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SOLVABILITY OF THREE POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER INTEGRO-DIFFERENTIAL EQUATIONS OF MIXED TYPE

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ABSTRACT. This paper investigates the existence of extreme solutions of the three point boundary value problem for a class of second order integro-differential equations of mixed type. By using the method of upper and lower solutions and monotone iterative technique, we establish the existence results of extreme solutions. An example is also provided to illustrate the efficiency of the obtained results.

1. Introduction. Theory of integro-differential equations in the field of modern applied mathematics has made considerable headway, because all the structure of its emergence has deep physical background and realistic mathematical model (see [**1, 2**]). One of the ideas in the study of certain higher order boundary value problems for differential equations is to reduce them to boundary value problems for lower order integro-differential equations [**3, 4**]. During the past years, many authors have paid attention to the research of three point boundary value problems for second order differential equations because of its potential applications, see, for example, [**1, 5-7, 16**]. In [**8, 9**], J.J. Nieto and R. Rodriguez-Lopez introduced a new concept of lower and upper solutions, they consider the periodic boundary value problems for the following first order functional differential equation

$$(1.1)\qquad \begin{cases} u'(t) = g(t, u(t), u(\theta(t))), \ t \in [0, T], \\ u(0) = u(T). \end{cases}$$

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Similar method has also already succeeded in employing to nonlinear impulsive integro-differential equations [**10**].

Motivated by [**8-10**], we study the existence of solution for a class of second order integro-differential equation of mixed type subject to nonlocal boundary conditions. More specifically, we consider the following three-point boundary value problem

$$\begin{cases} -u''(t) = f(t, u(t), [Tu](t), [Su](t)), & t \in J = [0,1], \\ u(0) = 0, \quad u(1) = au(\eta) - \lambda, \end{cases} \tag{1.2}$$

where $f \in C(J \times R^3, R)$, $0 < a \leq 1$, $0 < \eta < 1$, $\lambda \geq 0$,

$$[Tu](t) = \int_0^t K(t,s)u(s)ds, \qquad [Su](t) = \int_0^1 H(t,s)u(s)ds,$$

$K \in C(D, R^+), D = \{(t,s) \in J \times J : t \geq s\}, H \in C(J \times J, R^+), R^+ = [0, +\infty)$.

The special case $[Tu](t) = [Su](t) = 0$ and $\lambda = 0$ of the above problem so called nonlocal boundary value problems of ODE, was initiated by Il'in and Moiseev [**11**]. Recently, several papers have been devoted to the study of nonlocal boundary value problems of ODE, see [**12-15**]. Nonlocal boundary value problems can usually accurately describe a lot of important physics phenomena and mathematical models, for example, in view of error of actual measurement and relative factor disturbance, Robin boundary value condition that $u(0) = u'(1) = 0$ can been revised to $u(0) = 0, u(1) = u(\rho)$ where ρ is a constant such that $|1-\rho|$ is sufficient small. To our knowledge, few paper paid attention to nonlocal boundary value problems for the integro-differential equations.

In this paper, we are concerned with the existence of extreme solutions for equation (1.2). The paper is organized as follows. In Section 2, we establish several comparison principles. In section 3, we first introduce a new concept of lower and upper solutions, and then give a proof for the existence theorem related to a linear problem associated to equation (1.2). In Section 4, by using the method of upper and lower solutions and monotone iterative technique, we obtain the existence of extreme solutions for equation (1.2). Finally, an example is provided to verity the required assumptions.

2. A comparison principle. In the following, we denote

$$c(t) = \frac{\sin(\pi t)}{a\sin(\pi\eta)}, \quad r = \pi^2,$$

$$k_0 = \max\{K(t,s) : (t,s) \in D\} \geq 0,$$
$$h_0 = \max\{H(t,s) : (t,s) \in J \times J\} \geq 0.$$

We now present the main results of this section.

Theorem 2.1. *Assume that $u \in E = C^2(J,R)$ satisfies*

$$\begin{cases} -u''(t) + Mu(t) + N[Tu](t) + L[Su](t) \leq 0, & t \in J, \\ u(0) \leq 0, \quad u(1) \leq au(\eta) - \rho, \end{cases} \tag{2.1}$$

where $0 \leq a \leq 1,\ 0 < \eta < 1,\ \rho \geq 0$, constants $M > 0,\ N \geq 0, L \geq 0$ such that

$$M + Nk_0 + Lh_0 \leq 2. \tag{2.2}$$

Then $u(t) \leq 0$ for $t \in J$.

Proof. Suppose, to the contrary, that $u(t) > 0$ for some $t \in J$. Then from the boundary conditions, we have that there exists $t^* \in (0,1)$ such that

$$u_0 = u(t^*) = \max_{t \in J} u(t) > 0, \tag{2.3}$$

$$u'(t^*) = 0, \quad u''(t^*) \leq 0. \tag{2.4}$$

We consider the following two possible cases:

(1) $u(t) \geq 0$ for all $t \in J$;

(2) there exist $t_1,\ t_2 \in J$ such that $u(t_1) > 0$ and $u(t_2) < 0$.

case (1) (2.1) implies that $u(0) = 0$ and $u''(t) \geq 0$ for $t \in J$. From $u(0) = 0$ and $u(t) \geq 0$ for $t \in J$, we get that $u'(0) \geq 0$. Therefore, $u'(t) \geq u'(0) \geq 0$. It follows that $u(1) = \max_{t \in J} u(t) > 0$.

If $a = 1$, then $u(1) \leq u(\eta) \leq u(1)$. It follows that $u(t) \equiv c > 0$(c is a constant) for $t \in [\eta, 1]$ and $\rho = 0$. Let $t \in [\eta, 1]$, from the first inequality of (2.1), we obtain that

$$0 < Mc \leq Mc + N[Tu](t) + L[Su](t) \leq u''(t) = 0.$$

This is a contradiction.

If $0 < a < 1$, then it is easy to obtain that $u(\eta) > u(1)$, which contradicts $u(1) = \max_{t\in J} u(t)$.

If $a = 0$, then $u(1) \le 0$, which contradicts $u(1) > 0$.

case (2) Let $t_* \in J$ such that $u(t_*) = \min_{t\in J} u(t) < 0$.

If $t_* < t^*$, from the first inequality of (2.1), we have

$$u''(t) \ge (M+Nk_0t+Lh_0)u(t_*) \ge (M+Nk_0+Lh_0)u(t_*), \quad t \in J. \tag{2.5}$$

Integrating the above inequality from $s(t_* \le s \le t^*)$ to t^*, we obtain

$$-u'(s) \ge (t^* - s)(M + Nk_0 + Lh_0)u(t_*), \quad t_* \le s \le t^*,$$

and then integrate from t_* to t^*, we get

$$\begin{aligned}-u(t_*) < u(t^*) - u(t_*) &\le \int_{t_*}^{t^*} (s - t^*)(M + Nk_0 + h_0)u(t_*)ds \\ &= -\frac{M + Nk_0 + Lh_0}{2}u(t_*)(t^* - t_*)^2 \\ &\le -\frac{M + Nk_0 + Lh_0}{2}u(t_*).\end{aligned}$$

Hence

$$u(t_*)(2 - M - Nk_0 - Lh_0) > 0.$$

This is a contradiction.

If $t_* > t^*$, Integrating inequality (2.5) from $t^*(t^* \le s \le t_*)$ to s, we obtain

$$u'(s) \ge (s - t^*)(M + Nk_0 + Lh_0)u(t_*), \quad t^* \le s \le t_*,$$

and then integrating again from t^* to t_*, we get

$$u(t_*) - u(t^*) \ge \frac{M + Nk_0 + Lh_0}{2}u(t_*)(t_* - t^*)^2,$$

therefore

$$\begin{aligned}-u(t_*) < u(t^*) - u(t_*) &\le -\frac{M + Nk_0 + Lh_0}{2}u(t_*)(t_* - t^*)^2 \\ &\le -\frac{M + Nk_0 + Lh_0}{2}u(t_*).\end{aligned}$$

Hence

$$u(t_*)(2 - M - Nk_0 - Lh_0) > 0.$$

This is also a contradiction. The proof is complete. $\square$

Corollary 2.1. *Assume that $u \in E$ satisfies*

$$\begin{cases} -u''(t) + Mu(t) + N[Tu](t) + L[Su](t) \\ +[(M+r)c(t) + N[Tc](t) + L[Sc](t)](u(1) - au(\eta) + \rho) \le 0, \quad t \in J, \\ u(0) \le 0, \quad u(1) > au(\eta) - \rho, \end{cases}$$

where $0 < a \le 1,\ 0 < \eta < 1,\ \rho \ge 0$, constants $M > 0,\ N \ge 0,\ L \ge 0$ satisfying (2.2), then $u(t) \le 0$ for $t \in J$.

Proof. Put

$$y(t) = u(t) + c(t)(u(1) - au(\eta) + \rho), \quad t \in J,$$

then $y(t) \ge u(t)$ for all $t \in J$. Noting that $y''(t) = u''(t) - rc(t)(u(1) - au(\eta) + \rho),\ t \in J$, we have

$$\begin{aligned} -y''(t) + My(t) + N[Ty](t) + L[Sy](t) & \\ = -u''(t) + Mu(t) + N[Tu](t) + L[Su](t) + [(M+r)c(t) & \\ + N[Tc](t) + L[Sc](t)](u(1) - au(\eta) + \rho) & \\ \le 0, & \\ y(0) = u(0) \le 0, & \\ ay(\eta) - \rho = au(\eta) + ac(\eta)(u(1) - au(\eta) + \rho) - \rho = u(1) = y(1). & \end{aligned}$$

Hence by Theorem 2.1, $y(t) \le 0$ for all $t \in J$, which implies that $u(t) \le 0$ for $t \in J$. This ends the proof. $\square$

3. Linear problem.

Theorem 3.1. *Let $\sigma \in C(J),\ 0 < a \le 1,\ 0 < \eta < 1, \lambda \ge 0$, constants $M > 0,\ N \ge 0,\ L \ge 0$ satisfy (2.2). Consider the problem*

$$\begin{cases} -u''(t) + Mu(t) + N[Tu](t) + L[Su](t) = \sigma(t), \quad t \in J, \\ u(0) = 0, \quad u(1) = au(\eta) - \lambda. \end{cases} \tag{3.1}$$

Suppose that there exist $\alpha, \beta \in E$ such that

(h_1) $\alpha \le \beta$ *on* J.

(h_2)

$$\begin{cases} -\alpha''(t) + M\alpha(t) + N[T\alpha](t) + L[S\alpha](t) \le \sigma(t) - a^*(t), \ t \in J, \\ \alpha(0) \le 0, \end{cases}$$

where

$$a^*(t) = \begin{cases} 0, & \alpha(1) \le a\alpha(\eta) - \lambda, \\ ((r+M)c(t) + N[Tc](t) + L[Sc](t))(\alpha(1) - a\alpha(\eta) \\ \qquad +\lambda), & \alpha(1) > a\alpha(\eta) - \lambda. \end{cases}$$

(h_3)

$$\begin{cases} -\beta''(t) + M\beta(t) + N[T\beta](t) + L[S\beta](t) \ge \sigma(t) + b^*(t), \quad t \in J, \\ \beta(0) \ge 0, . \end{cases}$$

where

$$b^*(t) = \begin{cases} 0, \beta(1) \ge a\beta(\eta) - \lambda, \\ ((M+r)c(t) + N[Tc](t) + L[Sc](t))(a\beta(\eta) - \beta(1) - \lambda), \\ \qquad\qquad \beta(1) < a\beta(\eta) - \lambda.. \end{cases}$$

Then, there exists a unique solution u to problem (3.1). *Moreover,* $\alpha \le u \le \beta$.

Proof. We first show that the solution of equation (3.1) is unique. Let $u_1, \ u_2$ be the solution of (3.1) and set $v = u_1 - u_2$. Thus

$$\begin{cases} -v''(t) + Mv(t) + N[Tv](t) + L[Sv](t) = 0, \quad t \in J, \\ v(0) = 0, \quad v(1) = av(\eta). \end{cases}$$

By Theorem 2.1(the special case of $\rho = 0$), we have that $v \le 0$ for $t \in J$, that is, $u_1 \le u_2$ on J. Similarly, one can obtain $u_2 \le u_1$ on J. Hence $u_1 = u_2$.

Next, we prove that if u is a solution of equation (3.1), then $\alpha \le u \le \beta$. Let $m = \alpha - u$.

If $\alpha(1) \le a\alpha(\eta) - \lambda$, then $a^*(t) = 0$ on J. So we have

$$\begin{cases} -m''(t) + Mm(t) + N[Tm](t) + L[Sm](t) \le 0, & t \in J, \\ m(0) \le 0, \quad m(1) \le am(\eta). \end{cases}$$

By Theorem 2.1, we have that $m = \alpha - u \le 0$ on J.

If $\alpha(1) > a\alpha(\eta) - \lambda$, then $a^*(t) = ((r+M)c(t) + N[Tc](t) + L[Sc](t))(\alpha(1) - a\alpha(\eta) + \lambda))$. Thus

$$\begin{aligned} -m''(t) &+ Mm(t) + N[Tm](t) + L[Sm](t) \\ &= -\alpha''(t) + M\alpha(t) + N[T\alpha](t) + L[S\alpha](t) \\ &\quad + u''(t) - Mu(t) - N[Tu](t) - L[Su](t) \\ &\le \sigma(t) - a^*(t) - \sigma(t) \\ &= -a^*(t) \\ &= -((r+M)c(t) + N[Tc](t) + L[Sc](t))(m(1) - am(\eta)). \end{aligned}$$

It is easy to see that $m(0) \le 0, \quad m(1) > am(\eta)$. By Corollary 2.1, we have that $m = \alpha - u \le 0$ on J. Analogously, $u \le \beta$ on J.

Finally, we show that equation (3.1) has a solution by five steps as follows.

Step 1 Let

$$\bar{\alpha}(t) = \begin{cases} \alpha(t), & \alpha(1) \le a\alpha(\eta) - \lambda, \\ \alpha(t) + c(t)[\alpha(1) - a\alpha(\eta) + \lambda], & \alpha(1) > a\alpha(\eta) - \lambda.. \end{cases}$$

$$\bar{\beta}(t) = \begin{cases} \beta(t), & \beta(1) \ge a\beta(\eta) - \lambda, \\ \beta(t) - c(t)[a\beta(\eta) - \beta(1) - \lambda], & \beta(1) < a\beta(\eta) - \lambda. \end{cases}$$

We shall show that $\bar{\alpha}, \ \bar{\beta}$ are the lower and upper solutions of (3.1) respectively, and

$$\alpha \le \bar{\alpha} \le \bar{\beta} \le \beta, \quad \text{for } t \in J. \tag{3.2}$$

Obviously, $\alpha(0) = \bar{\alpha}(0), \ \alpha(1) = \bar{\alpha}(1), \ \beta(0) = \bar{\beta}(0), \ \beta(1) = \bar{\beta}(1)$ and

$$a\bar{\alpha}(\eta) - \lambda = \begin{cases} a\alpha(\eta) - \lambda, & \alpha(1) \le a\alpha(\eta) - \lambda, \\ \alpha(1), & \alpha(1) > a\alpha(\eta) - \lambda. \end{cases}$$

$$a\bar{\beta}(\eta) - \lambda = \begin{cases} a\beta(\eta) - \lambda, & \beta(1) \geq a\beta(\eta) - \lambda, \\ \beta(1), & \beta(1) < a\beta(\eta) - \lambda. \end{cases}$$

Hence

$$\bar{\alpha}(0) \leq 0, \quad \bar{\alpha}(1) \leq a\bar{\alpha}(\eta) - \lambda, \tag{3.3}$$

$$\bar{\beta}(0) \geq 0, \quad \bar{\beta}(1) \geq a\bar{\beta}(\eta) - \lambda. \tag{3.4}$$

If $\alpha(1) \leq a\alpha(\eta) - \lambda$, then $\alpha = \bar{\alpha}$ on J. So

$$-\bar{\alpha}''(t) + M\bar{\alpha}(t) + N[T\bar{\alpha}](t) + L[S\bar{\alpha}](t) \leq \sigma(t), \ t \in J.$$

If $\alpha(1) > a\alpha(\eta) - \lambda$, then $\bar{\alpha}(t) = \alpha(t) + c(t)[\alpha(1) - a\alpha(\eta) + \lambda]$. Thus

$$\begin{aligned} -\bar{\alpha}''(t) &+ M\bar{\alpha}(t) + N[T\bar{\alpha}](t) + L[S\bar{\alpha}](t) \\ &= -\alpha''(t) + M\alpha(t) + N[T\alpha](t) + L[S\alpha](t) \\ &\quad + ((M + r)c(t) + N[Tc](t) + L[Sc](t))(\alpha(1) - a\alpha(\eta) + \lambda) \\ &\leq \sigma(t). \end{aligned}$$

Combining the above two cases and (3.3), we see that $\bar{\alpha}$ is a lower solution of (3.1). Similarly, $\bar{\beta}$ is an upper solution of (3.1).

It is easy to see that $\alpha \leq \bar{\alpha}, \ \bar{\beta} \leq \beta$ on J. We show that $\bar{\alpha} \leq \bar{\beta}$ on J. We need to consider the following four cases.

Case 1 $\alpha(1) \leq a\alpha(\eta) - \lambda$ and $\beta(1) \geq a\beta(\eta) - \lambda$.

Case 2 $\alpha(1) \leq a\alpha(\eta) - \lambda$ and $\beta(1) < a\beta(\eta) - \lambda$.

Case 3 $\alpha(1) > a\alpha(\eta) - \lambda$ and $\beta(1) \geq a\beta(\eta) - \lambda$.

Case 4 $\alpha(1) > a\alpha(\eta) - \lambda$ and $\beta(1) < a\beta(\eta) - \lambda$.

Here we only consider Case 4, other cases are similar and so the proof is omitted. Let $m = \bar{\alpha} - \bar{\beta}$ for $t \in J$, then $m(0) \leq 0$, $m(1) \leq am(\eta)$ and

$$\begin{aligned} -m''(t) &+ Mm(t) + N[Tm](t) + L[Sm](t) \\ &= -\bar{\alpha}''(t) + M\bar{\alpha}(t) + N[T\bar{\alpha}](t) + L[S\bar{\alpha}](t) \\ &\quad + \bar{\beta}''(t) - M\bar{\beta}(t) - N[T\bar{\beta}](t) - L[S\bar{\beta}](t) \\ &= -\alpha''(t) + M\alpha(t) + N[T\alpha](t) + L[S\alpha](t) + a^*(t) + \beta''(t) \\ &\quad - M\beta(t) - N[T\beta](t) - L[S\beta](t) - b^*(t) \\ &\leq \sigma(t) - \sigma(t) = 0. \end{aligned}$$

By Theorem 2.1, we obtain that $m \leq 0$ on J, that is, $\bar{\alpha} \leq \bar{\beta}$ on J. Thus (3.2) holds.

Step 2 We consider the equation

$$\tag{3.5} \begin{cases} -u''(t) + Mu(t) + N[Tu](t) + L[Su](t) = \sigma(t), & t \in J, \\ u(0) = 0, & u(1) = \mu. \end{cases}$$

Next, we show that equation (3.5) has a unique solution $u(t, \mu)$.

It is easy to check that equation (3.5) is equivalent to the integral equation

$$u(t) = \mu t + \int_0^1 G(t,s)[\sigma(s) - Mu(s) - N[Tu](s) - L[Su](s)]ds$$

where

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Define a mapping $A : C(J) \to C(J)$ by

$$Au(t) = \mu t + \int_0^1 G(t,s)[\sigma(s) - Mu(s) - N[Tu](s) - L[Su](s)]ds.$$

For any $x,\ y \in C(J)$, we have

$$\begin{aligned} &(Ax)(t) - (Ay)(t) \\ &= \int_0^1 G(t,s)[M(y(s) - x(s)) + N[T(y-x)](s) + L[S(y-x)](s)]ds. \end{aligned}$$

Noting that $\max_{t \in J} \int_0^1 G(t,s)ds = 1/8$, by (2.2), we obtain

$$\begin{aligned} \max_{t \in J} |(Ax)(t) - (Ay)(t)| \leq (M + Nk_0 + Lh_0) \max_{t \in J} |x(t) - y(t)| \\ \max_{t \in J} \int_0^1 G(t,s)ds \leq \frac{1}{4} \max_{t \in J} |x(t) - y(t)|, \end{aligned}$$

and so

$$|Ax - Ay|_0 \leq \frac{1}{4}|x - y|_0.$$

It shows that $A : C(J) \to C(J)$ is a contraction mapping. Thus there exists a $u \in C(J)$ such that $Au = u$. The equation (3.5) has a unique solution.

Step 3 We show that for any $t \in J$, the unique solution $u(t,\mu)$ of (3.5) is continuous in μ. Let $u(t,\mu_1)$, $u(t,\mu_2)$ be the solution of

(3.6)
$$\begin{cases} -u''(t) + Mu(t) + N[Tu](t) + L[Su](t) = \sigma(t), & t \in J, \\ u(0) = 0, & u(1) = \mu_1 \end{cases}$$

and

(3.7)
$$\begin{cases} -u''(t) + Mu(t) + N[Tu](t) + L[Su](t) = \sigma(t), & t \in J, \\ u(0) = 0, & u(1) = \mu_2, \end{cases}$$

respectively. Then

(3.8)
$$u(t,\mu_i) = \mu_i t + \int_0^1 G(t,s)[\sigma(s) - Mu(s,\mu_i) - N[Tu](s,\mu_i) - L[Su](s,\mu_i)]ds, \quad i = 1,2.$$

From (3.8), we have

$$\max_{t\in J} |u(t,\mu_1) - u(t,\mu_2)| \le (M + Nk_0 + Lh_0) \max_{t\in J} |u(t,\mu_1) - u(t,\mu_2)| \max_{t\in J} \int_0^1 G(t,s)ds + \mid \mu_1 - \mu_2 \mid = \frac{M + Nk_0 + Lh_0}{8} \max_{t\in J} |u(t,\mu_1) - u(t,\mu_2)| + \mid \mu_1 - \mu_2 \mid .$$

Hence

$$\max_{t\in J} |u(t,\mu_1) - u(t,\mu_2)| \le \frac{8}{8 - M - Nk_0 - Lh_0} \mid \mu_1 - \mu_2 \mid .$$

Step 4 We show that

$$\bar{\alpha}(t) \le u(t,\mu) \le \bar{\beta}(t) \tag{3.9}$$

for any $t \in J$ and $\mu \in [a\bar{\alpha}(\eta) - \lambda,\ a\bar{\beta}(\eta) - \lambda]$, where $u(t,\mu)$ is unique solution of (3.5).

Let $m(t) = \bar{\alpha}(t) - u(t,\mu)$. From $\mu \in [a\bar{\alpha}(\eta) - \lambda,\ a\bar{\beta}(\eta) - \lambda]$ and (3.3), we have that $m(1) = \bar{\alpha}(1) - \mu \le a\bar{\alpha}(\eta) - \lambda - \mu \le 0$, $m(0) = \bar{\alpha}(0) \le 0$, and

$$\begin{aligned}
-m''(t) &+ Mm(t) + N[Tm](t) + L[Sm](t) \\
&= -\bar{\alpha}''(t) + M\bar{\alpha}(t) \\
&\quad + N[T\bar{\alpha}](t) + L[S\bar{\alpha}](t) + u''(t,\mu) - Mu(t,\mu) - N[Tu](t,\mu) \\
&\quad - L[Su](t,\mu) \le \sigma(t) - \sigma(t) = 0.
\end{aligned}$$

By Theorem 2.1, we obtain that $m \le 0$ on J, that is, $\bar{\alpha}(t) \le u(t,\mu)$ on J. Similarly, $u(t,\lambda) \le \bar{\beta}(t)$ on J.

Step 5 Let

$$D = [a\bar{\alpha}(\eta) - \lambda,\ a\bar{\beta}(\eta) - \lambda], \quad P(\mu) = au(\eta,\mu) - \lambda,$$

where $u(t,\mu)$ is unique solution of (3.5). From step 4, we have

$$P(D) \subset D.$$

Since D is a compact convex set and P is continuous, it follows by Schaefer's fixed point theorem that P has a fixed point μ_0 in D such that $au(\eta,\mu_0) - \lambda = \mu_0$. Obviously, $u(t,\mu_0)$ is unique solution of (3.1). This ends the proof. □

4. Main results.

In this section, we first give the following definition.

Definition 4.1. A function $\alpha \in E$ is called a lower solution of equation (1.2) if

$$\begin{cases} -\alpha''(t) \le f(t,\alpha(t),[T\alpha](t),[S\alpha](t)) - a(t), & t \in J, \\ \alpha(0) \le 0, \end{cases}$$

where

$$a(t) = \begin{cases} 0, & \alpha(1) \le a\alpha(\eta) - \lambda, \\ ((M+r)c(t) + N[Tc](t) + L[Sc](t))(\alpha(1) - a\alpha(\eta) + \lambda), & \\ & \alpha(1) > a\alpha(\eta) - \lambda. \end{cases}$$

Definition 4.2. A function $\beta \in E$ is called an upper solution of equation (1.2) if

$$\begin{cases} -\beta''(t) \geq f(t, \beta(t), [T\beta](t), [S\beta](t)) + b(t), & t \in J, \\ \beta(0) \geq 0, \end{cases}$$

where

$$b(t) = \begin{cases} 0, & \beta(1) \geq a\beta(\eta) - \lambda, \\ ((M+r)c(t) + N[Tc](t) + L[Sc](t))(a\beta(\eta) - \beta(1) - \lambda), & \\ & \beta(1) < a\beta(\eta) - \lambda. \end{cases}$$

We note that the above $a(t)$ and $b(t)$ are respectively $a^*(t)$ and $b^*(t)$ defined as in Theorem 3.1. Here, the reason is that we introduce $a(t)$ and $b(t)$ for the nonlinear problem (1.2), while $a^*(t)$ and $b^*(t)$ are introduced for the linear problem (3.1).

Our main result is the following theorem.

Theorem 4.1. *Suppose that* $0 < a \leq 1,\ 0 < \eta < 1,\ \lambda \geq 0$ *and the following conditions are satisfied*

(i) $\alpha,\ \beta$ *are lower and upper solutions for boundary value problem* (1.2) *respectively,* $\alpha(t) \leq \beta(t)$ *on* J.

(ii) *The constants* $M > 0,\ N \geq 0,\ L \geq 0$ *in the Definition* 4.1 *and* 4.2 *satisfy* (2.2) *and*

$$f(t,x,y,z) - f(t,\bar{x},\bar{y},\bar{z}) \geq -M(x-\bar{x}) - N(y-\bar{y}) - L(z-\bar{z}),$$

for $\alpha(t) \leq \bar{x} \leq x \leq \beta(t),\ [T\alpha](t) \leq \bar{y} \leq y \leq [T\beta](t)),\ [S\alpha](t) \leq \bar{z} \leq z \leq [S\beta](t)),\ t \in J$.

Then, there exist monotone sequences $\{\alpha_n\},\ \{\beta_n\}$ *with* $\alpha_0 = \alpha,\ \beta_0 = \beta$ *such that* $\lim_{n\to\infty} \alpha_n(t) = e(t),\ \lim_{n\to\infty} \beta_n(t) = r(t)$ *uniformly on* J, *and* $e,\ r$ *are the minimal and the maximal solutions of* (1.2) *respectively, such that*

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \alpha_n \leq e \leq x \leq r \leq \beta_n \leq \cdots \leq \beta_2 \leq \beta_1 \leq \beta_0$$

on J, *where* x *is any solution of (1.2) such that* $\alpha(t) \leq x(t) \leq \beta(t)$ *on* J.

Proof. Let $[\alpha,\beta]=\{u\in E:\alpha(t)\le u(t)\le\beta(t),t\in J\}$. For any $\gamma\in[\alpha,\beta]$, we consider the equation

$$(4.1)\quad \begin{cases} -u''(t)+Mu(t)+N[Tu](t)+L[Su](t) \\ = f(t,\gamma(t),[T\gamma](t),[S\gamma](t))+M\gamma(t)+N[T\gamma](t)+L[S\gamma](t), \\ \hfill t\in J, \\ u(0)=0, \qquad\qquad u(1)=au(\eta)-\lambda. \end{cases}$$

Since α is a lower solution of (1.2), from (ii), we have that

$$\begin{aligned} -\alpha''(t)&+M\alpha(t)+N[T\alpha](t)+L[S\alpha](t) \\ &\le f(t,\alpha(t),[T\alpha](t),[S\alpha](t))+M\alpha(t)+N[T\alpha](t) \\ &\quad +L[S\alpha](t)-a(t) \\ &\le f(t,\gamma(t),[T\gamma](t),[S\gamma](t)) \\ &+M\gamma(t)+N[T\gamma](t)+L[S\gamma](t)-a^*(t)\alpha(0)\le 0. \end{aligned}$$

Similarly, we have that

$$\begin{aligned} -\beta''(t)&+M\beta(t)+N[T\beta](t)+L[S\beta](t) \\ &\ge f(t,\gamma(t),[T\gamma](t),[S\gamma](t)) \\ &\quad +M\gamma(t)+N[T\gamma](t)+L[S\gamma](t)+b^*(t)\beta(0)\ge 0, \end{aligned}$$

where $a^*,\ b^*$ are defined in Theorem 3.1.

By Theorem 3.1, the equation (4.1) has a unique solution $u\in E$. We define an operator A by $u=A\gamma$, then A is an operator from $[\alpha,\beta]$ to E.

We shall show that

(a) $\alpha_0\le A\alpha_0,\quad A\beta_0\le\beta_0$.

(b) A is nondecreasing in $[\alpha_0,\beta_0]$.

To prove (a), let $p=\alpha_0-\alpha_1$, where $\alpha_1=A\alpha_0$. We finish (a) by two cases.

Case 1. $\alpha_0(1)\le a\alpha_0(\eta)-\lambda$. Since α_0 is a lower solution of (1.2), then we have

$$(4.2)\qquad -\alpha_0''(t)\le f(t,\alpha(t),[T\alpha_0](t),[S\alpha_0](t))-a(t),$$

where $a(t) = 0$. Hence, for $t \in J$, we get

$$\begin{aligned}
-p''(t) &+ Mp(t) + N[Tp](t) + L[Sp](t) \\
&= -\alpha_0''(t) + \alpha_1''(t) + M\alpha_0(t) - M\alpha_1(t) + N[T\alpha_0](t) \\
&\quad - N[T\alpha_1](t) + L[S\alpha_0](t) - L[S\alpha_1](t) \\
&\le f(t, \alpha_0(t), [T\alpha_0](t), [S\alpha_0](t)) - f(t, \alpha_0(t), [T\alpha_0](t), [S\alpha_0](t)) \\
&= 0.
\end{aligned}$$

It is easy to verify that

$$p(0) \le 0, \quad p(1) \le ap(\eta).$$

By Theorem 2.1, $p(t) \le 0$, which implies $\alpha_0 \le A\alpha_0$.

Case 2. $\alpha_0(1) > a\alpha_0(\eta) - \lambda$, which implies that (4.2) holds for

$$a(t) = [(M + r)c(t) + N[Tc](t) + L[Sc](t)](\alpha_0(1) - a\alpha_0(\eta) + \lambda).$$

Hence, $p(0) \le 0,\ p(1) > ap(\eta) - \lambda$ and

$$\begin{aligned}
-p''(t) &+ Mp(t) + N[Tp](t) + L[Sp](t) \\
&= -\alpha_0''(t) + \alpha_1''(t) + M\alpha_0(t) - M\alpha_1(t) + N[T\alpha_0](t) - N[T\alpha_1](t) \\
&\quad + L[S\alpha_0](t) - L[S\alpha_1](t) \\
&\le f(t, \alpha_0(t), [T\alpha_0](t), [S\alpha_0](t)) - a(t) \\
&\quad - f(t, \alpha_0(t), [T\alpha_0](t), [S\alpha_0](t)) \\
&= -a(t) = -((M + r)c(t) + N[Tc](t) + L[Sc](t))(p(1) - ap(\eta)).
\end{aligned}$$

It follows by Corollary 2.1 that $p(t) \le 0$, which implies $\alpha_0 \le A\alpha_0$. Similarly, $A\beta_0 \le \beta_0$.

To prove (b). We show that $A\mu_1 \le A\mu_2$ if $\alpha_0 \le \mu_1 \le \mu_2 \le \beta_0$.

Let $\mu_1^* = A\mu_1,\ \mu_2^* = A\mu_2$ and $p = \mu_1^* - \mu_2^*$, then by (ii), we have

$$\begin{aligned}
-p''(t) &+ Mp(t) + N[Tp](t) + L[Sp](t) \\
&= f(t, \mu_1(t), [T\mu_1](t), [S\mu_1](t)) + M\mu_1(t) + N[T\mu_1](t) \\
&\quad + L[S\mu_1](t) - f(t, \mu_2(t), [T\mu_2](t), [S\mu_2](t)) - M\mu_2(t) \\
&\quad - N[T\mu_2](t) - L[S\mu_2](t) \\
&\le 0.
\end{aligned}$$

And

$$p(0) = 0, \quad p(1) = ap(\eta).$$

By Theorem 2.1, $p(t) \leq 0$, which implies $A\mu_1 \leq A\mu_2$. Define the sequences $\{\alpha_n\}$, $\{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\alpha_{n+1} = A\alpha_n$, $\beta_{n+1} = A\beta_n$ for $n = 0, 1, 2, \ldots$. From (a) and (b), we have

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \alpha_n \leq \beta_n \leq \cdots \leq \beta_2 \leq \beta_1 \leq \beta_0$$

on $t \in J$, and each α_n, $\beta_n \in E$ satisfies

$$\begin{cases} -\alpha_n''(t) + M\alpha_n(t) + N[T\alpha_n](t) + L[\alpha_n](t) \\ = f(t, \alpha_{n-1}(t), [T\alpha_{n-1}](t), [S\alpha_{n-1}](t)) \\ +M\alpha_{n-1}(t) + N[T\alpha_{n-1}](t) + L[S\alpha_{n-1}](t), \quad t \in J, \\ \alpha_n(0) = 0, \quad \alpha_n(1) = a\alpha_n(\eta) - \lambda. \end{cases}$$

$$\begin{cases} -\beta_n''(t) + M\beta_n(t) + N[T\beta_n](t) + L[S\beta_n](t) \\ = f(t, \beta_{n-1}(t), [T\beta_{n-1}](t), [S\beta_{n-1}](t)) \\ +M\beta_{n-1}(t) + N[T\beta_{n-1}](t) + L[S\beta_{n-1}](t), \qquad t \in J, \\ \beta_n(0) = 0, \qquad \beta_n(1) = a\beta_n(\eta) - \lambda. \end{cases}$$

Therefore there exist e, r such that $\lim_{n\to\infty} \alpha_n(t) = e(t)$, $\lim_{n\to\infty} \beta_n(t) = r(t)$ uniformly on J. Clearly, e, r are solutions of (1.2).

Finally, we prove that if $x \in [\alpha_0, \beta_0]$ is any solution of (1.2), then $e(t) \leq x(t) \leq r(t)$ on J. To this end, we assume, without loss of generality, that $\alpha_n(t) \leq x(t) \leq \beta_n(t)$ for some n, since $\alpha_0(t) \leq x(t) \leq \beta_0(t)$. From property (b), we can get

$$\alpha_{n+1}(t) \leq x(t) \leq \beta_{n+1}(t), \quad t \in J.$$

Hence we can conclude that

$$\alpha_k(t) \leq x(t) \leq \beta_k(t), \quad for \ \ all \ k \geq n.$$

Passing to the limit as $k \to \infty$, we obtain $e(t) \leq x(t) \leq r(t)$, $\quad t \in J$. This ends the proof. $\square$

Example 4.1. Consider the following equation

$$\begin{cases} -x''(t) = \frac{1}{2}(1-x) + \int_0^1 sx(s)ds \\ x(0) = 0, \qquad x(1) = x(\frac{3}{4}) - \frac{1}{2}. \end{cases}$$

It is clear that $\alpha = -1-5t$ is a lower solution of (4.2). Let $M = 1/2, L = 0$, then $f(t,u,v) = \frac{1}{2}(1-u) + \int_0^1 v(s)ds$ satisfies (ii) of Theorem 4.1. Next we show that $\beta = t - 2t^2$ is a upper solution of (4.2).

Obviously $\beta(0) = 0,\ \beta(1) = -1 < \beta(\frac{3}{4}) - \frac{1}{2} = -\frac{7}{8}$ and

$$\begin{aligned} -\beta''(t) = 4 &\geq f(t, \beta(t), [S\beta](t)) + b(t) \\ &= \frac{1}{2}(1 - t - 2t^2) + \int_0^1 t(t-2t^2)dt + \frac{1}{8}(M+\pi^2)\frac{\sin(\pi t)}{\sin(\frac{3}{4}\pi)}. \end{aligned}$$

By Theorem 4.1, (4.2) has at least a solution $x(t) \in [\alpha, \beta]$. Moreover, we get from (4.2) that $x(t)$ satisfies

$$2x'''(t) - x'(t) = 0.$$

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REFERENCES

1. R.Y. Ma, *Existence theorem for a second order boundary value problem*, J. Math. Anal. Appl., **212** (1997),430–442.

2. V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, (1989).

3. A.R. Aftabizadeh, *Existence and uniqueness theorems for fourth order boundary value problems*, J. Math. Anal. Appl., **116** (1986), 415–426.

4. A.R. Aftabizadeh, and J. Wiener, *Existence and uniqueness theorems for third order boundary value problems*, Rend. Sem. Math. Univ. Padova, **75** (1986), 129–141.

5. C.P. Gupta, and V. Lakshmikantham, *Existence and uniqueness theorems for a third-order three-point boundary value problems*, Nonl. Anal., **16** (1991), 949–957.

6. C.P. Gupta, *Solvability of a three-point boundary value problem for a second order ordinary differential equations*, J. Math. Anal. Appl., **168** (1992), 540–551.

7. C.P. Gupta, and S.I. Trofimchuk, *A sharper condition for solvability of a three-point second-order BVP*, J. Math. Anal. Appl., **205** (1997), 586–597.

8. J.J. Nieto, and R. Rodrigue-Lopez, *Existence and approximation of solutions for nonlinear functional differential equations with periodic boundary value conditions*, Int. J. Comput. Math., **40** (2000),433–442.

9. ——— *Remark on periodic boundary value problems for func- tional differential equations*, Int.J.Comput.Math., **158** (2003), 339–353.

10. Zhiming He, Xiaoming He, *Periodic boundary value problems for first order impulsive intergro-differential equations of mixed type*, J. Math. Anal. Appl., **296** (2004), 8–20.

11.V. A. Il'in and E.I. Moiseev, *Nonlocal boundary-value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects, Differential Equations*, **23(7)** (1987) 803–810.

12. G. Infante and J.R.L. Webb, *Three-point boundary value problems with solutions that change sign*, J..Integ..Eqns..Appl., **15** (2002), 37–57.

13. J.R.L. Webb, *Positive solutions of some three-point boundary value problems via fixed point index theory*, Nonlinear Anal., **47** (2001), 4319–4332.

14. R. Ma, *Positive solutions of a nonlinear three-point boundary-value problem*, Electronic Journal of Differential Equations, **34** (1999), 1–8.

15. S. Marano, *A remark on a second-order three-point boundary value problem*, J..Math..Anal..Appl. **183** (1994),518–522.

16. T. Jankowski, *Solvability of three point boundary value problems for second order ordinary differential equations with deviating arguments*, J. Math. Anal. Appl. **312** (2005), 620–636.

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