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OBLIQUE DUALS ASSOCIATED WITH RATIONAL SUBSPACE GABOR FRAMES

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ABSTRACT. Given parameters $a, b > 0$ such that the product $a\,b$ is rational, we consider subspace Gabor frames $\mathbf{G}(b, a, g)$ and $\mathbf{G}(b, a, k)$ and ask for conditions under which the subspace generated by the second frame contains a function which acts as a Gabor dual for the first one ("oblique dual"). A necessary and sufficient condition for the existence of such a function is given in terms of an inequality between certain matrix-valued functions constructed using the Zak transform of the generators g and k. The uniqueness of the oblique dual is also characterized.

1. Introduction. The theory of frames, generalizing the notion an orthonormal basis in a Hilbert space to systems that might be overcomplete, was first introduced by R. J. Duffin and A. C. Schaeffer in [8] (see also [18]). We briefly recall some terminology, notation and basic facts about this tool which plays an important role in modern theories such as that of wavelet and Gabor expansions. If $\mathcal{N}$ is a countable index set and $\mathcal{H}$ is an infinite-dimensional separable Hilbert space with inner product $\langle ., . \rangle$, we say that a collection $X = \{x_n\}_{n\in\mathcal{N}}$ in $\mathcal{H}$ is a *subspace frame* (for its closed linear span $\mathcal{M}$) if there exist constants $C_1, C_2 > 0$, called the *frame bounds*, such that

$$(1.1)\qquad C_1\,\|x\|^2 \le \sum_{n\in\mathcal{N}} |\langle x, x_n\rangle|^2 \le C_2\,\|x\|^2, \quad x \in \mathcal{M}.$$

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We say that X is a *Bessel collection*, or has the *Bessel property*, with Bessel constant C_2, if the second inequality in (1.1) holds for all $x \in \mathcal{M}$ (or, equivalently, for all $x \in \mathcal{H}$). If X is a Bessel collection, we can define the *analysis operator* or *frame transform* associated with X, $T_X : \mathcal{M} \to \ell^2(\mathcal{N})$, by

$$T_X(x) = \{\langle x, x_n \rangle\}_{n\in\mathcal{N}}, \quad x \in \mathcal{M}. \tag{1.2}$$

Its adjoint, the *synthesis operator*, $T_X^* : \ell^2(\mathcal{N}) \to \mathcal{M}$, is defined by

$$T_X^*(\{c_n\}_{n\in\mathcal{N}}) = \sum_{n\in\mathcal{N}} c_n\, x_n, \quad \{c_n\}_{n\in\mathcal{N}} \in \ell^2(\mathcal{N}). \tag{1.3}$$

The operator $S = T_X^* T_X : \mathcal{M} \to \mathcal{M}$ is called the *frame operator* and is given explicitly as

$$Sx = \sum_{n\in\mathcal{N}} \langle x, x_n \rangle\, x_n, \quad x \in \mathcal{M}. \tag{1.4}$$

If X is a frame for $\mathcal{M}$, then S is a positive, bounded and invertible operator from $\mathcal{M}$ onto $\mathcal{M}$. The collection $\{S^{-1}x_n\}_{n\in\mathcal{N}}$ is called the *standard dual frame* of the frame X. It provides a reconstruction formula for the elements of $\mathcal{M}$ in terms of their inner products with the frame elements:

$$x = \sum_{n\in\mathcal{N}} \langle x, x_n \rangle\, S^{-1}x_n = \sum_{n\in\mathcal{N}} \langle x, S^{-1}x_n \rangle\, x_n, \quad x \in \mathcal{M}.$$

More generally, a *dual* for the the frame X is a Bessel collection $Y = \{y_n\}_{n\in\mathcal{N}}$ in $\mathcal{H}$ (with the y_n not necessarily in $\mathcal{M}$) satisfying

$$\sum_{n\in\mathcal{N}} \langle x, y_n \rangle\, x_n = x, \quad x \in \mathcal{M}.$$

A collection $\{y_n\}$ as above is sometimes called an *oblique dual*. Of particular interest are frames which are generated by a family (usually a group) of unitary operators acting on a particular function, called the *generator*. For example, a (one-dimensional) shift-invariant frame sequence is a subspace frame of the form $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$, where $\phi \in L^2(\mathbb{R})$, which is generated by applying integer translations to the function ϕ. Other examples are wavelet (or affine) systems which are defined using appropriate translations and scaling operators. In this paper, we will mainly investigate one-dimensional *Gabor* (also called *Weyl-Heisenberg*) systems which are generated using modulation and translation operators. Given two real numbers ω and t, we define the unitary operators E_ω (modulation) and T_t (translation) by

$$(E_\omega f)(x) = e^{2\pi i\omega x} f(x) \quad \text{and} \quad (T_t f)(x) = f(x-t), \quad f \in L^2(\mathbb{R}).$$

For convenience, given two positive parameters a and b, we also write

$$g_{mb,na} = E_{mb} T_{na}\, g, \quad m, n \in \mathbb{Z},$$

when $g \in L^2(\mathbb{R})$ and let $\mathbf{G}(b, a, g)$ denote the collection $\{g_{mb,na}\}_{m,n\in\mathbb{Z}}$. In the case of Gabor systems, the generator g is often called a *window function.* For such a function g, we define the space $\mathcal{M}(b, a, g)$ to be the closed linear span in $L^2(\mathbb{R})$ of the collection $\mathbf{G}(b, a, g)$. If $\mathbf{G}(b, a, g)$ is a frame for $\mathcal{M}(b, a, g)$, then we say that $\mathbf{G}(b, a, g)$ is a *subspace Gabor frame.* (We refer the reader to the papers [1, 2, 11, 12, 13, 14] for results related to subspace Gabor frames.) Let S be the frame operator associated with a subspace Gabor frame $\mathbf{G}(b, a, g)$. Then $S^{-1}g \in \mathcal{M}(b, a, g)$, and $S^{-1}(g_{mb,na}) = (S^{-1}g)_{mb,na}$. Therefore, $\mathbf{G}(b, a, S^{-1}g)$ is the standard dual of $\mathbf{G}(b, a, g)$. In general, unless the collection $\mathbf{G}(b, a, g)$ is a so-called Riesz sequence, there will exist many dual sequences associated with our subspace frame. We will, however, only consider here dual systems which are of the form $\mathbf{G}(b, a, h)$, for some function $h \in L^2(\mathbb{R})$. Such duals will be called *Gabor duals.* Thus, the function $h \in L^2(\mathbb{R})$ generates a Gabor dual for the subspace Gabor frame $\mathbf{G}(b, a, g)$ if and only if $\mathbf{G}(b, a, h)$ is a Bessel collection and we have

$$f = \sum_{m,n\in\mathbb{Z}} \langle f,\, E_{mb}\, T_{na}\, h\rangle\, E_{mb}\, T_{na}\, g, \quad f \in \mathcal{M}(b, a, g).$$

In the case of one-dimensional shift-invariant systems, it is natural, of course, to seek duals which are generated by the integer shifts of a single function. For $f \in L^2(\mathbb{R})$, let

$$\hat{f}(\gamma) = \int_{\mathbb{R}} e^{-2\pi i\gamma x} f(x)\, dx, \quad \gamma \in \mathbb{R},$$

denote the Fourier transform of f. The following result was proved by O. Christensen and Y. C. Eldar.

Theorem 1.1. ([5]) *Let $\phi, \phi_1 \in L^2(\mathbb{R})$ and assume that the sequences $\{T_k\phi\}_{k\in\mathbb{Z}}$ and $\{T_k\phi_1\}_{k\in\mathbb{Z}}$ are subspace frames associated with the subspace $\mathcal{V}$ and $\mathcal{V}_1$, respectively. Let*

$$\Phi(\gamma) = \sum_{k\in\mathbb{Z}} |\hat{\phi}(\gamma+k)|^2, \quad \gamma \in \mathbb{R}, \quad \textit{and} \quad \mathcal{N}(\Phi) = \{\gamma \in \mathbb{R},\ \phi(\gamma) = 0\}.$$

If there exists a constant $A > 0$ such that

$$\left|\sum_{k\in\mathbb{Z}} \hat{\phi}(\gamma+k)\,\overline{\hat{\phi}_1(\gamma+k)}\right| \geq A \tag{1.5}$$
$$\textit{a. e. on the set } \{\gamma \in \mathbb{R},\ \Phi(\gamma) \neq 0\},$$

then the following holds:

(i) *There exists a function $\tilde{\phi} \in \mathcal{V}_1$ generating a Bessel collection $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$ such that*

$$f = \sum_{k\in\mathbb{Z}} \langle f, T_k\tilde{\phi}\rangle\, T_k\phi, \quad f \in \mathcal{V}. \tag{1.6}$$

(ii) *One choice of $\tilde{\phi} \in \mathcal{V}_1$ satisfying (1.6) is given in the Fourier domain by*

$$\widehat{\tilde{\phi}}(\gamma) = \begin{cases} \dfrac{\hat{\phi}_1(\gamma)}{\sum_{k\in\mathbb{Z}} \hat{\phi}(\gamma+k)\,\overline{\hat{\phi}_1(\gamma+k)}}, & \textit{on } \{\gamma \in \mathbb{R},\ \Phi(\gamma) \neq 0\}, \\ 0, & \textit{on } \{\gamma \in \mathbb{R},\ \Phi(\gamma) = 0\}. \end{cases} \tag{1.7}$$

(iii) *There exists a unique function $\tilde{\phi} \in \mathcal{V}_1$ such that (1.6) is satisfied if and only if $\mathcal{N}(\Phi) = \mathcal{N}(\Phi_1)$. If this last condition holds, then $\{T_k\phi_1\}_{k\in\mathbb{Z}}$ is a frame for $\mathcal{V}$ and an oblique dual of $\{T_k\phi\}_{k\in\mathbb{Z}}$ on $\mathcal{V}$.*

Our main goal in this paper is to study the analogous problem for one-dimensional subspace Gabor frames. Thus the question we would like to answer is the following. Suppose that $\mathbf{G}(b,a,g)$ and $\mathbf{G}(b,a,k)$ are two subspace Gabor frames. Under what conditions can we find a function h in $\mathcal{M}(b,a,k)$ such that $\mathbf{G}(b,a,h)$ is a Gabor dual for $\mathbf{G}(b,a,g)$? Moreover, if such a Gabor dual exists, when is it unique? Our analysis of the problem will make an extensive use of the Zak transform and, for this reason, we will only consider rational Gabor systems in this paper, i. e. those for which the parameters a,b are such that $a\,b$ is a rational number.

The paper is organized as follows. In section 2, we introduce the Zak transform and prove some preliminary results. In particular, we give a characterization in terms of certain vector-valued functions defined using the Zak transform for the functions belonging to the space $\mathcal{M}(b,a,g)$ spanned by a Gabor system $\mathbf{G}(b,a,g)$ satisfying the Bessel condition (Proposition 2.5) or generating a subspace Gabor frame (Corollary 2.6). In this last situation, we give a similar characterization for the functions h in $\mathcal{M}(b,a,g)$ with the property that the Gabor system $\mathbf{G}(b,a,h)$ itself satisfies the Bessel condition (Corollary 2.7). In section 3, we prove the existence part of the problem mentioned above using certain matrix-valued functions (in particular the so-called Zibulski-Zeevi matrix) constructed again using the Zak transform. First we show that the existence of a Gabor dual belonging to a subspace generated by a possibly different Gabor window is equivalent to the existence of a specific factorization of these matrices (Theorem 3.2) or to a certain matrix inequality (Theorem 3.5). Finally, the uniqueness of the oblique dual is characterized in the last section (Theorem 4.2).

We refer the reader to the books [7, 9, 10, 15, 16, 18] as well as the papers [6, 17] for information on frames and, more particularly, wavelet and Gabor frames.

2. Some characterizations using the Zak transform Let g be a window in $L^2(\mathbb{R})$ and consider the Gabor system $\mathbf{G}(b,a,g)$ where $a\,b$ is rational, i. e. $a\,b = \frac{p}{q}$ where p,q are positive integers satisfying $\gcd(p,q) = 1$. If $n \in \mathbb{Z}$, we can write n uniquely as $n = i + \ell q$ with

$\ell \in \mathbb{Z}$ and $i \in \{0, 1, \dots, q-1\}$. Therefore, we have that

$$\begin{aligned} E_{mb}T_{na}g(x) &= g_{mb,\frac{np}{bq}}(x) = g_{mb,\frac{ip}{qb}+\frac{\ell p}{b}}(x) \\ &= e^{2\pi imbx}g\left(x - \frac{ip}{qb} - \frac{\ell p}{b}\right) = g^i_{mb,\frac{\ell p}{b}}(x), \end{aligned}$$

where $g^i(x) := g\big(x - \frac{ip}{qb}\big)$. The *Zak transform* $\mathcal{Z}_b : L^2(\mathbb{R}) \to L^2([0,1] \times [0,1])$ is defined by the formula

$$\mathcal{Z}_b g(x, w) = b^{-\frac{1}{2}} \sum_{k \in \mathbb{Z}} g\left(\frac{x-k}{b}\right) e^{2\pi ikw}, \quad (x, w) \in [0,1] \times [0,1],$$

and it is an isometric isomorphism between the two Hilbert spaces. It is easily checked that

$$\mathcal{Z}_b(g_{mb,\frac{n}{b}})(x, w) = e^{2\pi imx} e^{-2\pi inw} \mathcal{Z}_b g(x, w), \quad m, n \in \mathbb{Z}.$$

Therefore, defining the function $E_{m,n}$ of two variables by

$$E_{m,n}(x, w) = e^{2\pi imx}\, e^{2\pi inw}, \quad (x, w) \in [0,1] \times [0,1],$$

we have

$$\mathcal{Z}_b(g^i_{mb,\frac{\ell p}{b}})(x, w) = E_{m,-\ell p}(x, w)\, \mathcal{Z}_b g^i(x, w). \tag{2.1}$$

Note that a mapping with values in $\mathbb{C}^n$ (resp. $\mathcal{M}_{m,n}$, the space of complex matrices of size $m \times n$) is called *measurable* if all its components (resp. entries) are measurable. Clearly, all the mappings defined in the next definition are measurable.

Definition 2.1. Given a window function g in $L^2(\mathbb{R})$, we will associate with g a collection of q vector-valued functions G^i, $i = 0, \dots, q-1$ with values in $\mathbb{C}^p$ whose component functions are defined by

$$G^i_k(x, w) = \left(\mathcal{Z}_b g^i\right)(x, w + k/p), \quad k = 0, \dots, p-1.$$

for (x, w) in $[0,1] \times [0, 1/p]$. We also define the *Zibulski-Zeevi matrix* on the set $[0, 1/q] \times [0, 1/p]$, as the matrix-valued function $\mathcal{G}$, with values in the space $\mathcal{M}_{q,p}$ of complex matrices of size $q \times p$, with entries

$$\mathcal{G}_{ik}(x, w) = \left(\mathcal{Z}_b g^i\right)(x, w + k/p), \;\; i = 0, \dots q-1, \;\; k = 0, \dots, p-1. \tag{2.2}$$

The mapping $\mathcal{L}$ that maps g to the $q \times p$ matrix-valued function $\mathcal{G}$ is an isometric isomorphism (as a mapping from $L^2(\mathbb{R}) = L^2(\mathbb{R}, \mathbb{C})$ to the space of matrix-valued functions $L^2([0,1/q] \times [0,1/p], \mathcal{M}_{q,p})$ if the matrix norm of a $q \times p$ matrix $C = (c_{jk})$ is the Hilbert-Schmidt norm defined as

$$\|C\|^2_{\mathcal{M}_{q,p}} = \sum_{j=0}^{q-1} \sum_{k=0}^{p-1} |c_{jk}|^2$$

and the norm of matrix-valued function $\mathcal{G}$ is defined by the formula

$$\|\mathcal{G}\|^2 = \int_0^{1/q} \int_0^{1/p} \|\mathcal{G}(x,w)\|^2_{\mathcal{M}_{q,p}}\, dw\, dx.$$

Note that the rows of $\mathcal{G}$ are the row vectors G^i, $i = 0, \dots q-1$. We will also denote by $\mathcal{K}$ the operator which maps a function $f \in L^2(\mathbb{R})$ to the vector-valued function $F = \mathcal{K}f = (F_0, \dots, F_{p-1})$ whose p components are in $L^2([0,1] \times [0,1/p])$ and are defined by

$$F_k(x,w) = (\mathcal{K}f)_k\,(x,w) = (\mathcal{Z}_b f)\,(x, w + k/p)\,, \quad k = 0, \dots, p-1.$$

It is easily checked that the mapping $\mathcal{K} : L^2(\mathbb{R}, \mathbb{C}) \to L^2([0,1] \times [0,1/p], \mathbb{C}^p)$ is also an isometric isomorphism if $\mathbb{C}^p$ is equipped with its standard Euclidean norm.

The following result characterizing the Bessel condition is due to Zibulski and Zeevi (see also [2]).

Theorem 2.2 ([19]). *Let $a, b > 0$ with $ab = p/q$ and $\gcd(p,q) = 1$. Consider $g \in L^2(\mathbb{R})$ and let $\mathcal{G}$ be defined by formula (2.2). Then, the following are equivalent.*

(a) *There exists a positive constant B such that the inequality*

$$\sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g\rangle|^2 \le B\|f\|^2, \quad f \in \mathcal{M}(b,a,g),$$

holds.

(b) $\mathcal{G}^* \mathcal{G} \le Bp\, I$ *a. e. where I denotes the $p \times p$ identity matrix.*

(c) $\mathcal{G}\, \mathcal{G}^* \le Bp\, I$ *a. e. where I denotes the $q \times q$ identity matrix.*

It follows immediately from the previous proposition that a necessary and sufficient condition for a collection $\mathbf{G}(b,a,g)$ to satisfy the Bessel condition (in the case that $ab=p/q$ with $\gcd(p,q)=1$) is that the Zak transform of g, $\mathcal{Z}_b g$, belongs to $L^\infty([0,1]\times[0,1])$.

We will need the following result from [12] (see also [2]) which characterizes subspace Gabor frames in the rational case.

Theorem 2.3 ([12]). *Let $a,b>0$, with $ab=p/q$, $\gcd(p,q)=1$. The collection $\mathbf{G}(b,a,g)$ is a subspace Gabor frame with frame bounds A and B for the subspace $\mathcal{M}(b,a,g)$ if and only if*

$$Ap\,\xi \le \xi^2 \le Bp\,\xi, \quad a.\ e.\ on\ [0,1]\times[0,1/p],$$

where $\xi = \mathcal{G}\,\mathcal{G}^*$.

Note that if ξ is a positive semi-definite matrix, the matrix inequalities $Ap\,\xi \le \xi^2 \le Bp\,\xi$ are equivalent to

$$(2.3) \qquad Ap\,\langle x,\,x\rangle \le \langle \xi x,\,x\rangle \le Bp\,\langle x,\,x\rangle, \quad x\in \ker\xi^{\perp},$$

as follows easily from the spectral theorem.

Lemma 2.4. *Let $a,b>0$ and $ab=p/q$ with $\gcd(p,q)=1$. Let $g\in L^2(\mathbb{R})$ and assume that $\mathbf{G}(b,a,g)$ is a Bessel collection. Then $h\in L^2(\mathbb{R})$ is orthogonal to $\mathcal{M}(b,a,g)$ if and only if, for a. e. $(x,w)\in[0,1]\times[0,1/p]$, we have*

$$\langle H(x,w), G^i(x,w)\rangle_{\mathbb{C}^p} = 0, \quad i=0,\dots,p-1,$$

where the vector-valued functions G^i, $i=0,\dots,q-1$ and $H=\mathcal{K}h$ are as in Definition 2.1.

Proof. Let us prove the necessity part of the equivalence first. Let $h\in L^2(\mathbb{R})$, and assume that $h\perp\mathcal{M}(b,a,g)$. This is equivalent to

$$\langle h, E_{mb}T_{lp/b}\,g^i\rangle = 0, \quad m,l\in\mathbb{Z},\ i=0,\dots,q-1,$$

or, using the fact that the Zak transform is an isometric isomorphism together with (2.1), to

$$\int_0^1 \int_0^1 \mathcal{Z}_b h(x,w)\, \overline{\mathcal{Z}_b g^i(x,w)}\, \overline{E_{m,-\ell p}(x,w)}\, dw\, dx = 0, \quad m,l \in \mathbb{Z},$$

or, to

$$\int_0^1 \int_0^{1/p} \sum_{k=0}^{p-1} \mathcal{Z}_b h\left(x, w+\frac{k}{p}\right) \overline{\mathcal{Z}_b g^i\left(x, w+\frac{k}{p}\right)}\, e^{-2\pi i m x}\, e^{2\pi i \ell p w}\, dw\, dx = 0$$

for all $m,l \in \mathbb{Z}$. Using our definitions of H and G^i, $i = 0, \ldots, q-1$, we can rewrite this last set of equations as

$$\int_0^1 \int_0^{1/p} \langle H(x,w),\, G^i(x,w)\rangle_{\mathbb{C}^p}\, e^{-2\pi i m x}\, e^{2\pi i l p w}\, dw\, dx = 0,\ m,l \in \mathbb{Z}. \tag{2.4}$$

Using Theorem 2.2, it follows easily that the function $\langle H,\, G^i\rangle_{\mathbb{C}^p}$ belongs to $L^2([0,1]\times[0,1/p])$ and, since the collection $\{e^{2\pi i m x}\, e^{-2\pi i l p w}\}_{m,l\in\mathbb{Z}}$ forms an orthogonal basis for $L^2([0,1]\times[0,1/p])$, we deduce from (2.4) that for a. e. (x,w) belonging to the set $[0,1]\times[0,1/p]$, we have

$$\langle H(x,w), G^i(x,w)\rangle_{\mathbb{C}^p} = 0, \quad i = 0, \ldots, q-1.$$

Since this argument can clearly be reversed, our proof is now completed. □

We now characterize the functions in $\mathcal{M}(b,a,g)$ using the Zak transform.

Proposition 2.5. *Let $a, b > 0$ with $ab = p/q$ and* $\gcd(p,q) = 1$. *Let $g \in L^2(\mathbb{R})$ such that $\mathbf{G}(b,a,g)$ forms a Bessel collection. Then, $f \in L^2(\mathbb{R})$ belongs to $\mathcal{M}(b,a,g)$ if and only if $F = \mathcal{K}f$ has the representation*

$$F = \sum_{i=0}^{q-1} a_i\, G^i \quad \text{a. e. on } [0,1]\times[0,1/p], \tag{2.5}$$

where the functions $a_i : [0,1] \times [0,1/p] \to \mathbb{C}$, $i = 0, \dots, q-1$, *are measurable and satisfy*

$$\int_{[0,1]} \int_{[0,1/p]} \left\| \sum_{i=0}^{q-1} a_i(x,w)\, G^i(x,w) \right\|_{\mathbb{C}^p}^2 dw\, dx < \infty. \tag{2.6}$$

In particular, if $a_i \in L^2([0,1] \times [0,1/p])$, $i = 0, \dots, q-1$, *then there exists a function* f *which belongs to* $\mathcal{M}(b,a,g)$ *such that*

$$\mathcal{K}f = \sum_{i=0}^{q-1} a_i G^i.$$

Proof. Let us first assume that $f \in L^2(\mathbb{R})$ is such that $\mathcal{K}f = \sum_{i=0}^{q-1} a_i\, G^i$ where the coefficients a_i, $i = 0, \dots, q-1$, are measurable functions satisfying (2.6) and let $h \in L^2(\mathbb{R})$ with $h \perp \mathcal{M}(b,a,g)$. We have, letting $H = \mathcal{K}h$,

$$\begin{aligned}
\langle h, f \rangle &= \int_0^1 \int_0^1 \mathcal{Z}_b h(x,w)\, \overline{\mathcal{Z}_b f(x,w)}\, dw\, dx \\
&= \int_0^1 \int_0^{1/p} \sum_{k=0}^{p-1} \mathcal{Z}_b h(x, w + k/p)\, \overline{\mathcal{Z}_b f(x, w+k/p)}\, dw\, dx \\
&= \int_0^1 \int_0^{1/p} \langle H(x,w),\, F(x,w) \rangle_{\mathbb{C}^p}\, dw\, dx \\
&= \sum_{i=0}^{q-1} \int_0^1 \int_0^{1/p} \overline{a_i(x,w)}\, \langle H(x,w),\, G^i(x,w) \rangle_{\mathbb{C}^p}\, dw\, dx = 0
\end{aligned}$$

using Lemma 2.4. It follows that such a function f must belong to $\mathcal{M}(b,a,g)$.

To obtain the converse result, let us consider indices $i_1, \dots, i_r$ with $0 \le i_1 \le i_2 \le \dots \le i_r \le q-1$ for $1 \le r \le q$ and let us define $E_{i_1,\dots,i_r}$ to be the subset of $[0,1] \times [0,1/p]$ where $G^{i_1}, \dots, G^{i_r}$ are linearly independent and with the property that

$$\operatorname{span}(G^{i_1}, \dots, G^{i_r}) = \operatorname{span}(G^0, \dots, G^{q-1}).$$

Note that each set $E_{i_1,\dots,i_r}$ is measurable since it can be expressed as the intersection of the measurable sets

$$\begin{aligned} &\{\det(\langle G^k, G^l\rangle_{\mathbb{C}^p})_{k,l\in\{i_1,\dots,i_r\}} \neq 0\} \\ \text{and} \quad &\{\det(\langle G^k, G^l\rangle_{\mathbb{C}^p})_{k,l\in\{j,i_1,\dots,i_r\}} = 0\}, \end{aligned}$$

where j varies over all the indices in $\{0,\dots,q-1\}$ different from $i_1,\dots,i_r$. Defining K to be the subset of $[0,1]\times[0,1/p]$ where $G^i = 0$, for all $i = 0,\dots,q-1$, we have

$$[0,1]\times[0,1/p] = K \bigcup \left[\bigcup_{r=1}^{q-1} \bigcup_{0\le i_1\le i_2\le\dots\le i_r\le q-1} E_{i_1,\dots,i_r}\right]. \tag{2.7}$$

The sets $E_{i_1,\dots,i_r}$ are not necessarily disjoint, but we can replace them with a pairwise disjoint collection of measurable sets $F_{i_1,\dots,i_r}$ with $F_{i_1,\dots,i_r} \subseteq E_{i_1,\dots,i_r}$ such that the equality (2.7), with $E_{i_1,\dots,i_r}$ replaced by $F_{i_1,\dots,i_r}$, holds as a disjoint union. Assume that the measure of the set $F_{i_1,\dots,i_r}$ is not zero. Let $f \in L^2(\mathbb{R})$, and let $F = \mathcal{K}f$. Note that we can write

$$F = \sum_{i=0}^{q-1} a_i\, G^i + H \tag{2.8}$$

uniquely on the set $F_{i_1,\dots,i_r}$, where $a_i = 0$ for $i \notin \{i_1,\dots,i_r\}$, and $\langle H, G^i\rangle = 0$ for $i = 0,\dots,q-1$. Furthermore, the coefficients a_i, $i = 0,\dots,q-1$, are measurable. This is clear for $i \notin \{i_1,\dots,i_r\}$ and for $i \in \{i_1,\dots,i_r\}$, the coefficients a_i are obtained by solving the non-singular linear system of equations

$$\langle F, G^{i_k}\rangle_{\mathbb{C}^p} = \sum_{j=1}^{r} a_{i_j} \langle G^{i_j}, G^{i_k}\rangle_{\mathbb{C}^p}, \quad k = 1,\dots,r,$$

where all the coefficients involved are measurable, and thus the a_i are also measurable. Letting all the coefficients $a_i = 0$ on K, we can thus obtain the representation (2.8) for F on the whole set $[0,1]\times[0,1/p]$ with the coefficients a_i, $i = 0,\dots,q-1$, being measurable and with $\langle H, G^i\rangle = 0$ for each such i. Letting $Q = \sum_{i=0}^{q-1} a_i\, G^i$ and using the fact that Q is just the orthogonal projection of F onto the subspace spanned by the vectors G^i, $i = 0, \ldots, q-1$, we deduce that

$$\|Q\|_{\mathbb{C}^p} \leq \|F\|_{\mathbb{C}^p} \quad \text{on } [0,1] \times [0, 1/p].$$

In particular, the vector-valued function Q is square-integrable on $[0,1] \times [0,1/p]$ and there exists thus a function $q \in L^2(\mathbb{R})$ such that $\mathcal{K}q = Q$. The first part of the proof shows that $q \in \mathcal{M}(b,a,g)$. If we now assume that $f \in \mathcal{M}(b,a,g)$, we have also $f - q \in \mathcal{M}(b,a,g)$ and, since $\mathcal{K}(f-q) = H$, it follows from Lemma 2.4 that $f - q \perp \mathcal{M}(b,a,g)$. Hence, $f - q = 0$ or $f = q$, which yields the identity (2.5). To prove the last statement of the result, we use (c) of Theorem 2.2, to obtain the inequality

$$\Big\| \sum_{i=0}^{q-1} a_i \, G^i \Big\|^2_{\mathbb{C}^p} = \sum_{i,j=0}^{q-1} a_i \, \overline{a_j} \, \langle G^i, \, G^j \rangle_{\mathbb{C}^p} = \sum_{i,j=0}^{q-1} a_i \, \overline{a_j} \, (\mathcal{G}\,\mathcal{G}^*)_{ij} \leq Bp \sum_{i=0}^{q-1} |a_i|^2$$

which yields (2.6). This completes the proof. □

Corollary 2.6. *Let $a, b > 0$ with $ab = p/q$ and* $\gcd(p,q) = 1$. *Let $g \in L^2(\mathbb{R})$ such that $\mathbf{G}(b,a,g)$ forms a subspace Gabor frame. Then, $f \in L^2(\mathbb{R})$ belongs to $\mathcal{M}(b,a,g)$ if and only if*

$$F = \sum_{i=0}^{q-1} a_i \, G^i \quad \text{a. e. on } [0,1] \times [0,1/p], \tag{2.9}$$

where $F = \mathcal{K}f$ and the functions $a_i : [0,1] \times [0,1/p] \to \mathbb{C}$, $i = 0, \ldots, q-1$, belong to $L^2([0,1] \times [0,1/p])$.

Proof. The sufficiency part of the proof follows from the last part of Proposition 2.5. To prove the necessity part of the statement consider the frame operator S associated with the collection $\mathbf{G}(b,a,g)$:

$$Sf = \sum_{m,n \in \mathbb{Z}} \langle f, \, g_{mb,na} \rangle \, g_{mb,na}, \quad f \in \mathcal{M}(b,a,g).$$

A computation similar to the one leading to formula (3.5) yields

$$\mathcal{K}(Sf) = \frac{1}{p} \sum_{i=0}^{q-1} \langle F, G^i \rangle_{\mathbb{C}^p} \, G^i := \frac{1}{p} \, \mathcal{S} F \quad \text{on } [0,1] \times [0, 1/p],$$

where $\mathcal{S}$ denotes the frame operator associated with the finite collection $\{G^i\}_{i=0}^{q-1}$. Since, for $f \in \mathcal{M}(b, a, g)$, we have

$$F = \mathcal{K}(SS^{-1}f) = \frac{1}{p} \sum_{i=0}^{q-1} \langle \mathcal{K}(S^{-1}f), G^i \rangle_{\mathbb{C}^p} \, G^i = \frac{1}{p} \, \mathcal{S}\left(\mathcal{K}(S^{-1}f)\right),$$

it follows that $\mathcal{S}^{-1}F = \frac{1}{p} \, \mathcal{K}(S^{-1}f)$ and thus the functions $\langle \mathcal{S}^{-1}F, G^i \rangle_{\mathbb{C}^p}$, where $i = 0, \ldots, q-1$, are measurable on $[0,1] \times [0, 1/p]$. Note that, since for $i, j = 0, \ldots q-1$, we have $(\mathcal{G}\,\mathcal{G}^*)_{ij} = \langle G^i, G^j \rangle_{\mathbb{C}^p}$, a finite sequence $\{b_i\}_{i=0}^{q-1} \in \mathbb{C}^q$ belongs to the kernel of $\mathcal{G}\,\mathcal{G}^*$ if and only if

$$\sum_{i=0}^{q-1} \overline{b_i} \, G^i = 0 \tag{2.10}$$

as

$$\sum_{i,j=0}^{q-1} (\mathcal{G}\,\mathcal{G}^*)_{ij} \; b_j \, \overline{b_i} = \| \sum_{i=0}^{q-1} \overline{b_i} \, G^i \|_{\mathbb{C}^p}^2.$$

In particular, the sequence $\{\langle G^i, \mathcal{S}^{-1}F \rangle_{\mathbb{C}^p}\}_{i=0}^{q-1}$ belongs to $\ker(\mathcal{G}\,\mathcal{G}^*)^{\perp}$ since

$$\sum_{i=0}^{q-1} \langle G^i, \mathcal{S}^{-1}F \rangle_{\mathbb{C}^p} \, \overline{b_i} = \langle \sum_{i=0}^{q-1} \overline{b_i} \, G^i, \mathcal{S}^{-1}F \rangle_{\mathbb{C}^p} = 0$$

if $\{b_i\}_{i=0}^{q-1} \in \ker(\mathcal{G}\,\mathcal{G}^*)$. Letting $a_i = \langle \mathcal{S}^{-1}F, G^i \rangle_{\mathbb{C}^p}$, for $i = 0, \ldots, q-1$, and using Theorem 2.3, it follows that

$$\begin{aligned} \sum_{i=0}^{q-1} |\langle \mathcal{S}^{-1}F, G^i \rangle_{\mathbb{C}^p}|^2 &= \sum_{i=0}^{q-1} |\overline{a_i}|^2 \le \frac{1}{Ap} \sum_{i,j=0}^{q-1} (\mathcal{G}\,\mathcal{G}^*)_{ij} \; \overline{a_j} \, a_i \\ &= \frac{1}{Ap} \, \| \sum_{i=0}^{q-1} a_i \, G^i \|_{\mathbb{C}^p}^2 = \frac{1}{Ap} \, \| \mathcal{S}\,\mathcal{S}^{-1} F \|_{\mathbb{C}^p}^2 = \frac{1}{Ap} \, \| F \|_{\mathbb{C}^p}^2. \end{aligned}$$

This shows that $F = \sum_{i=0}^{q-1} a_i\, G^i$, where the coefficients a_i, $i = 0, \dots, q-1$, are square-integrable, which proves our claim. □

Corollary 2.7. *Let $a, b > 0$ and $ab = p/q$ and $\gcd(p,q) = 1$. Let $g \in L^2(\mathbb{R})$ be such that $\mathbf{G}(b,a,g)$ forms a subspace Gabor frame. Then, the following are equivalent:*

(a) *The function $f \in L^2(\mathbb{R})$ belongs to $\mathcal{M}(b,a,g)$ and $\mathbf{G}(b,a,f)$ has the Bessel property.*

(b) *We have*

$$F = \sum_{i=0}^{q-1} a_i\, G^i \quad a.\ e.\ on\ [0,1] \times [0,1/p], \tag{2.11}$$

where $F = \mathcal{K}f$ and the functions $a_i : [0,1] \times [0,1/p] \to \mathbb{C}$, $i = 0, \dots, q-1$, belong to $L^\infty([0,1] \times [0,1/p])$.

Proof. If (b) holds, it follows from (2.11) that all the components of F belong to $L^\infty([0,1] \times [0,1/p])$ since a_i and all the components of G^i belong to that same space, for $i = 0, \dots, q-1$. Hence, $\mathcal{Z}_b f \in L^\infty([0,1] \times [0,1])$ which is equivalent to $\mathbf{G}(b,a,f)$ being a Bessel collection. The fact that $f \in \mathcal{M}(b,a,g)$ follows from Proposition 2.5. Hence (a) holds. Conversely, if $f \in \mathcal{M}(b,a,g)$ is such that $\mathbf{G}(b,a,f)$ has the Bessel property, note that the collection $\mathbf{G}(b,a,S^{-1}f)$ also satisfies that same property, where S denotes the frame operator associated with the frame $\mathcal{M}(b,a,g)$. Indeed if, $h \in L^2(\mathbb{R})$, we have

$$\begin{aligned}
\sum_{m,n \in \mathbb{Z}} |\langle h,\, E_{mb} T_{na} S^{-1} f \rangle|^2 &= \sum_{m,n \in \mathbb{Z}} |\langle h,\, S^{-1} E_{mb} T_{na} f \rangle|^2 \\
&= \sum_{m,n \in \mathbb{Z}} |\langle S^{-1} h,\, E_{mb} T_{na} f \rangle|^2 \le C\, \|S^{-1}\|^2\, \|h\|^2,
\end{aligned}$$

where C denotes the Bessel constant associated with $\mathcal{M}(b,a,f)$. In particular, this shows that $\mathcal{Z}_b\left(S^{-1}f\right)$ belongs to $L^\infty([0,1] \times [0,1])$ and thus all the components of $\mathcal{K}\left(S^{-1}f\right)$ belong to $L^\infty([0,1] \times [0,1/p])$. Proceeding as in the proof of Corollary 2.6, we can write, using the fact that $f \in \mathcal{M}(b, a, g)$,

$$F = \sum_{i=0}^{q-1} a_i \, G^i \quad \text{a. e. on } [0,1] \times [0, 1/p],$$

where, for each $i = 0, \ldots, q-1$,

$$a_i = \langle \mathcal{S}^{-1} F, \, G^i \rangle_{\mathbb{C}^p} = \frac{1}{p} \, \langle \mathcal{K}\left(S^{-1} f\right), \, G^i \rangle_{\mathbb{C}^p} \in L^\infty([0,1] \times [0, 1/p]),$$

which yields (b). $\square$

3. Existence of oblique Gabor duals Our main goal in this section will be to answer the following question about Gabor dual systems in the rational case $ab = \frac{p}{q}$ where $\gcd(p, q) = 1$. Suppose that two subspace Gabor frames $\mathbf{G}(b, a, g)$ and $\mathbf{G}(b, a, k)$ are given. When can we find a function h in the subspace generated by the second Gabor frame such that the collection $\mathbf{G}(b, a, h)$ is a Gabor dual for the first one? Before dealing with this problem, we first need to introduce some notation that we will use throughout this section. We associate as before with the function g generating the first Gabor system a collection G^i, $i = 0, \ldots q-1$ of vector-valued functions (with values in $\mathbb{C}^p$)) defined as in Definition 2.1. We also associate with our second generator k a collection K^i, $i = 0, \ldots q-1$, of vector-valued functions defined in the same way. We then define two matrix-valued function ξ and η on the set $[0,1] \times [0, 1/p]$ with values in $\mathcal{M}_{q,q}$, the space of complex matrices of size $q \times q$, by the formula

$$\xi_{ij} = \langle G^i, G^j \rangle_{\mathbb{C}^p}, \quad \eta_{ij} = \langle K^i, G^j \rangle_{\mathbb{C}^p}, \quad i, j = 0, \ldots, q-1. \tag{3.1}$$

We first begin by rephrasing the duality conditions for our Gabor systems in terms of the Zak transform. Note that a different characterization for Gabor duality in the case of frames for $L^2(\mathbb{R})$ can be found in [4].

Proposition 3.1. *Let $a, b > 0$ with $ab = p/q$ and* $\gcd(p, q) = 1$*. Let g and h be functions in $L^2(\mathbb{R})$ generating subspace Gabor frames $\mathcal{M}(b, a, g)$ and $\mathcal{M}(b, a, h)$, respectively. Then, $\mathcal{M}(b, a, h)$ is a dual* *frame for the subspace Gabor frame $\mathcal{M}(b,a,g)$ if and only if the equations*

$$G^k = \frac{1}{p}\sum_{i=0}^{q-1} \langle G^k,\, H^i\rangle_{\mathbb{C}^p}\, G^i, \quad k=0,\dots,q-1, \tag{3.2}$$

hold a. e. on $[0,1]\times[0,\frac{1}{p}]$, where $G^i = \mathcal{K}\left(g(\cdot - \frac{ip}{qb})\right)$ (as in Definition 2.1) and $H^i = \mathcal{K}\left(h(\cdot - \frac{ip}{qb})\right)$, for $i=0,\dots,q-1$.

Proof. Under the assumptions given, let us assume that $\mathcal{M}(b,a,h)$ is a dual frame for $\mathcal{M}(b,a,g)$. Let $f\in\mathcal{M}(b,a,g)$. Then,

$$f = \sum_{m,n\in\mathbb{Z}} \langle f,\, h_{mb,\frac{np}{bq}}\rangle\, g_{mb,\frac{np}{bq}} = \sum_{m,\ell\in\mathbb{Z}}\sum_{i=0}^{q-1} \langle f,\, h^i_{mb,\frac{\ell p}{b}}\rangle\, g^i_{mb,\frac{\ell p}{b}}, \tag{3.3}$$

where $g^i = g(\cdot - \frac{ip}{qb})$ and $h^i = h(\cdot - \frac{ip}{qb})$ for $i=0,\dots,q-1$. Hence,

$$\begin{aligned}\mathcal{Z}_b f &= \sum_{m,\ell\in\mathbb{Z}}\sum_{i=0}^{q-1} \langle \mathcal{Z}_b f,\, \mathcal{Z}_b h^i_{mb,\ell p/b}\rangle\, \mathcal{Z}_b g^i_{mb,\ell p/b} \\ &= \sum_{i=0}^{q-1} \mathcal{Z}_b g^i \sum_{m,\ell\in\mathbb{Z}} \langle \mathcal{Z}_b f\overline{\mathcal{Z}_b h^i}, E_{m,-lp}\rangle\, E_{m,-lp} \qquad (3.4)\end{aligned}$$

Since

$$\langle \mathcal{Z}_b f\overline{\mathcal{Z}_b h^i}, E_{m,-lp}\rangle = \int_0^1\int_0^1 \mathcal{Z}_b f(x,w)\,\overline{\mathcal{Z}_b h^i(x,w)}\, e^{-2\pi i m x}\, e^{2\pi i \ell p w}\, dw\, dx$$

$$= \int_0^1\int_0^{1/p}\sum_{k=0}^{p-1} \mathcal{Z}_b f\left(x, w+\frac{k}{p}\right)\overline{\mathcal{Z}_b h^i\left(x,w+\frac{k}{p}\right)}\, e^{-2\pi i m x}\, e^{2\pi i \ell p w}\, dw\, dx$$

$$= \int_0^1\int_0^{1/p} \langle F(x,w),\, H^i(x,w)\rangle_{\mathbb{C}^p}\, e^{-2\pi i m x}\, e^{2\pi i \ell p w}\, dw\, dx,$$

where $F=\mathcal{K}f$, we deduce from (3.4) and the fact that $\{\sqrt{p}\, E_{m,-lp}\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis for $L^2([0,1]\times[0,1/p])$, that

$$\mathcal{Z}_b f = \frac{1}{p}\sum_{i=0}^{q-1} \langle F,\, H^i\rangle_{\mathbb{C}^p}\, \mathcal{Z}_b g^i,$$

or equivalently, that

$$F = \frac{1}{p} \sum_{i=0}^{q-1} \langle F,\, H^i \rangle_{\mathbb{C}^p}\, G^i \quad \text{on } [0,1] \times [0,1/p]. \tag{3.5}$$

The equations (3.2) follow from (3.5) by letting $F = G^k$, for $k = 0, \ldots, q-1$. Conversely, if the equations (3.2) hold, then so does (3.5) for every $F = \mathcal{K} f$ if $f \in \mathcal{M}(b,a,g)$, by Proposition 2.5. Reversing the previous argument, we obtain (3.3), which completes the proof. ◻

Our next result is a charaterization for the existence of an oblique Gabor dual in a particular Gabor subspace in terms of a factorization involving the matrix-valued functions introduced in (3.1).

Theorem 3.2. *Let $g, k \in L^2(\mathbb{R})$ generate subspace Gabor frames $\mathbf{G}(b,a,g)$ and $\mathbf{G}(b,a,k)$, respectively. Then, there exists a function $h \in \mathcal{M}(b,a,k)$ such that the collection $\mathbf{G}(b,a,h)$ is a Gabor dual for the Gabor frame $\mathbf{G}(b,a,g)$ if and only if there exists a matrix-valued function $\gamma : [0,1/q] \times [0,1/p] \to \mathcal{M}_{q,q}$ whose entries all belong to $L^\infty([0,1/q] \times [0,1/p])$ such that the identity*

$$\xi = \xi\, \gamma\, \eta \tag{3.6}$$

holds a. e. on $[0,1/q] \times [0,1/p]$.

Proof. Let us first assume the existence of $h \in \mathcal{M}(b,a,k)$ with the property $\mathbf{G}(b,a,h)$ is a Gabor dual for the Gabor frame $\mathbf{G}(b,a,g)$. Define for each $i = 0, \ldots, q-1$ the vector-valued functions $H^i : [0,1] \times [0,1/p] \to \mathbb{C}^p$ by

$$H^i_k(x,w) = \left(\mathcal{Z}_b h^i\right)(x, w + k/p)\,,$$
$$(x,w) \in [0,1] \times [0,1/p],\ k = 0, \ldots, p-1.$$

By Proposition 3.1, we have

$$G^k = \frac{1}{p} \sum_{i=0}^{q-1} \langle G^k,\, H^i \rangle_{\mathbb{C}^p}\, G^i, \quad k = 0, \ldots, q-1, \tag{3.7}$$

and, by Corollary 2.7, there exist, for each $i = 0, \dots, q-1$, coefficients $\gamma_{ij} \in L^\infty([0,1] \times [0,1/p])$, which define the matrix-valued function γ, such that

$$H^i = p \sum_{j=0}^{q-1} \gamma_{ij}\, K^j, \quad \text{a. e. on } [0,1] \times [0,1/p]. \tag{3.8}$$

In particular, it follows that

$$\langle G^k, H^i \rangle_{\mathbb{C}^p} = \langle G^k, p \sum_{j=0}^{q-1} \gamma_{ij}\, K^j \rangle_{\mathbb{C}^p} = p \sum_{j=0}^{q-1} \overline{\gamma_{ij}}\, \langle G^k,\, K^j \rangle_{\mathbb{C}^p}. \tag{3.9}$$

Using equation (3.7), we have thus

$$G^k = \sum_{i=0}^{q-1} \sum_{j=0}^{q-1} \overline{\gamma_{ij}}\, \langle G^k,\, K^j \rangle_{\mathbb{C}^p}\, G^i, \quad k = 0, \dots, q-1. \tag{3.10}$$

Hence, for each $k, l = 0, \dots, q-1$,

$$\xi_{lk} = \langle G^l,\, G^k \rangle_{\mathbb{C}^p} = \sum_{i=0}^{q-1} \sum_{j=0}^{q-1} \gamma_{ij}\, \langle K^j,\, G^k \rangle_{\mathbb{C}^p}\, \langle G^l,\, G^i \rangle_{\mathbb{C}^p} = (\xi\, \gamma\, \eta)_{lk},$$

yielding (3.6) on the set $[0,1] \times [0,1/p]$, and thus also on the smaller set $[0,1/q] \times [0,1/p]$. Conversely, if (3.6) holds for some matrix-valued function $\gamma : [0,1/q] \times [0,1/p] \to \mathcal{M}_{q,q}$ with entries in $L^\infty([0,1/q] \times [0,1/p])$, we can easily obtain (3.7), where the vector-valued function H^i, $i = 0, \dots, q-1$, are defined as in (3.8) (but on the smaller set $[0,1/q] \times [0,1/p]$). Since the mapping $\mathcal{L}$ constructed in Definition 2.1 is an isomorphism, there exists a unique function $h \in L^2(\mathbb{R})$ such that $H^i = \mathcal{K}\left(h(. - \frac{ip}{qb})\right)$ on the set $[0,1/q] \times [0,1/p]$. The almost-everywhere boundedness of each vector-function H^i, $i = 0, \dots q-1$, on $[0,1/q] \times [0,1/p]$ is easily seen to be equivalent to the boundedness of $\mathcal{K}h$ on the set $[0,1] \times [0,1/p]$. Furthermore, the representation of each H^i given in (3.8) on the set $[0,1/q] \times [0,1/p]$ easily implies a similar representation for the vector-function $\mathcal{K}h$ on the set $[0,1] \times [0,1/p]$. This shows, using Corollary 2.7, that $h \in \mathcal{M}(b,a,k)$ and that $\mathbf{G}(b,a,h)$ is a Bessel collection. Furthermore, using Proposition 3.1, we deduce that

$\mathbf{G}(b,a,h)$ is a Gabor dual for the Gabor frame $\mathbf{G}(b,a,g)$ as claimed. $\square$

Of course, the characterization obtained in the previous theorem can be seen as somewhat vacuous unless a condition for the existence of the matrix-valued function γ is given. Our next goal is to give such a condition which will be expressed in terms of a certain matrix inequality. We first need some preliminary lemmas.

Lemma 3.3. *Let $g, k \in L^2(\mathbb{R})$ generate subspace Gabor frames $\mathbf{G}(b,a,g)$ and $\mathbf{G}(b,a,k)$, respectively, and let $\xi, \eta : [0,1/q] \times [0,1/p] \to \mathcal{M}_{q,q}$ be defined as in* (3.1). *Then,*

(a) *There exists a constant $D > 0$ such that $D\,\xi - \eta^*\,\eta \geq 0$ a. e. on $[0,1/q] \times [0,1/p]$.*

(b) *$\eta = 0$ on $\ker(\xi)$.*

(c) *η^* maps $\mathbb{C}^q$ to $\ker(\xi)^\perp$.*

(d) *ξ maps $\ker(\xi)^\perp$ to itself and the mapping $\xi : \ker(\xi)^\perp \to \ker(\xi)^\perp$ is an isomorphism.*

(e) *The mapping $P : [0,1/q] \times [0,1/p] \to \mathcal{M}_{q,q}$ is measurable, where P denotes the orthogonal projection onto $\ker(\xi)$.*

Proof. If $u \in \mathbb{C}^q$, we have

$$\langle \eta^*\,\eta\,u,\,u\rangle = \|\eta\,u\|^2_{\mathbb{C}^q} = \sum_{i=0}^{q-1}\left|\sum_{j=0}^{q-1}\langle K^i, G^j\rangle_{\mathbb{C}^p}\,u_j\right|^2 = \sum_{i=0}^{q-1}\left|\langle K^i, \sum_{j=0}^{q-1}\overline{u_j}\,G^j\rangle_{\mathbb{C}^p}\right|^2$$
$$\leq \left(\sum_{i=0}^{q-1}\|K^i\|^2_{\mathbb{C}^p}\right)\left\|\sum_{j=0}^{q-1}\overline{u_j}\,G^j\right\|^2_{\mathbb{C}^p}.$$

On the other hand,

$$\langle \xi u,\,u\rangle = \sum_{i,j=0}^{q-1}\langle G^i, G^j\rangle_{\mathbb{C}^p}\,u_j\,\overline{u_i} = \|\sum_{j=0}^{q-1}\overline{u_j}\,G^j\|^2_{\mathbb{C}^p}.$$

Hence, we have $D\,\xi - \eta^*\,\eta \geq 0$ with $D = \sum_{i=0}^{q-1} \|K^i\|^2_{\mathbb{C}^p}$, which proves (a). If $u \in \ker(\xi)$, we have, using (a), that

$$\|\eta\, u\|^2_{\mathbb{C}^q} = \langle \eta^*\,\eta\, u,\, u\rangle_{\mathbb{C}^q} \leq D\,\langle \xi\, u,\, u\rangle_{\mathbb{C}^q} = 0,$$

which proves (b). If $v \in \mathbb{C}^q$ and $u \in \ker(\xi)$, we have, using (b), that

$$\langle \eta^*\, v,\, u\rangle_{\mathbb{C}^q} = \langle\, v,\, \eta\, u\rangle_{\mathbb{C}^q} = 0.$$

Hence, $\eta^*\, v \in \ker(\xi)^\perp$ and (c) holds. Since $\xi^* = \xi$, ξ maps $\ker(\xi)^\perp$ to itself and since the restriction of ξ to $\ker(\xi)^\perp$ is clearly injective, (d) follows. Finally, since $P = \lim_{N\to\infty} \exp(-N\,\xi)$, P is measurable which proves (e). (This is an argument borrowed from [6].) □

Lemma 3.4. *Let (Ω, μ) be a measure space. Let $A : \Omega \to \mathcal{M}_{m,m}$ be a measurable matrix-valued function such that, for a. e. $\omega \in \Omega$, there exists a subspace $N(\omega)$ of $\mathbb{C}^m$ such that*

(a) *$P_N : \Omega \to \mathcal{M}_{m,m}$ is measurable, where $P_N(\omega)$ denotes the orthogonal projection onto $N(\omega)$.*

(b) *$A(\omega) = 0$ on $N(\omega)$.*

(c) *$A(\omega)$ maps $N(\omega)^\perp$ to itself and, if $\tilde{A}(\omega)$ denotes the restriction of $A(\omega)$ to $N(\omega)^\perp$, the mapping $\tilde{A}(\omega) : N(\omega)^\perp \to N(\omega)^\perp$ is an isomorphism.*

Then, the mapping $B : \Omega \to \mathcal{M}_{m,m}$ defined by

$$B(\omega) = \begin{cases} 0 & \text{on } N(\omega), \\ \tilde{A}^{-1}(\omega) & \text{on } N(\omega)^\perp, \end{cases}$$

is measurable.

Proof. Let $A_1 = A + P_N : \Omega \to \mathcal{M}_{m,m}$. Then, $A_1(\omega)$ is measurable and invertible (as a mapping from $\mathbb{C}^m$ to itself) for a. e. $\omega \in \Omega$. Hence, A_1^{-1} is measurable since, clearly, the standard formulas to compute the inverse of a square matrix only use measurable operations. Since $B = A_1^{-1} - P_N$, the result follows. □

The main result of this section, which we prove next, can be seen as the analogue for Gabor systems of the statement (i) and (ii) of Theorem 1.1 concerning shift-invariant systems.

Theorem 3.5. *Under the conditions of the previous theorem, there exists a matrix-valued function* $\gamma : [0,1/q] \times [0,1/p] \to \mathcal{M}_{q,q}$ *whose entries all belong to* $L^\infty([0,1/q] \times [0,1/p])$ *satisfying* (3.6) *if and only if there exists a constant* $C > 0$ *such that*

$$\xi \leq C\,\eta^*\eta \quad \textit{a. e. on } [0,1/q] \times [0,1/p]. \tag{3.11}$$

Proof. Assume first the existence of a matrix-valued function γ as above such that (3.6) holds. Since the entries of γ belong to $L^\infty([0,1/q]\times[0,1/p])$, it follows that the operator norm of γ is uniformly bounded a. e. on $[0,1/q] \times [0,1/p]$, i. e. there exists a constant $C_1 > 0$ such that, for almost every $(x,w) \in [0,1/q] \times [0,1/p]$, we have

$$\|\gamma(x,w)\,u\|_{\mathbb{C}^q} \leq C_1\,\|u\|_{\mathbb{C}^q}, \quad u \in \mathbb{C}^q.$$

Furthermore, it follows from (2.3) that the operator norm of ξ is uniformly bounded a. e. by $B\,p$. Hence, if $u \in \mathbb{C}^q$, we have, using (3.6),

$$\begin{aligned}\langle \xi^2\,u,\,u\rangle_{\mathbb{C}^q} = \langle \xi\,u,\,\xi\,u\rangle_{\mathbb{C}^q} = \langle \xi\,\gamma\,\eta\,u,\,\xi\,\gamma\,\eta\,u\rangle_{\mathbb{C}^q} &= \langle \eta\,u,\,\gamma^*\xi^2\,\gamma\,\eta\,u\rangle_{\mathbb{C}^q}\\ &\leq \|\eta\,u\|_{\mathbb{C}^q}\,\|\gamma^*\xi^2\,\gamma\,\eta\,u\|_{\mathbb{C}^q}\\ &\leq C_1^2\,B^2\,p^2\,\|\eta\,u\|_{\mathbb{C}^q}^2.\end{aligned}$$

Therefore, letting $D = C_1^2\,B^2\,p^2$, we have, using (2.3) again,

$$\langle \eta^*\,\eta\,u,\,u\rangle_{\mathbb{C}^q} = \|\eta\,u\|_{\mathbb{C}^q}^2 \geq D^{-1}\,\langle \xi^2\,u,\,u\rangle_{\mathbb{C}^q} \geq D^{-1}\,A\,p\,\langle \xi\,u,\,u\rangle_{\mathbb{C}^q},$$

which yields (3.11) with $C = D\,(A\,p)^{-1}$. Conversely, let us assume that the inequality (3.11) holds. Then, the mapping η is injective on $\ker(\xi)^\perp$. Indeed, if $u \in \ker(\xi)^\perp$ and $\eta\,u = 0$, (3.11) implies that $\langle \xi\,u, u\rangle = 0$ and thus that $u = 0$, using part (d) of Lemma 3.3. Using (c) in that same lemma, it follows that $\eta^*\eta$ maps $\ker(\xi)^\perp$ to itself, and thus the mapping $\eta^*\eta : \ker(\xi)^\perp \to \ker(\xi)^\perp$, which we will denote by α, is an isomorphism. Define the mapping $\beta : [0,1/q] \times [0,1/p] \to \mathcal{M}_{q,q}$ by

$$\beta = \begin{cases} 0 & \text{on } \ker(\xi),\\ \alpha^{-1} & \text{on } \ker(\xi)^\perp.\end{cases}$$

Using part (e) of Lemma 3.3 together with Lemma 3.4, it follows that β is measurable. Furthermore, if $u \in \ker(\xi)^{\perp}$, we have using (3.11) and Theorem 2.3, that

$$\langle \eta^{*}\eta\, u,\, u\rangle_{\mathbb{C}^q} \geq C^{-1}\, \langle \xi\, u,\, u\rangle_{\mathbb{C}^q} \geq C^{-1}\, A\, p\, \langle u,\, u\rangle_{\mathbb{C}^q}$$

which shows that

$$\langle \alpha^{-1}\, u,\, u\rangle_{\mathbb{C}^q} \leq C\, (A\, p)^{-1}\, \langle u,\, u\rangle_{\mathbb{C}^q}.$$

This implies that the operator norm of α^{-1} and thus that of β, is uniformly bounded a. e. on $[0, 1/q] \times [0, 1/p]$. In particular, all the entries of β belong to $L^{\infty}([0, 1/q] \times [0, 1/p])$. Define now $\gamma = \beta\, \eta^{*}$. Since $\mathbf{G}(b, a, g)$ and $\mathbf{G}(b, a, k)$ are both Bessel collections, it easily follows from Theorem 2.2 that η, and thus also η^{*}, are uniformly bounded a. e. on $[0, 1/q] \times [0, 1/p]$. Hence the entries of γ belong to $L^{\infty}([0, 1/q] \times [0, 1/p])$. Since we have $\eta\, u = 0$ for $u \in \ker(\xi)$ by part (b) of Lemma 3.3, it follows that

$$\xi\, \gamma\, \eta\, u = 0 = \xi\, u.$$

On the other hand, if $u \in \ker(\xi)^{\perp}$, we have

$$\xi\, \gamma\, \eta\, u = \xi\, \alpha^{-1}\, \eta^{*}\, \eta\, u = \xi\, u.$$

Hence, we deduce that $\xi = \xi\, \gamma\, \eta$, which completes the proof. ◻

Note that the condition (3.11) for the existence of the oblique Gabor dual obtained in the previous theorem provides the analogue of the condition (1.5) in Theorem 1.1.

4. Uniqueness of the oblique Gabor dual In this last section, we consider the problem of the uniqueness of the Gabor duals discussed in the previous section. Note that this uniqueness problem was discussed extensively in [12, 13, 14] in the case where the dual window is assumed either to belong to the space generated by the original window ("dual of type I") or having the property that the range of the frame transform associated with the dual window is contained in that of the original window ("dual of type II"). The results obtained in this section will generalize some the results obtained in these papers concerning Gabor duals of type I (in the rational case). We first need the following lemma.

Lemma 4.1 *Let $a, b > 0$ with $ab = p/q$ and $\gcd(p, q) = 1$. Let $g, k \in L^2(\mathbb{R})$ generate subspace Gabor frames $\mathbf{G}(b, a, g)$ and $\mathbf{G}(b, a, k)$, respectively and suppose that there exists a function $h \in \mathcal{M}(b, a, k)$ such that the collection $\mathbf{G}(b, a, h)$ is a Gabor dual for the Gabor frame $\mathbf{G}(b, a, g)$. Then, the function h satisfying these conditions is unique if and only if the validity of the equations*

$$0 = \sum_{i=0}^{q-1} \langle G^k, L^i \rangle_{\mathbb{C}^p} G^i, \quad k = 0, \dots, q-1, \tag{4.1}$$

a. e. on $[0, 1/q] \times [1, 1/p]$, for vector-valued functions $L^i \in span\{K^1, \dots K^{q-1}\}$ with coefficients in $L^\infty([0, 1/q] \times [0, 1/p])$ implies that $L^i = 0$, $i = 0, \dots, q-1$, a. e. on $[0, 1/q] \times [0, 1/p]$.

Proof. The result follows from the fact that if h and $\tilde{h}$ are two different Gabor duals satisfying the above conditions, then the function $l = h - \tilde{h}$ can be used to define a non-trivial solution of the equations 4.1 by letting

$$L^i = \mathcal{K}\left(l(. - \frac{ip}{qb}) \right), \quad i = 0, \dots, q-1. \tag{4.2}$$

Conversely, if a non-trivial solution L^i, $i = 0, \dots, q-1$, exists, we can use (4.2) to define a function $l \neq 0$ in $\mathcal{M}(b, a, k)$ such that $\tilde{h} = h + l$ is a Gabor dual different than h for $\mathbf{G}(b, a, g)$. We leave the details to the reader. □

Theorem 4.2 *Let $a, b > 0$ with $ab = p/q$ and $\gcd(p, q) = 1$. Let $g, k \in L^2(\mathbb{R})$ generate subspace Gabor frames $\mathbf{G}(b, a, g)$ and $\mathbf{G}(b, a, k)$, respectively and suppose that there exists a function $h \in \mathcal{M}(b, a, k)$ such that the collection $\mathbf{G}(b, a, h)$ is a Gabor dual for the Gabor frame $\mathbf{G}(b, a, g)$. Define the sets*

$$\Omega_g := \{(x, w) \in [0, 1/q] \times [0, 1/p],\ G^i(x, w) = 0,\ \textit{for } i = 0, \dots, q-1\}$$

and

$$\Omega_k := \{(x,w) \in [0,1/q]\times[0,1/p],\ K^i(x,w) = 0,\ \textit{for } i = 0,\dots,q-1\}.$$

Then, the function h satisfying the conditions above is unique if and only if

$$\Omega_g = \Omega_k \quad \textit{(up to a set of zero measure).} \tag{4.3}$$

and

$$\textit{rank}\{G^0,\dots,G^{q-1}\} = q \textit{ a. e. on the set } [0,1/q]\times[0,1/p]\setminus\Omega_g. \tag{4.4}$$

Proof. We first remark that the conditions for the existence of a dual Gabor frame h given in Theorem 3.5 clearly imply that $\Omega_k \subset \Omega_g$ up to a set of zero measure. We will first prove the necessity of the conditions (4.3) and (4.4). Suppose first that (4.3) fails. Then the set $\Omega_g \setminus \Omega_k$ has positive measure and there exist $i_0 \in \{0,\dots,q-1\}$ such that the set $E := \Omega_g \setminus \{K^{i_0} \neq 0\}$ has positive measure. Defining $L^i = 0$, for $i \neq i_0$ and $L^{i_0} = K^{i_0}\chi_E$, where χ_E denotes the characteristic function of E, we see that the equations (4.1) in Lemma 4.1 have a solution $L^0,\dots,L^{q-1}$, where the vector-valued functions L^i are not identically zero and can be expressed as linear combinations of $K^0,\dots,K^{q-1}$ with bounded coefficients. Hence, the condition (4.3) is necessary. Suppose now that (4.3) holds but (4.4) fails. Let $e_0,\dots,e_{q-1}$ denote the standard orthonormal basis of $\mathbb{C}^q$. Since the condition $\text{rank}\{G^0,\dots,G^{q-1}\} = q$ is equivalent to $\ker(\xi) = 0$, there exist $i_1 \in \{0,\dots,q-1\}$ such that $P(e_{i_1}) \neq 0$ on a measurable subset F_1 of $[0,1/q]\times[0,1/p]\setminus\Omega_g$, where P denotes the orthogonal projection onto $\ker(\xi)$ (which is measurable by (e) of Lemma 3.3). Letting $u = P(e_{i_1})$, we have $\|u\|_{\mathbb{C}^q} \leq 1$ and

$$\sum_{j=0}^{q-1} \overline{u_j}\, G^j = 0 \quad \text{on } [0,1/q]\times[0,1/p], \tag{4.5}$$

using equation (2.10). Taking into account the identity (4.3), we deduce the existence of a measurable subset F of F_1 and of an index

$j_1 \in \{0, \dots, q-1\}$ such that $K^{j_1} \neq 0$ on F. Letting $H^j = u_j\, K^{j_1}\, \chi_F$, for $j = 0, \dots, q-1$, we have $H^j \neq 0$ for at least one such j. Furthermore, using (4.5), we have

$$\sum_{j=0}^{q-1} \langle G^i,\, H^j \rangle_{\mathbb{C}^p}\, G^j = \langle G^i,\, K^{j_1} \rangle_{\mathbb{C}^p} \left(\sum_{j=0}^{q-1} \overline{u_j}\, G^j \right) = 0, \quad i = 0, \dots, q-1,$$

which implies the non-uniqueness of h, using Lemma 4.1 again. Hence, both conditions (4.3) and (4.4) are necessary for the uniqueness of the Gabor dual in $\mathcal{M}(b, a, k)$. To prove the converse, let us assume that both (4.3) and (4.4) hold and let $L^0, \dots, L^{q-1}$ be vector-valued functions in $\operatorname{span}\{K^1, \dots K^{q-1}\}$ with coefficients functions in $L^\infty([0, 1/q] \times [0, 1/p])$ such that the equations (4.1) hold. Using (4.3), we deduce that $L^0 = \cdots = L^{q-1} = 0$ on Ω_g. On the other hand, using (4.4), we deduce that

$$\langle L^i,\, G^j \rangle_{\mathbb{C}^p} = 0, \quad i, j = 0, \dots, q-1, \text{on } [0, 1/q] \times [0, 1/p] \setminus \Omega_g$$

since $G^0, \dots, G^{q-1}$ are linearly independent on that set. Writing

$$L^i = \sum_{j=0}^{q-1} \rho_{ij}\, K^j, \quad i = 0, \dots, q-1,$$

where $\rho_{ij} \in L^\infty([0, 1/q] \times [0, 1/p])$ and considering the corresponding matrix-valued function ρ with values in $\mathcal{M}_{q,q}$, we have

$$\sum_{j=0}^{q-1} \rho_{ij} \langle K^j,\, G^l \rangle_{\mathbb{C}^p} = 0, \quad i, l = 0, \dots, q-1,$$

or, equivalently, $\rho\, \eta = 0$. Using the matrix inequality (3.11), we obtain that η is invertible on $[0, 1/q] \times [0, 1/p] \setminus \Omega_g$ since ξ is, and thus that $\rho = 0$ on that set. This shows that $L^0 = \cdots = L^{q-1} = 0$ on $[0, 1/q] \times [0, 1/p] \setminus \Omega_g$ also, which proves our claim. ◻

Remark 4.3. The condition (4.4) is equivalent to $A\, p\, I \leq \xi$ a. e. on the set $\{\xi \neq 0\}$.

Note that, because of the condition (4.4) involving the rank of a $q \times p$ matrix, uniqueness of the Gabor dual can never occur in the theorem above when $p < q$ or $a\,b < 1$ (unless $g = k = 0$). On the other hand, if $a\,b$ is an integer we have the analogue of the first part of (iii) of Theorem 1.1 as the following corollary shows.

Corollary 4.4. *Suppose that the assumptions of the previous theorem are satisfied and assume, furthermore, that $a\,b = p \in \mathbb{N}$. If a Gabor dual h for $\mathbf{G}(b, a, g)$ exists in $\mathbf{G}(b, a, k)$, then it is unique if and only if* (4.3) *holds.*

Proof. Note that, when $q = 1$, the condition (4.4) is always satisfied. $\square$

Of course, when $k = g$, the condition (4.3) automatically holds and thus uniqueness always occurs in the previous corollary. This result was obtained in [14] (see also [12, 13]). We point out that, for subspace Gabor frames, uniqueness of the dual in $\mathcal{M}(b, a, k)$ (when $p \geq q$) can occur even in the case that $\mathcal{M}(b, a, k) \neq \mathcal{M}(b, a, g)$ since the condition (3.11) can be satisfied together with (4.3) and (4.4) without the span of the vectors K^i being contained in that of the G^i. The only exception is when $a\,b = 1$ since in that case the condition (4.3) is easily seen to be equivalent to $\mathcal{M}(b, a, k) = \mathcal{M}(b, a, g)$. This is thus the only case where the analogue for Gabor systems of the second part in statement (iii) of Theorem 1.1 is true.

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