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A FAST WAVELET COLLOCATION METHOD FOR INTEGRAL EQUATIONS ON POLYGONS

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Dedicated to Professor Kendall E. Atkinson on the occasion of his 65th birthday with friendship and esteem

ABSTRACT. We develop a fast wavelet collocation method for solving Fredholm integral equations of the second kind with weakly singular kernels on *polygons*, following the general setting introduced in a recent paper [**11**]. Specifically, we construct wavelet functions and *multiscale* collocation functionals having vanishing moments on polygons. Critical issues for numerical implementation of this method are considered, such as a practical matrix compression scheme, numerical integration of weakly singular integrals, error controls of numerical quadratures and numerical solutions of the resulting compressed linear systems. The estimate of the computational complexity ensures the proposed method is a fast algorithm. Numerical experiments are presented to demonstrate the proposed methods and confirm the theoretical estimates.

1. Introduction. In a recent paper [**11**], a general setting was proposed for fast wavelet collocation methods for solving Fredholm integral equations of the second kind with weakly singular kernels. The main purpose of this paper is to develop a concrete fast wavelet collocation method for the integral equations on polygons, following the general recipe given in [**11**]. Specifically, we construct piecewise linear wavelets on polygons and their corresponding collocation functionals, both having vanishing moments of order two. We also consider several critical issues for numerical implementation of this method, such as a practical matrix compression scheme, numerical integration of weakly singular integrals, error controls of numerical quadratures and fast numerical solutions of the resulting compressed linear systems.

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As it is well-known, using a conventional collocation method to discretize Fredholm integral equations will lead to a linear system with a full coefficient matrix. It is computationally costly to generate the full coefficient matrix and to solve the corresponding discrete equations numerically, especially, when higher dimensional equations are considered. Therefore, fast algorithms are highly desirable. Recently, wavelet methods have been playing important roles in seeking fast algorithms for numerical solutions of integral equations.

Wavelet Galerkin methods and Petrov-Galerkin methods were studied in [**1, 8–10, 16, 17, 25, 26, 29, 30**]; see also references cited therein. A fast multilevel method was developed in [**12**] using a multilevel decomposition of the approximation space. References [**17, 25**] studied compression strategies with slightly different focuses. Paper [**17**] studied wavelet Galerkin methods using periodic wavelets based on refinement equations for periodic problems, while [**25**] developed wavelet Galerkin methods using piecewise polynomial orthogonal wavelets. These methods use L^2 analysis and therefore the vanishing moments of the multiscale basis functions naturally lead to matrix compression schemes. One advantage of using piecewise polynomial wavelets is that the wavelet functions have close forms which provide convenience for computation. The implementation of wavelet Galerkin methods based on refinement equations was done in [**15**] and that of those based on piecewise polynomial wavelets was considered in [**19**].

Collocation methods, due to lower computational cost for evaluations of integrals, for example, see [**2, 5**], receive more favorable attention from engineering fields. Yet less attention has been paid to wavelet-based collocation methods. For a large class of collocation methods, one of the appropriate spaces to work in is L^∞, and this causes challenging technical obstacles for identifying good matrix compression strategies. The recent work in [**11**] provides a framework of fast wavelet collocation methods for solving integral equations on invariant domains associated with families of contractive mappings. Paper [**14**] studies the actual computation of one-dimensional integral equations using such methods. We make use of the general setting introduced in [**11**] with a specialization to a polygonal domain which is a union of triangles (a special example of invariant sets) and present special results for this case.

We organize this paper in five sections. In Section 2, we describe wavelet collocation methods for Fredholm integral equations of the second kind on polygons. The multilevel wavelet basis and collocation functionals on polygons are constructed. In Section 3, we propose a *block* truncation scheme, which is a variation of the theoretical truncation scheme proposed in [**11**]. The block truncation scheme is much more convenient to implement for practical use. We consider in Section 4 the effect of numerical computation of integrals with weakly singular kernels. An efficient adaptive quadrature rule for computing singular double integrals is proposed and a control scheme of errors incurred by the numerical integration is introduced. The optimal order of convergence and computational complexity is estimated. In Section 5, we present several numerical examples which demonstrate the accuracy and efficiency of the proposed methods.

2. Wavelet collocation schemes on polygons. In this section, we describe our wavelet collocation schemes. Let E be a polygonal domain in $\mathbf{R}^2$. Set $\mathbf{X} := L^\infty(E)$ and $\mathbf{V} := C(E)$. Suppose that K is a weakly singular kernel so that the operator $\mathcal{K} : \mathbf{X} \to \mathbf{V}$ defined by

$$(\mathcal{K}u)(s) := \int_E K(s,t)u(t)\,dt, \quad s \in E$$

is compact on $\mathbf{X}$. We consider the Fredholm integral equation of the second kind

$$(\mathcal{I} - \mathcal{K})u = f, \tag{2.1}$$

where $\mathcal{I}$ is the identity operator on $\mathbf{X}$, $f \in \mathbf{X}$ is a given function and $u \in \mathbf{X}$ is the unknown to be determined. By the Fredholm alternative theorem, when the corresponding homogeneous equation has only the trivial solution, equation (2.1) has an unique solution in $\mathbf{X}$. Equations of this type have many important applications including boundary integral equations [**2, 3**] and radiosity equations [**4, 6**]. There have been many theoretical studies as well as numerical methods on such integral equations, cf. [**3, 21**].

Following the general construction described in [**11**], we next present piecewise linear wavelets on polygons and the corresponding multiscale collocation functionals. These basis functions and collocation functionals are then used to build a wavelet collocation scheme to solve equation (2.1).

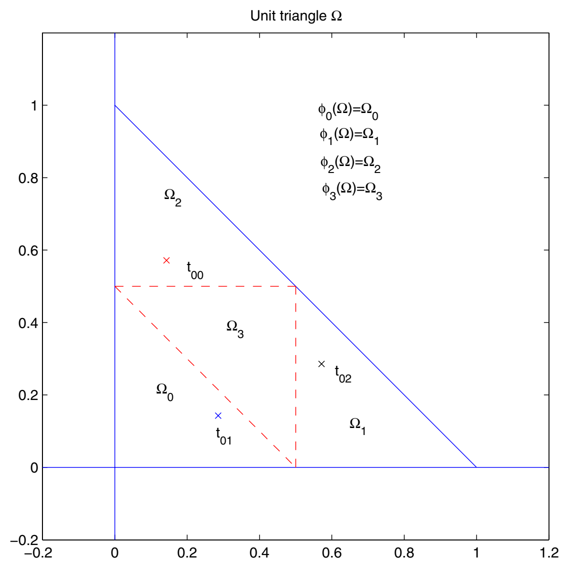


FIGURE 2.1. The unit triangle Ω.

2.1 *Construction on the unit triangle.* In this subsection, we construct piecewise linear wavelets on the unit triangle

$$\Omega := \{(x, y) \in \mathbf{R}^2 : x, y \geq 0, x + y \leq 1\}.$$

Let $\mathbf{Z}_n := \{0, 1, \ldots, n-1\}$. A family of contractive mappings $\Phi = \{\phi_i : i \in \mathbf{Z}_4\}$ with

$$\begin{aligned} \phi_0(x,y) &= \left(\frac{x}{2}, \frac{y}{2}\right), & \phi_1(x,y) &= \left(\frac{x+1}{2}, \frac{y}{2}\right), \\ \phi_2(x,y) &= \left(\frac{x}{2}, \frac{y+1}{2}\right), & \phi_3(x,y) &= \left(\frac{1-x}{2}, \frac{1-y}{2}\right), \end{aligned} \tag{2.2}$$

subdivides Ω into four triangles $\Omega_i := \phi_i(\Omega)$, $i \in \mathbf{Z}_4$, see Figure 2.1. Observe that

$$\operatorname{meas}(\Omega_i \cap \Omega_{i'}) = 0, \quad \text{for} \quad i, i' \in \mathbf{Z}_4,\ i \neq i', \quad \text{and} \quad \bigcup_{i \in \mathbf{Z}_4} \Omega_i = \Omega. \tag{2.3}$$

With the mappings ϕ_i, $i \in \mathbf{Z}_4$, we define operators $\mathcal{T}_i : \mathbf{X} \to \mathbf{X}$, $i \in \mathbf{Z}_4$ by

$$(\mathcal{T}_i \circ g)(t) = g(\phi_i^{-1}(t))\chi_{\Omega_i}(t), \quad t \in \Omega,$$

where χ_S is the characteristic function of the set S.

We choose three points, see Figure 2.1, in Ω as

$$t_{00} = \left(\frac{1}{7}, \frac{4}{7}\right), \quad t_{01} = \left(\frac{2}{7}, \frac{1}{7}\right), \quad t_{02} = \left(\frac{4}{7}, \frac{2}{7}\right), \tag{2.4}$$

and note the set $G'_0 := \{t_{0j} : j \in \mathbf{Z}_3\}$ is refinable with respect to the family of mappings Φ, i.e., $G'_0 \subset \Phi(G'_0)$, where for a set $D \in \mathbf{R}^2$

$$\Phi(D) := \bigcup_{i \in \mathbf{Z}_4} \phi_i(D),$$

and admits a unique Lagrange linear interpolation, see [**10, 22**]. We also choose three linear polynomials

$$\begin{aligned} w_{00}(x,y) &= -x + 2y, \\ w_{01}(x,y) &= -2x - 3y + 2, \\ w_{02}(x,y) &= 3x + y - 1, \end{aligned}$$

and nine piecewise linear polynomials

$$\begin{aligned} w_{10}(x,y) &= \frac{1}{8}(-2 + 3x + y)(\chi_{\Omega_0} + \chi_{\Omega_1} + \chi_{\Omega_2}) \\ &\quad + \frac{1}{8}(22 - 45x - 15y)\chi_{\Omega_3}, \\ w_{11}(x,y) &= \frac{1}{8}(1 - 2x - 3y)(\chi_{\Omega_0} + \chi_{\Omega_1} + \chi_{\Omega_2}) \\ &\quad + \frac{1}{8}(-23 + 30x + 45y)\chi_{\Omega_3}, \\ w_{12}(x,y) &= \frac{1}{8}(-1 - x + 2y)(\chi_{\Omega_0} + \chi_{\Omega_1} + \chi_{\Omega_2}) \\ &\quad + \frac{1}{8}(7 + 15x - 30y)\chi_{\Omega_3}, \end{aligned}$$

$$\begin{aligned}
w_{13}(x,y) &= \frac{1}{8}(2-3x-5y)(\chi_{\Omega_0}+\chi_{\Omega_1}+\chi_{\Omega_3}) \\
&\quad + \frac{1}{8}(-14+45x+11y)\chi_{\Omega_2}, \\
w_{14}(x,y) &= \frac{1}{8}(1+x-6y)(\chi_{\Omega_0}+\chi_{\Omega_1}+\chi_{\Omega_3}) \\
&\quad + \frac{1}{8}(-15-15x+26y)\chi_{\Omega_2}, \\
w_{15}(x,y) &= \frac{1}{8}(2-7x-y)(\chi_{\Omega_0}+\chi_{\Omega_2}+\chi_{\Omega_3}) \\
&\quad + \frac{1}{8}(-30+41x+15y)\chi_{\Omega_1}, \\
w_{16}(x,y) &= \frac{1}{8}(-1-2x+3y)(\chi_{\Omega_0}+\chi_{\Omega_2}+\chi_{\Omega_3}) \\
&\quad + \frac{1}{8}(31-34x-45y)\chi_{\Omega_1}, \\
w_{17}(x,y) &= \frac{1}{8}(-5+6x+7y)(\chi_{\Omega_1}+\chi_{\Omega_2}+\chi_{\Omega_3}) \\
&\quad + \frac{1}{8}(11-26x-41y)\chi_{\Omega_0}, \\
w_{18}(x,y) &= \frac{1}{8}(-3+5x+2y)(\chi_{\Omega_1}+\chi_{\Omega_2}+\chi_{\Omega_3}) \\
&\quad + \frac{1}{8}(-3-11x+34y)\chi_{\Omega_0}.
\end{aligned}$$

Note that

$$w_{0i}(t_{0l}) = \delta_{il}, \quad i,l \in \mathbf{Z}_3, \quad \text{and} \quad \int_\Omega h(t)w_{1j}(t)\,dt = 0, \quad j \in \mathbf{Z}_9,$$

where h is any linear polynomial on Ω. The linear functions w_{0i}, $i \in \mathbf{Z}_3$, are called the scaling functions and the nine piecewise polynomial functions are called the initial wavelets. The graphs of w_{0j}, $j \in \mathbf{Z}_3$ and w_{1k}, $k \in \mathbf{Z}_9$ are shown in Figure 2.2.

We next define a set of collocation functionals corresponding to w_{0j}, $j \in \mathbf{Z}_3$ by

$$L_0 := \{\ell_{0j} : \ell_{0j} = \delta_{t_{0j}},\ j \in \mathbf{Z}_3\},$$

where, for any $s \in E$, δ_s denotes the linear functional in $\mathbf{V}^*$ that for $v \in \mathbf{V}$, $\langle \delta_s, v\rangle = v(s)$. We shall need to evaluate δ_s on functions in

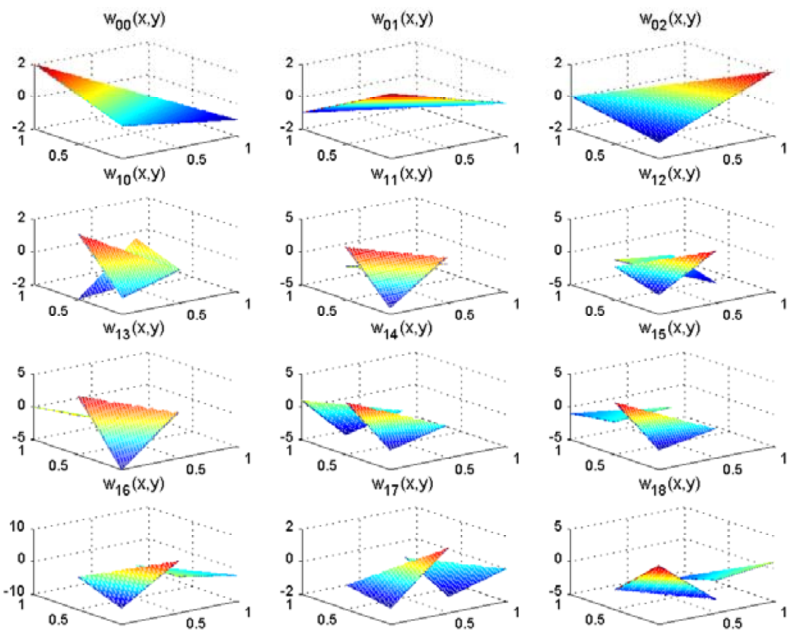


FIGURE 2.2. Scaling functions and wavelet functions on the unit triangle.

X. Therefore, as in [**7**] we take any norm preserving extension of δ_s to **X** and use the same notation for the extension. In particular, this extension allows us to evaluate piecewise polynomials on E. To find the collocation functionals $L_1 := \{\ell_{1l} : l \in \mathbf{Z}_9\}$ corresponding to w_{1j}, $j \in \mathbf{Z}_9$, we let

(2.5)

$$\ell_{1l} = \sum_{e \in \mathbf{Z}_{12}} c'_{le} \delta_{t_{1e}}, \quad t_{1e} = \phi_i(t_{0j}), \quad e = 3i + j, \quad i \in \mathbf{Z}_4, \quad j \in \mathbf{Z}_3,$$

and require for $l, j \in \mathbf{Z}_9$

$$\langle \ell_{1l}, 1 \rangle = 0, \quad \langle \ell_{1l}, x \rangle = 0, \quad \langle \ell_{1l}, y \rangle = 0, \quad \langle \ell_{1l}, w_{1j} \rangle = \delta_{lj}. \tag{2.6}$$

Thus, we obtain that

$$
\begin{aligned}
\ell_{10} &= -\frac{1}{2}\delta_{t_{1,2}} - \frac{1}{2}\delta_{t_{1,7}} + \delta_{t_{1,11}},\\
\ell_{11} &= -\frac{1}{2}\delta_{t_{1,3}} - \frac{1}{2}\delta_{t_{1,7}} + \delta_{t_{1,10}},\\
\ell_{12} &= -\frac{1}{2}\delta_{t_{1,2}} - \frac{1}{2}\delta_{t_{1,3}} + \delta_{t_{1,9}},\\
\ell_{13} &= \frac{1}{2}\delta_{t_{1,2}} - \frac{1}{2}\delta_{t_{1,3}} - \delta_{t_{1,7}} + \delta_{t_{1,8}},\\
\ell_{14} &= \frac{1}{2}\delta_{t_{1,2}} + \delta_{t_{1,6}} - \frac{3}{2}\delta_{t_{1,7}},\\
\ell_{15} &= -\frac{3}{2}\delta_{t_{1,3}} + \delta_{t_{1,5}} + \frac{1}{2}\delta_{t_{1,7}},\\
\ell_{16} &= -\frac{1}{2}\delta_{t_{1,2}} - \delta_{t_{1,3}} + \delta_{t_{1,4}} + \frac{1}{2}\delta_{t_{1,7}},\\
\ell_{17} &= \delta_{t_{1,1}} - \frac{3}{2}\delta_{t_{1,2}} + \frac{1}{2}\delta_{t_{1,3}},\\
\ell_{18} &= \delta_{t_{1,0}} - \delta_{t_{1,2}} + \frac{1}{2}\delta_{t_{1,3}} - \frac{1}{2}\delta_{t_{1,7}}.
\end{aligned}
$$

Functionals ℓ_{0i}, $i \in \mathbf{Z}_3$, are called the scaling functionals and ℓ_{1i}, $i \in \mathbf{Z}_9$ are called the initial wavelet functionals. In Figure 2.3, we plot ℓ_{0i}, $i \in \mathbf{Z}_3$ and ℓ_{1j}, $j \in \mathbf{Z}_9$, where each solid vertical bar stands for a point evaluation functional. The position of the bar indicates the location of the point, and the height of the bar stands for the coefficient of each point evaluation functional. Having the initial wavelets w_{1j}, $j \in \mathbf{Z}_9$, it is very easy to generate higher level wavelets w_{ij} for $i \geq 2$. We use $w(i)$ to denote number of piecewise polynomials w_{ij} at the resolution level i, for $i \geq 0$. Let

$$\mathbf{Z}_m^n := \mathbf{Z}_m \times \mathbf{Z}_m \times \cdots \times \mathbf{Z}_m$$

with n folds of $\mathbf{Z}_m$. For $\mathbf{e} := (e_0, \ldots, e_{n-1}) \in \mathbf{Z}_4^n$, we introduce a composite map

$$\phi_{\mathbf{e}} := \phi_{e_0} \circ \phi_{e_1} \circ \cdots \circ \phi_{e_{n-1}}$$

and a composite operator $\mathcal{T}_{\mathbf{e}}$

$$\mathcal{T}_{\mathbf{e}} := \mathcal{T}_{e_0} \circ \mathcal{T}_{e_1} \circ \cdots \circ \mathcal{T}_{e_{n-1}}.$$

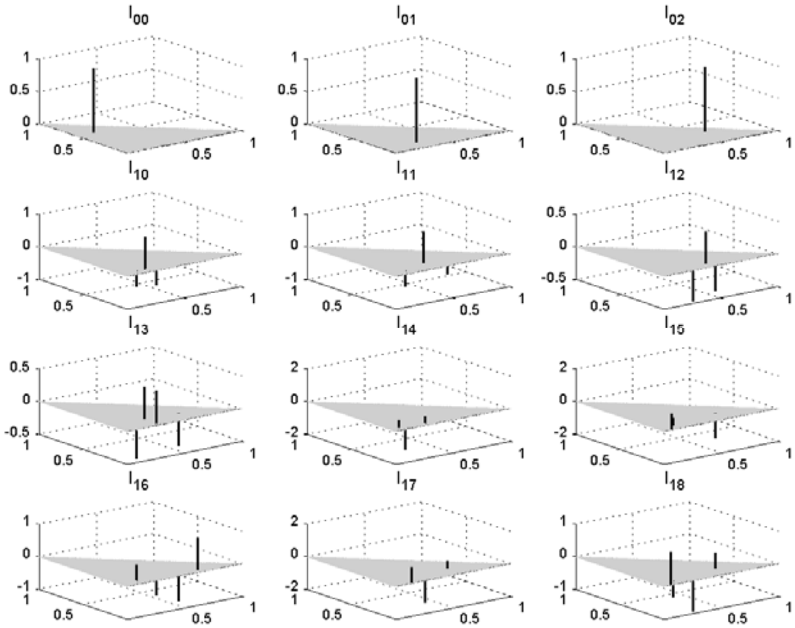


FIGURE 2.3. Collocation functionals in L_0 and L_1 on the unit triangle.

For convenience, we also define the number associated with $\mathbf{e}$

$$\mu(\mathbf{e}) := 4^{n-1}e_0 + \cdots + 4e_{n-2} + e_{n-1}.$$

Now, for $i \geq 2$, $j = 9\mu(\mathbf{e}) + l$, $\mathbf{e} \in \mathbf{Z}_4^{i-1}$, $l \in \mathbf{Z}_9$, let

$$w_{ij} = \mathcal{T}_{\mathbf{e}} w_{1l}. \tag{2.7}$$

Observe that the support of w_{ij}, $i \geq 2$, is contained in $S_{ij} := \phi_{\mathbf{e}}(\Omega)$, $j \in \mathbf{Z}_{w(i)}$.

To generate multiscale (wavelet) collocation functionals, we introduce for any $e \in \mathbf{Z}_4$ a linear operator $\mathcal{L}_e : \mathbf{X}^* \to \mathbf{X}^*$ defined for $v \in \mathbf{X}$ and $\ell \in \mathbf{X}^*$ by the equation

$$\langle \mathcal{L}_e \ell, v \rangle = \langle \ell, v \circ \phi_e \rangle.$$

Moreover, for $\mathbf{e} := (e_0, \dots, e_{n-1}) \in \mathbf{Z}_4^n$, define the composite operator

$$\mathcal{L}_{\mathbf{e}} := \mathcal{L}_{e_0} \circ \cdots \circ \mathcal{L}_{e_{n-1}}.$$

Now, for $i > 1$, $j = 9\mu(\mathbf{e}) + l$, $\mathbf{e} \in \mathbf{Z}_4^{i-1}$, $l \in \mathbf{Z}_9$, ℓ_{ij} is defined by

$$\ell_{ij} := \mathcal{L}_{\mathbf{e}} \ell_{1l}. \tag{2.8}$$

Note that the "support" of ℓ_{ij} is also contained in S_{ij}.

2.2 *Wavelet basis and collocation functionals on polygons.* We are now ready to construct wavelet basis and the corresponding collocation functionals on *polygons*. We begin with a nested sequence of finite dimensional subspaces of $\mathbf{X}$,

$$\mathbf{F}_n \subseteq \mathbf{F}_{n+1}, \quad n \in \mathbf{N}_0. \tag{2.9}$$

We shall construct the basis of $\mathbf{F}_n$, $n \in \mathbf{N}$, consisting of the basis of $\mathbf{F}_{n-1}$ and an additional set of functions that are finer in resolution, i.e.,

$$\mathbf{F}_n := \mathbf{F}_{n-1} \oplus \mathbf{W}_n = \mathbf{F}_0 \oplus \mathbf{W}_1 \oplus \cdots \oplus \mathbf{W}_n. \tag{2.10}$$

Let $\{\triangle_0, \dots, \triangle_{p-1}\}$ be a triangulation of a polygonal domain E, and assume that the triangles can intersect only at vertices or along their common edges. Let T_j, $j \in \mathbf{Z}_p$, be the unique affine mapping which maps Ω one-to-one onto $\triangle_j$. Let $\mathbf{U} := \{(i,j) : i \in \mathbf{N}_0,\ j \in \mathbf{Z}_{w(i)}\}$. Utilizing ℓ_{ij}, w_{ij}, $(i,j) \in \mathbf{U}$ defined on Ω, we can easily construct the (wavelet) collocation functionals and wavelet functions $\ell_{k,ij}$, $w_{k,ij}$, $k \in \mathbf{Z}_p$, $(i,j) \in \mathbf{U}$ on E by

$$\langle \ell_{k,ij}, v \rangle = \langle \ell_{ij}, v \circ T_k \rangle,$$

and

$$w_{k,ij}(x,y) = \begin{cases} w_{ij} \circ T_k^{-1}(x,y) & (x,y) \in \triangle_k, \\ 0 & (x,y) \in E \setminus \triangle_k, \end{cases} \tag{2.11}$$

where, $v \in \mathbf{X}$. Moreover, define

$$\mathbf{F}_0 = \operatorname{span}\{w_{k,0j} :\ j \in \mathbf{Z}_3,\ k \in \mathbf{Z}_p\},$$

and

$$\mathbf{W}_i = \operatorname{span}\{w_{k,ij} : \ j \in \mathbf{Z}_{w(i)}, \ k \in \mathbf{Z}_p\}, \quad i \geq 1.$$

We now present several important properties of the constructed wavelet basis functions and collocation functionals on polygons.

Proposition 2.1.

$$\dim \mathbf{W}_n = w(n) = pw(1) \cdot 4^{n-1}, \quad n \in \mathbf{N},$$
$$\dim \mathbf{F}_n := f(n) = pw(0) \cdot 4^n, \quad n \in \mathbf{N}_0.$$

For a set $A \subset \mathbf{R}^2$, we let $d(A)$ represent the diameter of A, i.e., $d(A) := \sup\{|s-t| : s, t \in A\}$, where $|.|$ denotes the Euclidean norm on the space $\mathbf{R}^2$. Let $S_{k,nm}$ denote the support of wavelet $w_{k,nm}$, $k \in \mathbf{Z}_p$, $(n, m) \in \mathbf{U}$. For each $n \in \mathbf{N}_0$, let

$$d_n := \max\{\mathrm{d}\,(S_{k,nm})\,, \ \ m \in \mathbf{Z}_{w(n)}, \ \ k \in \mathbf{Z}_p\}.$$

Proposition 2.2. *The constructed wavelet basis functions $w_{k,ij}$, $k \in \mathbf{Z}_p$, $(i, j) \in \mathbf{U}$ over the polygon E are locally supported and their supports are shrinking as level i increases, i.e.,*

$$d_n = \mathcal{O}(4^{-n/2}). \tag{2.12}$$

Let π_2 denote the space of polynomials of degree less than 2.

Proposition 2.3. *The constructed wavelets and collocation functionals have vanishing moments of order two, namely, for any polynomial $h \in \pi_2$,*

$$\langle \ell_{k,ij}, h\rangle = 0, \quad (w_{k,ij}, h) = 0, \quad (i, j) \in \mathbf{U}, \quad i \geq 1, \quad k \in \mathbf{Z}_p.$$

Now let

$$[\psi_j, j \in \mathbf{Z}_{12}] := [w_{00}, w_{01}, w_{02}, \mathcal{T}_0 w_{00}, \mathcal{T}_0 w_{01}, \mathcal{T}_1 w_{01}, \mathcal{T}_1 w_{02}, \mathcal{T}_2 w_{00}, \mathcal{T}_2 w_{02}, \mathcal{T}_3 w_{00}, \mathcal{T}_3 w_{01}, \mathcal{T}_3 w_{02}].$$

We then have for some c_{jl}, $j \in \mathbf{Z}_9$, $l \in \mathbf{Z}_{12}$ that

$$w_{1j} = \sum_{l \in \mathbf{Z}_{12}} c_{jl}\psi_l, \quad j \in \mathbf{Z}_9. \tag{2.13}$$

We use the coefficients c_{jl}, $j \in \mathbf{Z}_9$, $l \in \mathbf{Z}_{12}$ in (2.13) to form matrix $\mathbf{C} := [c_{jl} : j \in \mathbf{Z}_9,\ l \in \mathbf{Z}_{12}]$. Likewise, we use the coefficients c'_{le} in (2.5) to form matrix $\mathbf{C}'$ with

$$\mathbf{C}' = [c'_{le} : l \in \mathbf{Z}_9,\ \ e \in \mathbf{Z}_{12}]. \tag{2.14}$$

Proposition 2.4. *Both basis functions and collocation functionals are uniformly bounded.*

Proof. For $(i,j) \in \mathbf{U}$, $i \geq 2$, $j = 9\mu(\mathbf{e}) + l$, $l \in \mathbf{Z}_9$, (2.11) yields

$$|\langle \ell_{k,ij}, v\rangle| = |\langle \ell_{ij}, v \circ T_k\rangle| = |\langle \ell_{1l}, v \circ T_k \circ \phi_{\mathbf{e}}\rangle| \leq \|\mathbf{C}'\|_\infty \|v\|_\infty,$$

and

$$\|w_{k,ij}\|_\infty = \|w_{ij} \circ T_k^{-1}\|_\infty \leq \|\mathbf{C}\|_\infty \left(\max_{j \in \mathbf{Z}_{12}} \|\psi_j\|_\infty \right),$$

proving the result. $\square$

Proposition 2.5. *For any $i, i' \in \mathbf{N}_0$,*

$$\langle \ell_{k',i'j'}, w_{k,ij}\rangle = \delta_{k'k}\delta_{ii'}\delta_{jj'}, \tag{2.15}$$
$$(i,j),(i',j') \in \mathbf{U}, \quad i \leq i', \quad k,k' \in \mathbf{Z}_p,$$

$$\sum_{j \in \mathbf{Z}_{w(i)}} |\langle \ell_{k',i'j'}, w_{k,ij}\rangle| \leq \gamma, \quad (i,j),(i',j') \in \mathbf{U}, \quad i > i', \tag{2.16}$$

where $\gamma := 81/28$. Moreover,

$$\gamma < 4^{\kappa/2} - 1, \tag{2.17}$$

where κ is the order of the piecewise polynomials of the approximation spaces and when piecewise linear polynomials are used, $\kappa = 2$.

Proof. We present a proof for (2.16) only. When $k' \neq k$, $\langle \ell_{k',i'j'}, w_{k,ij} \rangle = 0$ by the definitions of $\ell_{k',i'j'}$ and $w_{k,ij}$. If $k' = k$, in view of (2.11), we have that $\langle \ell_{k',i'j'}, w_{k,ij} \rangle = \langle \ell_{i'j'}, w_{ij} \rangle$. Thus it suffices to prove

$$\sum_{j \in \mathbf{Z}_{w(i)}} |\langle \ell_{i'j'}, w_{ij} \rangle| \leq \gamma, \quad (i,j),(i',j') \in \mathbf{U}, \quad i > i'. \tag{2.18}$$

Let $\mathbf{C_1} = [c_{ij},\ i \in \mathbf{Z}_9,\ j \in \mathbf{Z}_3]$,where c_{ij} are elements of $\mathbf{C}$. Note $\|\mathbf{C_1}\|_1 = 27/28$, and $\|\mathbf{C}'\|_\infty = 3$. Set

$$\gamma = \max\{\|\mathbf{C_1}\|_1, \|\mathbf{C}'\|_\infty \|\mathbf{C_1}\|_1\}. \tag{2.19}$$

By Lemma 5.2 of [**11**], which says $\sum_{j \in \mathbf{Z}_{w(i)}} |\langle \ell_{i'j'}, w_{ij} \rangle| \leq \gamma$, we have (2.18). $\square$

2.3 *The collocation scheme.* To describe the collocation scheme, we let $\mathcal{P}_n : \mathbf{X} \mapsto \mathbf{F}_n$ be the projection defined by

$$\langle \ell_{k,ij}, \mathcal{P}_n x \rangle = \langle \ell_{k,ij}, x \rangle, \quad x \in \mathbf{X}, \quad k \in \mathbf{Z}_p, \quad (i,j) \in \mathbf{U}_n, \tag{2.20}$$

where $\mathbf{U}_n := \{(i,j) : j \in \mathbf{Z}_{w(i)}, i \in \mathbf{Z}_{n+1}\}$. The collocation scheme for solving equation (2.1) is to seek a vector $\mathbf{u}_n := [u_{k,ij} :\ k \in \mathbf{Z}_p,\ (i,j) \in \mathbf{U}_n]$, such that the function

$$u_n := \sum_{k \in \mathbf{Z}_p} \sum_{(i,j) \in \mathbf{U}_n} u_{k,ij} w_{k,ij} \tag{2.21}$$

in $\mathbf{F}_n$, satisfies

$$(\mathcal{I} - \mathcal{P}_n \mathcal{K}) u_n = \mathcal{P}_n f. \tag{2.22}$$

The corresponding linear system has the form

$$(\mathbf{E}_n - \mathbf{K}_n)\mathbf{u}_n = \mathbf{f}_n, \tag{2.23}$$

where

(2.24)
$$\mathbf{E}_n := [E_{k'i'j',kij} : k,k' \in \mathbf{Z}_p, (i,j),(i',j') \in \mathbf{U}_n] = [\langle \ell_{k',i'j'}, w_{k,ij} \rangle],$$

(2.25)
$$\mathbf{K}_n := [K_{k'i'j',kij} : k,k' \in \mathbf{Z}_p, (i,j),(i',j') \in \mathbf{U}_n] = [\langle \ell_{k',i'j'}, \mathcal{K} w_{k,ij} \rangle]$$

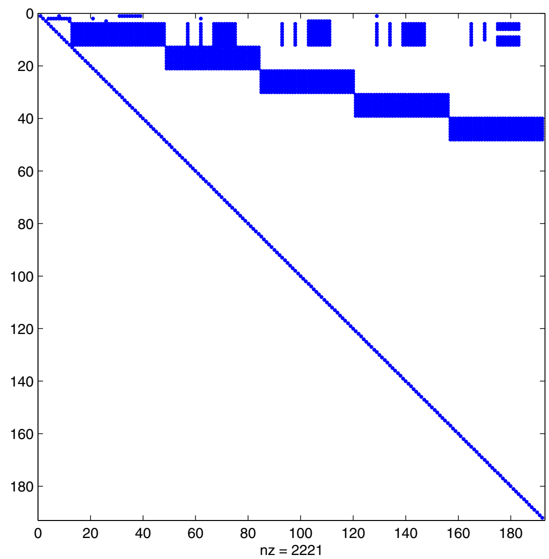


FIGURE 2.4. The matrix $\mathbf{E}_3$.

and

$$\mathbf{f}_n = [f_{k'ij} : k' \in \mathbf{Z}_p, (i', j') \in \mathbf{U}_n] = [\langle \ell_{i'j'}, f \circ T_{k'} \rangle] \, .$$

From (2.15), it is clear that $\mathbf{E}_n$ is upper triangular. Figure 2.4 demonstrates computed full matrix $\mathbf{E}_3$ after discretizing the integral equation (2.1) with $E = \Omega$ using the piecewise linear wavelets. According to [**11**], after discretizing the integral equation using the constructed wavelet basis and collocation functionals, the matrix $\mathbf{K}_n$ is numerically sparse and the matrix appears as a "finger" shape. Figure 2.5 depicts the computed full matrix $\mathbf{K}_3$ after discretizing (2.1) with $E = \Omega$ and

$$K(s,t) = \frac{1}{|s-t|}, \quad s,t \in \Omega. \tag{2.26}$$

3. Block truncation schemes. In this section, we present a *block truncation scheme* which allows us to approximate matrix $\mathbf{K}_n$ by a sparse matrix.

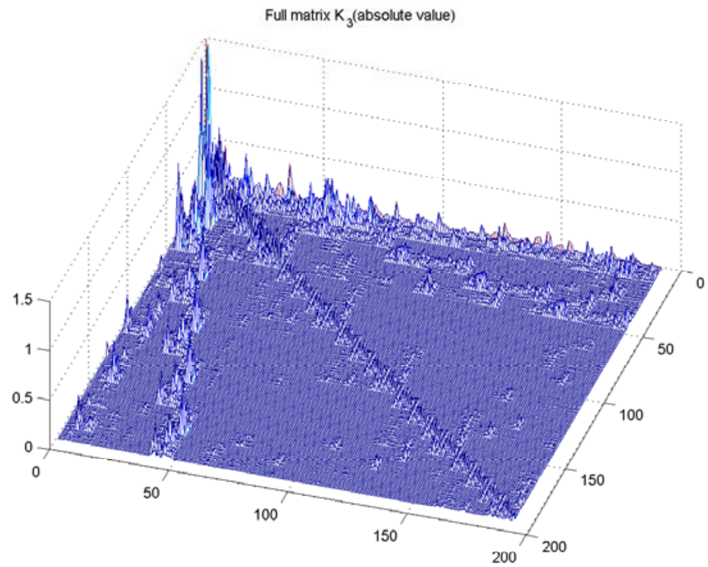


FIGURE 2.5. The full matrix $\mathbf{K}_3$.

Let μ be a lattice point in $\mathbf{N}_0^2$, viz, $\mu := [\mu_i \in \mathbf{N}_0 : i \in \mathbf{Z}_2]$. As usual, we set $|\mu| = \sum_{i\in\mathbf{Z}_2} \mu_i$ and for a function $v \in \mathbf{X}$ we use the standard multi-index notation for derivatives

$$D_x^\mu v(x) = \frac{\partial^{|\mu|} v(x)}{\partial x_0^{\mu_0} \partial x_1^{\mu_1}}, \quad x \in \mathbf{R}^2.$$

We assume that the weakly singular kernel K has the following properties: For $\mu, \rho \in \mathbf{N}_0^2$, $s, t \in E$, $s \neq t$, the kernel K has continuous partial derivatives $D_s^\mu D_t^\rho K(s,t)$ for $|\mu| \leq 2$, $|\rho| \leq 2$. Moreover, there exist positive constants σ with $0 \leq \sigma < 2$ and θ_1 such that for $|\mu| = |\rho| = 2$ there holds

$$|D_s^\mu D_t^\rho K(s,t)| \leq \frac{\theta_1}{|s-t|^{\sigma+|\mu|+|\rho|}}, \quad s \neq t. \tag{3.1}$$

According to Lemma 3.1 of [**11**] the value of $|K_{k'i'j',kij}|$ depends on the distance $\operatorname{dist}(\hat{S}_{k'i'j'}, S_{kij})$ between the support $\hat{S}_{k'i'j'}$ of $\ell_{k'i'j'}$ and the support S_{kij} of w_{kij}. Specifically, if there exists a constant $z > 1$ such that

$$\operatorname{dist}(\hat{S}_{k'i'j'}, S_{kij}) \geq z(d_i + d_{i'}),$$

where d_i is the maximum of diameters of $S_{k,ij}$, for all $j \in \mathbf{Z}_{w(i)}$, $k \in \mathbf{Z}_p$, see (2.12), then there exists a positive constant c (we use c for a generic constant throughout this paper) so that

$$|K_{k'i'j',kij}| \leq c(d_i d_{i'})^{\kappa} \sum_{s \in \hat{S}_{k'i'j'}} \int_{S_{kij}} \frac{1}{|s-t|^{2\kappa+\sigma}}\, dt.$$

This estimate leads to the following theoretical truncation scheme for $\mathbf{K}_n$ in [**11**]:

Let $\widetilde{\mathbf{K}}_n := [\widetilde{K}_{k'i'j',kij} : (i',j'), (i,j) \in \mathbf{U}_n,\ k, k' \in \mathbf{Z}_p]$ be the truncation matrix whose entries are defined in terms of a matrix truncation parameter $\varepsilon_{i'i}$ by

$$\tilde{K}_{k'i'j',kij} := \begin{cases} K_{k'i'j',kij} & \text{if } \operatorname{dist}(\widehat{S}_{k',i'j'}, S_{k,ij}) \leq \varepsilon_{i'i}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

The truncation parameters $\varepsilon_{i'i}$ are chosen such that for some positive constants z' and $z > 1$,

$$\varepsilon_{i'i} = \max\{z' 4^{[-i+b'(n-i')]/2},\ z(d_i + d_{i'})\}, \quad i, i' \in \mathbf{Z}_{n+1}, \tag{3.3}$$

where

$$\frac{\kappa - \sigma'}{2\kappa - \sigma'} < b' \leq 1, \quad \text{with} \quad 0 < \sigma' < 2 - \sigma,$$

and n is the highest level of resolution of the approximation space. Note the truncation parameter $\varepsilon_{i'i}$ (3.3) changes as i, i' change. This truncation scheme requires computing $\operatorname{dist}(\hat{S}_{k'i'j'}, S_{kij})$ explicitly in order to determine if the element $K_{k'i'j',kij}$ needs to be calculated. Computing such quantities requires significant computational cost. The block truncation scheme which we describe next avoids computing such quantities.

Note that in solving the integral equation (2.1) by the proposed wavelet collocation method, the element $K_{k'i'j',kij}$, $k, k' \in \mathbf{Z}_p$, (i,j), $(i',j') \in \mathbf{U}_n$, can be computed by

$$K_{k'i'j',kij} = J_{T_k} \int_{\Omega} \big\langle \ell_{i'j'}, K\big(T_{k'}(\cdot), T_k(t)\big)\big\rangle\, w_{ij}(t)\, \mathrm{d}t, \tag{3.4}$$

where J_{T_k} is the Jacobian of map T_k. In view of (3.4), when computing the element $K_{k'i'j',kij}$ of $\mathbf{K}_n$, one only needs to consider the corresponding w_{ij} and $\ell_{i'j'}$ defined on Ω. Therefore, we will present our block truncation schemes in terms of w_{ij} and $\ell_{i'j'}$. For $(i,j) \in \mathbf{U}_n$, with $i \geq 2$, there exists a unique pair of $\mathbf{e} \in \mathbf{Z}_4^{i-1}$ and $l \in \mathbf{Z}_9$ such that $j = 9\mu(\mathbf{e}) + l$ and $w_{ij} = \mathcal{T}_\mathbf{e} w_{1l}$. Likewise, for $(i',j') \in \mathbf{U}_n$, with $i' \geq 2$, there exists a unique pair of $\mathbf{e}' \in \mathbf{Z}_4^{i'-1}$ and $l' \in \mathbf{Z}_9$ such that $j' = \mu(\mathbf{e}')r + l'$ and $\ell_{i'j'} = \mathcal{L}_{\mathbf{e}'}\ell_{1l'}$. Recall that the support of w_{ij} is $S_{ij} := \phi_\mathbf{e}(\Omega)$, and the "support" of $\ell_{i'j'}$ is $\hat{S}_{i'j'} := \phi_{\mathbf{e}'}(\Omega)$. The two supports $\hat{S}_{i'j'}$ and S_{ij} are both right triangles transformed from Ω. For S_{ij}, we denote its vertex of right angle by $P_0 = (x_0, y_0)$, and counterclockwise, denote the other two vertices by $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. It is not difficult to find that the centroid of S_{ij} is at

$$(C_x, C_y) := \left(x_0 + \frac{x_1 - x_0}{3},\ y_0 + \frac{x_1 - x_0}{3} \right).$$

We assign a multi-index $\mathbf{q}_{ij} := (q_1, q_2)$ to S_{ij} with

$$q_1 = \left\lfloor \frac{C_x}{4^{(1-i)/2}} \right\rfloor \quad \text{and} \quad q_2 = \left\lfloor \frac{C_y}{4^{(1-i)/2}} \right\rfloor . \tag{3.5}$$

The index $\mathbf{q}_{ij}$ is always associated with the support S_{ij} and this should be clear with the context. To ease the burden of notation, we will use $\mathbf{q}$ instead of $\mathbf{q}_{ij}$. Let $\lambda : \mathbf{Z}_4^{i-1} \mapsto \mathbf{Z}^2_{4^{(i-1)/2}}$ be the function defined by (3.5). Then, we have that $\mathbf{q} = \lambda(\mathbf{e})$. We shall assign a multi-index $\mathbf{q}$ to $\mathbf{e} = (e_0, e_1, \ldots, e_{i-2})$ associated with w_{ij} and $\mathbf{q}'$ to $\mathbf{e}' = (e'_0, e'_1, \ldots, e'_{i'-2})$ associated with $\ell_{i'j'}$ as follows:

(1) If $i \geq i'$, let $\mathbf{e}_c := (e_0, e_1, \ldots, e_{i'-2})$ and

$$\mathbf{q}' = (q'_1, q'_2) = \lambda(\mathbf{e}'), \quad \mathbf{q} = (q_1, q_2) = \lambda(\mathbf{e}_c). \tag{3.6}$$

(2) If $i < i'$, let $\mathbf{e}'_c := (e'_0, e'_1, \ldots, e'_{i-2})$ and

$$\mathbf{q}' = (q'_1, q'_2) = \lambda(\mathbf{e}'_c), \quad \mathbf{q} = (q_1, q_2) = \lambda(\mathbf{e}). \tag{3.7}$$

Note that each element $K_{k'i'j',kij}$ corresponds to a pair of $(\mathbf{q}', \mathbf{q})$. Consequently, we can define a block $\mathbf{K}^{i',i}_{\mathbf{q}'\mathbf{q}}$ by

$$\mathbf{K}^{i',i}_{\mathbf{q}'\mathbf{q}} := \{K_{k'i'j',kij} : k', k \in \mathbf{Z}_p,\ K_{k'i'j',kij} \text{ corresponds to the same pair } (\mathbf{q}', \mathbf{q})\}. \tag{3.8}$$

For convenience of later discussion, we define a submatrix of $\mathbf{E}_n$ by

$$\mathbf{E}_{i'i} := [E_{k'i'j',kij} : \ j' \in \mathbf{Z}_{w(i')}, \ j \in \mathbf{Z}_{w(i)}, \ k, k' \in \mathbf{Z}_p],$$

and similarly, the submatrix $\mathbf{K}_{i'i}$ of $\mathbf{K}_n$ and $\widetilde{\mathbf{K}}_{i'i}$ of $\widetilde{\mathbf{K}}_n$. Hence, we have that

$$\mathbf{K}_{i'i} = \left[\mathbf{K}^{i',i}_{\mathbf{q}'\mathbf{q}} : k', k \in \mathbf{Z}_p, \mathbf{q}', \mathbf{q} \in \mathbf{Z}^2_{4^{(i_0-1)/2}}\right],$$

where $i_0 := \min\{i', i\}$.

Figure 3.1 shows the idea of the block truncation scheme. If the "support" $\widehat{S}_{i'j'}$ of the collocation functional $\ell_{i'j'}$ is in a finer level comparing with the support S_{ij} of the wavelet function w_{ij}, then the support of the same level as that of the wavelet w_{ij} containing $\widehat{S}_{i'j'}$ is considered when $\mathbf{q}'$ is assigned. A similar idea is applied if the support S_{ij} of the wavelet w_{ij} is in a finer level.

Let $S := \phi_{\mathbf{e}_c}(\Omega)$ if $i \geq i'$ and $S := \phi_{\mathbf{e}'_c}(\Omega)$ if $i < i'$. We have the following lemmas.

Lemma 3.1. *Let $K_{k'i'j',kij}$ be an entry of $\mathbf{K}^{i',i}_{\mathbf{q}'\mathbf{q}}$, for (i', j'), $(i, j) \in \mathbf{U}_n$, $k', k \in \mathbf{Z}_p$. Then,*

(3.9)
$$\frac{|\mathbf{q} - \mathbf{q}'| - \sqrt{2}}{4^{(i'-1)/2}} \leq \operatorname{dist}(\hat{S}_{i'j'}, S) \leq \frac{|\mathbf{q} - \mathbf{q}'|}{4^{(i'-1)/2}}, \quad \textit{for} \quad i \geq i',$$

or

$$\frac{|\mathbf{q} - \mathbf{q}'| - \sqrt{2}}{4^{(i-1)/2}} \leq \operatorname{dist}(S_{ij}, S) \leq \frac{|\mathbf{q} - \mathbf{q}'|}{4^{(i-1)/2}}, \quad \textit{for} \quad i < i'.$$

Lemma 3.2. *Let $K_{k'i'j',kij}$ be an entry of $\mathbf{K}^{i',i}_{\mathbf{q}'\mathbf{q}}$, for (i', j'), $(i, j) \in \mathbf{U}_n$, $k', k \in \mathbf{Z}_p$. Then,*

$$\frac{|\mathbf{q} - \mathbf{q}'| - \sqrt{2}}{4^{(i_0-1)/2}} \leq \operatorname{dist}(\hat{S}_{i'j'}, S_{ij}) < \frac{|\mathbf{q} - \mathbf{q}'| + \sqrt{2}}{4^{(i_0-1)/2}}. \tag{3.10}$$

Proof. We present only a proof for the case $i \geq i'$ since the proof for the case $i < i'$ is similar. By (3.6) we have that

$$\mathbf{q} = \lambda(\mathbf{e}_c), \quad \text{with} \quad \mathbf{e}_c = (e_0, \ldots, e_{i'-2}).$$

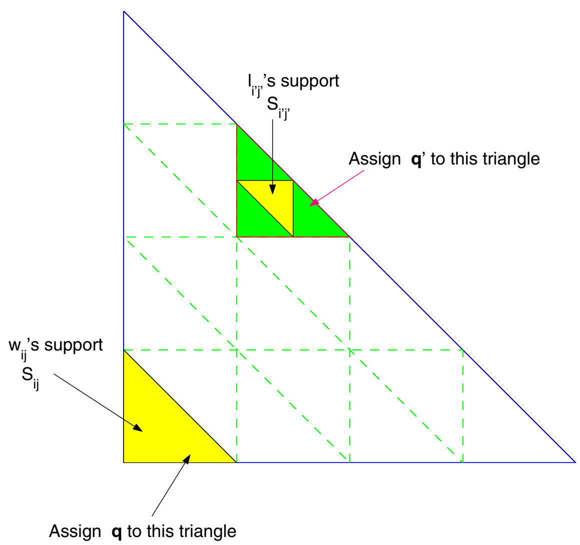


FIGURE 3.1. Block truncation scheme.

Observing that $S_{ij} \subset S = \phi_{\mathbf{e}_c}(\Omega)$, we conclude that

$$\operatorname{dist}(\hat{S}_{i'j'}, S) \leq \operatorname{dist}(\hat{S}_{i'j'}, S_{ij}) < \operatorname{dist}(\hat{S}_{i'j'}, S) + \frac{\sqrt{2}}{4^{(i'-1)/2}}.$$

Applying Lemma 3.1, we have the desired estimate (3.10). $\square$

We are now ready to present the block truncation scheme. Given $\mathbf{r} := [r_{i'i} : r_{i'i} > 0,\ i', i \in \mathbf{Z}_{n+1}]$, the block truncation scheme can be described as

$$\widetilde{\mathbf{K}}_{i',i} = \left[\mathbf{K}^{i',i}_{\mathbf{q}'\mathbf{q}}(\mathbf{r}) : \mathbf{q}', \mathbf{q} \in \mathbf{Z}^2_{4^{(i_0-1)/2}}\right]$$

with

$$\mathbf{K}^{i',i}_{\mathbf{q}'\mathbf{q}}(\mathbf{r}) = \begin{cases} \mathbf{K}^{i',i}_{\mathbf{q}'\mathbf{q}} & |\mathbf{q} - \mathbf{q}'| \leq r_{i'i}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

As in [**11**], we let $\widetilde{\mathcal{K}}_n : \mathbf{F}_n \mapsto \mathbf{F}_n$ be a linear operator relative to the basis $\{w_{k,ij} : (i,j) \in \mathbf{U}_n, k \in \mathbf{Z}_p\}$ and the corresponding wavelet functionals, with $\mathbf{E}_n^{-1}\widetilde{\mathbf{K}}_n$ as its matrix representation. It can be shown as [**11**] that, for $n \geq M$ with M being some positive integer,

$$(\mathcal{I} - \widetilde{\mathcal{K}}_n)\tilde{u}_n = \mathcal{P}_n f \tag{3.12}$$

has a unique solution $\tilde{u}_n$. The next theorem shows that solutions of (3.12) by this block truncation scheme has the same order of convergence and computational complexity as those by adopting the theoretical truncation scheme (3.3), provided that the truncation parameter $\mathbf{r}$ is properly chosen. For a positive integer κ, by $W^{\kappa,\infty}(E)$ we denote the set of all functions v on E such that $D^\mu v \in \mathbf{X}$ and we define the norm $\|v\|_{\kappa,\infty} := \max\{\|D^\mu v\|_\infty : |\mu| \leq \kappa\}$ on $W^{\kappa,\infty}(E)$. Also, we use $\mathcal{N}(\mathbf{A}_n)$ for the number of nonzero entries in matrix $\mathbf{A}_n$.

Theorem 3.3. *Let $u \in W^{\kappa,\infty}(E)$. For some $z' > 0$ and $z > 0$, choose*

(3.13)
$$r_{i'i} := \max\{z' 4^{[b'(n-i')+(i'-i-1)]/2},\ z(4^{(-i+i')/2}+1)\}+\sqrt{2}, \quad \text{for} \quad i' < i$$

and

$$r_{i'i} := \max\{z' 4^{[b'(n-i')-1]/2}, z(4^{(i-i')/2}+1)\}+\sqrt{2}, \quad \text{for} \quad i' \geq i. \tag{3.14}$$

Then there exists a positive constant c such that

$$\|u - \tilde{u}_n\|_\infty \leq c f(n)^{-\kappa/2} \log f(n) \|u\|_{\kappa,\infty} \tag{3.15}$$

and

$$\mathcal{N}(\mathbf{E}_n - \tilde{\mathbf{K}}_n) = \mathcal{O}\left(f(n) \log f(n)\right). \tag{3.16}$$

Proof. It was proved in [**11**] that, if the truncation parameter $\varepsilon_{i'i}$ is chosen by (3.3), and scheme (3.2) is applied, then estimate (3.15) holds. It suffices to show that, when the block truncation scheme (3.11) is used with the parameters $r_{i'i}$ satisfying (3.13) and (3.14),

$$\widetilde{K}_{k'i'j',kij} := K_{k'i'j',kij} \quad \text{if} \quad \text{dist}\,(\hat{S}_{k',i'j'}, S_{k,ij}) \leq \varepsilon_{i'i}. \tag{3.17}$$

When dist $(\hat{S}_{k',i'j'}, S_{k,ij}) \le \varepsilon_{i'i}$, we easily obtain that

$$|\mathbf{q}-\mathbf{q}'| \le \sqrt{2} + 4^{(i_0-1)/2}\varepsilon_{i'i}$$

by using Lemma 3.2. We choose the righthanded side of the above formula to be $r_{i'i}$ by using (3.3) and (2.12). Thus, (3.17) holds when the block truncation scheme is performed and hence (3.15) follows.

It remains to prove (3.16). To this end, we set

$$\begin{aligned}
\Lambda_1 &:= \{(i',j'),(i,j) \in \mathbf{U}_n : |\mathbf{q}'-\mathbf{q}| \le r_{i'i}\},\\
\Lambda_2 &:= \{(i',j'),(i,j) \in \mathbf{U}_n : \operatorname{dist}(\hat{S}_{i'j'}, S_{ij}) \le \varepsilon_{i'i}\},\\
\Lambda_3 &:= \{(i',j'),(i,j) \in \mathbf{U}_n : \varepsilon_{i'i} < \operatorname{dist}(\hat{S}_{i'j'}, S_{ij}) \text{ and } |\mathbf{q}'-\mathbf{q}| \le r_{i'i}\}.
\end{aligned}$$

By Lemma 3.2, the inequality $|\mathbf{q}'-\mathbf{q}| \le r_{i'i} - \sqrt{2}$ implies that dist $(\hat{S}_{i'j'}, S_{ij}) \le \varepsilon_{i'i}$. Thus, we have that

$$\operatorname{card}(\Lambda_1) = \operatorname{card}(\Lambda_2) + \operatorname{card}(\Lambda_3).$$

It is proved in [**11**] that

$$\operatorname{card}(\Lambda_2) = \mathcal{O}(f(n)\log f(n)). \tag{3.18}$$

Now for any fixed $i', i \in \mathbf{Z}_{n+1}$, there are at most $c4^{i_0-1}$ sub-blocks $\mathbf{K}^{i',i}_{\mathbf{q}\mathbf{q}'}$ in $\mathbf{K}_{i',i}$ for some positive constant c, satisfying both $\varepsilon_{i'i} <$ dist $(\hat{S}_{i'j'}, S_{ij})$ and $|\mathbf{q}'-\mathbf{q}| \le r_{i'i}$. While each sub-block $\mathbf{K}^{i',i}_{\mathbf{q}'\mathbf{q}}$ has at most $2\cdot 9^2 \cdot p \cdot 4^{|i-i'|}$ entries. Therefore,

$$\operatorname{card}(\Lambda_3) \le 4\cdot 9^2 \cdot c \sum_{i\in\mathbf{Z}_{n+1}} \sum_{i'\in\mathbf{Z}_{i+1}} 4^{|i-i'|+i_0-1} = \mathcal{O}\big(f(n)\log f(n)\big).$$

This with (3.18) establishes (3.16). $\square$

Remark. When $i \ge i'$, for any given triangle $\hat{S}_{i'j'}$ associated with the index $\mathbf{q}'$, the block truncation scheme is expected to catch any triangle S associated with the index $\mathbf{q}$ surrounding $\hat{S}_{i'j'}$. On the other hand, we do not want too many triangles to be caught by the scheme. A simple analysis shows that for triangles surrounding $\hat{S}_{i'j'}$, $\sqrt{2} \le |\mathbf{q}'-\mathbf{q}| < \sqrt{3}$

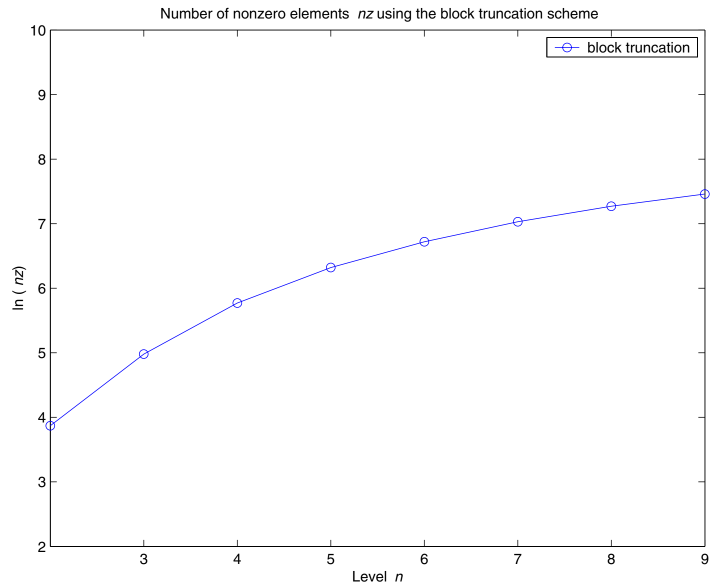


FIGURE 3.2. Number of nonzero elements vs. level in a logarithmic scale when using the block truncation scheme with $r_{i'i} = \sqrt{3}$.

and hence $\sqrt{2} < r_{i'i} \leq \sqrt{3}$, $i', i \in \mathbf{Z}_{n+1}$ would be a very good choice. It remains true for the case when $i < i'$.

Figure 3.2 depicts the number of nonzero elements vs. level in a logarithmic scale when using the block truncation scheme with $r_{i'i} = \sqrt{3}$. The matrix $\mathbf{K}_n$ after compression is quite sparse when n is relatively large. (See Figure 3.3. The sparse matrix is computed for the integral equation (2.1) with $E = \Omega$ and K is given by (2.26).) Special sparse matrix storage is used when doing the actual computation. Not only does this speed up computing, but also this dramatically saves memory. In fact when n is relatively large, we are forced to use the sparse storage scheme to store the matrix. On our computer `cluster.math.wvu.edu`, with AMD Athlon(tm) MP Processor 1600+ and 1 Gigabyte of RAM, if serial codes are used, we have to use sparse storage when $n \geq 6$. In Figure 3.4, we plot both $\widetilde{\mathbf{K}}_3$ and $\mathbf{K}_3$ in Example 5.4.

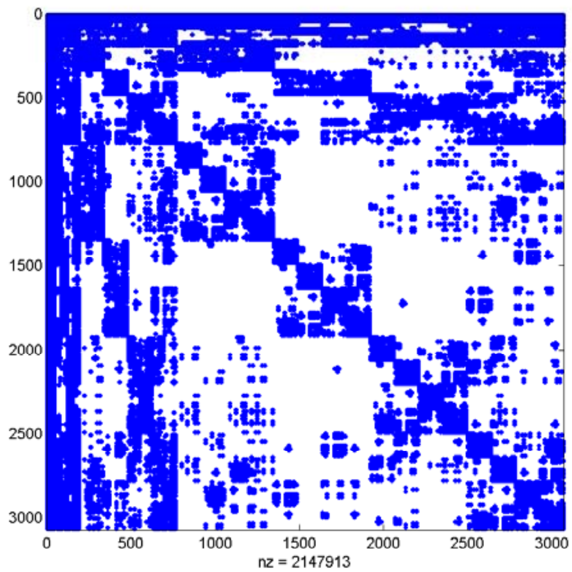


FIGURE 3.3. The sparse matrix $\mathbf{K}_5$ after compression.

4. A numerical quadrature rule and its error analysis. All entries of the matrix $\widetilde{\mathbf{K}}_n$ need to be computed numerically by evaluating weakly singular integrals. The numerical computation of these integrals is the most costly task in the numerical solutions of the integral equations. Therefore, it is of great importance to design a quadrature rule that leads to fast computation.

In this section, we describe a numerical quadrature rule to compute the entries of $\widetilde{\mathbf{K}}_n$ and consider the effect of errors introduced by the numerical integrations. For this purpose, we assume that for any lattice point $\mu \in \mathbf{N}_0^2$, the kernel function $K(s,t)$, $s,t \in \mathbf{R}^2$ satisfies the following condition

$$|D_t^\mu K(s,t)| \le c|s-t|^{-\sigma-|\mu|}, \quad s \ne t. \tag{4.1}$$

Recall that, in solving the integral equation (2.1) by the proposed wavelet collocation method, the element $K_{k'i'j',kij}$, $k,k' \in \mathbf{Z}_p$, $(i,j),(i',j') \in \mathbf{U}_n$ is computed by using (3.4), where the wavelet function w_{ij} is a piecewise polynomial supported only on a small triangle

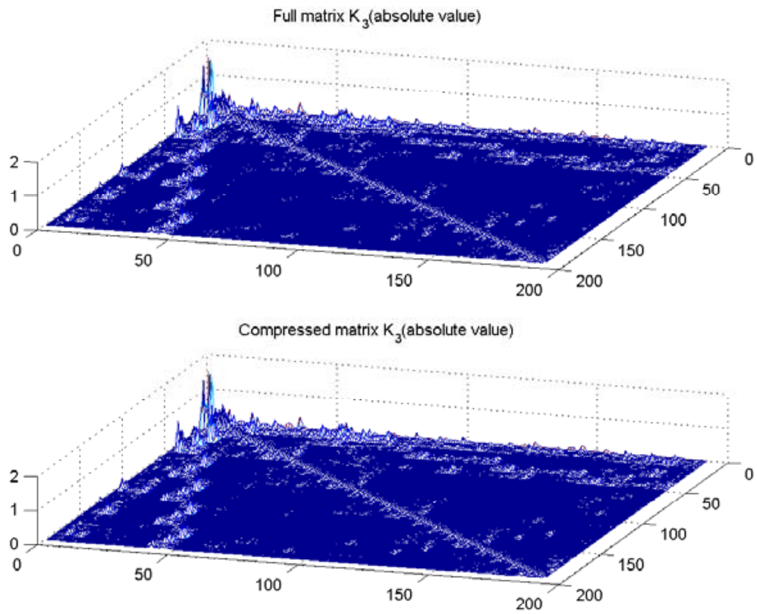


FIGURE 3.4. The full matrix $\mathbf{K}_3$ and the compressed matrix $\widetilde{\mathbf{K}}_3$.

$S_{ij} \subset \Omega$. We now represent the set S_{ij} in the form $\{(x, y) : a_1 \leq x \leq a_1',\ a_2 \leq y \leq a_2' - x\}$ with $0 \leq a_1 < a_1' \leq 1$, $0 \leq a_2, a_2' \leq 1$. Thus, each entry of matrix $\mathbf{K}_n$ in (2.23) involves an integral in the form

$$I := \int_{a_1}^{a_1'} \int_{a_2}^{-x+a_2'} h_{ij}(s,t)\, dy\, dx \tag{4.2}$$

with

$$h_{ij}(s,t) := K(s,t)w_{ij}(t),$$

where $s, t \in \Omega$, $s := (\xi, \zeta)$, $t := (x, y)$. For different entry of $\mathbf{K}_n$, their a_1, a_1', a_2, a_2' and s are different. Nevertheless, the quadrature rule developed in this section provides a uniform precision for all the entries of $\mathbf{K}_n$.

The basic idea of our quadrature method is applying Fubini's theorem to the above integral. For each single integral, we apply the composite

Gaussian quadrature rule which was developed in [**14**] by using an idea from [**29**] and [**21**]. To this end, we rewrite (4.2) as

$$I = \int_{a_1}^{a_1'} h(x)\,\mathrm{d}x, \quad \text{with} \quad h(x) := \int_{a_2}^{-x+a_2'} h_{ij}(t)\,\mathrm{d}y. \tag{4.3}$$

We now describe a nonuniform partition of the interval $[a_1, a_1')$. Set

$$\hat{t}_0 = 0, \quad \hat{t}_e = \theta^{m-e}, \quad e-1 \in \mathbf{Z}_m, \quad \theta \in (0,1),$$

where m is a prescribed integer (usually it takes a small value, say $m \leq 10$). According to [**27**], $0.1 < \theta < 0.2$ is optimal. With this partition of $[0,1]$, we choose two collections of nodes with $\xi \in [0,1]$ as the shifting parameter:

$$\pi_x^r := \{t_e^r = \xi + \hat{t}_e,\ e \in \mathbf{Z}_{m+1}\}, \quad \text{and} \quad \pi_x^l := \{t_e^l = \xi - \hat{t}_e,\ e \in \mathbf{Z}_{m+1}\}.$$

Note that $t_0^r = t_0^l = \xi$, $t_m^r = 1 + \xi$ and $t_m^l = \xi - 1$. We denote by $\pi_x(w_{ij})$ the set of x-coordinates of the corners of the triangular shaped subregions of the support S_{ij} associated with the function w_{ij}. Rearrange the elements of

$$\pi_x(w_{ij}) \cup \pi_x^r \cup \pi_x^l \cup \{a_1, a_1'\} \cap [a_1, a_1'] \tag{4.4}$$

in an increasing order and name them by $a_1 = t_0 < t_1 < \cdots < t_{m'} = a_1'$ and set

$$\widehat{J}_\alpha := [t_\alpha, t_{\alpha+1}), \quad \alpha \in \mathbf{Z}_{m'}. \tag{4.5}$$

To compute (4.3), we define a piecewise polynomial $S_x(h)$ on $[a_1, a_1']$ by the following rule: $S_x(h)$ is the polynomial which interpolates h at the zeros of the Legendre polynomial of total degree k_e on $\widehat{J}_\alpha$, where

$$k_e := e + 2 \quad \text{if} \quad \widehat{J}_\alpha \subset [t_e^r, t_{e+1}^r) \quad \text{or} \quad \widehat{J}_\alpha \subset [t_{e+1}^l, t_e^l), \quad e \in \mathbf{Z}_m. \tag{4.6}$$

Let

$$t_\alpha^+ := \frac{t_{\alpha+1} + t_\alpha}{2}, \qquad t_\alpha^- := \frac{t_{\alpha+1} - t_\alpha}{2}, \quad \alpha \in \mathbf{Z}_{m'},$$

and let $\tau_{l,e}$, $l \in \mathbf{Z}_{k_e}$ denote the zeros of the Legendre polynomial of degree k_e on $(-1, 1)$. Using these parameters we define the quadrature nodes and weights by

$$\hat{\tau}_{l,e}^{\alpha} = t_{\alpha}^{-}\tau_{l,e} + t_{\alpha}^{+}$$

and

$$\hat{w}_{l,e}^{\alpha} = \int_{t_\alpha}^{t_{\alpha+1}} \prod_{l' \neq l, l' \in \mathbf{Z}_{k_e}} \frac{x - \hat{\tau}_{l',e}^{\alpha}}{\hat{\tau}_{l,e}^{\alpha} - \hat{\tau}_{l',e}^{\alpha}}\, dx, \quad \alpha \in \mathbf{Z}_{m'}, \quad l \in \mathbf{Z}_{k_e}. \tag{4.7}$$

For simplicity, we will drop the subscript e in the notation $\hat{\tau}_{l,e}^{\alpha}$ and $\hat{w}_{l,e}^{\alpha}$. That is, we use $\hat{\tau}_l^{\alpha}$ and $\hat{w}_l^{\alpha}$ for $\hat{\tau}_{l,e}^{\alpha}$ and $\hat{w}_{l,e}^{\alpha}$, respectively. We then have the quadrature rule obtained by replacing the integrand h by $S_x(h)$ in (4.3):

$$\hat{I}_m = \sum_{\alpha=0}^{m'-1} \sum_{l=0}^{k_e-1} \hat{w}_l^{\alpha} h(\hat{\tau}_l^{\alpha}). \tag{4.8}$$

To compute $\hat{I}_m$, we need to evaluate $h(\hat{\tau}_l^{\alpha})$. Now

$$h(\hat{\tau}_l^{\alpha}) = \int_{a_2}^{a_2''} h_{ij}\big(\hat{\tau}_l^{\alpha}, y\big)\, \mathrm{d}y, \tag{4.9}$$

with $a_2'' = a_2' - \hat{\tau}_l^{\alpha}$.

We compute (4.9) using the same idea as to compute (4.3). Correspondingly, we have

$$\pi_y^u := \{s_j^u = \zeta + \hat{t}_e,\ e \in \mathbf{Z}_{m+1}\}, \quad \text{and} \quad \pi_y^d := \{s_e^d = \zeta - \hat{t}_e,\ e \in \mathbf{Z}_{m+1}\}.$$

We denote by $\pi_y(w_{ij})$ the set of y-coordinates of the intersection points of the vertical line $x = \hat{\tau}_l^{\alpha}$ with the boundaries of the triangular shaped subregions of the support S_{ij} of the function w_{ij}. Rearrange the elements of

$$\pi_y(w_{ij}) \cup \pi_y^u \cup \pi_y^d \cup \{a_2, a_2''\} \cap [a_2, a_2''] \tag{4.10}$$

in an increasing order and name them by $a_2 = s_0 < s_1 < \cdots < s_{m''} = a_2''$ to form a partition of $[a_2, a_2'']$ with ζ as the shifting parameter and set

$$J_\beta := [s_\beta, s_{\beta+1}) : \beta \in \mathbf{Z}_{m''}. \tag{4.11}$$

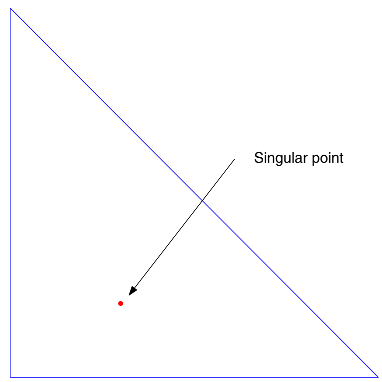


FIGURE 4.1. Domain of integration for Example 4.1.

We define the piecewise polynomial $S_y(h_{ij})$ which interpolates h_{ij} on $[a_2, a_2'']$ such that on $[s_\beta, s_{\beta+1}), \beta \in \mathbf{Z}_{m''}$, $S_y(h_{ij})$ is a polynomial of degree k_e, which can be defined similarly as in (4.6), in the same way as we define $S_x(h)$. We use τ_l^α and w_l^α to denote the quadrature nodes and weights in the y direction, respectively. In (4.9), we replace $h_{ij}(\hat{\tau}_l^\alpha, \cdot)$ by $S_y(h_{ij})$, and this gives us a quadrature method for computing $h(\hat{\tau}_l^\alpha)$. The resulting quadrature value of I after approximating $h_{ij}(\hat{\tau}_l^\alpha, \cdot)$ by $S_y(h_{ij})$ in (4.8) is denoted by I_m.

We next present a numerical example that demonstrates the efficiency of the quadrature rule. The order of convergence is computed by $|I_m - I_{m+1}|/|I_{m+1} - I_{m+2}|$ for $m \geq 1$. We note that

$$\frac{|I_m - I_{m+1}|}{|I_{m+1} - I_{m+2}|} \approx c,$$

which indicates that the quadrature error is decreasing exponentially, cf. Lemma 4.7.

Example 4.1. Consider the integral

$$I = \int_0^1 \int_0^{1-x} \frac{x + y + 1}{\sqrt{(x - 0.3)^2 + (y - 0.2)^2}} \, \mathrm{d}y \, \mathrm{d}x.$$

The singular point $(0.3, 0.2)$ is inside the domain Ω of integration, see Figure 4.1, and $\sigma = 1$ for the given integrand. We tabulate our numerical results in Table 4.1. It is clearly seen that the quadrature rule gives accurate results, and the order of convergence is also verified. $\theta = 0.17$ is used in this example and all other examples in this paper.

TABLE 4.1. Numerical quadrature results for Example 4.1.

m	Numerical I_m	$\lvert I_m - I_{m+1}\rvert$	Order of convergence
1	3.42569712973761929		
2	3.64089019720423062	2.15e-01	
3	3.70794909693406188	6.71e-02	3.21
4	3.71993130161368305	1.20e-02	5.60
5	3.72198907967352980	2.06e-03	5.82
6	3.72234233204242752	3.53e-04	5.83
7	3.72240262535117647	6.03e-05	5.86
8	3.72241283633917743	1.02e-05	5.90
9	3.72241457684138661	1.74e-06	5.87
10	3.72241487370465405	2.97e-07	5.86
11	3.72241492439826888	5.07e-08	5.86
12	3.72241493312499959	8.73e-09	5.81

When the elements of the matrix $\widetilde{\mathbf{K}}_n$ are computed by using the numerical quadrature developed above, new quadrature errors are introduced into the error of the numerical solution. Next we analyze the quadrature error and design an error control method so that the convergence order of the numerical solution is preserved with almost linear computational cost (measured in terms of the number of quadrature nodes).

Recall that the compressed matrix $\widetilde{\mathbf{K}}_n$ is obtained by the block truncation strategy defined with the truncation parameters $r_{i'i}$, $i', i \in \mathbf{Z}_{n+1}$. For the given truncation parameter $r_{i'i}$, we introduce an index set for $k', k \in \mathbf{Z}_p$, $(i', j') \in \mathbf{U}_n$,

$$\mathbf{Z}_{k'i'j',ki} := \{j : j \in \mathbf{Z}_{w(i)}, K_{k'i'j',kij} \text{ is a nonzero entry of } \mathbf{K}_{i'i}^{\mathbf{q}'\mathbf{q}} \text{ and } |\mathbf{q} - \mathbf{q}'| \le r_{i'i}\}.$$

That is, it is the set of indices of elements in the $k'i'j'$ row of the block $\widetilde{\mathbf{K}}_{i'i}$ such that the associated wavelet w_{kij} is supported on $\triangle_k$. We also define for $\ell \in \mathbf{Z}_9$,

$$\mathbf{Z}^{\ell}_{k'i'j',ki} := \{j \in \mathbf{Z}_{k'i'j',ki} : j = 9\mu(\mathbf{e}) + \ell\}.$$

Observe that $\mathbf{Z}^{\ell}_{k'i'j',ki} \subset \mathbf{Z}_{k'i'j',ki}$. For $j_1,\ j_2 \in \mathbf{Z}^{\ell}_{k'i'j',ki}$ with $j_1 \neq j_2$,

$$\text{meas}\,(\text{supp}\,(w_{k,ij_1}) \cap \text{supp}\,(w_{k,ij_2})) = 0,$$

and for any $\ell \in \mathbf{Z}_9$,

$$T_k^{-1}\Bigg(\bigcup_{j\in \mathbf{Z}^{\ell}_{k'i'j',ki}} \text{supp}\,(w_{k,ij})\Bigg) \subset \Omega.$$

We now define for $\ell \in \mathbf{Z}_9$ and $(i',j') \in \mathbf{U}_n$

$$\bar{w}_{k'i'j',ki\ell}(t) := \begin{cases} w_{kij}(t) & \text{if } t \in \text{int}\,(\text{supp}\,(w_{kij})) \text{ for some } j \in \mathbf{Z}^{\ell}_{k'i'j',ki} \\ 0 & \text{otherwise} \end{cases}$$

and set

$$\bar{h}_{k'i'j',ki\ell}(s,t) := K(s,t)\bar{w}_{k'i'j',ki\ell}(t).$$

Note that in the block $\widetilde{\mathbf{K}}_{i'i}$, for fixed $k'i'j'$, there are p such functions as $k \in \mathbf{Z}_p$. When the entry $K_{k'i'j',kij}$ of $\widetilde{\mathbf{K}}_n$ is computed, it is calculated according to (3.4) through change of variables. We therefore write $\bar{w}_{i'j',i\ell}(t)$ and $\bar{h}_{i'j',i\ell}(s,t)$ for $\bar{w}_{k'i'j',ki\ell}(t)$ and $\bar{h}_{k'i'j',ki\ell}(s,t)$, respectively, after a change of variables.

We next analyze the quadrature error of the proposed integration method. Let

$$I(\bar{h}_{i'j',i\ell}) := \int_0^1 \int_0^{1-x} \bar{h}_{i'j',i\ell}(s,t)\,dy\,dx,$$

$I_m(\bar{h}_{i'j',i\ell})$ be the approximate quadrature value of $I(\bar{h}_{i'j',i\ell})$ obtained by the proposed integration method, and

$$E := I(\bar{h}_{i'j',i\ell}) - I_m(\bar{h}_{i'j',i\ell}).$$

We need the following technical lemmas. Since the proof of Lemma 4.2 is similar to that of Lemma 4.1 in [**14**], we present it without a proof.

Lemma 4.2. *Let $\{k_l = l+2 : l \in \mathbf{N}_0\}$ be an increasing sequence of positive integers and $l' \in \mathbf{N}_0$. Then, for a fixed positive number q, there exists a positive constant c such that*

$$\frac{2^{2qk_l}}{(2k_l)!\theta^{(2-\sigma)\hat{l}+2k_l(1+l-\hat{l})}} \le c,$$

where $\hat{l} = \max\{l, l'\}$.

For simplicity, we introduce the index sets associated with the subintervals for the x variable

$$\Gamma_\alpha := \{e \in \mathbf{Z}_{m'} : \widehat{J}_e \subseteq [t^r_\alpha, t^r_{\alpha+1}] \text{ or } \widehat{J}_e \subseteq [t^l_\alpha, t^l_{\alpha+1}]\},$$

$$A := \{\alpha \in \mathbf{Z}_m : (\widehat{J}_e \cap [t^r_\alpha, t^r_{\alpha+1}]) \subset [0,1] \text{ or } (\widehat{J}_e \cap [t^l_\alpha, t^l_{\alpha+1}]) \subset [0,1],\ e \in \mathbf{Z}_{m'}\},$$

and

$$\Gamma := \{(e,\alpha) : e \in \Gamma_\alpha,\ \alpha \in A\}.$$

Likewise, we define the index sets associated with the subintervals for the y variable

$$\Lambda_\beta := \{e' \in \mathbf{Z}_{m''} : J_{e'} \subseteq [s^u_\beta, s^u_{\beta+1}] \text{ or } J_{e'} \subseteq [s^d_\beta, s^d_{\beta+1}]\},$$

$$B(x) := \{\beta \in \mathbf{Z}_{m''} : (J_{e'} \cap [s^u_\beta, s^u_{\beta+1}]) \subset [0, 1-x] \text{ or } (J_{e'} \cap [s^d_\beta, s^d_{\beta+1}]) \subset [0, 1-x],\ x \in (0,1),\ e' \in \mathbf{Z}_{m''}\}$$

and

$$\Lambda(x) = \{(e',\beta) : e' \in \Lambda_\beta :,\ \beta \in B(x)\}, \quad x \in (0,1).$$

We now return to the estimate of $I(\bar{h}_{i'j',i\ell})$. For notational convenience, for a fixed s, we let

$$g(x) = \int_0^{1-x} \bar{h}_{i'j',i\ell}(s, t(x,y))\, dy, \quad \text{for} \quad x \in [0,1]$$

and we will abuse the notation by writing

$$\bar{h}_{i'j',i\ell}(x,y) \quad \text{for} \quad \bar{h}_{i'j',i\ell}(s,t(x,y)).$$

Hence, we have that

$$I(\bar{h}_{i'j',i\ell}) = \int_0^1 g(x)\,dx = \sum_{(e,\alpha)\in\Gamma} \int_{\hat{J}_e} g(x)\,dx.$$

Applying the Gaussian quadrature rule to the integral, we have that

$$\int_{\hat{J}_e} g(x)\,dx = \sum_{l\in\mathbf{Z}_{k_\alpha}} \hat{w}_l^e g(\hat{\tau}_l^e) + \frac{1}{(2k_\alpha)!}\int_{\hat{J}_e} Q_{e,\alpha}(x) D_x^{2k_\alpha} g(x)\,dx,$$

where,

$$Q_{e,\alpha}(x) := \prod_{l\in\mathbf{Z}_{k_\alpha}} (x-\hat{\tau}_l^e)^2, \quad x\in\widehat{J}_e.$$

Setting

$$I_1 := \sum_{(e,\alpha)\in\Gamma}\sum_{l\in\mathbf{Z}_{k_\alpha}} \hat{w}_l^e g(\hat{\tau}_l^e)$$

and

(4.12)

$$E_1 := \sum_{(e,\alpha)\in\Gamma} \frac{1}{(2k_\alpha)!}\int_{\hat{J}_e} D_x^{2k_\alpha}\left(\int_0^{1-x} \bar{h}_{i'j',i\ell}(x,y)\,dy\right) Q_{e,\alpha}(x)\,dx,$$

we have that

$$I(\bar{h}_{i'j',i\ell}) = I_1 + E_1.$$

Next we compute

$$g(\hat{\tau}_l^e) = \int_0^{1-\hat{\tau}_l^e} \bar{h}_{i'j',i\ell}(\hat{\tau}_l^e,y)\,dy.$$

To this end, by applying the composite Gaussian quadrature rule with the nonuniform partition of $[0,1-\hat{\tau}_l^e]$ and using interpolating polynomials, we observe that

$$g(\hat{\tau}_l^e) = \sum_{(e',\beta)\in\Lambda(\hat{\tau}_l^e)}\Bigg[\sum_{l'\in\mathbf{Z}_{k_\beta}} w_{l'}^{e'}\bar{h}_{i'j',i\ell}(\hat{\tau}_l^e,\tau_{l'}^{e'}) + \frac{1}{(2k_\beta)!}\int_{J_{e'}} P_{e',\beta}(y) D_y^{2k_\beta}\bar{h}_{i'j',i\ell}(\hat{\tau}_l^e,y)\,dy\Bigg],$$

where

$$P_{e',\beta}(y) := \prod_{l'\in\mathbf{Z}_{k_\beta}} (y-\tau_{l'}^{e'})^2, \quad y \in J_{e'}.$$

Hence

$$I_1 = I_m + E_2,$$

with

$$I_m := \sum_{(e,\alpha)\in\Gamma} \sum_{l\in\mathbf{Z}_{k_\alpha}} \sum_{(e',\beta)\in\Lambda(\hat{\tau}_l^e)} \sum_{l'\in\mathbf{Z}_{k_\beta}} \hat{w}_l^e w_{l'}^{e'} \bar{h}_{i'j',i\ell}(\hat{\tau}_l^e, \tau_{l'}^{e'})$$

and

$$E_2 = \sum_{(e,\alpha)\in\Gamma} \sum_{l\in\mathbf{Z}_{k_\alpha}} \hat{w}_l^e \Bigg(\sum_{(e',\beta)\in\Lambda(\hat{\tau}_l^e)} \frac{1}{(2k_\beta)!} \times \int_{J_{e'}} P_{e',\beta}(y) D_y^{2k_\beta} \bar{h}_{i'j',i\ell}(\hat{\tau}_l^e, y)\, dy \Bigg).$$

To estimate E_2, we observe that E_2 is in fact the quadrature formula in the x direction for the integral

(4.13)

$$E_3 := \sum_{(e,\alpha)\in\Gamma} \int_{\hat{J}_e} \Bigg(\sum_{(e',\beta)\in\Lambda(x)} \frac{1}{(2k_\beta)!} \int_{J_{e'}} P_{e',\beta}(y) D_y^{2k_\beta} \bar{h}_{i'j',i\ell}(x, y)\, dy \Bigg)\, dx.$$

with the error

$$E_4 := \sum_{(e,\alpha)\in\Gamma} \frac{1}{(2k_\alpha)!} \int_{\hat{J}_e} D_x^{2k_\alpha} \Bigg[\sum_{(e',\beta)\in\Lambda(x)} \frac{1}{(2k_\beta)!} \times \int_{J_{e'}} P_{e',\beta}(y) D_y^{2k_\beta} \bar{h}_{i'j',i\ell}(x, y)\, dy \Bigg] Q_{e,\alpha}(x)\, dx.$$

Hence, we conclude that

$$E_2 = E_3 - E_4.$$

We next estimate E_1, E_3 and E_4. For convenience, let

$$\nu = \left[\left(d_i + d_{i'} + \frac{r_{i'i} + \sqrt{2}}{4^{(i_0-1)/2}} \right) 4^{(i-1)/2} \right].$$

We first estimate E_3.

Lemma 4.3. *The following estimate holds*:

$$|E_3| \leq c\theta^{m(2-\sigma)}\nu^\kappa.$$

Proof. Recall that

$$P_{e',\beta}(y) \leq \gamma(\beta), \quad \text{for} \quad e' \in \Lambda_\beta, \quad y \in J_{e'}$$

where

$$\gamma(\beta) := \theta^{2k_\beta(m-\beta-1)}(1-\theta)^{2k_\beta}.$$

From (4.13), we obtain that

$$|E_3| \leq \sum_{(e,\alpha)\in\Gamma} \sum_{\beta\in B(0)} \frac{\gamma(\beta)}{(2k_\beta)!} \int_{\hat{J}_e} \left(\sum_{e'\in\Lambda_\beta} \int_{J_{e'}} |D_y^{2k_\beta} \bar{h}_{i'j',i\ell}(x,y)|\, dy \right) dx,$$

where we have used $B(x) \subset B(0)$ for $0 \leq x \leq 1$. Note that $w_{ij}(t) = \mathcal{T}_{\mathbf{e}} w_{1\ell}(t)$ with $\mathbf{e} \in \mathbf{Z}_4^{i-1}$ and $j = 9\mu(\mathbf{e}) + \ell$ and that the wavelet function w_{ij} is a piecewise polynomial of order $\kappa = 2$. Denoting the set of points of discontinuity of w_{ij} by $\pi(w_{ij})$, the set of points of discontinuity of $\pi(\bar{h}_{i'j',i\ell})$ is

$$\pi(\bar{h}_{i'j',i\ell}) = \cup\{\pi(w_{ij}) : j \in \mathbf{Z}^\ell_{k'i'j',ki}\}.$$

For $t = (x,y) \in \operatorname{supp}(\bar{h}_{i'j',i\ell}) \setminus (\{s\} \cup \pi(\bar{h}_{i'j',i\ell}))$, using assumption (4.1) we have that

$$|D_y^{2k_\beta} \bar{h}_{i'j',i\ell}(s,t)| \leq c2^{2k_\beta}|s-t|^{-(\sigma+2k_\beta)} \sum_{b\in\mathbf{Z}_\kappa} |s-t|^b 4^{b(i-1)/2}.$$

Noting when $\bar{h}_{i'j',i\ell}(s,t) \neq 0$,

$$|s-t| \leq d_i + d_{i'} + \frac{r_{i'i} + \sqrt{2}}{4^{(i_0-1)/2}}.$$

Observing that the set $B(0)$ is independent of x, we thus have that

(4.14)
$$
\begin{aligned}
|E_3| &\le c\sum_{\beta\in B(0)}\frac{2^{2k_\beta}\nu^\kappa}{(2k_\beta)!}\gamma(\beta)\sum_{(e,\alpha)\in\Gamma}\int_{\hat{J}_e}\left(\sum_{e'\in\Lambda_\beta}\int_{J_{e'}}|s-t|^{-(\sigma+2k_\beta)}\,dy\right)dx\\
&= c\sum_{\beta\in B(0)}\frac{2^{2k_\beta}\nu^\kappa}{(2k_\beta)!}\gamma(\beta)\int_0^1\left(\sum_{e'\in\Lambda_\beta}\int_{J_{e'}}|s-t|^{-(\sigma+2k_\beta)}\,dy\right)dx.
\end{aligned}
$$

Introduce two rectangular domains

$$\widetilde{\square}_1 := [0,1]\times[s^u_\beta, s^u_{\beta+1}], \qquad \widetilde{\square}_2 := [0,1]\times[s^d_\beta, s^d_{\beta+1}],$$

and observe that

$$\bigcup_{e'\in\Lambda_\beta}([0,1]\times J_{e'}) \subset (\widetilde{\square}_1\cup\widetilde{\square}_2).$$

For $l=1,2$, let

$$E_{3,l} = \sum_{\beta\in B(0)}\frac{2^{2k_\beta}\nu^\kappa}{(2k_\beta)!}\gamma(\beta)\int_{\widetilde{\square}_l}|s-t|^{-(\sigma+2k_\beta)}\,dy\,dx.$$

We then have that

$$|E_3| \le c(E_{3,1}+E_{3,2}).$$

Since

$$
\begin{aligned}
&\int_{\widetilde{\square}_l}|s-t|^{-(\sigma+2k_\beta)}\,dy\,dx\\
&\qquad\le \int_0^{2\pi}\int_{\theta^{m-\beta}}^{\sqrt{2}} r^{1-\sigma-2k_\beta}\,dr\,d\vartheta \le c\theta^{(m-\beta)(2-\sigma-2k_\beta)},
\end{aligned}
$$

we conclude that

$$E_{3,l} \le c\sum_{\beta\in B(0)}\frac{2^{2k_\beta}(1-\theta)^{2k_\beta}\theta^{m(2-\sigma)}}{(2k_\beta)!\theta^{(2-\sigma)\beta+2k_\beta}}\nu^\kappa$$

On the other hand, applying Lemma 4.2 with $q=1$ yields

$$E_{3,l} \le c\theta^{m(2-\sigma)}\nu^\kappa.$$

Combining the above two estimates gives

$$|E_3| \leq c\theta^{m(2-\sigma)}\nu^{\kappa},$$

proving the lemma. $\square$

To estimate E_1, we need the next simple fact, which might be proved by induction.

Lemma 4.4. *Let $\tilde{n} \geq 1$ be an integer. Suppose that $f \in C^{\tilde{n}}([0,1] \times [0,1])$. Then*

$$D_x^{\tilde{n}} \int_0^{1-x} f(x,y)\,dy = \int_0^{1-x} D_1^{\tilde{n}} f(x,y)\,dy - \sum_{j=0}^{\tilde{n}-1} \sum_{k=0}^{j} \binom{j}{k} (-1)^k D_1^{\tilde{n}-k-1} D_2^k f(x,1-x),$$

where $D_j^k f$ denotes the kth partial derivative of f with respect to the jth variable.

Lemma 4.5. *The following estimate holds*:

$$|E_1| \leq c\theta^{m(2-\sigma)}\nu^{\kappa}.$$

Proof. Using Lemma 4.4, we obtain that

$$\left| D_x^{2k_\alpha} \left(\int_0^{1-x} \bar{h}_{i'j',i\ell}(x,y)\,dy \right) \right| \leq \int_0^{1-x} |D_x^{2k_\alpha} \bar{h}_{i'j',i\ell}(x,y)|\,dy + \sum_{\tilde{j}=0}^{2k_\alpha-1} \sum_{\tilde{k}=0}^{\tilde{j}} \binom{\tilde{j}}{\tilde{k}} |D_1^{2k_\alpha-1-\tilde{k}} D_2^{\tilde{k}} \bar{h}_{i'j',i\ell}(x,1-x)|.$$

Note that the point $\tilde{t} = (x, 1-x)$ for $0 < x < 1$ is on the hypotenuse of the unit triangle Ω. The value of $\bar{h}_{i'j',i\ell}(x,1-x)$ is understood as the limit when a point $t \in \Omega$ approaches $\tilde{t}$. Thus,

$$|E_1| \leq E_{1,1} + E_{1,2}$$

where

$$E_{1,1} := \sum_{(e,\alpha)\in\Gamma} \frac{1}{(2k_\alpha)!} \int_{\hat{J}_e} \int_0^{1-x} |D_x^{2k_\alpha} \bar{h}_{i'j',i\ell}(x,y)| Q_{e,\alpha}(x)\, dy\, dx,$$

$$E_{1,2} := \sum_{(e,\alpha)\in\Gamma} \frac{1}{(2k_\alpha)!} \times \int_{\hat{J}_e} \sum_{\tilde{j}=0}^{2k_\alpha-1} \sum_{\tilde{k}=0}^{\tilde{j}} \binom{\tilde{j}}{\tilde{k}} |D_1^{2k_\alpha-1-\tilde{k}} D_2^{\tilde{k}} \bar{h}_{i'j',i\ell}(x,1-x)| Q_{e,\alpha}(x)\, dx.$$

For $t=(x,y)\in \operatorname{supp}(\bar{h}_{i'j',i\ell}) \setminus (\{s\}\cup \pi(\bar{h}_{i'j',i\ell}))$, using assumption (4.1) we have that

$$|D_x^{2k_\alpha} \bar{h}_{i'j',i\ell}(s,t)| \le c2^{2k_\alpha} |s-t|^{-(\sigma+2k_\alpha)} \sum_{a\in\mathbf{Z}_\kappa} |s-t|^a 4^{a(i-1)/2}$$

and

$$|D_1^{2k_\alpha-1-\tilde{k}} D_2^{\tilde{k}} \bar{h}_{i'j',i\ell}(s,t)| \le c2^{2k_\alpha-1} |s-t|^{-(\sigma+2k_\alpha-1)} \sum_{a+b\in\mathbf{Z}_\kappa} |s-t|^{(a+b)} 4^{(a+b)(i-1)/2}.$$

Noting $Q_{e,\alpha}(x)\le \gamma(\alpha)$ for $e\in\Gamma_\alpha$, $x\in\hat{J}_e$ and when $\bar{h}_{i'j',i\ell}(s,t)\neq 0$,

$$\theta^{m-\alpha} \le |s-t| \le d_i + d_{i'} + \frac{r_{i'i}+\sqrt{2}}{4^{(i_0-1)/2}}.$$

We then have

$$E_{1,1} \le \sum_{\alpha\in A} \frac{c2^{2k_\alpha}\gamma(\alpha)\nu^\kappa}{(2k_\alpha)!} \sum_{e\in\Gamma_\alpha} \int_{\hat{J}_e} \int_0^1 |s-t|^{-(\sigma+2k_\alpha)}\, dy\, dx$$

and

$$E_{1,2} \le \sum_{\alpha\in A} \frac{c2^{4k_\alpha}\gamma(\alpha)\nu^\kappa}{(2k_\alpha)!} \sum_{e\in\Gamma_\alpha} \int_{\hat{J}_e} |s-\tilde{t}|^{-(\sigma+2k_\alpha-1)}\, dx.$$

In view of the proof of $|E_3|$, see (4.14), we readily have

$$E_{1,1} \le c\theta^{m(2-\sigma)}\nu^\kappa.$$

Now

$$\sum_{e\in\Gamma_\alpha}\int_{\hat{J}_e}|s-\tilde{t}|^{-(\sigma+2k_\alpha-1)}\,dx \le \theta^{(m-\alpha)(1-\sigma-2k_\alpha)}(\theta^{m-\alpha-1}-\theta^{m-\alpha})$$
$$\le c\theta^{(m-\alpha)(2-\sigma-2k_\alpha)}.$$

Then

$$E_{1,2}\le c\sum_{\alpha\in A}\frac{2^{4k_\alpha}(1-\theta)^{2k_\alpha}\theta^{m(2-\sigma)}}{(2k_\alpha)!\theta^{(2-\sigma)\alpha+2k_\alpha}}\nu^\kappa.$$

Applying Lemma 4.2 with $q=2$ yields

$$E_{1,2}\le c\theta^{m(2-\sigma)}\nu^\kappa,\quad \text{and hence}\quad |E_1|\le c\theta^{m(2-\sigma)}\nu^\kappa. \qquad \square$$

In the next lemma, we estimate E_4.

Lemma 4.6. *The following estimate holds*:

$$|E_4|\le c\theta^{m(2-\sigma)}\nu^\kappa.$$

Proof. To estimate E_4, we first compute

$$\mathcal{D}:=D_x^{2k_\alpha}\Bigg[\sum_{(e',\beta)\in\Lambda(x)}\frac{1}{(2k_\beta)!}\int_{J_{e'}}P_{e',\beta}(y)D_y^{2k_\beta}\bar{h}_{i'j',i\ell}(x,y)\,dy\Bigg],\ x\in\hat{J}_e.$$

Note for the domain $J_{e'}=[s_{e'},s_{e'+1})$ of the integration, one ending point may be dependent on x. For a fixed $i\ge 1$, the level of w_{ij} and a fixed x with $0<x<1$, we classify the points in the partition $\{s_{e'}: e'\in\mathbf{Z}_{m''}\}$ of $[0,1-x]$ into two groups. Group one contains points which are constant with respect to x and group two consists of points dependent on x, which is denoted by Π_x^i. In review of the construction of w_{ij}, we have that

$$\Pi_x^i:=\left\{s_{e'_l}:=\frac{l}{2^i}-x:\ l=1,2,\dots,2^i,\ i\ge 1,\ 0<x<\frac{l}{2^i}\right\}.$$

For each of the points

$$s_{e'_l} := \frac{l}{2^i} - x, \quad l = 1, 2, \dots, 2^i - 1,$$

there are always two subintervals of $[1 - x)$ associated with it, namely,

$$J_{e'_{l-1}} = \left[s_{e'_{l-1}}, \frac{l}{2^i} - x\right), \qquad J_{e'_l} = \left[\frac{l}{2^i} - x, s_{e'_{l+1}}\right).$$

When $l = 2^i$, there is only one interval $J_{m''-1} = [s_{m''-1}, 1 - x)$ associated with it. We denote by k_{β_l} the degree of the interpolating polynomial on the interval $J_{e'_l}$. When the two integration limits are constant, we have that

$$D_x^{2k_\alpha} \int_{s_{e'}}^{s_{e'+1}} P_{e',\beta}(y) D_y^{2k_\beta} \bar{h}_{i'j',i\ell}(x, y)\, dy = \int_{s_{e'}}^{s_{e'+1}} P_{e',\beta}(y) D_x^{2k_\alpha} D_y^{2k_\beta} \bar{h}_{i'j',i\ell}(x, y)\, dy.$$

By Lemma 4.4 with

$$f(x, y) := P_{e',\beta}(y) D_y^{2k_\beta} \bar{h}_{i'j',i\ell}(x, y)$$

we conclude that

$$\begin{aligned} D_x^{2k_\alpha} \int_{s_{e'}}^{l/2^i - x} & P_{e',\beta}(y) D_y^{2k_\beta} \bar{h}_{i'j',i\ell}(x, y)\, dy \\ &= \int_{s_{e'}}^{l/2^i - x} P_{e',\beta}(y) D_x^{2k_\alpha} D_y^{2k_\beta} \bar{h}_{i'j',i\ell}(x, y)\, dy \\ &\quad - \sum_{\tilde{j}=0}^{2k_\alpha - 1} \sum_{\tilde{k}=0}^{\tilde{j}} \binom{\tilde{j}}{\tilde{k}} (-1)^{\tilde{k}} D_1^{2k_\alpha - 1 - \tilde{k}} D_2^{\tilde{k}} f(x, l/2^i - x). \end{aligned} \tag{4.15}$$

Since

$$f(x, y) = P_{e',\beta}(y) D_y^{2k_\beta} K(x, y) w_{ij}(x, y)$$

by applying the Leibniz formula twice, the second term in (4.15) becomes

$$
\begin{aligned}
&S(e',\beta,l/2^i-x)\\
&:= -\sum_{j_1=0}^{2k_\alpha-1}\sum_{\tilde{k}=0}^{j_1}\binom{j_1}{\tilde{k}}\sum_{j_2=0}^{\tilde{k}}\binom{\tilde{k}}{j_2}\sum_{h_1+h_2\in\mathbf{Z}_\kappa}\Bigg[\binom{2k_\alpha-1-\tilde{k}}{h_1}\binom{2k_\beta+\tilde{k}-j_2}{h_2}\\
&\qquad\times(-1)^{\tilde{k}}D_1^{2k_\alpha-1-\tilde{k}-h_1}D_2^{2k_\beta+\tilde{k}-j_2-h_2}K(x,l/2^i-x)\\
&\qquad\qquad\times D_1^{h_1}D_2^{h_2}w_{ij}(x,l/2^i-x)D^{j_2}P_{e',\beta}(l/2^i-x)\Bigg].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\mathcal{D}=&\sum_{(e',\beta)\in\Lambda(x)}\frac{1}{(2k_\beta)!}\int_{J_{e'}}P_{e',\beta}(y)D_x^{2k_\alpha}D_y^{2k_\beta}\bar{h}_{i'j',i\ell}(x,y)\,dy\\
&+\sum_{l=1}^{2^i}\frac{1}{(2k_{\beta_{l-1}})!}\,S(e'_{l-1},\beta_{l-1},l/2^i-x)\\
&-\sum_{l=1}^{2^i-1}\frac{1}{(2k_{\beta_l})!}\,S(e'_l,\beta_l,l/2^i-x).
\end{aligned}
$$

Again, the evaluation of $\bar{h}_{i'j',i\ell}$ at the point $\tilde{t}=(x,l/2^i-x)$ is understood as the limit as a point $t\in\operatorname{supp}w_{ij}$ approaches $\tilde{t}$. Therefore, we obtain that

$$|E_4|\le E_{4,1}+E_{4,2},$$

where

$$
\begin{aligned}
E_{4,1}=&\sum_{(e,\alpha)\in\Gamma}\frac{1}{(2k_\alpha)!}\sum_{(e',\beta)\in\Lambda(\eta_x^e)}\frac{\left|D_x^{2k_\alpha}D_y^{2k_\beta}\bar{h}_{i'j',i\ell}(\eta_x^e,\zeta_y^{e'})\right|}{(2k_\beta)!}\\
&\qquad\times\int_{\hat{J}_e\times J_{e'}}P_{e',\beta}(y)Q_{e,\alpha}(x)\,dy\,dx
\end{aligned}
$$

for some $(\eta_x^e, \zeta_y^{e'}) \in \hat{J}_e \times J_{e'}$, and

$$E_{4,2} = \sum_{(e,\alpha)\in\Gamma} \frac{1}{(2k_\alpha)!} \int_{\hat{J}_e} \left[\sum_{l=1}^{2^i} \frac{1}{(2k_{\beta_{l-1}})!} \left| S(e'_{l-1}, \beta_{l-1}, l/2^i - x) \right| \right.$$
$$\left. + \sum_{l=1}^{2^i-1} \frac{1}{(2k_{\beta_l})!} \left| S(e'_l, \beta_l, l/2^i - x) \right| \right] Q_{e,\alpha}(x)\, dx.$$

We next estimate $E_{4,1}$. For $t \in \operatorname{supp}(\bar{h}_{i'j',i\ell}) \setminus (\{s\} \cup \pi(\bar{h}_{i'j',i\ell}))$, using assumption (4.1) we have that

$$|D_x^{2k_\alpha} D_y^{2k_\beta} \bar{h}_{i'j',i\ell}(s,t)|$$
$$\le c \sum_{a+b\in\mathbf{Z}_\kappa} \rho_{a,b} |s-t|^{-(\sigma+2k_\alpha+2k_\beta-a-b)} 4^{(a+b)(i-1)/2},$$

where

$$\rho_{a,b} := \frac{(2k_\alpha)!(2k_\beta)!}{a!b!(2k_\alpha - a)!(2k_\beta - b)!}.$$

Let $t' := (\eta_x^e, \zeta_y^{e'})$ and $\hat{\alpha} := \max\{\alpha, \beta\}$. Observing that when $\bar{h}_{i'j',i\ell}(s,t') \ne 0$,

$$\theta^{m-\hat{\alpha}} \le |s - t'| \le d_i + d_{i'} + \frac{r_{i'i} + \sqrt{2}}{4^{(i_0-1)/2}},$$

we then obtain that

$$|D_x^{2k_\alpha} D_y^{2k_\beta} \bar{h}_{i'j',i\ell}(s,t')| \le c\nu^\kappa 2^{2k_\alpha} 2^{2k_\beta} \theta^{-(m-\hat{\alpha})(\sigma+2k_\alpha+2k_\beta)}.$$

Note that $B(\eta_x^e) \subset B(0)$ and thus

$$E_{4,1} \le c \sum_{\alpha\in A} \sum_{\beta\in B(0)} \lambda(\alpha,\beta)\gamma(\alpha)\gamma(\beta) \sum_{e\in\Gamma_\alpha} \sum_{e'\in\Lambda_\beta} \int_{\hat{J}_e\times J_{e'}} dy\, dx,$$

where

$$\lambda(\alpha,\beta) := \frac{2^{2k_\alpha} 2^{2k_\beta} \theta^{-(m-\hat{\alpha})(\sigma+2k_\alpha+2k_\beta)} \nu^\kappa}{(2k_\alpha)!(2k_\beta)!}.$$

We now introduce four rectangular domains

$$\square_1 := [t^r_\alpha, t^r_{\alpha+1}] \times [s^u_\beta, s^u_{\beta+1}], \quad \square_2 := [t^r_\alpha, t^r_{\alpha+1}] \times [s^d_\beta, s^d_{\beta+1}],$$
$$\square_3 := [t^l_\alpha, t^l_{\alpha+1}] \times [s^u_\beta, s^u_{\beta+1}], \quad \square_4 := [t^l_\alpha, t^l_{\alpha+1}] \times [s^d_\beta, s^d_{\beta+1}]$$

and observe that

$$\bigcup_{e\in\Gamma_\alpha, e'\in\Lambda_\beta} \left(\hat{J}_e \times J_{e'}\right) \subset \bigcup_{j=1}^{4} \square_j.$$

For $j = 1, 2, 3, 4$ we let

$$E_{4,1}(j) := \sum_{\alpha\in A} \sum_{\beta\in B(0)} \lambda(\alpha,\beta)\gamma(\alpha)\gamma(\beta) \text{ meas}(\square_j).$$

Thus, we have that

$$E_{4,1} \le c \sum_{j=1}^{4} E_{4,1}(j).$$

Noting that

$$\text{meas}(\square_j) \le \int_0^{2\pi} \int_{\theta^{m-\hat{\alpha}}}^{\theta^{m-\hat{\alpha}-1}} r\, dr\, d\vartheta = c\theta^{2(m-\hat{\alpha}-1)}$$

we obtain that

$$E_{4,1}(j) \le c \sum_{\alpha\in A} \sum_{\beta\in B(0)} \frac{2^{2k_\alpha} 2^{2k_\beta} (1-\theta)^{2k_\alpha} (1-\theta)^{2k_\beta} \theta^{m(2-\sigma)} \nu^\kappa}{(2k_\alpha)!(2k_\beta)!\theta^{(2-\sigma)\hat{\alpha}+2k_\alpha(1+\alpha-\hat{\alpha})+2k_\beta(1+\beta-\hat{\alpha})}}.$$

Applying Lemma 4.2 with $q = 1$ yields

$$E_{4,1}(j) \le c\theta^{m(2-\sigma)}\nu^\kappa \quad \text{and thus} \quad E_{4,1} \le c\theta^{m(2-\sigma)}\nu^\kappa.$$

It remains to estimate $E_{4,2}$. To this end, We first estimate $|S(e', \beta, l/2^i - x)|$, which is associated with the interval $J_{e'} := [s_{e'}, s_{e'+1}) = [s_{e'}, l/2^i - x)$. We first note that

$$|D^{j_2} P_{e',\beta}(l/2^i - x)| \le \frac{(2k_\beta)!}{(2k_\beta - j_2)!} |J_{e'}|^{2k_\beta - j_2}, \quad 0 \le j_2 \le 2k_\beta$$

where $|J_{e'}|$ is the length of the interval $J_{e'}$. Recall that the wavelet function w_{ij} is a piecewise polynomial of order $\kappa = 2$; we thus have

$$|D_1^{h_1} D_2^{h_2} w_{ij}(x, l/2^i - x)| \le c4^{(h_1+h_2)(i-1)/2}.$$

For $\tilde{t} = (x, l/2^i - x) \in \operatorname{supp}(\bar{h}_{i'j',i\ell}) \backslash (\{s\} \cup \pi(\bar{h}_{i'j',i\ell}))$, using assumption (4.1) we have that

$$|D_1^{2k_\alpha - 1 - \tilde{k} - h_1} D_2^{2k_\beta + \tilde{k} - j_2 - h_2} K(x, l/2^i - x)| \\ \le |s - \tilde{t}|^{-(\sigma + 2k_\alpha + 2k_\beta - 1 - j_2 - h_1 - h_2)}.$$

Note that when $\bar{h}_{i'j',i\ell}(s, \tilde{t}) \neq 0$, we have that

$$\theta^{m-\hat{\alpha}} \le |s - \tilde{t}| \le d_i + d_{i'} + \frac{r_{i'i} + \sqrt{2}}{4^{(i_0-1)/2}}, \tag{4.16}$$

and thus

$$\frac{2\,|J_{e'}|}{|s - \tilde{t}|} \le \frac{2\theta^{m-\beta-1}(1-\theta)}{\theta^{m-\beta}} = \frac{2(1-\theta)}{\theta}.$$

Therefore,

$$|S(e', \beta, l/2^i - x)| \le c\nu^{\kappa-1} |s - \tilde{t}|^{-(\sigma + 2k_\alpha - 1)} 2^{2k_\alpha} (2k_\beta)! \\ \times \sum_{j_1=0}^{2k_\alpha - 1} \sum_{\tilde{k}=0}^{j_1} \binom{j_1}{\tilde{k}} \sum_{j_2=0}^{\tilde{k}} \binom{\tilde{k}}{j_2} \frac{[2(1-\theta)]^{2k_\beta - j_2}}{\theta^{2k_\beta - j_2}(2k_\beta - j_2)!}.$$

By Lemma 4.2, we find that

$$\frac{[2(1-\theta)]^{2k_\beta - j_2}}{\theta^{2k_\beta - j_2}(2k_\beta - j_2)!} \le c, \quad \text{for some constant} \quad c.$$

Consequently,

$$|S(e', \beta, l/2^i - x)| \le c\nu^{\kappa-1} |s - \tilde{t}|^{-(\sigma + 2k_\alpha - 1)} 2^{6k_\alpha} (2k_\beta)!.$$

Now note that when $\bar{h}_{i'j',i\ell}(s, \tilde{t}) \neq 0$, we only need to add those terms of the two summations in $E_{4,2}$ for which the second inequality of (4.16) holds. The number of those terms is at most

$$2 \cdot 2 \cdot \left(d_i + d_{i'} + \frac{r_{i'i} + \sqrt{2}}{4^{(i_0-1)/2}} \right) \Big/ \left(4^{-i/2} \right) = 8\nu.$$

Therefore,

$$E_{4,2} \leq \sum_{(e,\alpha)\in\Gamma} \frac{c\nu^{\kappa} 2^{6k_\alpha}\gamma(\alpha)}{(2k_\alpha)!} \int_{\hat{J}_e} |s-\tilde{t}|^{-(\sigma+2k_\alpha-1)}\, dx$$
$$\leq c \sum_{\alpha\in A} \frac{2^{6k_\alpha}(1-\theta)^{2k_\alpha}\theta^{m(2-\sigma)}}{(2k_\alpha)!\theta^{(2-\sigma)\alpha+2k_\alpha}}\, \nu^{\kappa}.$$

Applying Lemma 4.2 with $q = 3$ yields

$$E_{4,2} \leq c\theta^{m(2-\sigma)}\nu^{\kappa}.$$

This with the estimate for $E_{4,1}$ gives the desired estimate for E_4. $\square$

The estimates of E_1, E_3 and E_4 together establish the following lemma.

Lemma 4.7. *Suppose that condition* (4.1) *holds. Then there exists a positive constant c such that for $i \in \mathbf{Z}_{n+1}$, $\ell \in \mathbf{Z}_9$, $(i', j') \in \mathbf{U}_n$*

$$E \leq c\theta^{(2-\sigma)m}\left[\left(d_i + d_{i'} + \frac{r_{i'i}+\sqrt{2}}{4^{(i_0-1)/2}}\right)4^{(i-1)/2}\right]^{\kappa}. \tag{4.17}$$

We denote by $\tilde{\tilde{\mathbf{K}}}_n$ the compressed matrix with entries computed numerically by the quadrature method. Accordingly, the submatrix $\tilde{\tilde{\mathbf{K}}}_{i'i}$ of $\tilde{\tilde{\mathbf{K}}}_n$ is defined similarly to $\mathbf{K}_{i'i}$ of $\mathbf{K}_n$. By using the definition of the ℓ^{∞} norm of matrices and the definition of $\bar{h}_{i'j',i\ell}$, we translate the error estimate for each entry presented in Lemma 4.7 to the error for a block.

Lemma 4.8. *There exists a constant c such that, for i', $i \in \mathbf{Z}_{n+1}$ and $n \in \mathbf{N}$,*

$$\|\widetilde{\mathbf{K}}_{i'i} - \tilde{\tilde{\mathbf{K}}}_{i'i}\|_{\infty}$$
$$\leq c\left[\left(4^{(-i+1)/2} + 4^{(-i'+1)/2} + \frac{r_{i'i}+\sqrt{2}}{4^{(i_0-1)/2}}\right)4^{(i-1)/2}\right]^{\kappa}\theta^{m(2-\sigma)}.$$

Let $m_{i'i}$ be the integer m appearing in the proposed quadrature rule for the computation of the entries of block $\widetilde{\widetilde{\mathbf{K}}}_{i^\pi}$. We choose $m_{i'i}$ according to

$$m_{i'i} \geq c(2i + i'), \quad i, i' \in \mathbf{Z}_{n+1} \tag{4.18}$$

for some positive constant c. Let $\tilde{\tilde{u}}_n$ be the approximate solution of the integral equation (2.1) computed according to $\widetilde{\widetilde{\mathbf{K}}}_n$.

Theorem 4.9. *Suppose that the entries of* $\widetilde{\widetilde{\mathbf{K}}}_n$ *are computed by the proposed quadrature rule with* $m_{i'i}$ *given by* (4.18)*. Then there exists a positive constant* c *and a positive integer* N *such that, for all* $n > N$,

$$\|u - \tilde{\tilde{u}}_n\|_\infty \leq cf(n)^{-\kappa/2} \log f(n) \|u\|_\infty. \tag{4.19}$$

Proof. By the proof of Theorem 4.4 of [**11**], the estimate (4.19) holds if there exists a constant c such that for i', $i \in \mathbf{Z}_{n+1}$ and for all $n \in \mathbf{N}$

$$\|\mathbf{K}_{i',i} - \widetilde{\widetilde{\mathbf{K}}}_{i',i}\|_\infty \leq c\varepsilon_{i'i}^{-(2\kappa-\sigma')}(d_i d_{i'})^\kappa, \quad i', i \in \mathbf{Z}_{n+1}, \tag{4.20}$$

where $\varepsilon_{i'i}$ is defined in (3.3). Using the triangle inequality, it suffices to prove that there exists a constant c such that

$$\|\widetilde{\mathbf{K}}_{i'i} - \widetilde{\widetilde{\mathbf{K}}}_{i'i}\|_\infty \leq c\varepsilon_{i^\pi}^{-(2\kappa-\sigma')}(d_i d_{i'})^\kappa, \quad i', i \in \mathbf{Z}_{n+1}.$$

Our choice of $m_{i'i}$ ensures that

$$\theta^{(2-\sigma)m_{i'i}} \leq c\left(\frac{r_{i'i} + \sqrt{2}}{4^{(i_0-1)/2}}\right)^{-(3\kappa-\sigma')} 4^{-(2\kappa i + \kappa i')/2}.$$

This with

$$c_1(4^{(-i+1)/2} + 4^{(-i'+1)/2}) \leq \varepsilon_{i'i} \leq \frac{r_{i'i} + \sqrt{2}}{4^{(i_0-1)/2}} \leq c_2$$

for some positive constant c_1 and c_2 implies that

$$\left[\left(4^{(-i+1)/2}+4^{(-i'+1)/2}+\frac{r_{i'i}+\sqrt{2}}{4^{(i_0-1)/2}}\right)4^{(i-1)/2}\right]^{\kappa}\theta^{(2-\sigma)m_{i'i}}$$
$$\leq c\varepsilon_{i'i}^{-(2\kappa-\sigma')}4^{-\kappa(i'+i)/2}.$$

Thus, the result of this theorem follows directly from Lemma 4.8. □

In the next theorem, we estimate the total number $\mathcal{M}_n$ of function evaluations needed for computing all entries of matrix $\tilde{\tilde{\mathbf{K}}}_n$.

Theorem 4.10. *Let* $m_{i'i}$, i', $i \in \mathbf{Z}_{n+1}$ *be the smallest integer satisfying condition* (4.18). *Then there exists a positive constant* c *such that*

$$\mathcal{M}_n \leq cf(n)\log^5 f(n).$$

Proof. The proof of this theorem is similar to that for Theorem 4.5 of [**14**]. Let $\mathcal{M}_{i'i}$ and $\mathcal{M}_{k'i'j',i}$ be the number of function evaluations for computing the entries of $\tilde{\tilde{\mathbf{K}}}_{i'i}$ and the $k'i'j'$ row of the block $\tilde{\tilde{\mathbf{K}}}_{i'i}$, respectively. To estimate $\mathcal{M}_{k'i'j',i}$, we let $\mathcal{M}(h)$ be the number of function evaluations used in computing $I_m(h)$. For fixed $k'i'j'$, let

$$h_{k'i'j',kij}(s,t) := K(s,t)w_{kij}(t).$$

Recalling the definition of the function $\bar{h}_{k'i'j',ki\ell}$, we have that

$$\begin{aligned}\mathcal{M}_{k'i'j',i} &= \sum_{k\in\mathbf{Z}_p}\sum_{j\in\mathbf{Z}_{k'i'j',ki}}\mathcal{M}(h_{k'i'j',kij})\\ &= \sum_{\ell\in\mathbf{Z}_9}\sum_{k\in\mathbf{Z}_p}\sum_{j\in\mathbf{Z}^{\ell}_{k'i'j',ki}}\mathcal{M}(h_{k'i'j',kij})\\ &= \sum_{k\in\mathbf{Z}_p}\sum_{\ell\in\mathbf{Z}_9}\mathcal{M}(\bar{h}_{k'i'j',ki\ell}).\end{aligned}$$

Noting that

$$\text{Card}\left\{\alpha : J_\alpha \subset [\theta^{m_{i'i-l}},\theta^{m_{i'i-l-1}}]\right\} \leq \frac{\theta^{m_{i'i-l-1}}-\theta^{m_{i'i-l}}}{4^{(-i+1)/2}}+2,$$

we observe that

$$\sqrt{\mathcal{M}_{k'i'j',i}} \leq 2 \sum_{k\in\mathbf{Z}_p} \sum_{\ell\in\mathbf{Z}_9} \sum_{l=0}^{m_{i'i-1}} k_l \biggl(\frac{\theta^{m_{i'i-l-1}} - \theta^{m_{i'i-l}}}{4^{(-i+1)/2}} + 2 \biggr).$$

Since $k_l = l+2$, for $l \in \mathbf{Z}_m$, there exists a positive constant c such that

$$\sqrt{\mathcal{M}_{k'i'j',i}} \leq c\biggl(\frac{1}{\theta} - 1\biggr) 4^{(i-1)/2} \biggl(\sum_{l=0}^{m_{i^\pi}-1} l\theta^{m_{i'i}-l} + 2 \sum_{l=0}^{m_{i^\pi}-1} \theta^{m_{i^\pi}-l} \biggr)$$
$$+ 2c \sum_{l=0}^{m_{i'i}-1} (l+2).$$

According to the truncation strategy, we only have to add those terms of the sum in the last formula for which

$$t_l = \theta^{m_{i^\pi}-l} \leq \frac{r_{i^\pi} + \sqrt{2}}{4^{(i_0-1)/2}} + d_{i'} + d_i.$$

By the choice of $r_{i'i}$, the following holds

$$\frac{r_{i^\pi}}{4^{(i_0-1)/2}} = \frac{\sqrt{2}}{4^{(i_0-1)/2}} + \max \Bigl\{ z' 4^{[b'(n-i')-i]/2},$$
$$z \left(4^{(-i+1)/2} + 4^{(-i'+1)/2} \right) \Bigr\}.$$

Using the assumption on the choice of $m_{i'i}$, we conclude that

$$\sqrt{\mathcal{M}_{k'i'j',i}} \leq c \Bigl[(n+n^2) 4^{(i-1)/2} \Bigl(4^{[b'(n-i')-i]/2}$$
$$+ 4^{(-i+1)/2} + 4^{(-i'+1)/2} \Bigr) + n^2 \Bigr].$$

Hence,

$$\mathcal{M}_{k'i'j',i} \leq cn^4 \left(4^{[b'(n-i')-1]} + 4^{(i-i')} + 1 \right).$$

This leads to

$$\mathcal{M}_n = \sum_{i\in\mathbf{Z}_{n+1}} \sum_{i'\in\mathbf{Z}_{n+1}} \mathcal{M}_{i'i} = \sum_{i\in\mathbf{Z}_{n+1}} \sum_{i'\in\mathbf{Z}_{n+1}} pw(i')\mathcal{M}_{k'i'j',i}.$$

Noticing that $w(i') = c4^{i'-1}$ for some constant c, we thus obtain that

$$\mathcal{M}_n \leq c \sum_{i\in\mathbf{Z}_{n+1}} \sum_{i'\in\mathbf{Z}_{n+1}} \left[n^4 \left(4^{b'n}4^{(1-b')i'} + 4^i + 4^{i'}\right)\right].$$

Recalling that $f(n) = \mathcal{O}(4^n)$, a simple computation yields the estimate of the theorem. $\square$

5. Numerical experiments. In this section, we present numerical examples to demonstrate the proposed methods for solving integral equation (2.1) with various domains E. In all of these examples, we use the kernel

$$K(x,y,\xi,\zeta) := \frac{1}{\sqrt{(x-\xi)^2 + (y-\zeta)^2}}, \quad (x,y),(\xi,\zeta) \in E,$$

and choose

$$f(x,y) := x^2 + y^2 - \int_E \frac{\xi^2+\zeta^2}{\sqrt{(x-\xi)^2+(y-\zeta)^2}}\,\mathrm{d}\xi\,\mathrm{d}\zeta, \quad (x,y) \in E$$

so that

$$u(x,y) = x^2 + y^2, \quad (x,y) \in E$$

is the exact solution to equation (2.1).

In our numerical examples, we use the multilevel iterative method developed in [**18**] to solve the linear system unless stated otherwise. The compression rate is computed as the ratio of the number of nonzero entries in the compressed matrix $\widetilde{\mathbf{K}}_n$ over the total number of entries of $\mathbf{K}_n$. The order of convergence of the approximation solution u_n in the space $\mathbf{F}_n$ is computed by $\log_4 \|u - u_n\|_\infty / \|u - u_{n+1}\|_\infty$, where n stands for the level number of resolution and u is the exact solution. The theoretical order of convergence is 1.

Example 5.1. *The adaptive quadrature rule.* In this example, we test if the proposed quadrature rule is good enough in solving the integral equation (2.1). We take $E = \Omega$. To exclude other possible factors affecting the solution, the linear system of equations are solved by standard Gaussian elimination method. No compression is assumed. The matrix entries obtained without using quadrature are computed by analytical formula. The results are tabulated in Table 5.1. We see that the results by using the proposed quadrature rule are very close to those obtained without quadrature rule.

TABLE 5.1. Comparison of uncompressed solutions using the wavelet collocation method, with and without numerical integration for the entries.

	Without quadrature		With quadrature		
n	L^∞-error	order	m	L^∞-error	order
1	1.513187e-1		7	1.513184e-1	
2	3.599605e-2	1.04	7	3.599379e-2	1.04
3	7.499082e-3	1.13	7	7.496564e-3	1.13
4	1.849162e-3	1.01	7	1.850269e-3	1.01
5	4.650610e-4	1.00	7	4.657219e-4	1.00

Example 5.2. *Block truncation scheme.* This example is designed to test if the proposed block truncation strategy is good enough. We solve the integral equation (2.1) with $E = \Omega$. All the elements are computed by analytical method and the linear system of equations are solved by standard Gaussian elimination method. The results are tabulated in Table 5.2. We can see that the compressed solutions by block truncation scheme follow very closely the uncompressed solutions. The block truncation parameter $r_{i'i}$ is set as $\sqrt{3}$.

TABLE 5.2. Comparison of solutions obtained by the full collocation method and by the compressed method.

Block truncation scheme with $r_{i'i} = \sqrt{3}$					
	Full collocation solution u_n		Compressed solution $\tilde{u}_n$		
n	L^∞-error	order	L^∞-error	order	compression rate
1	1.513187e-1		1.513187e-1		1.000000
2	3.599605e-2	1.04	3.599605e-2	1.04	1.000000
3	7.499082e-3	1.13	7.493692e-3	1.13	0.758301
4	1.849162e-3	1.01	1.851165e-3	1.01	0.416626
5	4.650610e-4	1.00	4.683404e-4	0.99	0.181021

Example 5.3. *Multilevel iteration method.* In this example, we test if the Gauss-Seidel type multilevel iteration method gives good solutions. The solutions are compared with those from standard Gaussian elimination. All the entries are computed analytically and no compression is assumed. The results are tabulated in Table 5.3. We see that both methods give very close solutions, where β is the number of iterations. The significant reduction of the computing time when using iterative method is very impressive.

TABLE 5.3. Comparison of uncompressed solutions using Gaussian elimination and multilevel iteration methods.

	Multilevel iterate				Gaussian elimination		
(k, ℓ)	β	$\lvert u_{k,\ell}^{(\beta)} - u\rvert_\infty$	order	time (sec.)	$\lvert u_{k+\ell} - u\rvert_\infty$	order	time (sec.)
(3,0)		7.499082e-3		0.16	7.499082e-3		0.16
(3,1)	4	1.845814e-3	1.01	0.81	1.849162e-3	1.01	10
(3,2)	8	4.650403e-4	0.99	8.74	4.650610e-4	1.00	632

In the next four examples, we combine the proposed block truncation scheme, the quadrature rule and multilevel iterative scheme to solve (2.1) with various domains E. The block truncation parameters are taken as $r_{i'i} = \sqrt{3}$, i', $i \in \mathbf{Z}_{n+1}$. When the matrix entries are computed numerically, the quadrature rule presented in Section 4 is adopted, and the number of subdivision m used in the quadrature is taken as $m = 9$. All the computations are done using parallel computing algorithms, which will be discussed in detail in a separate paper.

Example 5.4. In this example, the domain $E = \Omega$ is the unit triangle. The initial level k used for the multilevel iteration is chosen as $k = 3$. Note that when the level $n = k + \ell = 3 + 5 = 8$, the matrix size is 196608×196608. The results are listed in Table 5.4.

TABLE 5.4. Comparison of compressed solutions using multilevel iteration method, with and without quadrature rule.

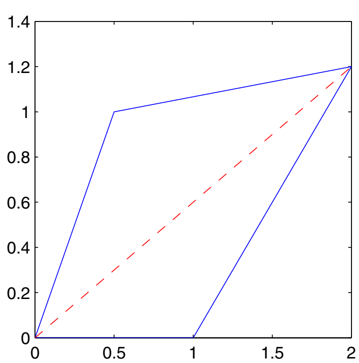
Results for the integral equation on the unit triangle Ω						
(k, ℓ)	Without quadrature		With quadrature			
$n{=}k{+}\ell$	L^∞-error	order	m	L^∞-error	order	compression rate
(1,0)	1.513187e-1		9	1.513184e-1		1.000000
(2,0)	3.599605e-2	1.04	9	3.599379e-2	1.04	1.000000
(3,0)	7.499082e-3	1.13	9	7.499123e-3	1.13	1.000000
(3,1)	1.850492e-3	1.01	9	1.850509e-3	1.01	0.431732
(3,2)	4.648565e-4	1.00	9	4.648294e-4	1.00	0.181965
(3,3)	1.171548e-4	0.99	9	1.174169e-4	0.99	0.067760
(3,4)	2.999366e-5	0.98	9	2.980252e-5	0.99	0.023004
(3,5)	8.010510e-6	0.95	9	7.920534e-6	0.96	0.007332

Example 5.5. In this example, we consider the triangle $\triangle$ with three vertices $(0, 0), (1, 0)$ and $(0.5, 1)$ and take $E = \triangle$. The results are listed in Table 5.5.

TABLE 5.5. Comparison of solutions obtained by the full collocation method and by our compressed collocation method.

Results for the integral equation on an oblique triangle $\triangle$						
(k,ℓ)	Full collocation solution u_n		Compressed solution $\tilde{u}_n$			
$n=k+\ell$	L^∞-error	order	m	L^∞-error	order	compression rate
(1,0)	1.100676e-1		9	1.100676e-1		1.000000
(2,0)	2.081240e-2	1.20	9	2.081280e-2	1.20	1.000000
(2,1)	4.972866e-3	1.03	9	4.974875e-3	1.03	0.758301
(2,2)	1.285098e-3	0.98	9	1.285873e-3	0.98	0.416626
(2,3)	3.230465e-4	1.00	9	3.257940e-4	0.99	0.181021
(2,4)			9	7.869679e-5	1.02	0.067701
(2,5)			9	2.195469e-5	0.92	0.023000
(2,6)			9	7.829137e-6	0.74	0.007331

Example 5.6. In this example, we consider the quadrilateral E with four vertices $(0,0)$, $(1,0)$, $(2,1.2)$, $(0.5,1)$, see Figure 5.1. The computational results are listed in Table 5.6.



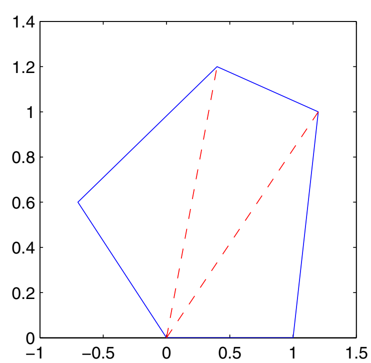


FIGURE 5.1. Computational domains of Example 5.6 and Example 5.7.

TABLE 5.6. Comparison of solutions obtained by the full collocation method and by our compressed collocation method.

Results for the integral equation on a quadrilateral							
(k, ℓ)	Full collocation solution u_n			Compressed solution $\tilde{u}_n$			
$n=k+\ell$	m_n	L^∞-error	order	m	L^∞-error	order	compression rate
(1,0)	9	3.607234e-1		9	3.607234e-1		1.000000
(2,0)	9	9.443242e-2	0.97	9	9.443194e-2	0.97	1.000000
(2,1)	9	2.282575e-2	1.02	9	2.272426e-2	1.03	0.758301
(2,2)	9	5.735749e-3	1.00	9	5.891051e-3	0.97	0.416626
(2,3)	9	1.456519e-3	0.99	9	1.492354e-3	0.99	0.181021

Example 5.7. In this example, we consider the pentagon E with five vertices (0,0), (1,0), (1.2,1), (0.4,1.2), (-0.7,0.6), see Figure 5.1. The computational results are listed in Table 5.7.

TABLE 5.7. Comparison of solutions obtained by the full collocation method and by our compressed collocation method.

Results for the integral equation on a pentagon							
(k, ℓ)	Full collocation solution u_n			Compressed solution $\tilde{u}_n$			
$n=k+\ell$	m_n	L^∞-error	order	m	L^∞-error	order	compression rate
(1,0)	9	2.015684e-1			2.01570e-1		1.000000
(2,0)	9	4.076680e-2	1.15	9	4.076633e-2	1.15	1.000000
(2,1)	9	1.017880e-2	1.00	9	1.019395e-2	1.00	0.758301
(2,2)	9	2.593854e-3	0.99	9	2.624460e-3	0.98	0.416626
(2,3)	9	6.440248e-4	1.00	9	6.675855e-4	0.99	0.181021
(2,4)				9	1.939918e-4	0.89	0.067701

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