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FAST SOLVERS OF INTEGRAL EQUATIONS OF THE SECOND KIND: QUADRATURE METHODS

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ABSTRACT. For the integral equation $u(x) = \int_0^1 K(x,y) \times u(y)dy + f(x)$ with m-smooth f and m'-smooth K, $m' \geq 2m$, quadrature fast (C, C^m) solvers are designed. The quadrature system is solved by two grid iteration method, in certain cases simply by the Gauss method. By a fast (C, C^m) solver we mean a fully discrete method which involves the values of f and K at $O(n_\star)$ points and produces at the cost of $O(n_\star)$ flops an approximate solution u_n of the optimal convergence order $\| u - u_n \|_0 \leq c n_\star^{-m} \| f \|_m$; moreover, we set a requirement about a fast evaluation of values of the approximate solution. Here $\| \cdot \|_0$ and $\| \cdot \|_m$ are the norms in $C[0,1]$ and $C^m[0,1]$, respectively, $n_\star = n_\star(n) \to \infty$ as $n \to \infty$.

1. Introduction.

Consider the integral equation

$$u(x) = \int_0^1 K(x,y)u(y)\,dy + f(x), \quad 0 \leq x \leq 1, \tag{1.1}$$

where $f \in C^m[0,1]$, $K \in C^{m'}([0,1] \times [0,1])$, $m' \geq 2m$. Assume that the homogenous integral equation corresponding to (1.1) has in $C[0,1]$ only the trivial solution. Introduce the norms

$$\|u\|_0 = \sup_{0 \leq x \leq 1} |u(x)|, \quad \|u\|_m = \sum_{k=0}^{m} \|u^{(k)}\|_0.$$

Our aim is to construct methods that produce approximate solutions u_n, $n \in \mathbf{N}$, such that

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• given the values of f and K at $O(n_\star)$ suitably chosen points, where $n_\star = n_\star(n) \to \infty$ as $n \to \infty$, the parameters of u_n are available at the cost of $O(n_\star)$ flops, and the accuracy

$$\|u - u_n\|_0 \le c n_\star^{-m} \|f\|_m \tag{1.2}$$

is achieved, where u is the solution of (1.1) and c is a constant that is independent of n and f;

• having determined the parameters of u_n, the value of u_n at any particular point $x \in [0,1]$ is available with the same accuracy as (1.2) at the cost of $O(1)$ flops.

We call such methods *fast* (C, C^m) *solvers* of equation (1.1) (although we allow u_n to be only piecewise continuous). Note that a fast (C, C^m) solver is a method of optimal accuracy order to solve (1.1): the convergence speed (1.2) is the best that one can achieve for $f \in C^m$, $K \in C^{2m}$ by a method of $O(n_\star)$ flops. It is optimal also in the sense of information: to obtain the convergence speed (1.2) for all $f \in C^m$, $\|f\|_m \le 1$, at least $O(n_\star)$ values of f and K must be involved. More precise and general formulations of these statements can be found in the recent paper by Werschulz [**23**]; see [**23**] also for references to earlier works on the optimality of methods and on the complexity of integral equations. Forgetting for a moment the requirement $m' \ge 2m$, one can achieve by an arithmetical work of $O(n_\star)$ flops, with no restriction to the amount of the sample points for f and K, as well as by any arithmetical work using $O(n_\star)$ sample values of f and K, the accuracy

$$\|u - u_n\|_0 \le c n_\star^{-m''} \|f\|_m, \quad m'' = \min\{m,\, m'/2\},$$

and not more in the worst case when f varies in $C^m[0,1]$ and K varies in $C^{m'}([0,1]\times[0,1])$ so that $\|(I-T)^{-1}\|_{\mathcal{L}(\mathcal{C}[\prime,\infty])} \le c_1$, $\|K\|_{C^{m'}([0,1]\times[0,1])} \le c_2$. For instance, in the case $m' = m$, the accuracy $O(n_\star^{-m/2})$ can be achieved and not more. This result was established already in 1967 by Emelyanov and Ilin [**8**]; let us mention also that they proposed a quadrature method with two- and multigrid iterations as examples of methods of the optimal accuracy $\|u - u_n\|_0 \le c n_\star^{-m/2} \|f\|_m$ at the cost of $O(n_\star)$ flops.

Thus, only in the case $m' \ge 2m$ the accuracy (1.2) can be achieved at the cost of $O(n_\star)$ flops using $O(n_\star)$ sample values of f and K.

In [**23**], a fast (C, C^m) solver of equation (1.1) has been constructed on the basis of piecewise polynomial Galerkin method and two grid iterations to solve the Galerkin system in which the kernel K has been approximated by a suitable piecewise polynomial function. In the present paper we show how different fast (C, C^m) solvers can be designed on the basis of the quadrature method (Nyström method); depending on m' and the method, the quadrature system is solved either directly by the Gauss elimination or by two grid iterations. See also the paper by Lin [**12**] where a quadrature preconditioner is constructed for the quadrature system so that finally the accuracy $O(n^{-m})$ of the approximate solution u_n is achieved in $O(n)$ flops using $O(n)$ units of information. Implicitly, a fast (C, C^m) solver with $n_\star = n$ is constructed, but conditions on f and K are not specified in [**12**]; the argument uses the existence of the continuous derivative $(\partial/\partial x)^{2m}(\partial/\partial y)^{2m}K(x, y)$, but it seems that the smoothness condition can be reduced to $K \in C^{2m}$ using Lemma 4.1 of the present paper. We quote also [**22**] where fast solvers are constructed for integral equations of the second kind on the real line on the basis of Daubechies orthogonal wavelet Galerkin method and the Sloan iteration improvement of the Galerkin solution; due to this improvement, one can use such small Galerkin systems that no compression, so traditional in wavelet techniques, is needed at all. Often also methods of the complexity $O(n_\star \log n_\star)$ flops, instead of the optimal $O(n_\star)$ ones, are held for fast solvers. Many of the wavelet methods for boundary integral equations belong to this class; but recently biorthogonal piecewise polynomial wavelet Galerkin methods of optimal complexity $O(n_\star)$ have been developed, see [**7**]. For possibly weakly singular, Cauchy singular and hypersingular periodic integral and pseudodifferential equations methods of complexity $O(n_\star \log n_\star)$ have been designed in [**14**] in the scale of Sobolev norms on the basis of the trigonometric Galerkin method and two grid iterations. See also [**13**].

In the present paper, we confine ourselves to a simplest version of the two grid iterations when a direct solution of the quadrature system by the Gauss method is too expensive. Actually different other versions of two and multi-grid iteration methods, cf., e.g., [**2, 3, 6, 8–10**], and other iteration methods could be used, among them GMRES and conjugate gradients. In the two grid iterations, the optimal accuracy is achieved in a finite number of iteration steps uniformly with respect

to n, whereas GMRES and conjugate gradients need $O(\log n)$ iteration steps, hence to keep the computation in $O(n)$ flops, only nested versions of GMRES and conjugate gradients are acceptable; unfortunately, this complicates the algorithms. A basis for all iteration methods is a fast matrix-vector multiplication. We will demonstrate how, with a suitable accuracy, the n^2-matrix of the quadrature system and an n-vector can be multiplied in $O(n)$ flops; our elementary algorithm is close to that of [**23**] and is effective only in case of smooth K, similarly as the Nyström method itself is. In the literature, many other fast matrix-vector multiplication algorithms, applicable to more general problems, e.g., to boundary integral equations with singularities, are known: the algorithms based on the panel clustering by Hackbusch, the fast multipole algorithms, wavelet compression algorithms, and others; they are developed mainly for the Galerkin discretisations and, moreover, the complexity of many of them is $O(n \log n)$ instead of $O(n)$.

The rest of the paper is organized as follows. In Section 2 we recall with short proofs basic results about the convergence speed of the quadrature method (the Nyström method); the accuracy $\|u - u_n\|_0 \le cn^{-m}\|f\|_m$ is achieved at the cost of $O(n^3)$ flops if we solve the quadrature system by the Gauss method. In Section 3 we recall/prove some convergence results for the two grid iteration method applied to the quadrature system. This enables us to reduce the computational cost from $O(n^3)$ to $O(n^2)$ flops. Finally, in Section 4 we reduce the computational cost to $O(n)$ flops using instead of $K(x, y)$ its piecewise polynomial interpolant $K_n(x, y)$. So we obtain a fast (C, C^m) solver with $n_\star = n$ (called fast solver 1 in the paper). A further fast (C, C^m) solver (fast solver 2, with $n_\star \asymp n^{m'/m}$) is designed looking for the solution of (1.1) in the form $u = v + f$ where v is the new unknown function. Finally, since $K_n(x, y)$ is a degenerate kernel, it is possible to represent the solution of the approximating integral equation $u_n(x) = \int_0^1 K_n(x, y)u_n(y)dy + f(x)$ in a closed form solving a linear system of equations. In this way we design a third fast (C, C^m) solver. Each of the fast solvers 1–3 has its algorithmical advantages and disadvantages. In particular, for sufficiently smooth K (for $m' \ge 3m$), the quadrature system may be solved directly by the Gauss method in the fast solver 2; a similar property holds for the fast solver 3 but not the fast solver 1.

2. Preliminaries: quadrature method.

2.1 *Compact convergence of operators.* By $c, c', c_1, \dots$, we denote generic constants that may have different values by different occurrences. For Banach spaces E and F, $\mathcal{L}(E,F)$ denotes the space of linear continuous operators from E into F, $\mathcal{L}(E) = \mathcal{L}(E,E)$,

$$\|A\|_{\mathcal{L}(E,F)} = \sup_{u\in E,\, \|u\|_E=1} \|Au\|_F \quad \text{for } A \in \mathcal{L}(E,F).$$

The concept of the compact convergence of linear operators has been introduced in the framework of so called (abstract) discrete convergence where an operator $A \in \mathcal{L}(E)$ acting in a Banach space E is approximated by operators $A_n \in \mathcal{L}(E_n)$ acting in (possibly finite dimensional) Banach spaces E_n, see [**16–18, 20**]. When treating the Nyström method for an integral equation, it is sufficient to restrict ourselves to the case where the spaces coincide: $E = E_n$, $n \in \mathbf{N}$. In this situation the concept of compact convergence of operators is closely related to the concept of collectively compact approximation in sense of Anselone, see [**1**] and the pioneering paper by Sobolev [**15**].

Definition 2.1. Let E be a Banach space and $T, T_n \in \mathcal{L}(E)$, $n \in \mathbf{N}$. We say that T_n *converges compactly* to T if the following two conditions are fulfilled:

- $\|Tu - T_n u\|_E \to 0$ as $n \to \infty$ for every $u \in E$;
- $(u_n) \subset E$, $\|u_n\|_E \le 1$, $n \in \mathbf{N} \Rightarrow (T_n u_n)$ is relatively compact in E, i.e., every subsequence of $(T_n u_n)$ contains a subsequence that converges in E.

Lemma 2.1. *Assume that the operators $T, T_n \in \mathcal{L}(E)$, $n \in \mathbf{N}$, are compact, T_n converges compactly to T and the homogenous equation $u = Tu$ has only the trivial solution $u = 0$. Then there are n_0 and a constant c such that for all $n \ge n_0$ the inverse operator $(I - T_n)^{-1} \in \mathcal{L}(E)$ exists and $\|(I - T_n)^{-1}\|_{\mathcal{L}(E)} \le c$.*

Proof. If the assertion fails then there exists a sequence (u_n), $n \in N' \subset \mathbf{N}$ such that $\|u_n\| = 1$, $\|u_n - T_n u_n\| \to 0$, $N' \ni n \to \infty$. By

Definition 2.1, sequence $(T_n u_n)$ is relatively compact; let $\|T_n u_n - v\| \to 0$, $n \in N'' \subset N'$. Then also $\|u_n - v\| \to 0$, $N'' \ni n \to \infty$, that implies $\|T_n(u_n - v)\| \to 0$ since $\|T_n\| \le \text{const}$ by the Banach-Steinhaus theorem. Thus $\|v\| = 1$ but $\|v - Tv\| = \lim_{n \in N''} \|u_n - T_n u_n\| = 0$ that contradicts the condition of the lemma. □

Actually it is possible to estimate $\|(I - T_n)^{-1}\|$ through $\|(I - T)^{-1}\|$.

2.2 *Quadrature formula.* Take a quadrature formula

$$\int_0^1 v(y)\,dy = \sum_{j=1}^{n} w_{j,n} v(x_{j,n}) + \varphi_n(v) \cong \sum_{j=1}^{n} w_{j,n} v(x_{j,n}) \tag{2.1}$$

with the weights $w_{j,n} \in \mathbf{R}$ and knots $x_{j,n}$, $0 \le x_{1,n} < \cdots < x_{n,n} \le 1$; the rest term $\varphi_n(v)$ is dropped in computations. We assume that

$$\sum_{j=1}^{n} |w_{j,n}| \le c, \quad n \in \mathbf{N}, \tag{2.2}$$

and that either

$$|\varphi_n(v)| \le c n^{-m} \|v\|_m, \quad n \in \mathbf{N}, \quad v \in C^m[0,1], \tag{2.3}$$

or

$$|\varphi_n(v)| \le c n^{-m'} \|v\|_{m'}, \quad n \in \mathbf{N}, \quad v \in C^{m'}[0,1]. \tag{2.4}$$

Here m and m' are the parameters from the smoothness conditions for f and K, respectively, see Section 1. For most quadrature formulae in the practice, $w_{j,n} \ge 0$, $j = 1, \ldots, n$, $n \in \mathbf{N}$, and then (2.2) is a consequence of (2.3) or (2.4); on the other hand, there are some quadratures, e.g., composite Newton-Cotes formulae of high order, that contain negative weights but (2.2) is still fulfilled.

Since $\|\varphi_n\|_{\mathcal{L}(C[0,1],\mathbf{K})} \le 1 + \sum_{j=1}^{n} |w_{j,n}|$, with $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$ depending on whether a real or complex space $C[0,1]$ is considered, it follows from (2.2) and (2.3)/(2.4) by the Banach-Steinhaus theorem that $\varphi_n(v) \to 0$ as $n \to \infty$ for every $v \in C[0,1]$; moreover,

$$\sup_{v \in M} |\varphi_n(v)| \to 0 \quad \text{as } n \to \infty \quad \text{for relatively compact sets } M \subset C[0,1]. \tag{2.5}$$

2.3 *Nyström and quadrature methods.* Replacing the integral in equation (1.1) by the quadrature (2.1) we obtain the approximating equation

$$u_n(x) = \sum_{j=1}^{n} w_{j,n} K(x, x_{j,n}) u_n(x_{j,n}) + f(x). \tag{2.6}$$

Its solution $u_n(x)$ is uniquely determined by the grid values $u_n(x_{j,n})$, $j = 1, \dots, n$; collocating (2.6) at $x = x_{i,n}$, we arrive at the system of linear algebraic equations

$$u_n(x_{i,n}) = \sum_{j=1}^{n} w_{j,n} K(x_{i,n}, x_{j,n}) u_n(x_{j,n}) + f(x_{i,n}), \quad i = 1, \dots, n, \tag{2.7}$$

to determine these grid values. This is the Nyström method: one solves system (2.7) and extends the solution to all $x \in [0,1]$ using (2.6). In this paper we distinguish the Nyström method and the quadrature method. Speaking about the quadrature method we mean the determination of the grid values $u_n(x_{i,n})$, $i = 1, \dots, n$, from system (2.7), without the Nyström interpolation (2.6). On a final stage we will use a cheaper prolongation of the grid solution to $[0,1]$ based on the piecewise polynomial interpolation (this procedure remains outside the quadrature method in its standard meaning).

Let us denote by T_n the Nyström approximation of the operator T,

$$(T_n v)(x) = \sum_{j=1}^{n} w_{j,n} K(x, x_{j,n}) v(x_{j,n}), \quad v \in C[0,1]. \tag{2.8}$$

There is no norm convergence $\|T - T_n\|_{\mathcal{L}(\mathcal{C})}$ to 0, nevertheless, different approaches have been proposed to prove the convergence of the Nyström method, see, e.g., [**1–5, 9, 11, 17, 19**], especially the works on collectively compact approximation theory [**1**] and discrete compact convergence theory [**17, 18, 20**]. Namely, it follows from (2.5) and the continuity of the kernel K that $T_n \to T$ compactly in $C[0,1]$, i.e.,

- $\|Tu - T_n u\|_0 \to 0$ as $n \to \infty$ for every $u \in C[0,1]$;
- $(u_n) \subset C[0,1]$, $\|u_n\|_0 \le 1 \Rightarrow (T_n u_n)$ is relatively compact in $C[0,1]$.

By Lemma 2.1, for all sufficiently large n, say for $n \geq n_0$, the operators $I-T_n$ are invertible in $C[0,1]$, and the inverses are uniformly bounded,

$$\|(I-T_n)^{-1}\|_{\mathcal{L}(C)} \leq c, \quad n \geq n_0. \tag{2.9}$$

We obtain the following well-known result.

Theorem 2.1. *Assume* (2.1)–(2.3). *Further, assume that* $K(x,y)$ *has for* $(x,y) \in [0,1]\times[0,1]$ *the continuous derivatives*

$$\frac{\partial}{\partial x}^{i} K(x,y), \ \frac{\partial}{\partial y}^{i} K(x,y), \quad i = 0,1,\dots,m,$$

and the homogenous integral equation $v = Tv$ *has only the trivial solution. Then, with any* $f \in C^m[0,1]$, *the quadrature system* (2.7) *has for all sufficiently large* n *a unique solution* $u_n(x_i)$, $i = 1,\dots,n$, *and for the Nyström extension* $u_n(x)$ *defined by* (2.6), *the error estimate*

$$\|u-u_n\|_0 \leq cn^{-m}\|f\|_m, \quad n \geq n_0, \tag{2.10}$$

holds where $u = (I-T)^{-1}f \in C^m[0,1]$ *is the solution of* (1.1) *and the constant* c *is independent of* n *and* f. *In particular, for the quadrature solution we have*

$$\max_{1\leq i\leq n} |u(x_{i,n}) - u_n(x_{i,n})| \leq cn^{-m}\|f\|_m, \quad n \geq n_0. \tag{2.11}$$

Proof. Under formulated conditions on K and f, the solution $u = (I-T)^{-1}f$ of equation (1.1) belongs to $C^m[0,1]$, and with $u_n = (I-T_n)^{-1}f$, we have

$$(I-T_n)(u-u_n) = Tu - T_nu.$$

Together with (2.3), (2.9) and the equality

$$(Tu)(x) - (T_nu)(x) = \varphi_n(K(x,\cdot)u(\cdot)), \quad 0 \leq x \leq 1, \tag{2.12}$$

this yields the estimate $\|u-u_n\|_0 \leq cn^{-m}\|u\|_m \leq c'n^{-m}\|f\|_m$. □

Remark 2.1. Assume, in addition to the conditions of Theorem 2.1, that $K(x,y)$ has continuous derivatives

$$\frac{\partial}{\partial x}^{i}\frac{\partial}{\partial y}^{j}K(x,y), \quad i\le m, \quad j\le m, \quad (x,y)\in[0,1]\times[0,1]. \tag{2.13}$$

Then

$$\|T-T_n\|_{\mathcal{L}(C^m)}\le cn^m, \tag{2.14}$$

and this can be used for an alternative convergence proof of the Nyström method under the strengthened smoothness conditions on the kernel; moreover, the error estimate in the C^m norm follows:

$$\|u-u_n\|_m\le cn^{-m}\|f\|_m, \quad n\ge n_0.$$

In the sequel we need the following perturbation result.

Theorem 2.2. *Let the conditions of Theorem* 2.1 *be fulfilled. Consider the perturbed quadrature system*

(2.15)

$$\tilde u_n(x_{i,n})=\sum_{j=1}^{n}w_{j,n}\left(K(x_{i,n},x_{j,n})+\tau_{i,j,n}\right)\tilde u_n(x_{j,n})+f(x_{i,n})+\sigma_{i,n},$$
$$i=1,\dots,n,$$

where

$$\max_{1\le i,j\le n}|\tau_{i,j,n}|\le cn^{-m}, \quad \max_{1\le i\le n}|\sigma_{i,n}|\le cn^{-m}\|f\|_m. \tag{2.16}$$

Then system (2.15) *is uniquely solvable for all sufficiently large* n*, and for the solutions of* (2.7) *and* (2.15) *the following estimate holds*

$$\max_{1\le i\le n}|u_n(x_{i,n})-\tilde u_n(x_{i,n})|\le cn^{-m}\|f\|_m, \quad n\ge n_0. \tag{2.17}$$

Proof. Denote by $A_n=(a_{i,j,n})_{i,j=1}^n$, $a_{i,j,n}=\delta_{i,j}-w_{j,n}K(x_{i,n},x_{j,n})$, the matrix of system (2.7). Introduce the matrix norm $\|A_n\|=$

$\max_{1\le i\le n}\sum_{j=1}^{n}|a_{i,j,n}|$ which is the operator norm with respect to the vector norm $\|u_n\| = \max_{1\le i\le n}|u_{i,n}|$ for $u_n = (u_{1,n},\dots,u_{n,n})$. With these designations, (2.9) implies

$$\|A_n^{-1}\| \le c, \quad n \ge n_0. \tag{2.18}$$

Having this stability estimate, the assertions of Theorem 2.2 follow by standard argument omitted here. □

Direct addressing of the quadrature method to equation (1.1) creates $n \times n$ system (2.7) which can be solved, e.g., by the Gauss method in $O(n^3)$ flops. In Section 3 we show how the amount of work can be reduced to $O(n^2)$ flops using two grid iterations; in Section 4 we reduce the computational cost to $O(n)$ flops by using a "cheaper" approximation of the kernel K which maintains the accuracy order (2.11). This will be used as the skeleton of a fast (C, C^m) solver with $n_\star = n$.

3. Preliminaries: Quadrature two grid iterations.

3.1 *Operator form of the two grid iterations.* Two grid iteration methods involve two approximation levels. On the fine level n, only the direct Nyström operator T_n is applied, whereas on the coarse level ν, also the inverse operator $(I - T_\nu)^{-1}$ is involved, see [**2, 3, 8, 10**] for the Nyström method and [**9, 14, 21, 23**] for related methods. In this section we specify to our needs some results of [**2, 3, 8**].

Let $\nu \in \mathbf{N}$ satisfy $n_0 \le \nu \le n$ with n_0 from (2.9). Using quadrature (2.1) on the level ν, introduce the Nyström operator T_ν similarly to (2.8),

$$(T_\nu v)(x) = \sum_{j=1}^{\nu} w_{j,\nu} K(x, x_{j,\nu}) v(x_{j,\nu}), \quad v \in C[0,1].$$

Applying $(I - T_\nu)^{-1}$ to both sides of the equation $u_n = T_n u_n + f$ (the Nyström equation (2.6)), we obtain the equivalent equation

$$u_n = S_n u_n + (I - T_\nu)^{-1} f \quad \text{with} \quad S_n = S_{n,\nu} = (I - T_\nu)^{-1}(T_n - T_\nu), \tag{3.1}$$

which can be solved by the iteration method

$$u_n^0 = 0, \quad u_n^k = S_n u_n^{k-1} + (I - T_\nu)^{-1} f, \quad k = 1, 2, \dots . \tag{3.2}$$

This is the two grid iteration method for the Nyström equation (2.6) (method number 1 in [**2, 3**]).

Lemma 3.1. *Let $\nu \in \mathbf{N}$ satisfy $n_0 \le \nu \le n$ with n_0 from* (2.9)*, and let the conditions of Theorem* 2.1 *be fulfilled. Then*

$$\|S_{n,\nu}\|_{\mathcal{L}(C,C^m)} \le c_1, \quad \|S_{n,\nu}\|_{\mathcal{L}(C^m,C)} \le c_2\nu^{-m}, \quad n_0 \le \nu \le n, \tag{3.3}$$

$$\|S_{n,\nu}^{2l-1}\|_{\mathcal{L}(C^m,C)} \le c_1^{l-1}c_2^l\nu^{-lm}, \quad \|S_{n,\nu}^{2l}\|_{\mathcal{L}(C^m)} \le c_1^l c_2^l \nu^{-lm}, \quad l = 1, 2, \dots, \tag{3.4}$$

where the constants c_1 and c_2 are independent of n and ν.

Proof. Due to (2.2), $\|T_n\|_{\mathcal{L}(C,C^m)} \le c$, $n \in \mathbf{N}$. To obtain the first one of inequalities (3.3), it remains to notice that, due to (2.9), also

$$\|(I - T_n)^{-1}\|_{\mathcal{L}(C^m)} \le c, \quad n \ge n_0. \tag{3.5}$$

Indeed, $(I - T_n)^{-1} = I + T_n(I - T_n)^{-1}$, and for $u \in C^m[0,1]$ we have

$$\begin{aligned}\|(I - T_n)^{-1}u\|_m &\le \|u\|_m + \|T_n(I - T)_n^{-1}u\|_m \\ &\le \|u\|_m + \|T_n\|_{\mathcal{L}(C,C^m)}\|(I - T_n)^{-1}\|_{\mathcal{L}(\mathcal{C})}\|u\|_0 \\ &\le c\|u\|_m, \quad n \ge n_0.\end{aligned}$$

In view of (2.9), the second one of inequalities (3.3) is a consequence of the relations $T_n - T_\nu = (T - T_\nu) - (T - T_n)$ and $\|T - T_n\|_{\mathcal{L}(C^m,C)} \le cn^{-m}$, $\|T - T_\nu\|_{\mathcal{L}(C^m,C)} \le c\,\nu^{-m}$ that directly follow from (2.3) and (2.12).

Inequalities (3.4) can be easily obtained by induction using (3.3). □

Assume the conditions of Lemma 3.1. With $u_n = (I - T_n)^{-1}f$, we have

$$\begin{aligned}u_n - u_n^k &= S_n(u_n - u_n^{k-1}) = S_n^2(u_n - u_n^{k-2}) = \dots \\ &= S_n^k(u_n - u_n^0) = S_n^k u_n,\end{aligned}$$

and due to (3.4) and (3.5),

$$\begin{aligned}\|u_n - u_n^{2l-1}\|_0 &\le c_1^{l-1} c_2^l \nu^{-lm} \|u_n\|_m \\ &\le c_1^{l-1} c_2^l c\, \nu^{-lm} \|f\|_m, \quad l = 1, 2, \dots .\end{aligned}$$

For $\nu \asymp n^\rho$, $0 < \rho < 1$, $l \ge 1/\rho$, this estimate yields

$$\|u_n - u_n^{2l-1}\|_0 \le c_1^{l-1} c_2^l c' n^{-m} \|f\|_m.$$

Together with Theorem 2.1, this implies the following result.

Theorem 3.1. *Assume the conditions of Theorem* 2.1. *Further, let* $\nu \asymp n^\rho$, $0 < \rho < 1$. *Then for* $k = k_\star = 2l - 1$ *with an* $l \ge 1/\rho$, *the optimal error estimate*

$$\|u - u_n^k\|_0 \le c n^{-m} \|f\|_m \tag{3.6}$$

holds true where $u = (I - T)^{-1} f$ *is the solution of* (1.1) *and* u_n^k *is computed via the two grid iteration method* (3.2).

For instance, in the case of $\rho = 1/3$, one of the most important values of ρ, estimate (3.6) holds already for $k = 5$. Moreover, this number can be reduced to $k = 3$ if $K(x, y)$ has continuous derivatives (2.13), see Remark 3.1 below. A general qualitative consequence is that the accuracy comparable with (2.10) is achieved by u_n^k for an iteration number k which remains to be bounded as $n \to \infty$.

Remark 3.1. If in addition to conditions of Lemma 3.1, $K(x, y)$ has continuous derivatives (2.13) then due to (2.14)

$$\|S_{n,\nu}\|_{\mathcal{L}(C^m)} \le c\, \nu^{-m}, \quad n_0 \le \nu \le n. \tag{3.7}$$

Remark 3.2. Instead of a priori fixing the number of iterations, the two grid iterations (3.2) may be stopped on the first $k = k_n$ for which

$$\max_{1 \le i \le n} |u_n^k(x_{i,n}) - u_n^{k-1}(x_{i.n})| \le \delta n^{-m} \max_{1 \le i \le n} |f(x_{i,n})| \tag{3.8}$$

where $\delta > 0$ is a parameter. Under the conditions of Theorem 3.1, (3.8) implies (3.6). Moreover, k_n remains bounded as $n \to \infty$.

Remark 3.3. Let the conditions of Theorem 3.1 be fulfilled. If, in the two grid iterations (3.2), the operations of the type $T_n v_n$, $T_\nu v_n$ and $(I - T_\nu)^{-1} v_n$ are applied approximately with an accuracy $O(n^{-m}\|v_n\|_0)$, then estimate (3.6) still holds true.

3.2 *Matrix form of the two grid iterations.* Let us present the matrix form of the two grid iterations (3.2) and discuss the implementation cost. On every iteration step we will compute the values of u_n^k on two grids, namely, on the fine grid $x_{i,n}$, $i = 1, \dots, n$, and on the coarse grid $x_{i,\nu}$, $i = 1, \dots, \nu$, and we also compute an auxiliary function $f_n^k := f + T_n u_n^{k-1} - T_\nu u_n^{k-1}$ on these two grids. For brevity, denote provisionally

$$\begin{aligned} u_i^k &= u_n^k(x_{i,n}), \quad f_i^k = f_n^k(x_{i,n}), \\ i &= 1, \dots, n \quad \text{(the values on the fine grid)}, \\ v_i^k &= u_n^k(x_{i,\nu}), \quad g_i^k = f_n^k(x_{i,\nu}), \\ i &= 1, \dots, \nu \quad \text{(same for the coarse grid)}. \end{aligned}$$

The following algorithm is a realization of the two grid iteration method (3.2); the number of iterations, $k_\star$, was defined in Theorem 3.1.

Algorithm 3.1. Given $\nu, n \in \mathbf{N}$, $\nu \asymp n^\rho$, $0 < \rho < 1$, fix integers $l \geq 1/\rho$, $k_\star = 2l - 1$, and set $u_i^0 = f_i^0 = 0$, $i = 1, \dots, n$, $v_i^0 = g_i^0 = 0$, $i = 1, \dots, \nu$. For $k = 1, \dots, k_\star$, do the following:

1. compute

$$f_i^k = f_i^{k-1} + f(x_{i,n}) + \sum_{j=1}^{n} w_{j,n} K(x_{i,n}, x_{j,n}) u_j^{k-1} - u_i^{k-1},$$
$$i = 1, \dots, n,$$

$$g_i^k = g_i^{k-1} + f(x_{i,\nu}) + \sum_{j=1}^{n} w_{j,n} K(x_{i,\nu}, x_{j,n}) u_j^{k-1} - v_i^{k-1},$$
$$i = 1, \dots, \nu;$$

2. compute v_i^k, $i=1,\dots,\nu$, by solving the $\nu\times\nu$ system

$$v_i^k = \sum_{j=1}^{\nu} w_{j,\nu} K(x_{i,\nu}, x_{j,\nu}) v_j^k + g_i^k, \quad i = 1,\dots,\nu; \tag{3.9}$$

3. compute

$$u_i^k = f_i^k + \sum_{j=1}^{\nu} w_{j,\nu} K(x_{i,n}, x_{j,\nu}) v_j^k, \quad i = 1,\dots,n;$$

the output is given by $u_i^{k_\star} = u_n^{k_\star}(x_{i,n})$, $i = 1,\dots,n$.

The computational cost of Algorithm 3.1 is of the same order as that for one iteration: $O(n^2)$ flops for the application of the $n\times n$, $n\times\nu$, $\nu\times n$ matrices, plus $O(n^{3\rho})$ flops for solving the $\nu\times\nu$ system (3.9) by the Gauss elimination or some other standard method of the same complexity. The solution of the system reduces to $O(n)$ flops if we take $\rho\le 1/3$. To obtain a fast solver with $n_\star = n$, we need to reduce to $O(n)$ flops also the cost of the application of the $n\times n$, $n\times\nu$, $\nu\times n$ matrices in Algorithm 3.1. At first look it may seem impossible since the underlying matrices are fully populated. Nevertheless, we will succeed due to the circumstance that an error of order $O(n^{-m})$ is acceptable in the matrix applications. Technically, instead of the kernel function K, we will use a suitable piecewise polynomial interpolant of K; at this stage, we are forced to strengthen the smoothness condition on K assuming that $K\in C^{m'}$, $m'\ge 2m$. By this approximation the information we need about K is reduced to $O(n)$ sample values as required in the definition of the fast (C, C^m) solver with $n_\star = n$.

4. Fast solvers.

4.1 *Approximation of the kernel function.* Let ξ_λ, $\lambda = 1,\dots,m'$, be some interpolation nodes in $[-1,1]$:

$$-1\le\xi_1<\xi_2<\cdots<\xi_{m'}\le 1.$$

Denote by

$$\ell_\lambda(\xi) = \frac{(\xi-\xi_1)\dots(\xi-\xi_{\lambda-1})(\xi-\xi_{\lambda+1})\dots(\xi-\xi_{m'})}{(\xi_\lambda-\xi_1)\dots(\xi_\lambda-\xi_{\lambda-1})(\xi_\lambda-\xi_{\lambda+1})\dots(\xi_\lambda-\xi_{m'})},$$
$$\lambda = 1,\dots,m',$$

the corresponding Lagrange fundamental polynomials of degree $m'-1$ that satisfy

$$\ell_\lambda(\xi_\mu)=\delta_{\lambda,\mu}\quad\text{(Kronecker symbol)},\quad \lambda,\mu=1,\dots,m'.$$

Take a number $n'\in\mathbf{N}$ (later we will assume that either $n'\asymp n^{m/m'}$ or $n'\asymp n$). Introduce subintervals $\mathcal{I}_p=\mathcal{I}_{p,n'}$, $p=1,\dots,n'$, such that the closure of $\mathcal{I}_p$ is $[(p-1)/n', p/n']$, $\mathcal{I}_p\cap\mathcal{I}_q=\varnothing$ for $p\neq q$ and $\cup_{p=1}^{n'}\mathcal{I}_p=[0,1]$. To be concrete, we adopt the following agreement: if $\xi_1=-1$, $\xi_{m'}<1$, then

$$\mathcal{I}_p=[(p-1)/n',\,p/n'),\quad p=1,\dots,n'-1,\quad \mathcal{I}_{n'}=[(n'-1)/n',\,1];$$

otherwise

$$\mathcal{I}_1=[0,\,1/n'],\quad \mathcal{I}_p=((p-1)/n',\,p/n'],\quad p=2,\dots,n'.$$

With the help of the affine transformation $[-1,1]\mapsto[(p-1)/n', p/n']$ we transfer ξ_λ into the points $\xi_{p,\lambda}$ of the intervals $[(p-1)/n', p/n']$:

$$\xi_{p,\lambda}=\xi_{p,\lambda,n'}=\frac{p-1}{n'}+\frac{\xi_\lambda+1}{2n'},\quad \lambda=1,\dots,m',:\,p=1,\dots,n'.$$

With the help of the inverse affine transformation $[(p-1)/n', p/n']\mapsto[-1,1]$ we transfer ℓ_λ into Lagrange fundamental polynomials $\ell_\lambda(-1+2(n'x-p+1))$ of degree $m'-1$ corresponding to the nodes $\xi_{p,\lambda}$, $\lambda=1,\dots,m'$; denote

$$L_{p,\lambda}(x)=L_{p,\lambda,n'}(x)=\left\{\begin{matrix}\ell_\lambda(-1+2(n'x-p+1)) & x\in\mathcal{I}_p\\ 0 & x\in[0,1]\setminus\mathcal{I}_p\end{matrix}\right\},$$
$$\lambda=1,\dots,m',$$

where $p=1,\dots,n'$. Let us approximate the kernel $K(x,y)$ by the piecewise polynomial function $K_{n'}(x,y)$ defined on every square $\mathcal{I}_p\times\mathcal{I}_q$, $p,q=1,\dots,n'$, independently as the interpolation polynomial of degree $m'-1$ with respect to x and y:

$$K_{n'}(x,y)=\sum_{\lambda,\mu=1}^{m'}K(\xi_{p,\lambda},\xi_{q,\mu})L_{p,\lambda}(x)L_{q,\mu}(y),\quad (x,y)\in\mathcal{I}_p\times\mathcal{I}_q.$$

Respectively, on whole $[0,1]\times[0,1]$, $K_{n'}(x,y)$ is given by

(4.1)

$$K_{n'}(x,y)=\sum_{p,q=1}^{n'}\sum_{\lambda,\mu=1}^{m'}K(\xi_{p,\lambda},\xi_{q,\mu})L_{p,\lambda}(x)L_{q,\mu}(y),\quad 0\le x,y\le 1.$$

In general, $K_{n'}(x,y)$ may have jumps as x crosses an inner line $x=\xi_{\lambda,p}$ or y crosses an inner line $y=\xi_{\mu,q}$; under the conditions of Lemma 4.1 below, the jumps are of the order $O((n')^{-m'})$. If $\xi_1=-1$, $\xi_{m'}=1$ then $K_{n'}(x,y)$ is continuous on $[0,1]\times[0,1]$.

Lemma 4.1. *Assume that $K(x,y)$ has for $(x,y)\in[0,1]\times[0,1]$ continuous derivatives*

$$\frac{\partial}{\partial x}^{l}K(x,y),\quad \frac{\partial}{\partial y}^{l}K(x,y),\quad l=0,1,\dots,m'. \tag{4.2}$$

Then

$$\sup_{0\le x,y\le 1}|K(x,y)-K_{n'}(x,y)|\le c(n')^{-m'}. \tag{4.3}$$

In particular,

$$\sup_{0\le x,y\le 1}|K(x,y)-K_{n'}(x,y)|\le cn^{-m}\quad \textit{if } n'\asymp n^{m/m'}. \tag{4.4}$$

Proof. Introduce the constants

$$\begin{aligned}\omega_{m'}&=\max_{-1\le\xi\le 1}|(\xi-\xi_1)\dots(\xi-\xi_{m'})|\\&=(2n')^{m'}\sup_{x\in\mathcal{I}_p}|(x-\xi_{p,1})\dots(x-\xi_{p,m'})|,\end{aligned}$$

$$\gamma_{m'}=\max_{-1\le\xi\le 1}\sum_{\lambda=1}^{m'}|\ell_\lambda(\xi)|=\sup_{x\in\mathcal{I}_p}\sum_{\lambda=1}^{m'}|L_{p,\lambda}(x)|,\quad p=1,\dots,n',$$

which characterize the quality of the interpolation nodes $\xi_1,\dots,\xi_{m'}$. For the one-dimensional interpolant

$$\overline{K}_{n'}(x,y)=\sum_{\lambda=1}^{m'}K(\xi_{p,\lambda},y)L_{p,\lambda}(x),\quad x\in\mathcal{I}_p,$$

with y as a parameter, the well-known error estimate yields

$$\sup_{x\in\mathcal{I}_p}|K(x,y)-\overline{K}_{n'}(x,y)| \leq \frac{1}{m'!}\sup_{x\in\mathcal{I}_p}\left|\frac{\partial}{\partial x}^{m'}K(x,y)\right|\sup_{x\in\mathcal{I}_p}|(x-\xi_{p,1})\dots(x-\xi_{p,m'})|,$$

$$\sup_{(x,y)\in\mathcal{I}_p\times\mathcal{I}_q}|K(x,y)-\overline{K}_{n'}(x,y)| \leq \frac{\omega_{m'}}{m'!}\sup_{(x,y)\in\mathcal{I}_p\times\mathcal{I}_q}\left|\frac{\partial}{\partial x}^{m'}K(x,y)\right|(2n')^{-m'}.$$

Further, for $(x,y)\in\mathcal{I}_p\times\mathcal{I}_q$,

$$\overline{K}_{n'}(x,y)-K_{n'}(x,y) = \sum_{\lambda=1}^{m'}\left(K(\xi_{p,\lambda},y)-\sum_{\mu=1}^{m'}K(\xi_{p,\lambda},\xi_{q,\mu})L_{q,\mu}(y)\right)L_{p,\lambda}(x),$$

and using again the one-dimensional interpolation error estimate we obtain

$$\left|K(\xi_{p,\lambda},y)-\sum_{\mu=1}^{m'}K(\xi_{p,\lambda},\xi_{q,\mu})L_{q,\mu}(y)\right| \leq \frac{\omega_{m'}}{m'!}\sup_{y\in\mathcal{I}_q}\left|\frac{\partial}{\partial y}^{m'}K(\xi_{p,\lambda},y)\right|(2n')^{-m'},$$

$$\sup_{(x,y)\in\mathcal{I}_p\times\mathcal{I}_q}|\overline{K}_{n'}(x,y)-K_{n'}(x,y)| \leq \frac{\omega_{m'}}{m'!}\sup_{(x,y)\in\mathcal{I}_p\times\mathcal{I}_q}\left|\frac{\partial}{\partial y}^{m'}K(x,y)\right|(2n')^{-m'}\gamma_{m'}.$$

Combining the estimates of $|K(x,y)-\overline{K}_{n'}(x,y)|$ and $|\overline{K}_{n'}(x,y)-K_{n'}(x,y)|$ we obtain (4.3) with

$$c=\frac{2^{-m'}\omega_{m'}}{m'!}\left(\max_{0\leq x,y\leq 1}\left|\frac{\partial}{\partial x}^{m'}K(x,y)\right|+\gamma_{m'}\max_{0\leq x,y\leq 1}\left|\frac{\partial}{\partial y}^{m'}K(x,y)\right|\right).$$

By the symmetry argument (4.3) also holds with

$$c = \frac{2^{-m'}\omega_{m'}}{m'!}\left(\gamma_{m'} \max_{0\le x,y\le 1}\left|\frac{\partial}{\partial x}^{m'} K(x,y)\right| + \max_{0\le x,y\le 1}\left|\frac{\partial}{\partial y}^{m'} K(x,y)\right|\right)$$

and with the minimum of these two constants. □

As a consequence of (2.2) and (4.4) we obtain that, in case $n' \asymp n^{m/m'}$,

$$\max_{1\le i\le n}\left|\sum_{j=1}^{n} w_{j,n}K(x_{i,n},x_{j,n})u_j - \sum_{j=1}^{n} w_{j,n}K_{n'}(x_{i,n},x_{j,n})u_j\right| \le cn^{-m}\max_{1\le i\le n}|u_i|, \tag{4.5}$$

and similar estimates hold for other matrix applications in Algorithm 3.1. Replacing in Algorithm 3.1 $K(x,y)$ by $K_{n'}(x,y)$, we still solve system (2.7) with the accuracy (3.6), see Theorem 2.2 and Remark 3.3. In particular, stability estimate (2.18) guarantees that the application of $(I-T_\nu)^{-1}$, i.e., the solution of system (3.9), is performed with the accuracy $O(n^{-m})$ when K is replaced by $K_{n'}$.

Designing fast (C, C^m) solvers 1–3 in subsections 4.2–4.4, we assume that $m' \ge 2m$ and $K(x,y)$ has continuous derivatives (4.2). Of course, we always assume also that the homogenous integral equation $u = Tu$ has only the trivial solution.

4.2 *Fast solver* 1. Set $n_\star = n$, $n' \asymp n^{m/m'}$. After replacing $K(x,y)$ by $K_{n'}(x,y)$, the number of sample points for $K(x,y)$ we need in the quadrature system (2.7) reduces to $O((n')^2) = O(n^{2m/m'})$. Moreover, the matrix-vector multiplications

$$\sum_{j=1}^{n} w_{j,n}K_{n'}(x_{i,n},x_{j,n})u_j, \quad \sum_{j=1}^{\nu} w_{j,\nu}K_{n'}(x_{i,n},x_{j,\nu})v_j, \quad i=1,\dots,n,$$

$$\sum_{j=1}^{n} w_{j,n}K_{n'}(x_{i,\nu},x_{j,n})u_j, \quad i=1,\dots,\nu,$$

can be performed in $O(n)$ flops. Indeed, consider the first of them which is most laborious. Denoting

$$J_p = J_{p,n,n'} = \{j \in \mathbf{N} :\ x_{j,n} \in \mathcal{I}_p\}, \quad p = 1,\dots,n',$$

we have

$$\begin{aligned}
&\sum_{j=1}^{n} w_{j,n} K_{n'}(x_{i,n}, x_{j,n}) u_j \\
&\quad = \sum_{j=1}^{n} w_{j,n} \sum_{p,q=1}^{n'} \sum_{\lambda,\mu=1}^{m'} K(\xi_{p,\lambda}, \xi_{q,\mu}) L_{\lambda,p}(x_{i,n}) L_{\mu,q}(x_{j,n}) u_j \\
&\quad = \sum_{\lambda=1}^{m'} L_{\lambda,p}(x_{i,n}) \sum_{q=1}^{n'} \sum_{\mu=1}^{m'} K(\xi_{p,\lambda}, \xi_{q,\mu}) \sum_{j \in J_q} L_{\mu,q}(x_{j,n}) w_{j,n} u_j, \\
&\qquad\qquad i \in J_p, \quad p = 1, \ldots, n'.
\end{aligned}$$

The computation of the quantities

$$u'_{\mu,q} := \sum_{j \in J_q} L_{\mu,q}(x_{j,n}) w_{j,n} u_j, \quad \mu = 1, \ldots, m', \quad q = 1, \ldots, n',$$

costs $O(\sum_{q=1}^{n'} (\sharp J_q)) = O(n)$ flops. Further, the computation of

$$u''_{\lambda,p} := \sum_{q=1}^{n'} \sum_{\mu=1}^{m'} K(\xi_{p,\lambda}, \xi_{q,\mu}) u'_{\mu,q}, \quad \lambda = 1, \ldots, m', \quad p = 1, \ldots, n',$$

costs $O((n')^2) = O(n)$ flops (recall that $m' \geq 2m$, $n' \asymp n^{m/m'}$). Finally, the computation of

$$\begin{aligned}
\sum_{\lambda=1}^{m'} L_{p,\lambda}(x_{i,n}) u''_{\lambda,p} &= \sum_{j=1}^{n} w_{j,n} K_{n'}(x_{i,n}, x_{j,n}) u_j, \\
i \in J_p, \quad p &= 1, \ldots, n',
\end{aligned}$$

costs again $O(n)$.

Algorithm 3.1 contains also the solution of $\nu \times \nu$-systems (3.9). Applying the Gauss elimination method, we remain in $O(n)$ flops for $\nu \asymp n^{\rho}$, $0 < \rho \leq 1/3$. We see that with $K_{n'}(x, y)$ in the role of $K(x, y)$, the two grid iteration Algorithm 3.1 can be realized in $O(n)$ flops. Together with the accuracy (3.6) achieved by $u_n^{k_\star}$, this means that we have a good basis for a fast (C, C^m) solver. We only have

to specify how the values of $u_n^{k_\star}$ are computed between the quadrature knots $x_{i,n}$, $i = 1, \dots, n$, at the cost of $O(1)$ flops per value, maintaining the accuracy order (3.6). A local interpolation by polynomials of degree $m-1$ using m neighboring knots $x_{i,n}$ is suitable: due to (2.3), $\max_{2\le i\le n}(x_{i,n} - x_{i-1,n}) \le cn^{-1}$, and together with (2.11), (3.6) we obtain that the error of the local interpolant $\Pi_{m,n}u_n^{k_\star}$ is estimated by

$$\sup_{0\le x\le 1} |u(x) - (\Pi_{m,n}u_n^{k_\star})(x)| \le cn^{-m}\|u\|_m \le c'n^{-m}\|f\|_m.$$

Thus, Algorithm 3.1 based on quadrature formula (2.1)–(2.3), with $\nu \asymp n^\rho$, $0 < \rho \le 1/3$, and $K_{n'}(x,y)$ in the role of $K(x,y)$, $n' \asymp n^{m/m'}$, together with the local interpolation of the computed iteration solution $u_n^{k_\star}(x_{i,n})$, $i = 1, \dots, n$, by polynomials of degree $m-1$, constitute a fast (C, C^m) solver of integral equation (1.1) with $n_\star = n$. We quote it as fast solver 1.

4.3 *Fast solver* 2. Looking for the solution of integral equation (1.1) in the form $u = v + f$, the function v satisfies the equation

$$v = Tv + g, \quad g = Tf. \tag{4.6}$$

It follows from the existence and continuity on $[0,1]\times[0,1]$ of derivatives (4.2) that $g \in C^{m'}[0,1]$ and $v \in C^{m'}[0,1]$ for the solution of (4.6). Solving (4.6) by the quadrature method it is reasonable to apply a quadrature of order (2.4). So we assume in this subsection that quadrature formula (2.1) satisfies conditions (2.2) and (2.4). Then the quadrature method and two grid iteration Algorithm 3.1 applied to integral equation (4.6) are of accuracy $O(n^{-m'})$ provided that the values $g(x_{i,n}) = \int_0^1 K(x_{i,n}, y)f(y)\,dy$, $i = 1, \dots, n$, have been computed with an accuracy $O(n^{-m'})$, see Theorems 2.2 and 3.1. Note that $O(n^{-m'}) = O(n_\star^{-m})$ with $n_\star \asymp n^{m'/m}$. Respectively, we may spend $O(n_\star) = O(n^{m'/m})$ flops to compute $g(x_{i,n})$, $i = 1, \dots, n$, and solve the quadrature $n \times n$ system. This time we use the exact values $K(x,y)$ in the matrix of the system; the interpolation approximation of $K(x,y)$ is delegated to compute the righthand terms of the quadrature system. Put $n' \asymp n$ and define

$$\widetilde{K}_n(x_{i,n}, y) = \sum_{q=1}^{n'}\sum_{\mu=1}^{m'} K(x_{i,n}, \xi_{q,\mu})L_{q,\mu}(y), \quad 0 \le y \le 1, \quad i = 1, \dots, n,$$

$$g_n(x_{i,n}) = \int_0^1 \widetilde{K}_n(x_{i,n}, y) f(y)\, dy$$
$$= \sum_{q=1}^{n'} \sum_{\mu=1}^{m'} K(x_{i,n}, \xi_{q,\mu}) \int_{\mathcal{I}_q} L_{q,\mu}(y) f(y)\, dy, \quad i = 1, \dots, n,$$

with $\xi_{q,\mu}$ and $L_{q,\mu}$ introduced in Section 4.1. Then

$$\max_{1\le i\le n} |g(x_{i,n}) - g_n(x_{i,n})| \le \max_{1\le i\le n} \int_0^1 |K(x_{i,n}, y) - \widetilde{K}_n(x_{i,n}, y)|\, dy \|f\|_0$$
$$\le c n^{-m'} \|f\|_0.$$

Moreover, $g_n(x_{i,n})$, $i = 1, \dots, n$, can be computed with the accuracy $O(n^{-m'})$ in $O(n^{m'/m})$ flops. Indeed, divide $\mathcal{I}_q$, $q = 1, \dots, n'$, into subintervals of the length $h \asymp n^{-m'/m}$, then we have all together $O(n^{m'/m})$ subintervals of $[0, 1]$. On those subintervals we approximate the integrals of $L_{q,\mu}(y) f(y)$ by an $O(1)$ point quadrature formula that is sharp for polynomials of degree $m' + m - 2$. Then the computation of all integrals

$$f_{q,\mu} := \int_{\mathcal{I}_q} L_{q,\mu}(y) f(y)\, dy \approx \tilde{f}_{q,\mu}, \quad \mu = 1, \dots, m', \quad q = 1, \dots, n',$$

costs $O(n^{m'/m})$ flops, and we achieve an accuracy comparable with the accuracy of the best approximation of f by polynomials of degree $m-1$ on subintervals of the length h. For $f \in C^m[0, 1]$,

$$|f_{q,\mu} - \tilde{f}_{q,\mu}| \le c \frac{1}{n'} h^m \|f\|_m \le c' n^{-m'-1} \|f\|_m \tag{4.7}$$

where $1/n'$ is the length of $\mathcal{I}_q$. Other operations in the computation of $g_n(x_{i,n})$, $i = 1, \dots, n$, cost $O(n^2)$ flops that is also $O(n^{m'/m})$ since $m' \ge 2m$.

Thus we are in the position to compute the approximations

$$g_{i,n} := \sum_{q=1}^{n'} \sum_{\mu=1}^{m'} K(x_{i,n}, \xi_{q,\mu}) \tilde{f}_{q,\mu}, \quad i = 1, \dots, n,$$

with the accuracy

$$\max_{1\le i\le n} |g(x_{i,n}) - g_{i,n}| \le cn^{-m'}\|f\|_m$$

at the cost of $O(n^{m'/m})$ flops. After that we solve the quadrature system

$$v_n(x_{i,n}) = \sum_{j=1}^{n} w_{j,n}K(x_{i,n}, x_{j,n})v_n(x_{j,n}) + g_{i,n}, \quad i = 1, \dots, n, \tag{4.8}$$

either directly by the Gauss method (case $m' \ge 3m$, then $O(n^3) = O(n^{m'/m})$) or by the two grid iteration Algorithm 3.1 (case $2m \le m' < 3m$) with $\nu \asymp n^\rho$, $0 < \rho \le m'/(3m)$. All this costs $O(n^{m'/m})$ flops and we obtain approximations $v_{i,n} \approx v(x_{i,n})$, $i = 1, \dots, n$, of the accuracy

$$\max_{1\le i\le n} |v(x_{i,n}) - v_{i,n}| \le cn^{-m'}\|f\|_m.$$

Using the local interpolation by polynomials of degree $m' - 1$, we extend the grid solution $v_{i,n}$, $i = 1, \dots, n$, up to a piecewise polynomial function $\Pi_{m',n}v_n$ of the accuracy

$$\sup_{0\le x\le 1} |v(x) - (\Pi_{m',n}v_n)(x)| \le cn^{-m'}(\|v\|_{m'} + \|f\|_m) \le c'n^{-m'}\|f\|_m.$$

The function $u_n = \Pi_{m',n}v_n + f$ satisfies the conditions of the fast (C, C^m) solver of equation (1.1) with $n_\star \asymp n^{m'/m}$; if the computation of a value of f costs more than $O(1)$ flops, we replace f by its piecewise polynomial interpolant of degree $m - 1$ exploiting once more the grid values of f which already have been used when the integrals $f_{q,\mu}$ on subintervals of the length $h \asymp n^{-m'/m}$ were computed. We quote the described fast (C, C^m) solver as fast solver 2.

Compared with fast solver 1, the main advantage of fast solver 2 is that in the case $m' \ge 3m$ we may solve (4.8) directly by the Gauss method, without two grid iterations. Further, only the computation of $g_{i,n}$, $i = 1, \dots, n$, needs the interpolation approximation of K, whereas in the matrix of the quadrature system (4.8) we now use the values of K itself. On the other hand, fast solver 1 does not need the computation of integrals $f_{q,\mu}$.

4.4 *Fast solver* 3. Introduce the equation

$$u_n(x) = \int_0^1 K_n(x,y)u_n(y)\,dy + f(x), \quad 0 \le x \le 1, \tag{4.9}$$

where $K_n(x,y)$ is defined by (4.1) with $n' = n$:

$$K_n(x,y) = \sum_{p,q=1}^{n} \sum_{\lambda,\mu=1}^{m'} K(\xi_{p,\lambda,n}, \xi_{q,\mu,n}) L_{p,\lambda,n}(x) L_{q,\mu,n}(y), \quad 0 \le x,y \le 1.$$

Since

$$\sup_{0\le x\le 1} \int_0^1 |K(x,y) - K_n(x,y)|\,dy \le \sup_{0\le x,y\le 1} |K(x,y) - K_n(x,y)| \le cn^{-m'} \tag{4.10}$$

the estimate

$$\sup_{0\le x\le 1} |u(x) - u_n(x)| \le cn^{-m'}\|f\|_0 \tag{4.11}$$

holds for the solutions of equations (1.1) and (4.9). To obtain a fast (C, C^m) solver with $n_\star \asymp n^{m'/m}$, we have to solve equation (4.9) with an accuracy $cn^{-m'}\|f\|_m$ at the cost of $O(n^{m'/m})$flops.

The kernel $K_n(x,y)$ is degenerate, and the solution of equation (4.9) has the form

$$u_n(x) = f(x) + \sum_{p=1}^{n} \sum_{\lambda=1}^{m'} v_{p,\lambda,n} L_{p,\lambda,n}(x);$$

the coefficients $v_{p,\lambda,n}$ can be determined as the solution of the system of linear equations

$$v_{p,\lambda,n} = \frac{1}{n} \sum_{q=1}^{n} \sum_{\mu=1}^{m'} \left\{ \sum_{\mu'=1}^{m'} K(\xi_{p,\lambda,n}, \xi_{q,\mu,n}) \gamma_{\mu,\mu'} \right\} v_{q,\mu,n} + g_{p,\lambda,n}, \quad \lambda = 1, \dots, m', \quad p = 1, \dots, n, \tag{4.12}$$

where

$$g_{p,\lambda,n} = \sum_{q=1}^{n} \sum_{\mu=1}^{m'} K(\xi_{p,\lambda,n}, \xi_{q,\mu,n}) f_{q,\mu,n},$$

$$\gamma_{\mu,\mu'} = \int_{-1}^{1} \ell_\mu(\xi) \ell_{\mu'}(\xi)\, d\xi, \quad \mu, \mu' = 1, \dots, m',$$

$$f_{q,\mu,n} = \int_{\mathcal{I}_q} L_{q,\mu,n}(y) f(y)\, dy, \quad \mu = 1, \dots, m', \quad q = 1, \dots, n.$$

The integrals $f_{q,\mu,n}$ were arrived at in Section 4.3, and we compute their quadrature approximations $\tilde{f}_{q,\mu,n}$ of accuracy (4.7) at the cost of $O(n^{m'/m})$ flops in a similar way as we did there. The integrals $\gamma_{\mu,\mu'}$ can be computed analytically, but the exact values of those can be obtained also numerically by applying a quadrature formula which is sharp for polynomials of degree $2m'-2$. An example of such a quadrature formula is given by the m' point Gauss formula

$$\int_{-1}^{1} v(\xi) d\xi \approx \sum_{\lambda=1}^{m'} w_\lambda v(\xi_\lambda)$$

where ξ_λ, $\lambda = 1, \dots, m'$, are the Gauss nodes on $[-1, 1]$ (the zeros of the Legendre polynomial of degree m') and w_λ are corresponding Gauss weights. Moreover, if we use in the constructions of Section 4.1 from the very beginning the Gauss nodes ξ_λ, $\lambda = 1, \dots, m'$, then $\gamma_{\mu,\mu'} = w_\mu \delta_{\mu,\mu'}$, $\mu, \mu' = 1, \dots, m'$, and system (4.12) simplifies to the form

$$v_{p,\lambda,n} = \frac{1}{n} \sum_{q=1}^{n} \sum_{\mu=1}^{m'} w_\mu K(\xi_{p,\lambda,n}, \xi_{q.\mu,n}) v_{q,\mu,n}$$

$$+ \sum_{q=1}^{n} \sum_{\mu=1}^{m'} K(\xi_{p,\lambda,n}, \xi_{q.\mu,n}) f_{q,\mu,n},$$

$$\lambda = 1, \dots, m', \quad p = 1, \dots, n.$$

System (4.12), with $f_{q,\mu,n}$ approximated by $\tilde{f}_{q,\mu,n}$, can be solved by the same strategy as system (4.8) in Section 4.3: by the Gauss method in the case $m' \geq 3m$ and by the two grid iteration method with

$\nu \asymp n^{\rho}$, $0 < \rho \le m'/(3m)$, $k_\star \ge 1/\rho$, in the case $2m \le m' < 3m$, see Appendix A for algorithmical details. By this strategy, the solving of system (4.12) costs $O(n^{m'/m})$ flops, and we obtain an approximate solution $\tilde{v}_{p,\lambda,n} \approx v_{p,\lambda,n}$, $\lambda = 1, \dots, m'$, $p = 1, \dots, n$, of accuracy

$$\max_{\substack{1 \le \lambda \le m' \\ 1 \le p \le n}} |\tilde{v}_{p,\lambda,n} - v_{p,\lambda,n}| \le c n^{-m'} \|f\|_m.$$

For

$$\tilde{u}_n(x) = f(x) + \sum_{p=1}^{n} \sum_{\lambda=1}^{m'} \tilde{v}_{p,\lambda,n} L_{p,\lambda,n}(x) \tag{4.13}$$

this implies

$$\sup_{0 \le x \le 1} |u_n(x) - \tilde{u}_n(x)| \le c n^{-m'} \|f\|_m$$

and, together with (4.11),

$$\sup_{0 \le x \le 1} |u(x) - \tilde{u}_n(x)| \le c n^{-m'} \|f\|_m.$$

Notice that, for any x, the series in (4.13) contains maximally m' nonvanishing terms. Thus $\tilde{u}_n(x)$ satisfies the requirements on the fast (C, C^m) solver; again, if the computation of one value of f costs more than $O(1)$ flops, we replace in (4.13) f by its piecewise polynomial interpolant of degree $m-1$ exploiting the grid values of f already used during the calculation of integrals $f_{p,\lambda,n}$. We quote the designed fast (C, C^m) solver as fast solver 3.

It can be expected that fast solver 3 is more precise than fast solvers 1 and 2 (in the sense of a smaller constant in (1.2) at the same amount of arithmetical work). Unfortunately, the two grid iterations in fast solver 3 have a more complicated matrix form than those in fast solvers 1 and 2, cf. Section 3.2 and Appendix A.

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Appendix

A. Two grid iterations for equation (4.9). Let $\nu \asymp n^{\rho}$, $0 < \rho < 1$; we assume now that $l := n/\nu \in \mathbf{N}$. For simplicity of formulae we also assume that $\xi_1, \dots, \xi_{m'}$ are the nodes of the m' point Gauss quadrature formula on $[-1, 1]$; corresponding Gauss weights are denoted by $w_1, \dots, w_{m'}$. In this subsection we use the designations T_n and T_ν for the integral operators

$$(T_n u)(x) = \int_0^1 K_n(x, y)u(y)\, dy, \quad (T_\nu u)(x) = \int_0^1 K_\nu(x, y)u(y)\, dy$$

(rather than for the Nyström operators in Sections 2 and 3; no Nyström operators will be involved in the sequel). Looking for the solution of equation (4.9) in the form $u_n = v_n + f$, the unknown function v_n is determined from the equation $v_n = T_n v_n + T_n f$ equivalent to

$$v_n = (I - T_\nu)^{-1}(T_n v_n - T_\nu v_n + T_n f). \tag{4.14}$$

Due to (4.10),

$$\|(I - T_\nu)^{-1}(T_n - T_\nu)\|_{\mathcal{L}(L^\infty[0,1])} \le c(n^{-m'} + \nu^{-m'}) \le c' n^{-\rho m'},$$

hence (4.14) can be solved by the iteration method

$$\begin{gathered} v_n^0 = 0, \quad v_n^k = (I - T_\nu)^{-1}(T_n v_n^{k-1} - T_\nu v_n^{k-1} + T_n f), \\ k = 1, \dots, k_\star; \end{gathered} \tag{4.15}$$

the number of iteration steps $k_\star \ge 1/\rho$ is sufficient for the accuracy $\|v_n - v_n^k\|_0 \le c n^{-m'} \|f\|_0$, $n \ge n_0$. Since $(I - T_\nu)^{-1} = I + (I - T_\nu)^{-1} T_\nu$, we can represent the two grid iterations (4.15) in the form

$$\begin{gathered} v_n^0 = 0, \quad z_n^k = T_n v_n^{k-1} - T_\nu v_n^{k-1} + T_n f, \\ v_n^k = z_n^k + (I - T_\nu)^{-1} T_\nu z_n^k, \quad k = 1, \dots, k_\star. \end{gathered} \tag{4.16}$$

Denote

$$\begin{aligned} E_n &= \operatorname{span}\{L_{p,\lambda,n} : \lambda = 1, \dots, m',\ p = 1, \dots, n\}, \\ E_\nu &= \operatorname{span}\{L_{p,\lambda,\nu} : \lambda = 1, \dots, m',\ p = 1, \dots, \nu\} \subset E_n. \end{aligned}$$

Observe that $v_n^k, z_n^k \in E_n$, T_n maps $L^\infty[0,1]$ into E_n, T_ν maps E_n into E_ν, and $(I-T_\nu)^{-1}$ maps E_ν into E_ν. The (approximate) computation of

$$T_n f = \sum_{p=1}^{n} \sum_{\lambda=1}^{m'} g_{p,\lambda,n} L_{p,\lambda,n},$$

$$g_{p,\lambda,n} = \sum_{q=1}^{n} \sum_{\mu=1}^{m'} K(\xi_{p,\lambda,n}, \xi_{q.\mu,n}) f_{q,\mu,n} \in E_n$$

has been already discussed in details. The rest of the operations preserve the elements in E_n and can be reduced to linear procedures over the coefficients of v_n^k with respect to the basis $\{L_{p,\lambda,n} : \lambda = 1,\dots,m',\ p = 1,\dots,n\}$. To perform a two-grid iteration step (4.16), it is sufficient to explain how the following operations are realized algebraically: (i) the computation of $T_n v_n \in E_n$ for $v_n \in E_n$; (ii) the computation of $T_\nu v_n \in E_\nu$ for $v_n \in E_n$; (iii) the computation of $(I-T_\nu)^{-1} v_\nu \in E_\nu$ for $v_\nu \in E_\nu$; (iv) the representation of $v_\nu \in E_\nu \subset E_n$ in the basis $\{L_{p,\lambda,n} : \lambda = 1,\dots,m',\ p = 1,\dots,n\}$ of E_n.

(i) For $v_n = \sum_{p=1}^{n} \sum_{\lambda=1}^{m'} v_{p,\lambda,n} L_{p,\lambda,n}$, we easily obtain

$$T_n v_n = \sum_{p=1}^{n} \sum_{\lambda=1}^{m'} v'_{p,\lambda,n} L_{p,\lambda,n},$$

$$v'_{p,\lambda,n} = \frac{1}{n} \sum_{q=1}^{n} \sum_{\mu=1}^{m'} w_\lambda K(\xi_{p,\lambda,n}, \xi_{q,\mu,n}) v_{q,\mu,n}$$

(recall that $\xi_1,\dots,\xi_{m'}$ are the knots of the m' point Gauss quadrature formula on $[-1,1]$; for other ξ_λ, in general, the formula is more complicated).

(ii) For $v_n = \sum_{p=1}^{n} \sum_{\lambda=1}^{m'} v_{p,\lambda,n} L_{p,\lambda,n}$, we have

$$T_\nu v_n = \sum_{p,q=1}^{\nu} \sum_{\lambda,\mu=1}^{m'} K(\xi_{p,\lambda,\nu}, \xi_{q,\mu,\nu}) \sum_{q'=1}^{n} \sum_{\mu'=1}^{m'} v_{q',\mu',n} \vartheta_{q,\mu,\nu,q',\mu',n} L_{p,\lambda,\nu}$$

where

$$\vartheta_{q,\mu,\nu,q',\mu',n} = \int_0^1 L_{q,\mu,\nu}(y) L_{q',\mu',n}(y)\, dy = (1/n) w_{\mu'} L_{q,\mu,\nu}(\xi_{q',\mu',n})$$

(the integral actually extends to $[(q'-1/n),(q'/n)]$ and the Gauss quadrature is sharp). Notice that $L_{q,\mu,\nu}(\xi_{q',\mu',n}) = 0$ if the inclusion $[(q'-1/n),(q'/n)] \subset [(q-1/\nu),(q/\nu)]$ is violated. We obtain $T_\nu v_n = \sum_{p=1}^{\nu} \sum_{\lambda=1}^{m'} v'_{p,\lambda,\nu} L_{p,\lambda,v}$, where

$$v'_{p,\lambda,\nu} = \frac{1}{n} \sum_{q=1}^{\nu} \sum_{\mu=1}^{m'} K(\xi_{p,\lambda,\nu}, \xi_{q,\mu,\nu}) \times \sum_{q'=(q-1)l+1}^{ql} \sum_{\mu'=1}^{m'} w_{\mu'} L_{q,\mu,\nu}(\xi_{q',\mu',n}) v_{q',\mu',n}.$$

(iii) For $v_\nu = \sum_{p=1}^{\nu} \sum_{\lambda=1}^{m'} v_{p,\lambda,\nu} L_{p,\lambda,\nu}$, we have

$$u_\nu := (I - T_\nu)^{-1} v_\nu = \sum_{p=1}^{\nu} \sum_{\lambda=1}^{m'} u_{p,\lambda,\nu} L_{p,\lambda,\nu}$$

where $u_{p,\lambda,\nu}$, $\lambda = 1,\dots,m'$, $p = 1,\dots,\nu$, is the solution of the $m'\nu \times m'\nu$ system

$$u_{p,\lambda,\nu} = \frac{1}{\nu} \sum_{q=1}^{\nu} \sum_{\mu=1}^{m'} w_\mu K(\xi_{p,\lambda,\nu}, \xi_{q,\mu,\nu}) u_{q,\mu,n} + v_{p,\lambda,\nu}, \quad \lambda = 1,\dots,m', \quad p = 1,\dots,\nu.$$

(iv) Since $L_{p,\lambda,\nu}$ is a polynomial of degree $\le m'-1$ on every subinterval $\mathcal{I}_{q,n}$ (in particular, the zero polynomial if the inclusion $[(q-1/n),(q/n)] \subset [(p-1/\nu),(p/\nu)]$ is violated), then $L_{p,\lambda,\nu}$ coincides with its interpolant,

$$L_{p,\lambda,\nu} = \sum_{q=1}^{n} \sum_{\mu=1}^{m'} L_{p,\lambda,\nu}(\xi_{q,\mu,\nu}) L_{q,\mu,n}.$$

For $v_\nu = \sum_{p=1}^{\nu} \sum_{\lambda=1}^{m'} v_{p,\lambda,\nu} L_{p,\lambda,\nu} \in E_\nu$, we obtain the representations in the basis of E_n:

$$v_\nu = \sum_{q=1}^{n} \sum_{\mu=1}^{m'} \left(\sum_{\lambda=1}^{m'} L_{p(q),\lambda,\nu}(\xi_{q,\mu,\nu}) v_{p(q),\lambda,v} \right) L_{q,\mu,n}$$

where $p(q) = \min\{p \in \mathbf{N} : p \geq \nu/n\}$. On the last step we took into account that for fixed q, only one of numbers $L_{p,\lambda,\nu}(\xi_{q,\mu,\nu})$, $p = 1, \ldots, \nu$, does not vanish, namely the one with $\xi_{q,\mu,\nu} \in \mathcal{I}_{p,\nu}$.

Let us return to the iterations (4.16). We compute $T_n v_n^{k-1}$ with the help of (i), $T_\nu v_n^{k-1}$ with the help of (ii) and (iv), $T_\nu z_n^k$ with the help of (ii) and $(I - T_\nu)^{-1} T_\nu z_n^k$ with the help of (iii) and (iv). All this costs $O(n^2 + \nu^3)$ flops, hence $O(n^{m'/m})$ flops for $2m \leq m' < 3m$, $\nu \asymp n^\rho$, $0 < \rho \leq m'/(3m)$.

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