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A CLASS OF MAXIMAL OPERATORS RELATED TO ROUGH SINGULAR INTEGRALS ON PRODUCT SPACES

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ABSTRACT. This paper is concerned with studying the L^p boundedness of a class of maximal operators $\mathcal{S}_\Omega^{(\gamma)}$ related to rough singular integrals on product spaces. We obtain appropriate L^p bounds for such maximal operators and establish the optimality of our condition on the kernel for the L^2 boundedness of $\mathcal{S}_\Omega^{(2)}$. Our results improve substantially the main result obtained by Ding in [**8**].

1. Introduction and statement of results. Throughout this paper, we let ξ' denote $\xi/|\xi|$ for $\xi \in \mathbf{R}^n \setminus \{0\}$ and p' denote the exponent conjugate to p, that is, $1/p+1/p'=1$. Let $n, m \geq 2$. Suppose that $\mathbf{S}^{d-1}$ ($d = n$ or m) is the unit sphere of $\mathbf{R}^d$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$.

In [**7**], Chen and Lin studied the L^p boundedness of a class of maximal operators $\mathcal{M}_\Omega^{(\gamma)}$ defined by

$$\mathcal{M}_\Omega^{(\gamma)} f(x) = \sup_h \left| \int_{\mathbf{R}^n} f(x-y) h(|y|)\, \Omega(y/|y|)\, |y|^{-n}\, dy \right|,$$

where the supremum is taken over the set $\{h : \|h\|_{L^\gamma(\mathbf{R}^+, dr/r)} \leq 1\}$, $\gamma > 1$ and $\Omega \in L^1(\mathbf{S}^{n-1})$ is a function satisfying the cancelation condition

$$\int_{\mathbf{S}^{n-1}} \Omega(y')\, d\sigma(y') = 0. \tag{1.1}$$

Chen and Lin in [**7**] proved the L^p boundedness of the maximal operator $\mathcal{M}_\Omega^{(\gamma)}$ under a smoothness condition on Ω as described in the following theorem:

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Theorem A [**7**]. *Assume $n \geq 2$ and $\Omega \in C(\mathbf{S}^{n-1})$ satisfying* (1.1). *Then*

$$\|\mathcal{M}_\Omega^{(\gamma)}(f)\|_{L^p(\mathbf{R}^n)} \leq C_p \|f\|_{L^p(\mathbf{R}^n)}$$

for $n\gamma/(n\gamma-1) < p < \infty$, $1 \leq \gamma \leq 2$, and $f \in L^p$. Moreover, the range of p is the best possible.

On the other hand, the corresponding maximal operator of $\mathcal{M}_\Omega^{(\gamma)}$ on the product space $\mathbf{R}^n \times \mathbf{R}^m$ is defined by

$$(1.2) \quad \mathcal{S}_\Omega^{(\gamma)} f(x,y) = \sup_{h \in \mathcal{B}^{(\gamma)}} \left| \int_{\mathbf{R}^n \times \mathbf{R}^m} f(x-u, y-v) h(|u|, |v|)\, \Omega(u', v')\, |u|^{-n} |v|^{-n}\, du\, dv \right|,$$

where $\mathcal{B}^{(\gamma)}$ is the set of all radial functions $h(s,t)$ with

$$\|h\|_{L^\gamma(\mathbf{R}^+ \times \mathbf{R}^+, ds\, dt/(st))} \leq 1$$

and Ω is a function on $\mathbf{R}^n \times \mathbf{R}^m$ satisfying the following conditions:

$$(1.3) \qquad \begin{cases} \displaystyle\int_{\mathbf{S}^{n-1}} \Omega(u', \cdot)\, d\sigma(u') = 0, \\ \displaystyle\int_{\mathbf{S}^{m-1}} \Omega(\cdot, v')\, d\sigma(v') = 0, \end{cases}$$

$$(1.4) \qquad \Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}),$$

and

$$\Omega(tx, sy) = \Omega(x,y) \quad \text{for any} \quad t, s > 0.$$

Recently, Ding in [**8**] obtained the following L^2 boundedness of $\mathcal{S}_\Omega^{(\gamma)}$ when $\gamma = 2$:

Theorem A. *Assume that $n, m \geq 2$ and Ω satisfies* (1.3)–(1.4). *Then $\mathcal{S}_\Omega^{(2)}$ is bounded on $L^2(\mathbf{R}^n \times \mathbf{R}^m)$ if $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$.*

Here, a function Ω belongs to the class $L(\log L)^{\alpha}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ if

$$\|\Omega\|_{L(\log L)^{\alpha}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega(x,y)| \log^{\alpha}(2+|\Omega(x,y)|)\, d\sigma(x)\, d\sigma(y) < \infty.$$

A question which arises naturally in light of Theorem A is the following:

Question. *Does the L^p boundedness of $\mathcal{S}_{\Omega}^{(\gamma)}$ hold for some $p \neq 2$ under a condition in the form $\Omega \in L(\log L)^{\alpha}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$, and what is the best possible value of the exponent α so that the L^2 boundedness of $\mathcal{S}_{\Omega}^{(\gamma)}$ holds.*

The main purpose of this paper is to obtain an answer to this question. In fact, we prove the following:

Theorem 1.1. *Assume that $n, m \geq 2$ and Ω satisfies* (1.3)–(1.4). *Then*

(a) *If $\Omega \in L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$, $S_{\Omega}^{(\gamma)}$ is bounded on $L^p(\mathbf{R}^n\times\mathbf{R}^m)$ for $\gamma' \leq p < \infty$ if $1 < \gamma \leq 2$; and it is bounded on $L^{\infty}(\mathbf{R}^n\times\mathbf{R}^m)$ if $\gamma = 1$;*

(b) *There exists an Ω which lies in $L(\log L)^{1-\varepsilon}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ for all $\varepsilon > 0$ and satisfies* (1.3) *such that $S_{\Omega}^{(2)}$ is not bounded on $L^2(\mathbf{R}^n\times\mathbf{R}^m)$.*

We remark that, for any $q > 1$, the following inclusions hold and are proper:

$$C^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}) \subset L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}) \subset L(\log L)(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}),$$

and

$$L(\log L)^{\beta}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}) \subset L(\log L)^{\alpha}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}) \quad \text{for} \quad \alpha < \beta.$$

Clearly, part (a) of Theorem 1.1 represents a substantial improvement in both the range of p and Ω of the main result of Ding [**8**], while part (b) shows that the condition $\Omega \in L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ is the best possible in the case $\gamma = 2$.

The method employed in this paper allows us to treat a more general class of maximal operators than those given by (1.2). To give a full statement of our results, we let Φ and Ψ be suitable functions defined on $\mathbf{R}^+$. For an Ω satisfying (1.3)–(1.4), we define the operator $\mathcal{S}^{(\gamma)}_{\Omega,\Phi,\Psi}$ on $\mathbf{R}^n \times \mathbf{R}^m$ by

$$(1.5)\qquad \begin{aligned}(\mathcal{S}^{(\gamma)}_{\Omega,\Phi,\Psi}f)(x,y) = \sup_{b\in\mathcal{B}}\bigg|\int_{\mathbf{R}^n\times\mathbf{R}^m} & f(x-\Phi(|u|)u', y-\Phi(|v|)v') \\ & \times b\left(|u|,|v|\right)\Omega\left(u',v'\right)|u|^{-n}\,|v|^{-n}\,du\,dv\bigg|.\end{aligned}$$

Since $\mathcal{S}^{(\gamma)}_{\Omega,\Phi,\Psi} = \mathcal{S}^{(\gamma)}_{\Omega}$ when $\Phi(t)\equiv\Psi(t)\equiv t$, part (a) of Theorem 1.1 is a special case of the following theorem whose proof will be given in Section 4.

Theorem 1.2. *Assume that $n, m \geq 2$ and Ω satisfies* (1.3)–(1.4). *Let $S^{(\gamma)}_{\Omega,\Phi,\Psi}$ be given as in* (1.5) *with $1 \leq \gamma \leq 2$. Assume that Φ and Ψ are in $C^2([0,\infty))$, convex and increasing functions with $\Phi(0) = \Psi(0) = 0$.*

(a) *If $\Omega \in L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$, $S^{(\gamma)}_{\Omega,\Phi,\Psi}$ is bounded on $L^p(\mathbf{R}^n\times\mathbf{R}^m)$ for $\gamma' \leq p < \infty$ if $1 < \gamma \leq 2$; and it is bounded on $L^\infty(\mathbf{R}^n\times\mathbf{R}^m)$ if $\gamma = 1$;*

(b) *If $\Omega \in L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$, for some $q > 1$, $S^{(\gamma)}_{\Omega,\Phi,\Psi}$ is bounded on $L^p(\mathbf{R}^n\times\mathbf{R}^m)$ for $\max\{\gamma' n\delta/(\gamma' n + n\delta - \gamma'), \gamma' m\delta/(\gamma' m + m\delta - \gamma')\} < p < \infty$, where $\delta = \max\{2, q'\}$.*

Throughout the rest of the paper the letter C will stand for a constant but not necessarily the same one in each occurrence.

2. Proof of Theorem 1.1 (b). We follow a similar argument as in [**1**]. By duality, the operator $\mathcal{S}_\Omega^{(2)}$ is simply

$$\mathcal{S}_\Omega^{(2)} f(x,y) = \left(\int_{(0,\infty)\times(0,\infty)} \left| \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} f(x-r\xi, y-t\eta) \times \Omega(\xi,\eta)\, d\sigma(\xi)\, d\sigma(\eta) \right|^2 \frac{drdt}{rt} \right)^{1/2}.$$

It is obvious that $\mathcal{S}_\Omega^{(2)}$ is bounded on $L^2(\mathbf{R}^n \times \mathbf{R}^m)$ if and only if the multiplier

$$m(\xi,\eta) = \left(\int_{(0,\infty)\times(0,\infty)} \left| \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} e^{-2\pi i (t\xi'\cdot u + s\eta'\cdot v)} \times \Omega\,(u,v)\; d\sigma(u)\, d\sigma(v) \right|^2 \frac{dtds}{ts} \right)^{1/2}$$

is an L^∞ function, where $\xi' = \xi/\left|\xi\right|$ and $\eta' = \eta/\left|\eta\right|$. It is easy to see that

$$\begin{aligned}(m(\xi,\eta))^2 = \lim_{M\to\infty,\varepsilon_2\to 0}\ \lim_{N\to\infty,\varepsilon_1\to 0} \int_{(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})^2} \Omega\,(u,v)\,\overline{\Omega\,(x,y)} \\ \times \int_{\varepsilon_2}^{M} \left(e^{-2\pi i s\eta'\cdot(v-y)}\, \frac{ds}{s}\right) \\ \times \int_{\varepsilon_1}^{N} \left(e^{-2\pi i t\xi'\cdot(u-x)}\, \frac{dt}{t}\right) d\sigma(u)\, d\sigma(v)\, d\sigma(x)\, d\sigma(y).\end{aligned}$$

Notice that

$$\int_{\varepsilon_1}^{N} \left(e^{-2\pi i t\xi'\cdot(u-x)} - \cos(2\pi t)\right) \frac{dt}{t} \longrightarrow \log\left|\xi'\cdot(u-x)\right|^{-1} - i\,\frac{\pi}{2}\,\mathrm{sgn}\left(\xi'\cdot(u-x)\right)$$

as $N\to\infty$ and $\varepsilon_1 \to 0$, and the integral is bounded uniformly in ε_1 and $N, C\left(1+\left|\log\left|\xi'\cdot(u-x)\right|\right|\right)$. Now, if we choose Ω to be a real-valued function, by the cancelation conditions on Ω and invoking Lebesgue dominated convergence theorem, we obtain

(2.1)
$$(m(\xi,\eta))^2 = \int_{(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})^2} \Big(\Omega\,(u,v)\,\Omega\,(x,y)\log\left|\xi'\cdot(u-x)\right|^{-1} \times \log\left|\eta'\cdot(v-y)\right|^{-1} - \Big(\frac{\pi^2}{4}\,\mathrm{sgn}\,(\xi'\cdot(u-x)) \times \mathrm{sgn}\,(\eta'\cdot(v-y))\Big)\Big)\,d\sigma(u)\,d\sigma(v)\,d\sigma(x)\,d\sigma(y).$$

For simplicity, we shall construct the function Ω only in the case $n=m=2$, and we shall work on $[-1,1]^2$ instead of $\mathbf{S}^1\times\mathbf{S}^1$. By (2.1), we notice that Theorem 1.1 (b) is proved if we can construct an Ω on $[-1,1]^2$ with the following properties:

$$\int_{-1}^{1}\Omega\,(u,\cdot)\;du = \int_{-1}^{1}\Omega\,(\cdot,v)\;dv = 0; \tag{2.2}$$

$$\int_{[-1,1]^2}\left|\Omega\,(u,v)\right|\left(\log(2+\left|\Omega\,(u,v)\right|)\right)du\,dv = \infty; \tag{2.3}$$

$$\int_{[-1,1]^2}\left|\Omega\,(u,v)\right|\left(\log(2+\left|\Omega\,(u,v)\right|)\right)^{1-\varepsilon}du\,dv < \infty \quad \text{for each} \quad \varepsilon>0; \tag{2.4}$$

$$\mathcal{I}(1,1) = \int_{[0,1]^2}\int_{[0,1]^2}\Omega\,(u,v)\;\Omega(x,y) \times F(u,v,x,y)\,du\,dv\,dx\,dy = \infty; \tag{2.5}$$

$$\mathcal{I}(1,2) = \int_{[-1,1]^2\setminus[0,1]^2}\int_{[0,1]^2}\left|\Omega\,(u,v)\,\Omega(x,y) \times F(u,v,x,y)\right|du\,dv\,dx\,dy < \infty; \tag{2.6}$$

$$\mathcal{I}(2,1) = \int_{[0,1]^2}\int_{[-1,1]^2\setminus[0,1]^2}\left|\Omega\,(u,v)\;\Omega(x,y) \times F(u,v,x,y)\right|du\,dv\,dx\,dy < \infty; \tag{2.7}$$

$$\mathcal{I}(2,2) = \int_{[-1,1]^2\setminus[0,1]^2} \int_{[-1,1]^2\setminus[0,1]^2} |\,\Omega\,(u,v)\;\Omega(x,y) \times F(u,v,x,y)|\,du\,dv\,dx\,dy < \infty, \tag{2.8}$$

where

$$F(u,v,x,y) = \big(\log|x-u|^{-1}\big)\big(\log|y-v|^{-1}\big).$$

For $k \in \mathbf{N}$, let $I_k = [(1/k+1),(1/k))$ and

$$b_k = \sum_{j=3}^{\infty} \frac{k}{(j+1)\,[\log(k+j)]^3}.$$

Now, by definition of b_k, we have

$$\begin{aligned} b_k &= \sum_{j=3}^{k} \frac{k}{(j+1)\,[\log(k+j)]^3} + \sum_{j=k+1}^{\infty} \frac{k}{(j+1)\,[\log(k+j)]^3} \\ &\le \frac{k}{(\log k)^3}\left(\sum_{j=3}^{k}\frac{1}{(j+1)}\right) + k\left(\sum_{j=k+1}^{\infty}\frac{1}{(j+1)\,(\log j)^3}\right) \\ &\le C\,\frac{k}{(\log k)^2}. \end{aligned}$$

Define Ω on $[-1,1]^2$ by

$$\begin{aligned} \Omega\,(u,v) = \sum_{j=3}^{\infty}\sum_{k=3}^{\infty} \frac{jk}{[\log(k+j)]^3}\,\chi_{I_k\times I_j}(u,v) - \chi_{[-1,0]}(v)\left(\sum_{k=3}^{\infty} b_k\chi_{I_k}(u)\right) \\ - \chi_{[-1,0]}(u)\left(\sum_{k=3}^{\infty} b_k\chi_{I_k}(v)\right) + \chi_{[-1,0]^2}(u,v)\left(\sum_{k=3}^{\infty}\frac{b_k}{k(k+1)}\right), \end{aligned}$$

where χ_A represents the characteristic function of a set A.

Let us now turn to the proof of (2.2)–(2.8). First, the proof of (2.2) is straightforward. To prove (2.3), it suffices to show that

$$\int_{[0,1]^2} |\,\Omega\,(u,v)|\,(\log(2+|\,\Omega\,(u,v)|))\,du\,dv = \infty. \tag{2.9}$$

To see this, notice that

$$\begin{aligned}
&\int_{[0,1]^2} |\,\Omega\,(u,v)|\,(\log(2+|\,\Omega\,(u,v)|))\,du\,dv \\
&\qquad = \sum_{j=3}^{\infty}\sum_{k=3}^{\infty}\frac{jk}{[\log(k+j)]^3}\int_{I_k\times I_j}(\log(2+|\,\Omega\,(u,v)|))\;du\,dv \\
&\qquad \geq C\sum_{j=3}^{\infty}\sum_{k=3}^{\infty}\frac{(\log k+\log j)}{jk\,[\log(k+j)]^3} \\
&\qquad \geq C\sum_{k=3}^{\infty}\sum_{j=k}^{\infty}\frac{(\log k+\log j)}{jk\,[\log(k+j)]^3} \\
&\qquad \geq C\sum_{k=3}^{\infty}\frac{1}{k\log k}=\infty.
\end{aligned}$$

We now prove (2.4). We divide the integral over $[-1,1]^2$ into four parts: over $[0,1]^2$, $[-1,0]\times[0,1]$, $[0,1]\times[-1,0]$ and $[-1,0]\times[-1,0]$. By similar calculations as those in the proof of (2.9), we obtain the finiteness of the integral over $[0,1]^2$. On the other hand, by definition of Ω, we can see that the integral over $[-1,0]\times[0,1]$ equals to

$$\sum_{k=3}^{\infty}\frac{b_k(\log(2+b_k))^{1-\varepsilon}}{k(k+1)}<\infty.$$

Similarly, we can show that the integral over $[0,1]\times[-1,0]$ is finite. Finally, since

$$\left(\sum_{k=3}^{\infty}\frac{b_k}{k(k+1)}\right)\chi_{[-1,0]}\in L^{\infty},$$

we have that the integral over $[-1,0]\times[-1,0]$ is finite.

Now, we verify (2.5). Let us first prove $\mathcal{I}(1,1)=\infty$. By definition of $\mathcal{I}(1,1)$, we have

$$\begin{aligned}
&\mathcal{I}(1,1) \\
&\quad =\sum_{j=3}^{\infty}\sum_{k=3}^{\infty}\sum_{s=3}^{\infty}\sum_{l=3}^{\infty}a_{k,j}a_{l,s}\int_{I_k\times I_j}\int_{I_l\times I_s}F(u,v,x,y)\,dx\,dy\,du\,dv,
\end{aligned}$$

where
$$a_{k,j} = \frac{jk}{[\log(k+j)]^3}.$$

Notice that, for each $(u,v) \in I_k \times I_j$ and $(x,y) \in I_l \times I_s$, $F(u,v,x,y) \geq 0$. Thus,

$$\mathcal{I}(1,1) \geq \sum_{j=3}^{\infty} \sum_{k=3}^{\infty} \sum_{s \geq 2(j+1)}^{\infty} \sum_{l \geq 2(k+1)}^{\infty} a_{k,j} a_{l,s} \times \int_{I_k \times I_j} \int_{I_l \times I_s} F(u,v,x,y)\, dx\, dy\, du\, dv.$$

Now, for $(u,x) \in I_k \times I_l$ with $l \geq 2(k+1)$, we have $u \geq 2x$ and hence $\log |x-u|^{-1} \geq \log k$. Similarly, $\log |y-v|^{-1} \geq \log j$ for $(v,y) \in I_j \times I_s$ with $s \geq 2(j+1)$. Therefore,

$$\begin{aligned}
\mathcal{I}(1,1) &\geq C \sum_{j=3}^{\infty} \sum_{s \geq 2(j+1)}^{\infty} \sum_{k=3}^{\infty} \sum_{l \geq 2(k+1)}^{\infty} \frac{\log k \log j}{lkjs\, [\log(k+j)]^3\, [\log(l+s)]^3} \\
&\geq C \sum_{k=3}^{\infty} \sum_{j=3}^{\infty} \sum_{s \geq 2(j+1)}^{\infty} \frac{\log k \log j}{kjs\, [\log(k+j)]^3\, [\log(k+s)]^2} \\
&\geq C \sum_{k=3}^{\infty} \sum_{j=3}^{\infty} \frac{\log k \log j}{kj\, [\log(k+j)]^4} \\
&\geq C \sum_{k=3}^{\infty} \sum_{j \geq k}^{\infty} \frac{\log k \log j}{kj\, [\log(k+j)]^4} \\
&\geq C \sum_{k=3}^{\infty} \frac{\log k}{k} \left(\sum_{j \geq k}^{\infty} \frac{1}{j\, (\log j)^3} \right) \\
&\geq C \sum_{k=3}^{\infty} \frac{1}{k \log k} = \infty.
\end{aligned}$$

Next, we turn to the proof of (2.6). Divide $[-1,1]^2 \backslash [0,1]^2$ into three parts: $[-1,0] \times [0,1]$, $[0,1] \times [-1,0]$ and $[-1,0] \times [-1,0]$. We notice that the integral over $[-1,0] \times [0,1] \times [0,1]^2$ is dominated from above by

$$S=\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{s=3}^{\infty}a_{k,j}b_s\left|\mathcal{I}(k)\right|\mathcal{J}(j,s), \tag{2.10}$$

where

$$\mathcal{J}(j,s)=\int_{I_j\times I_s}\log|y-v|^{-1}\,dv\,dy,$$

and

$$\mathcal{I}(k)=\int_{I_k}\int_{-1}^{0}\log|x-u|^{-1}\,dx\,du.$$

By elementary calculations, it is easy to verify that the following inequalities hold for some positive constant C independent of k and j:

$$|\mathcal{I}(j)|\le C\,\frac{1}{j^2}; \tag{2.11}$$

$$\mathcal{J}(j,s)\le C\,\frac{\log j}{j^2s^2}\quad\text{if}\quad s>2j; \tag{2.12}$$

$$\mathcal{J}(j,s)\le C\,\frac{\log s}{j^2s^2}\quad\text{if}\quad j>2s; \tag{2.13}$$

$$\mathcal{J}(j,s)\le C\,\frac{\log s}{s^4}\quad\text{if}\quad j/2\le s\le 2j. \tag{2.14}$$

In view of (2.10)–(2.11), we have

$$S\le S_1+S_2+S_3, \tag{2.15}$$

where

$$S_1=\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{s>2j}\frac{js}{k\left[\log(k+j)\right]^3(\log s)^2}\,\mathcal{J}(j,s),$$

$$S_2=\sum_{k=3}^{\infty}\sum_{s=3}^{\infty}\sum_{j>2s}^{\infty}\frac{js}{k\left[\log(k+j)\right]^3(\log s)^2}\,\mathcal{J}(j,s)$$

$$S_3=\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{j/2\le s\le 2j}\frac{js}{k\left[\log(k+j)\right]^3(\log s)^2}\,\mathcal{J}(j,s).$$

By (2.12), we have

$$\begin{aligned}
S_1 &\le C \sum_{k=3}^{\infty} \sum_{j=3}^{\infty} \frac{\log j}{kj\,[\log(k+j)]^3} \sum_{s>2j} \frac{1}{s(\log s)^2} \\
&\le C \sum_{k=3}^{\infty} \sum_{j=3}^{\infty} \frac{1}{kj\,[\log(k+j)]^3} \\
&\le C\Big(\sum_{k=3}^{\infty} \frac{1}{k\,[\log(k+3)]^{3/2}}\Big)\Big(\sum_{j=3}^{\infty} \frac{1}{j\,[\log(3+j)]^{3/2}}\Big) < \infty.
\end{aligned}$$

The proof of $S_2 < \infty$ follows by (2.13) and the same argument as proving $S_1 < \infty$. To prove the finiteness of S_3, we invoke (2.14) to get

$$\begin{aligned}
S_3 &\le C \sum_{k=3}^{\infty} \sum_{j=3}^{\infty} \frac{j}{k\,[\log(k+j)]^3} \Big(\sum_{j/2\le s\le 2j} \frac{1}{s^3 \log s}\Big) \\
&\le C \sum_{k=3}^{\infty} \sum_{j=3}^{\infty} \frac{1}{kj\,[\log(k+j)]^3 \log j} \\
&\le C\Big(\sum_{k=3}^{\infty} \frac{1}{k(\log k)^2}\Big)\Big(\sum_{j=3}^{\infty} \frac{1}{j(\log j)^2}\Big) < \infty.
\end{aligned}$$

Thus, the integral over $[-1,0]\times[0,1]\times[0,1]^2$ is finite. Similarly, the integral over $[0,1]\times[-1,0]\times[0,1]^2$ is finite. Also, the integral over $[-1,0]\times[-1,0]\times[0,1]^2$ is bounded from above by

$$\begin{aligned}
C &\sum_{k=3}^{\infty} \sum_{j=3}^{\infty} a_{k,j}\,|\mathcal{I}(k)\mathcal{I}(j)| \\
&\le C \sum_{k=3}^{\infty} \sum_{j=3}^{\infty} \frac{1}{kj\,[\log(k+j)]^3} \\
&\le C\Big(\sum_{k=3}^{\infty} \frac{1}{k\,[\log(k+3)]^{3/2}}\Big)\Big(\sum_{j=3}^{\infty} \frac{1}{j\,[\log(3+j)]^{3/2}}\Big) < \infty,
\end{aligned}$$

which ends the proof of (2.6). By following a similar argument as proving (2.6), we obtain $\mathcal{I}(2,1) < \infty$. Now, it remains to verify (2.8).

Divide $[-1,1]^2\backslash[0,1]^2$ into three parts: $[-1,0]\times[0,1]$, $[0,1]\times[-1,0]$ and $[-1,0]\times[-1,0]$. As above, we shall only present the proof of the finiteness of the integral over $[-1,0]\times[0,1]\times[-1,0]\times[0,1]$ and over $[-1,0]\times[0,1]\times[0,1]\times[-1,0]$ because the proof of the other cases are similar. We start now by proving the finiteness of the integral over $[-1,0]\times[0,1]\times[-1,0]\times[0,1]$. Notice that the last integral is bounded from above by

$$C\sum_{k=3}^{\infty}\sum_{l=3}^{\infty}\frac{kl}{(\log k)^2(\log l)^2}\,\mathcal{J}(k,l)\left(\int_{-1}^{0}\int_{-1}^{0}\log|y-v|^{-1}\,dv\,dy\right)$$
$$\leq C\sum_{k=3}^{\infty}\sum_{l=3}^{\infty}\frac{kl}{(\log k)^2(\log l)^2}\,\mathcal{J}(k,l)=S^*.$$

As above, split S^* as

$$S^*=S_1^*+S_2^*+S_3^*,$$

where

$$S_1^*=\sum_{k=3}^{\infty}\sum_{l>2k}^{\infty}\frac{kl}{(\log k)^2(\log l)^2}\,\mathcal{J}(k,l);$$
$$S_2^*=\sum_{l=3}^{\infty}\sum_{k>2l}^{\infty}\frac{kl}{(\log k)^2(\log l)^2}\,\mathcal{J}(k,l);$$
$$S_3^*=\sum_{k=3}^{\infty}\sum_{k/2\leq l\leq 2k}\frac{kl}{(\log k)^2(\log l)^2}\,\mathcal{J}(k,l).$$

By (2.12), we have

$$S_1^*\leq C\sum_{k=3}^{\infty}\frac{1}{k(\log k)}\left(\sum_{l>2k}^{\infty}\frac{1}{l(\log l)^2}\right)$$
$$\leq C\sum_{k=3}^{\infty}\frac{1}{k(\log k)^2}<\infty.$$

Similarly, by (2.13) $S_2^*<\infty$. By (2.14),

$$S_3^*\leq C\sum_{k=3}^{\infty}\frac{k}{(\log k)^2}\sum_{k/2\leq l\leq 2k}\frac{1}{l^3(\log l)}$$
$$\leq C\sum_{k=3}^{\infty}\frac{k}{k(\log k)^3}<\infty.$$

This finishes the proof of the finiteness of the integral over $[-1,0]\times[0,1]\times[-1,0]\times[0,1]$. Now, we turn to the proof of the finiteness of the integral over $[-1,0]\times[0,1]\times[0,1]\times[-1,0]$. We notice that the integral over $[-1,0]\times[0,1]\times[0,1]\times[-1,0]$ is bounded from above by

$$C\sum_{k=3}^{\infty}\sum_{l=3}^{\infty}\frac{kl}{(\log k)^2(\log l)^2}\;|\mathcal{I}(k)\mathcal{I}(j)|$$
$$\leq C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\frac{1}{kj(\log k)^2(\log j)^2}<\infty.$$

This completes the proof of Theorem 1.1 (b). □

3. Some lemmas.

Lemma 3.1. *Let $\mu\in\mathbf{N}\cup\{0\}$, $a_\mu=2^{(\mu+1)}$ and $\Omega_\mu(\cdot,\cdot)$ be a function on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$ satisfying the conditions*:

(i) $\|\Omega_\mu\|_{L^2(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\leq a_\mu^2$,

(ii) $\|\Omega_\mu\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\leq 1$, *and*

(iii) Ω_μ *satisfies the cancelation conditions in* (1.3) *with* Ω *replaced by* Ω_μ. *Assume that* Φ,Ψ *are in* $C^2([0,\infty))$, *convex, and increasing functions with* $\Phi(0)=\Psi(0)=0$. *Let*

$$I_{\mu,k,j}(\xi,\eta)=\Bigg(\int_{[a_\mu^k,a_\mu^{k+1})\times[a_\mu^j,a_\mu^{j+1})}\bigg|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\Omega_\mu(x,y)$$
$$\times\, e^{-i(\Phi(t)\langle\xi,x\rangle+\Psi(s)\langle\eta,y\rangle)}\,d\sigma(x)\,d\sigma(y)\bigg|^2\frac{dtds}{ts}\Bigg)^{1/2}.$$

Then there exist positive constants C and α such that

$$|I_{\mu,k,j}(\xi,\eta)|\leq C(\mu+1);\tag{3.1}$$

$$|I_{\mu,k,j}(\xi,\eta)|\leq C(\mu+1)\,(\Phi(a_\mu^{k+1})\,|\xi|)^{\alpha/(\mu+1)}\,(\Psi(a_\mu^{j+1})\,|\eta|)^{\alpha/(\mu+1)};\tag{3.2}$$

$$|I_{\mu,k,j}(\xi,\eta)| \le C(\mu+1)\,(\Phi(a_\mu^k)\,|\xi|)^{-\alpha/(\mu+1)}\,(\Psi(a_\mu^j)\,|\eta|)^{-\alpha/(\mu+1)}; \tag{3.3}$$

$$|I_{\mu,k,j}(\xi,\eta)| \le C(\mu+1)\,(\Phi(a_\mu^{k+1})\,|\xi|)^{\alpha/(\mu+1)}\,(\Psi(a_\mu^j)\,|\eta|)^{-\alpha/(\mu+1)}; \tag{3.4}$$

$$|I_{\mu,k,j}(\xi,\eta)| \le C(\mu+1)\,(\Phi(a_\mu^k)\,|\xi|)^{-\alpha/(\mu+1)}\,(\Psi(a_\mu^{j+1})\,|\eta|)^{\alpha/(\mu+1)}, \tag{3.5}$$

where C is a constant independent of k, j, ξ, η and μ.

Proof. First, by condition (ii) on Ω_μ it is easy to see that (3.1) holds. Next, by the cancelation properties of Ω_μ and by a simple change of variables we have

$$|I_{\mu,k,j}(\xi,\eta)|^2 \le \int_{[1,a_\mu)\times[1,a_\mu)} \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\,\Omega_\mu(x,y)| \times \left| e^{-i\Phi(a_\mu^k t)\langle\xi,x\rangle} - 1\right| d\sigma(x)\,d\sigma(y)\right)^2 \frac{dtds}{ts}.$$

Since Φ is increasing we get

$$|I_{\mu,k,j}(\xi,\eta)| \le C(\mu+1)\,|\Phi(a_\mu^{k+1})\xi|. \tag{3.6}$$

Similarly,

$$|I_{\mu,k,j}(\xi,\eta)| \le C(\mu+1)\,|\Psi(a_\mu^{j+1})\eta|. \tag{3.7}$$

Now, by Schwarz's inequality we have

$$\begin{aligned} &\left| \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega_\mu(x,y) e^{-i(\Phi(t)<\xi,x>+\Psi(s)\langle\eta,y\rangle)}\,d\sigma(x)\,d\sigma(y)\right|^2 \\ &\quad\le \int_{\mathbf{S}^{m-1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\mu(x,y)\,e^{-i\Phi(a_\mu^k t)\langle\xi,x\rangle}\,d\sigma(x)\right|^2 d\sigma(y) \\ &\quad= \int_{\mathbf{S}^{m-1}} \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \Omega_\mu(x,y)\,\overline{\Omega_\mu(u,y)} \times e^{-i\Phi(a_\mu^k t)\langle\xi,x-u\rangle}\,d\sigma(x)\,d\sigma(u)\right) d\sigma(y). \end{aligned}$$

Therefore,

$$(3.8)\quad |I_{\mu,k,j}(\xi,\eta)|^2 \le \int_{\mathbf{S}^{m-1}} \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \Omega_\mu(x,y)\,\overline{\Omega_\mu(u,y)} \times J_{\mu,k}(\xi,x,u)\,d\sigma(x)\,d\sigma(u) \right) d\sigma(y),$$

where

$$J_{\mu,k}(\xi,x,u) = \int_1^{a_\mu} e^{-i\Phi(a_\mu^k t)\langle \xi, x-u\rangle}\,\frac{dt}{t}.$$

We now show that

$$(3.9)\qquad |J_{\mu,k}(\xi,x,u)| \le C(\mu+1)\,\left|\Phi(a_\mu^k)\xi\right|^{-1/4} |\langle \xi', x-u\rangle|^{-1/4}$$

for some positive constant C independent of μ.

The proof of (3.9) follows by a simple application of van der Corput's lemma. In fact, we notice first that

$$J_{\mu,k}(\xi,x,u) = \int_1^{a_\mu} H'(t)\,\frac{dt}{t},$$

where

$$H(t) = \int_1^t e^{-i\Phi(a_\mu^k w)\langle \xi, x-u\rangle}\,dw, \quad 1 \le t \le a_\mu.$$

By the assumptions on Φ and the mean value theorem, we have

$$\frac{d}{dw}\left(\Phi(a_\mu^k w)\right) = a_\mu^k\,\Phi'(a_\mu^k w) \ge \frac{\Phi(a_\mu^k w)}{w} \ge \frac{\Phi(a_\mu^k)}{t} \quad \text{for} \quad 1 \le w \le t \le a_\mu.$$

Thus, by van der Corput's lemma,

$$|H(t)| \le |\Phi(a_\mu^k)\xi|^{-1}\,|\langle \xi', x-u\rangle|^{-1}\,t,$$

for $1 \le t \le a_\mu$. Hence by integration by parts,

$$|J_{\mu,k}(\xi,x,u)| \le C(\mu+1)\,|\Phi(a_\mu^k)\xi|^{-1}\,|\langle \xi', x-u\rangle|^{-1}.$$

By combining this estimate with the trivial estimate,

$$|J_{\mu,k}(\xi, x, u)| \le (\ln 2)(\mu + 1),$$

we get (3.9). By Schwarz's inequality, condition (i) on Ω_μ and (3.8)–(3.9), we get

$$|I_{\mu,k,j}(\xi,\eta)|^2 \le C(\mu+1)^2\, a_\mu^4\, |\Phi(a_\mu^k)\xi|^{-1/4} \times \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} |\langle \xi', x-u\rangle|^{-1/2}\, d\sigma(x)\, d\sigma(u) \right)^{1/2}.$$

Since the last integral is finite, we get

$$|I_{\mu,k,j}(\xi,\eta)| \le C(\mu+1)\, a_\mu^2\, |\Phi(a_\mu^k)\xi|^{-1/8}. \tag{3.10}$$

Similarly,

$$|I_{\mu,k,j}(\xi,\eta)| \le C(\mu+1)\, a_\mu^2\, |\Psi(a_\mu^j)\xi|^{-1/8}. \tag{3.11}$$

By (3.1), (3.6)–(3.7) and (3.10)–(3.11) we obtain (3.2)–(3.5). The proof of the lemma is complete. □

By the same argument as in [**17**, p. 57], we get the following:

Lemma 3.2. *Let φ be a nonnegative, decreasing function on $[0,\infty)$ with $\int_{[0,\infty)} \varphi(t)\, dt = 1$. Then*

$$\left| \int_{[0,\infty)} f(x - ty')\, \varphi(t)\, dt \right| \le M_{y'} f(x),$$

where

$$M_{y'} f(x) = \sup_{R\in\mathbf{R}} \frac{1}{R} \int_0^R |f(x - sy')|\, ds$$

is the Hardy-Littlewood maximal function of f in the direction of y'.

For $\mu \in \mathbf{N} \cup \{0\}$ and $u' \in \mathbf{S}^{n-1}$, let $\mathcal{M}_{\Phi,\mu,u'}(f)$ denote the maximal function defined by

$$\mathcal{M}_{\Phi,\mu,u'} f(x) = \sup_{k\in\mathbf{Z}} \left| \int_{a_\mu^k}^{a_\mu^{k+1}} f(x - \Phi(t)u')\, \frac{dt}{t} \right|.$$

Lemma 3.3. *Assume that Φ is in $C^2([0,\infty))$, convex, and increasing function with $\Phi(0)=0$. Then*

$$\|\mathcal{M}_{\Phi,\mu,u'}(f)\|_p \le C_p(\mu+1)\,\|f\|_p \tag{3.12}$$

for $1<p\le\infty$ and $f\in L^p$.

Proof. By a change of variable we have

$$\mathcal{M}_{\Phi,\mu,u'}f(x) \le \sup_{k\in\mathbf{Z}}\left(\int_{\Phi(a_\mu^k)}^{\Phi(a_\mu^{k+1})} |f(x-tu')|\,\frac{dt}{\Phi^{-1}(t)\,\Phi'(\Phi^{-1}(t))}\right).$$

Without loss of generality, we may assume that $\Phi(t)>0$ for all $t>0$. By Lemma 3.2 and since the function $1/(\Phi^{-1}(t)\Phi'(\Phi^{-1}(t)))$ is nonnegative, decreasing and its integral over $[\Phi(a_\mu^k),\,\Phi(a_\mu^{k+1})]$ is equal to $(\ln 2)(\mu+1)$, we obtain

$$\mathcal{M}_{\Phi,\mu,u'}f(x) \le C(\mu+1)\,M_{u'}f(x), \tag{3.13}$$

By the L^p boundedness of $M_{u'}f$ with bound independent of u' we get (3.12) and the proof of the lemma is concluded.

For $\mu\in\mathbf{N}\cup\{0\}$, let

$$E_{k,j,\mu} = \left\{(u,v)\in\mathbf{R}^n\times\mathbf{R}^m : a_\mu^k \le |u| < a_\mu^{k+1} \text{ and } a_\mu^j \le |v| < a_\mu^{j+1}\right\}.$$

For any $\Omega\in L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$, we define the maximal operator

$$\lambda^*_{\Omega,\mu}f(x,y) = \sup_{k,j\in\mathbf{Z}}\left|\lambda_{k,j,\Omega,\mu}*f(x,y)\right|, \tag{3.14}$$

where

$$\lambda_{k,j,\Omega,\mu}*f(x,y) = \int_{E_{k,j,\mu}} |f(x-\Phi(|u|)u',\,y-\Psi(|v|)v')|\,\frac{|\Omega(u',v')|}{|u|^n\,|v|^m}\,du\,dv. \quad \square$$

Lemma 3.4. *Let $\Omega\in L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ and let Φ and Ψ be in $C^2([0,\infty))$, convex and increasing functions with $\Phi(0)=\Psi(0)=0$. Then*

$$\left\|\lambda^*_{\Omega,\mu}(f)\right\|_p \le C_p(\mu+1)^2\,\|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\,\|f\|_p \tag{3.15}$$

for $1 < p \leq \infty$ *and* $f \in L^p$*, where* C_p *is independent of* Ω, μ *and* f*.*

Proof. Using polar coordinates we get

$$|\lambda_{k,j,\Omega,\mu} * f(x,y)| \leq \int_{[a_\mu^k, a_\mu^{k+1}) \times [a_\mu^j, a_\mu^{j+1})} \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega(u',v')| \\ \times |f(x-\Phi(t)u', y-\Psi(s)v')| \, d\sigma(u')\, d\sigma(v')\, \frac{dtds}{ts}.$$

Therefore,

$$\lambda^*_{\Omega,\mu} f(x,y) \leq C \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega(u',v')| \\ \times (\mathcal{M}_{\Psi,\mu,v'} \circ \mathcal{M}_{\Phi,\mu,u'})\, f(x,y)\, d\sigma(u')\, d\sigma(v'),$$

where "$\circ$" denotes the composition of operators. By Lemma 3.3 and noticing that

$$\left\|\lambda^*_{\Omega,\mu}(f)\right\|_p \\ \leq C \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega(u',v')| \left\|(\mathcal{M}_{\Psi,\mu,v'} \circ \mathcal{M}_{\Phi,\mu,u'})\,(f)\right\|_p \, d\sigma(u')\, d\sigma(v'),$$

we get (3.15) which ends the proof of the lemma. □

Let $\mathcal{M}_S$ be the spherical maximal operator defined by

$$\mathcal{M}_S f(x) = \sup_{r>0} \int_{\mathbf{S}^{n-1}} |f(x-r\theta)| \, d\sigma(\theta).$$

By applying Stein's and Bourgain's results, see [**16**] and [**6**], we have

Lemma 3.5. *Suppose that* $n \geq 2$ *and* $p > n'$*. Then* $\mathcal{M}_S(f)$ *is bounded on* $L^p(\mathbf{R}^n)$*.*

We shall need the spherical maximal operator $\mathcal{M}_{SP}$ defined on functions $f(x,y)$ on $\mathbf{R}^n \times \mathbf{R}^m$ by

(3.16)
$$\mathcal{M}_{SP} f(x,y) = \sup_{r,s>0} \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |f(x-r\theta, y-sv)| \, d\sigma(\theta)\, d\sigma(v).$$

Define the operators $\mathcal{M}_S^{(1)}$ and $\mathcal{M}_S^{(2)}$ on functions f on $\mathbf{R}^n \times \mathbf{R}^m$ by $(\mathcal{M}_S^{(1)} f)(x,y) = (\mathcal{M}_S^{(1)}) f(\cdot, y))\,(x)$ and $(\mathcal{M}_S^{(2)} f)(x,y) = (\mathcal{M}_S^{(2)}) f\,(x,\cdot))$ (y). By invoking Lemma 3.5 and the inequality

$$\mathcal{M}_{SP} f(x,y) \le \left(\mathcal{M}_S^{(2)} \circ \mathcal{M}_S^{(1)}\right) f(x,y),$$

we get the following:

Lemma 3.6. *Suppose that $n, m \ge 2$ and $p > \max\{n', m'\}$. Then $\mathcal{M}_{SP}(f)$ is bounded on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$.*

4. Proof of Theorem 1.2. We start with proving part (a) of Theorem 1.2. Assume that Ω satisfies (1.3) and belongs to $L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for $1 \le \gamma \le 2$. Decompose Ω as in [**2**], (see also [**4**]). For $\mu \in \mathbf{N}$, let $\mathbf{E}_\mu$ be the set of points $(x,y) \in \mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ which satisfy $2^\mu \le |\Omega\ (x,y)| < 2^{\mu+1}$. Also, we let $\mathbf{E}_0$ be the set of all those points $(x,y) \in \mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ which satisfy $|\Omega\ (x,y)| < 2$. For $\mu \in \mathbf{N} \cup \{0\}$, set $b_\mu = \Omega \chi_{\mathbf{E}_\mu}$ and $\omega_\mu = \|b_\mu\|_1$. Set $I = \{\mu \in \mathbf{N} : \omega_\mu \ge 2^{-4\mu}\}$ and define the sequence of functions $\{\Omega_\mu\}_{\mu \in I \cup \{0\}}$ by

$$\begin{aligned}\Omega_0(x,y) = &\sum_{\mu \in \{0\} \cup (\mathbf{N}-I)} b_\mu(x,y) - \sum_{\mu \in \{0\} \cup (\mathbf{N}-I)} \left(\int_{\mathbf{S}^{n-1}} b_\mu(x,y)\, d\sigma(x)\right) \\ &- \sum_{\mu \in \{0\} \cup (\mathbf{N}-I)} \left(\int_{\mathbf{S}^{m-1}} b_\mu(x,y)\, d\sigma(y)\right) \\ &+ \sum_{\mu \in \{0\} \cup (\mathbf{N}-I)} \int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} b_\mu(x,y)\, d\sigma(x)\, d\sigma(y),\end{aligned}$$

and for $\mu \in I$,

$$\begin{aligned}\Omega_\mu(x,y) = (\omega_\mu)^{-1}\Big(b_\mu(x,y) - &\int_{\mathbf{S}^{n-1}} b_\mu(x,y)\, d\sigma(x) - \int_{\mathbf{S}^{m-1}} b_\mu(x,y)\, d\sigma(y) \\ &+ \int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} b_\mu(x,y)\, d\sigma(x)\, d\sigma(y)\Big).\end{aligned}$$

Then one can easily verify that the following hold for all $\mu \in I \cup \{0\}$ and for some positive constant C:

$$\|\Omega_\mu\|_2 \le C a_\mu^2, \qquad \|\Omega_\mu\|_1 \le C; \tag{4.1}$$

$$\sum_{\mu \in I \cup \{0\}} (\mu+1)^{2/\gamma'} \omega_\mu \le C \, \|\Omega\|_{L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\,; \tag{4.2}$$

$$\int_{\mathbf{S}^{n-1}} \Omega_\mu\,(u,\cdot)\; d\sigma\,(u) = \int_{\mathbf{S}^{m-1}} \Omega_\mu\,(\cdot,v)\; d\sigma\,(v) = 0; \tag{4.3}$$

$$\Omega = \sum_{\mu \in I \cup \{0\}} \omega_\mu \Omega_\mu. \tag{4.4}$$

By (4.4) we have

$$\mathcal{S}_{\Omega,\Phi,\Psi} f(x,y) \le \sum_{\mu \in I \cup \{0\}} \omega_\mu \mathcal{S}_{\Omega_\mu,\Phi,\Psi} f(x,y). \tag{4.5}$$

By (4.5) it suffices to show that the inequality

$$\left\|\mathcal{S}_{\Omega_\mu,\Phi,\Psi} f\right\|_p \le C_p(\mu+1)^{2/\gamma'} \|f\|_p \quad \text{for all} \quad \gamma' \le p < \infty \quad \text{and} \quad f \in L^p \tag{4.6}$$

holds for $\gamma' \le p < \infty$ if $1 < \gamma \le 2$ and for $p = \infty$ if $\gamma = 1$. To prove (4.6), we need to consider three cases. We first prove (4.6) for the case $\gamma = 2$.

The case $\gamma = 2$. Since Φ is convex and increasing in $(0,\infty)$, $\Phi(t)/t$ is also increasing for $t > 0$. Therefore, for $\mu \in \mathbf{N} \cup \{0\}$, the sequence $\left\{\Phi(a_\mu^k) : k \in \mathbf{Z}\right\}$ is a lacunary sequence with $\Phi(a_\mu^{k+1})/\Phi(a_\mu^k) \ge a_\mu > 1$. Let $\{\psi_{k,\mu,\Phi}\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0,\,\infty)$ adapted to the interval $E_{k,\mu,\Phi} = [(\Phi(a_\mu^{k+1}))^{-1},\ (\Phi(a_\mu^{k-1}))^{-1}]$. To be precise, we require the following:

$$\psi_{k,\mu,\Phi} \in C^\infty,\ 0 \le \psi_{k,\mu,\Phi} \le 1,\ \sum_k \psi_{k,\mu,\Phi}\,(t) = 1,$$

$$\operatorname{supp} \psi_{k,\mu,\Phi} \subseteq E_{k,\mu,\Phi},\ \left|\frac{d^s \psi_{k,\mu,\Phi}\,(t)}{dt^s}\right| \le \frac{C_s}{t^s},$$

where C_s is independent of the lacunary sequence $\{\Phi(a_\mu^k) : k \in \mathbf{Z}\}$. Define the multiplier operators $S_{k,j,\mu}$ in $\mathbf{R}^n \times \mathbf{R}^m$ by

$$\widehat{(S_{k,j,\mu} f)}(\xi,\eta) = \psi_{k,\mu,\Phi}(|\xi|)\, \psi_{j,\mu,\Psi}(|\eta|)\, \hat{f}(\xi,\eta).$$

Then for any $f \in \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^m)$ and $l, s \in \mathbf{Z}$ we have

$$f(x,y) = \sum_{k,j\in\mathbf{Z}} (S_{k+l,j+s,\mu} f)(x,y).$$

By duality we have

$$\begin{aligned}\mathcal{S}^{(2)}_{\Omega_\mu,\Phi,\Psi} f(x,y) &= \left(\int_{(0,\infty)\times(0,\infty)} \left|F_{r,t,\Omega_\mu} f(x,y)\right|^2 \frac{drdt}{rt} \right)^{1/2} \\ &= \left(\sum_{k,j\in\mathbf{Z}} \int_{[a_\mu^k, a_\mu^{k+1})\times[a_\mu^j, a_\mu^{j+1})} \left|F_{r,t,\Omega_\mu}(x,y)\right|^2 \frac{drdt}{rt} \right)^{1/2},\end{aligned}$$

where

$$\begin{aligned}&F_{r,t,\Omega} f(x,y) \\ &\qquad = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} f(x-\Phi(r)\xi, y-\Psi(t)\,\eta)\, \Omega(\xi,\eta)\, d\sigma(\xi)\, d\sigma(\eta).\end{aligned}$$

By Minkowski's inequality it is easy to see that

$$\begin{aligned}&\mathcal{S}^{(2)}_{\Omega_\mu,\Phi,\Psi} f(x,y) \\ &\quad \le \left(\sum_{k,j\in\mathbf{Z}} \int_{[a_\mu^k, a_\mu^{k+1})\times[a_\mu^j, a_\mu^{j+1})} \left| \sum_{l,s\in\mathbf{Z}} H_{k+l,j+s,r,t,\mu,\Omega_\mu} f(x,y) \right|^2 \frac{drdt}{rt} \right)^{1/2} \\ &\quad \le \sum_{l,s\in\mathbf{Z}} \left(\sum_{k,j\in\mathbf{Z}} \int_{[a_\mu^k, a_\mu^{k+1})\times[a_\mu^j, a_\mu^{j+1})} \left| H_{k+l,j+s,r,t,\mu,\Omega_\mu} f(x,y) \right|^2 \frac{drdt}{rt} \right)^{1/2}\end{aligned}$$

where

$$\begin{aligned}&H_{l,s,t,r,\mu,\Omega} f(x,y) \\ &\quad = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega\,(\xi,\eta)(S_{l,s,\mu} f)(x-\Phi(r)\xi, y-\Psi(t)\,\eta)\, d\sigma(\xi)\, d\sigma(\eta).\end{aligned}$$

Now if we let

$$T_{l,s,\mu,\Omega_\mu} f(x,y) = \sum_{k,j\in\mathbf{Z}} \int_{[a_\mu^k, a_\mu^{k+1})\times[a_\mu^j, a_\mu^{j+1})} \left|H_{k+l,j+s,r,t,\mu,\Omega_\mu} f(x,y)\right|^2 \frac{drdt}{rt},$$

then we have

$$\mathcal{S}^{(2)}_{\Omega_\mu,\Phi,\Psi} f(x,y) \leq \sum_{l,s\in\mathbf{Z}} T_{l,s,\mu,\Omega_\mu}\, f(x,y). \tag{4.7}$$

Therefore, to prove (4.6), it suffices to prove

$$\left\|T_{l,s,\mu,\Omega_\mu}(f)\right\|_p \leq C_p(\mu+1)\, 2^{-\theta_p|l|}\, 2^{-\theta_p|s|}\, \|f\|_p \tag{4.8}$$

for some positive constants C_p, θ_p and for all $2 \leq p < \infty$.

The proof of (4.8) follows by interpolation between a sharp L^2 estimate and a cruder L^p estimate of $T_{l,s,\mu,\Omega_\mu}(f)$.

First, the L^2 boundedness of $T_{l,s,\mu,\Omega_\mu}(f)$ is provided by a simple application of Plancherel's theorem and using Lemma 3.1.

$$\begin{aligned}
&\left\|T_{l,s,\mu,\Omega_\mu}(f)\right\|_2^2 \\
&= \int_{\mathbf{R}^n\times\mathbf{R}^m} \sum_{k,j\in\mathbf{Z}} \int_{[a_\mu^k, a_\mu^{k+1})\times[a_\mu^j, a_\mu^{j+1})} \left|H_{k+l,j+s,r,t,\mu,\Omega_\mu} f(x,y)\right|^2 \frac{drdt}{rt}\, dx\, dy \\
&\leq \sum_{k,j\in\mathbf{Z}} \int_{\Delta_{k+l,j+s}} \int_{[a_\mu^k, a_\mu^{k+1})\times[a_\mu^j, a_\mu^{j+1})} \\
&\qquad \left|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega_\mu(x,y)\, e^{-i(\Phi(r)\langle\xi,x\rangle+\Psi(t)\langle\eta,y\rangle)}\, d\sigma(x)\, d\sigma(y)\right|^2 \frac{drdt}{rt} \\
&\left|\hat{f}(\xi,\eta)\right|^2 d\xi\, d\eta \\
&\leq C(\mu+1)^2\, 2^{-2\alpha|l|}\, 2^{-2\alpha|s|} \sum_{k,j\in\mathbf{Z}} \int_{\Delta_{k+l,j+s}} \left|\hat{f}(\xi,\eta)\right|^2 d\xi\, d\eta \\
&\leq C(\mu+1)^2\, 2^{-2\alpha|l|}\, 2^{-2\alpha|s|}\, \|f\|_2^2,
\end{aligned}$$

where

$$\Delta_{k,j} = \{(\xi,\eta)\in\mathbf{R}^n\times\mathbf{R}^m : (|\xi|, |\eta|) \in E_{k,\mu,\Phi}\times E_{j,\mu,\Psi}\}.$$

Therefore, we have

$$\left\|T_{l,s,\mu,\Omega_\mu}(f)\right\|_2 \le C(\mu+1)2^{-\alpha|l|}2^{-\alpha|s|}\left\|f\right\|_2. \tag{4.9}$$

On the other hand, we need to compute the L^p-norm of $T_{l,s,\mu,\Omega_\mu}(f)$ for $p>2$. By duality, there is a function g in $L^{(p/2)'}(\mathbf{R}^n\times\mathbf{R}^m)$ with $\|g\|_{(p/2)'}\le 1$ such that

$$\begin{aligned}
&\left\|T_{l,s,\mu,\Omega_\mu}(f)\right\|_p^2\\
&= \sum_{k,j\in\mathbf{Z}}\int_{\mathbf{R}^n\times\mathbf{R}^m}\int_{[a_\mu^k,a_\mu^{k+1})\times[a_\mu^j,a_\mu^{j+1})}\left|H_{k+l,j+s,r,t,\mu,\Omega_\mu}f(x,y)\right|^2\frac{dr\,dt}{rt}\\
&\qquad\qquad\times|g(x,y)|\,dx\,dy\\
&\le \|\Omega_\mu\|_1\sum_{k,j\in\mathbf{Z}}\int_{\mathbf{R}^n\times\mathbf{R}^m}\int_{[a_\mu^k,a_\mu^{k+1})\times[a_\mu^j,a_\mu^{j+1})}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\left|\,\Omega_\mu(\xi,\eta)\right|\\
&\quad\times|S_{k+l,j+s,\mu}\,f(x,y)|^2|g(x+\Phi(r)\xi,y+\Psi(t)\eta)|\,d\sigma(\xi)\,d\sigma(\eta)\,\frac{dr\,dt}{rt}\,dx\,dy\\
&\le C\sum_{k,j\in\mathbf{Z}}\int_{\mathbf{R}^n\times\mathbf{R}^m}|S_{k+l,j+s,\mu}\,f(x,y)|^2\,\lambda^*_{\Omega_\mu,\mu}(\tilde g)(-x,-y)\,dx\,dy\\
&\le C\left\|\sum_{l,s\in\mathbf{Z}}|S_{k+l,j+s,\mu}\,f|^2\right\|_{p/2}\left\|\lambda^*_{\Omega_\mu,\mu}(\tilde g)\right\|_{(p/2)'},
\end{aligned}$$

where $\tilde g(x,y)=g(-x,-y)$.

By (4.1), invoking Lemma 3.4 and using the Littlewood-Paley theory and Theorem 3 along with the remark that follows its statement in [**15**, p. 96], we have

$$\left\|T_{l,s,\mu,\Omega_\mu}(f)\right\|_p \le C_p(\mu+1)\left\|f\right\|_p \quad\text{for}\quad 2\le p<\infty. \tag{4.10}$$

Now, (4.8) follows by interpolating between (4.9) and (4.10). This completes the proof of (4.6) in the case $\gamma=2$. □

The case $\gamma = 1$. If $f \in L^\infty(\mathbf{R}^n \times \mathbf{R}^m)$ and $h \in L^1(\mathbf{R}^+ \times \mathbf{R}^+, ds\,dt/(st))$, then

$$\left| \int_0^\infty \int_0^\infty h(t,s) \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} f(x-\Phi(t)u, y-\Psi(s)v) \times \Omega_\mu(u,v)\, d\sigma(u)\, d\sigma(v) \frac{dtds}{ts} \right| \leq C \|f\|_{L^\infty} \|h\|_{L^1(\mathbf{R}^+\times\mathbf{R}^+, ds\,dt/(st))}$$

for every $(x,y) \in \mathbf{R}^n \times \mathbf{R}^m$. By taking the supremum on both sides of the above inequality over all radial functions h with

$$\|h\|_{L^1(\mathbf{R}^+\times\mathbf{R}^+, ds\,dt/(st))} \leq 1$$

yields

$$\mathcal{S}^{(1)}_{\Omega_\mu,\Phi,\Psi} f(x,y) \leq C \|f\|_{L^\infty(\mathbf{R}^n\times\mathbf{R}^m)}$$

for almost every $(x,y) \in \mathbf{R}^n \times \mathbf{R}^m$. Hence,

$$\left\| \mathcal{S}^{(1)}_{\Omega_\mu,\Phi,\Psi} f \right\|_{L^\infty(\mathbf{R}^n\times\mathbf{R}^m)} \leq C \|f\|_{L^\infty(\mathbf{R}^n\times\mathbf{R}^m)}. \tag{4.11}$$

The case $1 < \gamma < 2$. We shall use an idea employed in the one-parameter case in [**14**]. By duality,

$$\mathcal{S}^{(\gamma)}_{\Omega_\mu,\Phi,\Psi} f(x,y) = \left\| \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} f\big(x-\Phi(t)u, y - \Psi(s)v\big)\, \Omega_\mu(u,v)\, d\sigma(u)\, d\sigma(v) \right\|_{L^{\gamma'}(\mathbf{R}^+\times\mathbf{R}^+, ds\,dt/(st))}.$$

Thus,

$$\left\| \mathcal{S}^{(\gamma)}_{\Omega_\mu,\Phi,\Psi} f \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} = \|S(f)\|_{L^p(L^{\gamma'}(\mathbf{R}^+\times\mathbf{R}^+, ds\,dt/(st)), \mathbf{R}^n\times\mathbf{R}^m)},$$

where

$$S : L^p(\mathbf{R}^n \times \mathbf{R}^m) \longrightarrow L^p(L^{\gamma'}(\mathbf{R}^+ \times \mathbf{R}^+, ds\,dt/(st)), \mathbf{R}^n \times \mathbf{R}^m)$$

defined by

$$S(f)(x,y,t,s) = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} f(x-\Phi(t)u, y-\Psi(s)v)\Omega_\mu(u,v)\, d\sigma(u)\, d\sigma(v).$$

By (4.6), for $\gamma = 2$, and (4.11), we interpret that

$$\|S(f)\|_{L^p(L^2(\mathbf{R}^+\times\mathbf{R}^+, ds\,dt/(st)),\mathbf{R}^n\times\mathbf{R}^m)} \le C(\mu+1)\,\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \quad \text{for} \quad 2 \le p < \infty$$

and

$$\|S(f)\|_{L^\infty(L^\infty(\mathbf{R}^+\times\mathbf{R}^+, ds\,dt/(st)),\mathbf{R}^n\times\mathbf{R}^m)} \le C\,\|f\|_{L^\infty(\mathbf{R}^n\times\mathbf{R}^m)}.$$

Applying the real interpolation theorem for Lebesgue mixed normed spaces to the above results, see [**5**], we conclude that

$$\|S(f)\|_{L^p(L^{\gamma'}(\mathbf{R}^+\times\mathbf{R}^+, ds\,dt/(st)),\mathbf{R}^n\times\mathbf{R}^m)} \le C(\mu+1)^{2/\gamma'}\,\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \quad \text{for} \quad \gamma' \le p < \infty,$$

which in turn implies (4.6) for $1 < \gamma < 2$. The proof of Theorem 1.2 is complete. □

A proof of part (b) of Theorem 1.2 can be constructed by the above estimates and following the same argument as in [**1**]. Details are omitted.

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