)$ in C_u of degree m, namely $\|(k_x - P_{m,x})\,u\|_\infty = E_m(k_x)_{u,\infty}$. For every $f \in C_u$ we have

$$\begin{aligned}E_m(Kf)_{u,\infty} &\le \sup_{y\in\mathbf{R}} u(y)\left|\int_{\mathbf{R}} [k_x(y) - P_{m,x}(y)]\, f(x)w(x)\,\mathrm{d}x\right|\\ &\le \sup_{x\in\mathbf{R}} u(x)E_m(k_x)_{u,\infty}\|fu\|_\infty \int_{\mathbf{R}} \frac{\mathrm{d}x}{(1+|x|)^{2\beta}}.\end{aligned}$$

Since $\beta > 1/2$, it follows that

$$\frac{E_m(Kf)_{u,\infty}}{\|fu\|_\infty} \le \mathrm{C}\sup_{x\in\mathbf{R}} u(x)E_m(k_x)_{u,\infty},$$

from which we deduce

$$\lim_{m\to\infty}\sup_{f\in C_u}\frac{E_m(Kf)_{u,\infty}}{\|fu\|_\infty} = 0,$$

by virtue of (3.4). Namely K is compact (see [**21**, pages 44–45, 93–95]). □

Since Theorem 3.2 is a consequence of Lemma 3.4, we first prove this last one.

Proof of Lemma 3.4. Inequalities (3.18) and (3.19) can be proved using the same arguments as in the proof of Proposition 3.1. Let us first prove (3.18), i.e., K_m is (with respect to $m \in \mathbf{N}$) a uniformly bounded operator from C_u into C_u. By hypotheses, for any $y \in \mathbf{R}$, we get

$$\begin{aligned} |(K_m f)(y)|\, u(y) &\le \|fu\|_\infty \sup_{x\in\mathbf{R}} u(x)\, \|k_x u\|_\infty \sum_{|l|\le j} \frac{\lambda_l(w)}{u^2(x_l)} \\ &\le \|fu\|_\infty \sup_{x\in\mathbf{R}} u(x)\, \|k_x u\|_\infty \int_{\mathbf{R}} \frac{\mathrm{d}x}{(1+|x|)^{2\beta}} \\ &\le \mathrm{C}\, \|fu\|_\infty \end{aligned}$$

and (3.18) follows.

Now let us consider (3.19). Let $P_{n,x_l}(y)$ be the polynomial of best approximation polynomial of $k_{x_l}(y)$ of degree n, namely, $\|(k_{x_l} - P_{n,x_l})\, u\|_\infty = E_n\,(k_{x_l})_{u,\infty}$. For any $f \in C_u$ we have

$$\begin{aligned} E_n(K_m f)_{u,\infty} &\le \left| \sum_{|l|\le j} [k_{x_l}(y) - P_{n,x_l}(y)] f(x_l) \lambda_l(w) \right| \\ &\le \|fu\|_\infty \sup_{x\in\mathbf{R}} u(x) E_n(k_x)_{u,\infty} \int_{\mathbf{R}} \frac{\mathrm{d}x}{(1+|x|)^{2\beta}}. \end{aligned}$$

For $m, n > 0$ arbitrary integers, it follows that

$$\frac{E_n(K_m f)_{u,\infty}}{\|fu\|_\infty} \le \mathrm{C} \sup_{x\in\mathbf{R}} u(x) E_n(k_x)_{u,\infty}, \quad \mathrm{C} \ne \mathrm{C}(m),$$

and then, by (3.11), we deduce that

$$\lim_{n\to\infty} \sup_{m\in\mathbf{N}} \sup_{f\in C_u} \frac{E_n(K_m f)_{u,\infty}}{\|fu\|_\infty} = 0,$$

uniformly with respect to $m \in \mathbf{N}$, namely, $\{K_m\}_m$ is collectively compact.

In order to prove (3.20), we use Proposition 2.1, with $\sigma = u^2$. Then, for any $f \in C_u$, we get

$$\begin{aligned}(8.1)\quad &\|(Kf - K_m f)\, u\|_\infty \\ &\le \mathrm{C} \sup_{y\in\mathbf{R}} u(y) \left\{E_M(k_y f)_{u^2,\infty} + \mathrm{e}^{-\mathrm{A}m}\|k_y u\|_\infty \|fu\|_\infty\right\} \\ &\le \mathrm{C}\, \|fu\|_\infty \sup_{y\in\mathbf{R}} u(y) \left\{E_n(k_y)_{u,\infty} + \mathrm{e}^{-\mathrm{A}m}\|k_y u\|_\infty\right\} \\ &\quad + \mathrm{C}\, E_n(f)_{u,\infty} \sup_{y\in\mathbf{R}} u(y)\|k_y u\|_\infty,\end{aligned}$$

where $n = \lfloor M/2 \rfloor$. By assumption (3.11) and inequality (8.1), we obtain (3.20).

Finally, let us prove (3.21). By (8.1), (3.18) and (3.19), we obtain

$$\begin{aligned}&\|(K - K_m)\, K_m(f)u\|_\infty \\ &\le \mathrm{C}\, \|K_m(f)u\|_\infty \sup_{y\in\mathbf{R}} u(y) \left\{E_n(k_y)_{u,\infty} + \mathrm{e}^{-\mathrm{A}m}\|k_y u\|_\infty\right\} \\ &\quad + \mathrm{C}\, E_n(K_m f)_{u,\infty} \sup_{y\in\mathbf{R}} u(y)\|k_y u\|_\infty \\ &\le \mathrm{C}\, \|fu\|_\infty \sup_{y\in\mathbf{R}} u(y) \left\{E_n(k_y)_{u,\infty} + \mathrm{e}^{-\mathrm{A}m}\|k_y u\|_\infty\right\} \\ &\quad + \mathrm{C}\, \|fu\|_\infty \sup_{x\in\mathbf{R}} u(x) E_n(k_x)_{u,\infty} \sup_{y\in\mathbf{R}} u(y)\|k_y u\|_\infty\end{aligned}$$

and (3.21) follows. □

Proof of Theorem 3.2. By Lemma 3.4 and Theorem 4.1.1 in [**1**, page 106], the operator $I - \mu K_m$ is invertible for $m \ge m_0$. Moreover, the matrix A_j of the coefficients of system (3.9) is such that (see [**1**, page 113])

$$\|A_j\| \le \|I - \mu K_m\|$$

and

$$\left\|A_j^{-1}\right\| \le \left\|(I - \mu K_m)^{-1}\right\|$$

hence

$$\mathrm{cond}\,(A_j) \le \mathrm{cond}\,(I - \mu K_m)\,.$$

Finally, since for $m \geq m_0$ we have

$$\begin{aligned} \|[f^* - f_m^*]\, u\|_\infty &\leq \left\|(I - \mu K_m)^{-1}\right\| \|(K - K_m)\, f^*\|_{C_u} \\ &\leq \mathrm{C}\, \|(K - K_m)\, f^*\|_{C_u}, \end{aligned}$$

inequality (3.14) is an immediate consequence of (8.1). □

Proof of Corollary 3.3. We have only to prove inequality (3.17). By (3.14), for $m \geq m_0$, we get

$$\begin{aligned} &\|[f^* - f_m^*]\, u\|_\infty \\ &\leq \mathrm{C} \left\{ \|f^* u\|_\infty \sup_{y \in \mathbf{R}} u(y) E_n(k_y)_{u,\infty} + E_n(f^*)_{u,\infty} \sup_{y \in \mathbf{R}} u(y) \|k_y u\|_\infty \right\}, \end{aligned} \tag{8.2}$$

with $n = \lfloor M/2 \rfloor$ and $M \sim m$.

Let us consider the first summand at the right-hand side of (8.2). By (2.7) and the hypotheses, for $r > s$, we get

$$\begin{aligned} \sup_{y \in \mathbf{R}} u(y) E_n(k_y)_{u,\infty} &\leq \mathrm{C} \sup_{y \in \mathbf{R}} u(y) \int_0^{a_n/n} \frac{\Omega^r(k_y, \delta)_{u,\infty}}{\delta} \,\mathrm{d}\delta \\ &\leq \mathrm{C} \left(\frac{a_n}{n} \right)^s \sup_{y \in \mathbf{R}} u(y)\, \|k_y\|_{Z_s^\infty(u)}. \end{aligned} \tag{8.3}$$

On the other hand, $g \in Z_s^\infty(u)$ and $k_x \in Z_s^\infty(u)$ imply $f^* \in Z_s^\infty(u)$. Therefore we obtain

$$\begin{aligned} E_n(f^*)_{u,\infty} &\leq \mathrm{C} \int_0^{a_n/n} \frac{\Omega^r(f^*, \delta)_{u,\infty}}{\delta} \,\mathrm{d}\delta \\ &\leq \mathrm{C} \left(\frac{a_n}{n} \right)^s \|f^*\|_{Z_s^\infty(u)}. \end{aligned} \tag{8.4}$$

Combining estimates (8.3) and (8.4) in (8.2), inequality (3.17) follows. □

In order to prove Lemma 4.1, we need some known results.

We note that for a "small" $h > 0$ and $x, y \in \mathcal{I}_h = [-\mathrm{A}r\, h^{-1/(\alpha-1)}, \mathrm{A}r\, h^{-1/(\alpha-1)}]$, $\mathrm{A} > 1$ and $r \in \mathbf{Z}^+$, we have

$$|x-y| \le \mathrm{C}\, h \Longrightarrow u(x) \sim u(y) \tag{8.5}$$

for any $\beta \ge 0$, namely, for any value of the parameter β of the weight u in (2.1).

Moreover we recall that the r-th finite difference, defined by (2.2), of the product of two functions F and G is given by

(8.6)
$$\begin{aligned}\Delta_h^r(FG;x) &= \sum_{i=0}^{r} \binom{r}{i} \Delta_h^i\left(F; x+[r-i]\frac{h}{2}\right)\Delta_h^{r-i}\left(G; x-i\frac{h}{2}\right)\\ &= \Delta_h^r(F;x)G\left(x-r\frac{h}{2}\right)\\ &\quad+\sum_{i=1}^{r-1} \binom{r}{i} \Delta_h^i\left(F; x+[r-i]\frac{h}{2}\right)\Delta_h^{r-i}\left(G; x-i\frac{h}{2}\right).\end{aligned}$$

Finally, if a function F admits the i-th derivative, we have

(8.7)
$$\begin{aligned}\left|\Delta_h^i\,(F;x)\right| &= \left|\int_{-h/2}^{h/2}\cdots\int_{-h/2}^{h/2} F^{(i)}(x+s_1+\cdots+s_i)\,\mathrm{d}s_1\cdots\mathrm{d}s_i\right|\\ &= \left|\int_{T_i} F^{(i)}(x+\zeta)\,\mathrm{d}\zeta\right|\end{aligned}$$

and then

$$\left|\Delta_h^i\,(F;x)\right| = \frac{h^i\left|F^{(i)}(\overline{x})\right|}{i!}, \tag{8.8}$$

with $x, \overline{x} \in \mathcal{I}_h$ and $|x-\overline{x}| \le ih$.

The next lemma was proved in [**7**] (see Lemma 2.5).

Lemma 8.1. *Let $1 \le p \le \infty$, and let $u(x) = (1+|x|)^\beta \mathrm{e}^{-|x|^\alpha/2}$, with $\alpha > 1$, $\beta \ge 0$. For every $P \in \mathbf{P}_m$ and $\theta > 0$, we have*

$$\|Pu\|_{L^p(J_m)} \le \mathrm{C}\,\mathrm{e}^{-\mathrm{A}m}\|Pu\|_p, \tag{8.9}$$

with $J_m = \{x \in \mathbf{R} : \ |x| > (1+\theta)a_m\}$, C *and* A *positive constants independent of* m *and* P.

Proof of Lemma 4.1. Let $x \in \mathbf{R}$ be fixed. Since the Hilbert transform is bounded in L^p-spaces, $1 < p < \infty$, and since $\beta \ge 0$, we have

$$\left\| \mathcal{H}\left(\eta_x \sqrt{w}\right) u \right\|_p \le \mathrm{C} \left\| \mathcal{H}\left(\eta_x \sqrt{w}\right) \right\|_p \le \mathrm{C} \left\| \eta_x u \right\|_p. \tag{8.10}$$

Now we want to estimate the main part of the r-th modulus of smoothness of $\Psi_x = \mathcal{H}(\eta_x\sqrt{w})$ with step δ, where $r = \lfloor s + 1/p \rfloor + 1$. For any $0 < h \le \delta$, we can write

$$\begin{aligned} \left\| \Delta_h^r \left[\mathcal{H}\left(\eta_x \sqrt{w}\right) \right] u \right\|_{L^p(\mathcal{I}_h)} & \\ \le \left(\int_{\mathcal{I}_h} \left| u(y) \int_{\mathcal{I}_h} \frac{\Delta_h^r \left(\eta_x \sqrt{w}; t\right)}{t-y} \, \mathrm{d}t \right|^p \mathrm{d}y \right)^{1/p} & \\ + \left(\int_{\mathcal{I}_h} \left| u(y) \int_{\mathbf{R} \setminus \mathcal{I}_h} \frac{\Delta_h^r \left(\eta_x \sqrt{w}; t\right)}{t-y} \, \mathrm{d}t \right|^p \mathrm{d}y \right)^{1/p} & \\ =: A_1 + A_2. & \end{aligned} \tag{8.11}$$

Let us denote by χ_h the characteristic function of the interval $\mathcal{I}_h$. By the same arguments as used to derive (8.10), we have

$$\begin{aligned} A_1 &\le \left\| \mathcal{H}\left[\chi_h \Delta_h^r \left(\eta_x \sqrt{w}\right) \right] u \right\|_p \\ &\le \mathrm{C} \left\| \Delta_h^r \left(\eta_x \sqrt{w}\right) (1 + |\cdot|)^\beta \right\|_{L^p(\mathcal{I}_h)}. \end{aligned} \tag{8.12}$$

Moreover, using (8.6), (8.8) and (8.5), we obtain

$$\begin{aligned} &\left\| \Delta_h^r \left(\eta_x \sqrt{w}\right) (1 + |\cdot|)^\beta \right\|_{L^p(\mathcal{I}_h)} \\ &\le \mathrm{C} \left\| \Delta_h^r(\eta_x) u \right\|_{L^p(\mathcal{I}_h)} \\ &\quad + \mathrm{C} \sum_{i=0}^{r-1} \left(\int_{\mathcal{I}_h} \left| \Delta_h^i \left(\eta_x; t + [r-i] \frac{h}{2} \right) h^{r-i} \sqrt{w}^{(r-i)}(\bar{t}) \, (1 + |t|)^\beta \right|^p \mathrm{d}t \right)^{1/p} \\ &\le \mathrm{C} \left\| \Delta_h^r(\eta_x) u \right\|_{L^p(\mathcal{I}_h)} \\ &\quad + \mathrm{C} \sum_{i=0}^{r-1} h^{r-i} \left(\int_{\mathcal{I}_h} \left| \Delta_h^i \left(\eta_x; t + [r-i] \frac{h}{2} \right) u_{r-i}(t) \right|^p \mathrm{d}t \right)^{1/p} \\ &=: \mathrm{C} \left\{ \left\| \Delta_h^r (\eta_x) u \right\|_{L^p(\mathcal{I}_h)} + S_{r-1}(\eta_x) \right\}. \end{aligned} \tag{8.13}$$

Concerning $S_{r-1}(\eta_x)$, by (8.7), it follows that

$$S_{r-1}(\eta_x) \leq \sum_{i=0}^{r-1} h^{r-i} \left(\int_{\mathcal{I}_h} \left| \int_{T_i} \eta_x^{(i)}(t+\zeta)\, \mathrm{d}\zeta \right|^p u_{r-i}^p(t)\, \mathrm{d}t \right)^{1/p}$$
$$\leq \mathrm{C} \sum_{i=0}^{r-1} h^{r-i} \left(\int_{\mathcal{I}_h} \left| \int_{T_i} \eta_x^{(i)}(t+\zeta) u_{r-i}(t+\zeta)\, \mathrm{d}\zeta \right|^p \mathrm{d}t \right)^{1/p};$$

hence, by the Minkowski inequality, we get

$$S_{r-1}(\eta_x) \leq \mathrm{C} \sum_{i=0}^{r-1} h^{r-i} \int_{T_i} \left\| \eta_x^{(i)} u_{r-i} \right\|_{L^p(\mathcal{I}_h)} \mathrm{d}\zeta$$
$$\leq \mathrm{C}\, h^r \sum_{i=0}^{r-1} \|\eta_x^{(i)} u_{r-i}\|_p. \tag{8.14}$$

Finally, combining (8.12), (8.13) and (8.14), we obtain

$$A_1 \leq \mathrm{C} \left\{ \|\Delta_h^r(\eta_x)u\|_{L^p(\mathcal{I}_h)} + h^r \sum_{i=0}^{r-1} \|\eta_x^{(i)} u_{r-i}\|_p \right\}. \tag{8.15}$$

Now, let us consider the term A_2. Using the same arguments as in the proof of (8.12), we get

$$A_2 \leq \left\| \mathcal{H}\left[(1-\chi_h)\, \Delta_h^r \left(\eta_x \sqrt{w} \right) \right] u \right\|_p \leq \mathrm{C}\, \|\eta_x u\|_{L^p(\mathcal{J}_\delta)}, \tag{8.16}$$

where $\mathcal{J}_\delta := (-\infty, -\mathrm{A} r\, \delta^{1/(1-\alpha)}) \cup (\mathrm{A} r\, \delta^{1/(1-\alpha)}, +\infty)$, A is a positive constant and $h \leq \delta$.

For any fixed δ and for some $\theta > 0$, we choose

$$N = \left\lfloor \left(\frac{\mathrm{A} r}{1+\theta} \right)^\alpha \delta^{\alpha/(1-\alpha)} \right\rfloor \sim \delta^{\alpha/(1-\alpha)}.$$

Hence $\delta \sim a_N/N$ and $N \geq \delta^{-1}$. By (8.9), for every $P_N \in \mathbf{P}_N$ we have

$$\|\eta_x u\|_{L^p(\mathcal{J}_\delta)} \leq \|(\eta_x - P_N)u\|_{L^p(\mathcal{J}_\delta)} + \|P_N u\|_{L^p(\mathcal{J}_\delta)}$$
$$\leq \|(\eta_x - P_N)u\|_p + \mathrm{C}\, \mathrm{e}^{-\mathrm{A}\delta^{-1}} \|P_N u\|_p,$$

where $\mathrm{A} > 0$ is a constant. Taking the infimum on all $P_N \in \mathbf{P}_N$ and using (2.4), from (8.16) it follows that

$$A_2 \le \mathrm{C}\left\{\omega^r(\eta_x,\delta)_{u,p} + \mathrm{e}^{-\mathrm{A}\delta^{-1}}\|\eta_x u\|_p\right\}. \tag{8.17}$$

Combining (8.15) and (8.17) and taking the supremum on all $h \in (0,\delta]$, we obtain

$$\begin{aligned}\Omega^r\left(\mathcal{H}\left[\eta_x\sqrt{w}\right],\delta\right)_{u,p}\\ \le \mathrm{C}\left\{\omega^r(\eta_x,\delta)_{u,p} + \mathrm{e}^{-\mathrm{A}\delta^{-1}}\|\eta_x u\|_p + \delta^r\sum_{i=0}^{r-1}\|\eta_x^{(i)}u_{r-i}\|_p\right\}.\end{aligned} \tag{8.18}$$

Dividing both sides of (8.18) for $\delta^{s+1/p}$ and taking the supremum on all $\delta > 0$, we have $\mathcal{H}(\eta_x\sqrt{w}) \in Z^p_{s+1/p}(u)$ for any $x \in \mathbf{R}$.

Moreover, using the results in [**8**], it follows that $\mathcal{H}(\eta_x\sqrt{w}) \in Z_s^\infty(u)$ (see also [**10**]). Then, multiplying by $u(x)$ and taking the supremum on all $x \in \mathbf{R}$, we get (4.6). □

Proof of Lemma 4.2. We prove that Ψ_y satisfies (4.9) only for $y > 0$, the other case being similar. We can write

$$\begin{aligned}\int_{\mathbf{R}}\frac{\eta(x,t)\sqrt{w(t)}}{t-y}\,\mathrm{d}t &= \left\{\int_{y-1}^{y+1} + \int_{y+1}^{+\infty} + \int_{-\infty}^{y-1}\right\}\frac{\eta(x,t)\sqrt{w(t)}}{t-y}\,\mathrm{d}t\\ &=: A_1(x) + A_2(x) + A_3(x).\end{aligned} \tag{8.19}$$

Concerning $A_1(x)$, we have

$$\begin{aligned}A_1(x) &= \int_{y-1}^{y+1}\frac{\eta(x,t)\sqrt{w(t)} - \eta(x,y)\sqrt{w(y)}}{t-y}\,\mathrm{d}t\\ &= \int_{y-1}^{y+1}\frac{1}{t-y}\left\{\int_y^t\frac{\partial}{\partial v}\left[\eta(x,v)\sqrt{w(v)}\right]\mathrm{d}v\right\}\mathrm{d}t\\ &= \int_{y-1}^{y+1}\frac{1}{t-y}\left\{\int_y^t\widehat{\eta}(x,v)\sqrt{w(v)}\,\mathrm{d}v\right.\\ &\qquad\left. + \int_y^t\eta(x,v)\left[\sqrt{w(v)}\right]'\,\mathrm{d}v\right\}\mathrm{d}t\\ &=: B_1(x) + B_2(x).\end{aligned} \tag{8.20}$$

For $x \in \mathcal{I}_h$, with $r > s$ and $0 < h \le \delta$, we get
(8.21)

$$\begin{aligned} u(x)\Delta_h^r(B_1;x) &= \int_{y-1}^{y+1} \frac{1}{t-y} \int_t^y u(x)\Delta_h^r\left(\widehat{\eta}_v;x\right)\sqrt{w(v)}\,\mathrm{d}v\,\mathrm{d}t \\ &\le \int_{y-1}^{y+1} \frac{1}{t-y} \int_t^y \left\|\Delta_h^r\left(\widehat{\eta}_v\right)u\right\|_{L^\infty(\mathcal{I}_h)} \frac{u(v)}{(1+|v|)^\beta}\,\mathrm{d}v\,\mathrm{d}t \\ &\le \sup_{v\in\mathbf{R}} u(v)\Omega^r\left(\widehat{\eta}_v,\delta\right)_{u,\infty} \int_{y-1}^{y+1} \frac{1}{t-y} \int_y^t \frac{\mathrm{d}v}{(1+|v|)^\beta}\,\mathrm{d}t \\ &\le \mathrm{C} \sup_{v\in\mathbf{R}} u(v)\Omega^r\left(\widehat{\eta}_v,\delta\right)_{u,\infty} \end{aligned}$$

and, analogously,

$$\begin{aligned} (8.22)\quad & u(x)\Delta_h^r(B_2;x) \\ &= \int_{y-1}^{y+1} \frac{1}{t-y} \int_t^y u(x)\Delta_h^r\left(\eta_v;x\right)(1+|v|)^{\alpha-1}\sqrt{w(v)}\,\mathrm{d}v\,\mathrm{d}t \\ &\le \mathrm{C} \sup_{v\in\mathbf{R}} u_1(v)\Omega^r\left(\eta_v,\delta\right)_{u,\infty}. \end{aligned}$$

Combining (8.21) and (8.22) and taking the supremum on all $x \in \mathcal{I}_h$, we obtain

$$\|\Delta_h^r(A_1)u\|_{L^\infty(\mathcal{I}_h)} \le \mathrm{C} \sup_{v\in\mathbf{R}} \left\{ u(v)\Omega^r\left(\widehat{\eta}_v,\delta\right)_{u,\infty} + u_1(v)\Omega^r\left(\eta_v,\delta\right)_{u,\infty} \right\}.$$

Then, taking the supremum on all $h \in (0,\delta]$, it follows that
(8.23)

$$\Omega^r(A_1,\delta)_{u,\infty} \le \mathrm{C} \sup_{v\in\mathbf{R}} \left\{ u(v)\Omega^r\left(\widehat{\eta}_v,\delta\right)_{u,\infty} + u_1(v)\Omega^r\left(\eta_v,\delta\right)_{u,\infty} \right\}.$$

Now we estimate the term $A_2(x)$. We have

$$\begin{aligned} u(x)\left|\Delta_h^r(A_2;x)\right| &= \left| \int_{y+1}^{+\infty} \frac{u(x)\Delta_h^r\left(\eta_t;x\right)\sqrt{w(t)}}{t-y}\,\mathrm{d}t \right| \\ &\le \sup_{t\in\mathbf{R}} u(t)\Omega^r(\eta_t,\delta)_{u,\infty} \int_{y+1}^{+\infty} \frac{\mathrm{d}t}{(t-y)(1+t)^\beta} \\ &\le \mathrm{C} \sup_{t\in\mathbf{R}} u(t)\Omega^r(\eta_t,\delta)_{u,\infty}. \end{aligned}$$

Taking the supremum on all $x \in \mathcal{I}_h$ first and then the supremum on all $h \in (0, \delta]$, it follows that

$$\Omega^r(A_2, \delta)_{u,\infty} \leq \mathrm{C} \sup_{t \in \mathbf{R}} u(t) \Omega^r(\eta_t, \delta)_{u,\infty}. \tag{8.24}$$

Proceeding as was done for $A_2(x)$ we get

$$\Omega^r(A_3, \delta)_{u,\infty} \leq \mathrm{C} \sup_{t \in \mathbf{R}} u(t) \Omega^r(\eta_t, \delta)_{u,\infty}. \tag{8.25}$$

Combining (8.23), (8.24) and (8.25), we obtain

$$\Omega^r(\Psi_y, \delta)_{u,\infty} \leq \mathrm{C} \sup_{v \in \mathbf{R}} \left\{ u(v) \Omega^r \left(\widehat{\eta}_v, \delta\right)_{u,\infty} + u_1(v) \Omega^r \left(\eta_v, \delta\right)_{u,\infty} \right\} \tag{8.26}$$

for any $y \in \mathbf{R}$. Dividing both sides of (8.26) by δ^s, $r > s$, and taking the supremum on all $\delta > 0$, we get

$$\sup_{\delta > 0} \frac{\Omega^r(\Psi_y, \delta)_{u,\infty}}{\delta^s} < \mathrm{C}. \tag{8.27}$$

On the other hand, in an analogous way we can prove that

$$\|\Psi_y u\|_\infty \leq \mathrm{C} \sup_{v \in \mathbf{R}} \left\{ u(v) \left\|\widehat{\eta}_v u\right\|_\infty + u_1(v) \|\eta_v u\|_\infty \right\}. \tag{8.28}$$

Thus, combining (8.27) and (8.28), we obtain (4.9). □

Acknowledgments. The authors wish to thank the referees: their remarks contributed to the improvement of the first version of the manuscript.

REFERENCES

1. K.E. Atkinson, *The numerical solution of integral equations of the second kind*, Cambridge Mono. Appl. Comp. Math. **4**, Cambridge University Press, Cambridge, 1997.

2. A.S. Cvetković and G.V. Milovanović, *The Mathematica package "OrthogonalPolynomials"*, Facta Univ. Ser. Math. Inform. **19** (2004), 17–36.

3. M.C. De Bonis, B. Della Vecchia and G. Mastroianni, *Approximation of the Hilbert transform on the real line using Hermite zeros*, Math. Comp. **71** (2002), 1169–1188.

4. M.C. De Bonis, C. Frammartino and G. Mastroianni, *Numerical methods for some special Fredholm integral equations on the real line*, J. Comput. Appl. Math. **164–165** (2004), 225–243.

5. M.C. De Bonis and G. Mastroianni, *Some simple quadrature rules for evaluating the Hilbert transform on the real line*, Arch. Inequal. Appl. **1** (2003), 475–494.

6. ———, *Projection methods and condition numbers in uniform norm for Fredholm and Cauchy singular integral equations*, SIAM J. Numer. Anal. **44** (2006), 1–24.

7. Z. Ditzian and D.S. Lubinsky, *Jackson and smoothness theorems for Freud weights in L_p $(0 < p \leq \infty)$*, Constr. Approx. **13** (1997), 99–152.

8. Z. Ditzian and S. Tikhonov, *Ul'yanov and Nikol'skii-type inequalities*, J. Approx. Theory **133** (2005), 100–133.

9. Z. Ditzian and V. Totik, *Moduli of smoothness*, Springer Ser. Comp. Math. **9**, Springer-Verlag, New York, 1987.

10. L. Fermo, *Embedding theorems for functions with inner singularities*, Acta Sci. Math. (Szeged) **75** (2009), 547–573.

11. C. Frammartino, C. Laurita and G. Mastroianni, *On the numerical solution of Fredholm integral equations on unbounded intervals*, J. Comput. Appl. Math. **158** (2003), 355–378.

12. W. Gautschi, *Algorithm* 726: ORTHPOL–*A package of routines for generating orthogonal polynomials and Gauss-type quadrature rules*, ACM Trans. Math. Software **20** (1994), 21–62.

13. G.H. Golub, *Some modified matrix eigenvalue problems*, Siam Rev. **15** (1973), 318–334.

14. G.H. Golub and J.H. Welsch, *Calculation of gaussian quadrature rules*, Math. Comp. **23** (1969), 221–230.

15. A.L. Levin and D.S. Lubinsky, *Christoffel functions, orthogonal polynomials, and Nevai's conjecture for Freud weights*, Constr. Approx. **8** (1992), 463–535.

16. E. Levin and D.S. Lubinsky, *Orthogonal polynomials for exponential weights*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC **4**, Springer-Verlag, New York, 2001.

17. G. Mastroianni and I. Notarangelo, *A Lagrange-type projector on the real line*, Math. Comp. **79** (2010), 327–352.

18. S.G. Mikhlin and S. Prössdorf, *Singular integral operators*, translated from the German by A. Böttcher and R. Lehmann, Akademie-Verlag, Berlin, 1986.

19. I. Notarangelo, *Approximation of the Hilbert transform on the real line using Freud weights*, in *Approximation and computation*, W. Gautschi, G. Mastroianni and T.M. Rassias, eds., Springer Optimiz. Appl. **42**, Springer (2011), 233–252.

20. S. Prössdorf and B. Silbermann, *Numerical analysis for integral and related operator equations*, Akademie-Verlag, Berlin, 1991 and Birkhäuser Verlag, Basel, 1991.

21. A.F. Timan, *Theory of approximation of functions of a real variable*, translated from the Russian by J. Berry. Reprint of the 1963 English translation. Dover Publications, Inc., New York, 1994.

22. F.G. Tricomi, *Integral equations*, Dover Publications, Inc., New York, 1985.

Dipartimento di Matematica, Università degli Studi della Basilicata, V.le dell'Ateneo Lucano 10, I-85100 Potenza, Italy
Email address: mastroianni.csafta@unibas.it

Dipartimento di Matematica, Università degli Studi della Basilicata, V.le dell'Ateneo Lucano 10, I-85100 Potenza, Italy
Email address: incoronata.notarangelo@unibas.it