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SUPERCONVERGENCE IN THE MAXIMUM NORM OF A CLASS OF PIECEWISE POLYNOMIAL COLLOCATION METHODS FOR SOLVING LINEAR WEAKLY SINGULAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. A piecewise polynomial collocation method on graded grids for solving linear weakly singular integro-differential equations of Volterra type is studied. It is shown that a special choice of collocation parameters improves the convergence rate of the method, the error estimates for all values of the nonuniformity parameter of the grid are obtained.

1. Introduction and the main result. We consider the linear integro-differential equation

$$y'(t) = p(t)y(t) + q(t) + \int_0^t K(t,s)y(s)\,ds, \quad t \in [0,T], \quad T > 0, \tag{1}$$

with a given initial condition $y(0) = y_0$, $y_0 \in \mathbf{R} = (-\infty, \infty)$. We assume that

$$p, q \in C^{k,\nu}(0,T], \quad K \in \mathcal{W}^{k,\nu}(\Delta_T), \quad k \in \mathbf{N} = \{1,2,\dots\}, \quad \nu \in \mathbf{R} \setminus \mathbf{Z},\ \nu < 1. \tag{2}$$

Here $C^{k,\nu}(0,T]$, $k \in \mathbf{N}$, $\nu < 1$, is defined as collection of all k times continuously differentiable functions $x : (0,T] \to \mathbf{R}$ such that the estimation

$$\left|x^{(j)}(t)\right| \le c_j \begin{cases} 1 & \text{if } j < 1-\nu, \\ t^{1-\nu-j} & \text{if } j > 1-\nu \end{cases}$$

holds with a constant $c_j = c_j(x)$ for all $t \in (0,T]$ and $j = 0,1,\dots,k$. The set $\mathcal{W}^{k,\nu}(\Delta_T)$, with $k \in \mathbf{N}$, $\nu < 1$, $\Delta_T = \{(t,s) \in \mathbf{R}^2 : 0 \le t \le T,$

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$0 \le s < t\}$ consists of k times continuously differentiable functions $K : \Delta_T \to \mathbf{R}$ satisfying

$$\left|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)^j K(t,s)\right| \le c \begin{cases} 1 & \text{if } \nu+i<0,\\ (t-s)^{-\nu-i} & \text{if } \nu+i>0 \end{cases}$$

with a constant $c = c(K)$ for all $(t,s) \in \Delta_T$ and all integers $i, j \ge 0$ such that $i + j =\le k$.

It is well known, see [**1**], that under the assumptions (2) the equation (1) has a unique solution $y \in C^{k+1,\nu-1}(0,T]$.

For solving problem (1) we use piecewise polynomial collocation method on graded grids. Fix $r \in \mathbf{R}$, $r \ge 1$. For $N \in \mathbf{N}$ define a graded grid Π_N^r on the interval $[0,T]$ by

$$\Pi_N^r = \left\{t_0, t_1, \dots, t_N : t_n = T\left(\frac{n}{N}\right)^r, \quad n = 0, \dots, N\right\}$$

and denote $h_n = t_n - t_{n-1}$. We look for an approximate solution u_N to (1) in the form

$$u_N(t) = y_0 + \int_0^t v_N(s)ds, \quad v_N \in S_{m-1}^{(-1)}(\Pi_N^r), \tag{3}$$

where $S_{m-1}^{(-1)}(\Pi_N^r) := \{u : u|_{[t_{n-1},t_n]} \in \pi_{m-1}, \quad n = 1, \dots, N\}$. Here π_{m-1} denotes the set of polynomials of degree not exceeding $m-1$ and $u|_{[t_{n-1},t_n]}$ is the restriction of u to the subinterval $[t_{n-1}, t_n]$.

In order to determine v_N we choose m collocation points in every subinterval $[t_{n-1}, t_n]$, $n = 1, \dots, N$,

$$t_{nj} = t_{n-1} + \eta_j h_n, \quad j = 1, \dots, m, \quad n = 1, \dots, N, \tag{4}$$

where $\eta_1, \dots, \eta_m$ do not depend on N and satisfy $0 \le \eta_1 < \dots < \eta_m \le 1$. We require that the collocation equations

$$\begin{aligned} v_N(t_{nj}) &= f_1(t_{nj}) + p(t_{nj}) \int_0^{t_{nj}} v_N(s)\,ds \\ &\quad + \int_0^{t_{nj}} K(t_{nj}, s)\left(\int_0^s v_N(\tau)\,d\tau\right) ds, \\ &\quad j = 1, \dots, m;\ n = 1, \dots, N, \end{aligned} \tag{5}$$

where

$$f_1(t) = q(t) + y_0 p(t) + y_0 \int_0^t K(t,s)\,ds, \quad t \in [0,T]$$

hold.

Piecewise collocation methods for weakly singular integro-differential equations are extensively examined by many authors. The present paper is most closely related to recent papers [**1, 2**] by Brunner, Pedas and Vainikko and the paper [**3**] by Tao Tang, an extensive list of related papers is given in [**1**].

Brunner, Pedas, Vainikko have proved the following convergence result for method {(3), (5)}.

Theorem 1 [**2**]. *Assume* (2) *with* $k = m$ *and that the collocation points* (4) *are used. Then for all sufficiently large* $N \in \mathbf{N}$ *and for every choice of parameters* $0 \le \eta_1 < \cdots < \eta_m \le 1$, *the equations* (3) *and* (5) *determine unique approximations* u *and* v *to* y *and* y' *respectively. Then, for* $r \ge 1$, *we have*

$$\|u_N - y\|_\infty \le c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \le r < \dfrac{m}{2-\nu}, \\ N^{-m}(1+\log N) & \text{for } r = \dfrac{m}{2-\nu}, \\ N^{-m} & \text{for } r > \dfrac{m}{2-\nu}, \end{cases}$$

and

$$||u'_N - y'||_\infty \le c' \begin{cases} N^{-r(1-\nu)} & \text{for } 1 \le r < \dfrac{m}{1-\nu}, \\ N^{-m}(1+\log N) & \text{for } r = \dfrac{m}{1-\nu} = 1, \\ N^{-m} & \text{for } r = \dfrac{m}{1-\nu} > 1 \\ & \text{or } r > \dfrac{m}{1-\nu}. \end{cases}$$

Here $c = c(r,\nu,m)$ *and* $c' = c'(r,\nu,m)$ *are some positive constants which are independent of* N.

In this paper we show that by a careful choice of the collocation parameters η_j it is possible, assuming a little more regularity of functions p, q and K, to improve the convergence rate. Namely we prove the following result.

Theorem 2. *Assume* (2) *with* $k = m + 1$ *and that the parameters* η_j *are chosen so that the interpolatory quadrature approximation* $\int_0^1 \varphi(s)\,ds \approx \sum_{j=1}^m A_j \varphi(\eta_j)$, *with appropriate weights* $\{A_j\}$, *is exact for all polynomials of degree* m. *Then there exists* $N_0 \in \mathbf{N}$ *such that for all* $N \geq N_0$ *the error estimate*

$$||u_N - y||_\infty \leq c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leq r < \dfrac{m+1}{2-\nu}, \\ N^{-r(2-\nu)}(1+\log N) & \text{for } r = \dfrac{m+1}{2-\nu}, \\ N^{-m-1} & \text{for } r > \dfrac{m+1}{2-\nu} \end{cases}$$

holds, where $c = c(r, \nu, m)$ *is a positive constant which is independent of* N.

In order to prove the theorem, we need several technical results that will be presented in the next section.

2. Lemmas. First we present a result about approximating functions with piecewise polynomial interpolants on a graded grid. In order to state the result we introduce the following notations.

We define k interpolation points in every subinterval $[t_{n-1}, t_n]$ ($n = 1, \ldots, N$) of the grid Π_N^r by

$$t'_{nj} = t_{n-1} + \eta'_j h_n, \quad j = 1, \ldots, k, \quad n = 1, \ldots, N, \tag{6}$$

where $\eta'_1, \ldots, \eta'_k$ do not depend on N and satisfy $0 \leq \eta'_1 < \ldots < \eta'_k \leq 1$. To a continuous function $x : [0, T] \to \mathbf{R}$ we assign a piecewise polynomial function $P_k x \in S_{m-1}^{(-1)}$ which interpolates x at the points (6):

$$(P_k x)(t'_{nj}) = x(t'_{nj}), \quad j = 1, \ldots, k; \quad n = 1, \ldots, N. \tag{7}$$

Applying to $x \in C^{k,\nu}(0,T]$ the estimate (7.15) from [**4**, p. 116] we obtain that

$$\max_{t_{n-1}\le t\le t_n} |x(t)-(P_kx)(t)| \le c h_n^k \begin{cases} 1 & \text{if } k<1-\nu,\\ t_n^{1-\nu-k} & \text{if } k>1-\nu \end{cases} \tag{8}$$

for $n=1,\dots,N$, with a constant c which is independent of n and N.

Lemma 1. *For $x \in C^{k,\nu}(0,T]$ the following estimates hold*:

1) *if $k<1-\nu$, then*

$$\int_0^T |x(s)-(P_kx)(s)|\,ds \le cN^{-k} \quad \textit{for } r\ge 1; \tag{9}$$

2) *if $k>1-\nu$, then*

$$\int_0^T |x(s)-(P_kx)(s)|\,ds \le c\begin{cases} N^{-r(2-\nu)} & \textit{for } \ 1\le r<\dfrac{k}{2-\nu},\\ N^{-r(2-\nu)}(1+\log N) & \textit{for } \ r=\dfrac{k}{2-\nu},\\ N^{-k} & \textit{for } \ r>\dfrac{k}{2-\nu}. \end{cases} \tag{10}$$

Here $c=c(r,\nu,k)$ is a positive constant which is independent of N.

Proof. Using equality (8) we get

$$\begin{aligned}\int_0^T |x(s)-(P_kx)(s)|\,ds &= \sum_{j=1}^N \int_{t_{j-1}}^{t_j} |(P_kx)(s)-x(s)|\,ds\\ &\le c\sum_{j=1}^N (t_j-t_{j-1})^{k+1}\begin{cases} 1 & \text{if } k<1-\nu,\\ t_j^{1-\nu-k} & \text{if } k>1-\nu.\end{cases}\end{aligned}$$

Since

$$\begin{aligned} t_j-t_{j-1} &= T\left(\frac{j}{N}\right)^r - T\left(\frac{j-1}{N}\right)^r\\ &= TN^{-r}(j^r-(j-1)^r) \le TN^{-r}rj^{r-1}, \quad r\ge 1\end{aligned}$$

and

$$t_j^{1-\nu-k} = \left(T\left(\frac{j}{N}\right)^r\right)^{1-\nu-k} = T^{1-\nu-k}N^{-r(1-\nu-k)}j^{r(1-\nu-k)}$$

we have

$$\begin{aligned}(t_j - t_{j-1})^{k+1} &\le T^{k+1}r^{k+1}N^{-r(k+1)}j^{r(k+1)-k-1},\\ (t_j - t_{j-1})^{k+1}t_j^{1-\nu-k} &\le T^{2-\nu}r^{k+1}N^{-r(2-\nu)}j^{r(2-\nu)-k-1}.\end{aligned}$$

In case $k < 1-\nu$ we get

$$\begin{aligned}\int_0^T |x(s) - (P_kx)(s)|ds &\le cT^{k+1}r^{k+1}N^{-r(k+1)}\sum_{j=1}^{N} j^{r(k+1)-k-1}\\ &\le c'N^{-r(k+1)}N^{r(k+1)-k}\\ &= c'N^{-k}, \quad r \ge 1.\end{aligned}$$

In case $k > 1-\nu$ we get

$$\begin{aligned}\int_0^T |x(s) - (P_kx)(s)|\, ds &\le cT^{2-\nu}r^{k+1}N^{-r(2-\nu)}\sum_{j=1}^{N} j^{r(2-\nu)-k-1}\\ &\le cN^{-r(2-\nu)}\begin{cases}1 & \text{for } 1 \le r < \dfrac{k}{2-\nu}\\ 1+\log N & \text{for } r = \dfrac{k}{2-\nu}\\ N^{r(2-\nu)-k} & \text{for } r > \dfrac{k}{2-\nu}\end{cases}\\ &\le c\begin{cases}N^{-r(2-\nu)} & \text{for } 1 \le r < \dfrac{k}{2-\nu},\\ N^{-r(2-\nu)}(1+\log N) & \text{for } r = \dfrac{k}{2-\nu},\\ N^{-k} & \text{for } r > \dfrac{k}{2-\nu}.\end{cases}\end{aligned}$$

Lemma 1 is proved. □

In the proof of the next lemma we use repeatedly the following result

Proposition 1. *Let $\gamma < 0$, then for all $n \geq 2$ the estimate*

$$\sum_{i=1}^{n-1} i^{\beta}(n-i)^{\gamma} \leq c \begin{cases} 1 & \text{if } \beta+\gamma < -1 \text{ and } \beta < 0, \\ n^{\beta} & \text{if } \beta \geq 0 \text{ and } \gamma < -1, \\ n^{\beta+\gamma+1} & \text{if } \beta+\gamma \geq -1 \text{ and } \gamma > -1 \end{cases} \tag{11}$$

holds, where c is independent of n.

Proof. If $\beta+\gamma < -1$ and $\beta < 0$ we have

$$\sum_{i=1}^{n-1} i^{\beta}(n-i)^{\gamma} \leq \sum_{i=1}^{n-1} \left(i^{\beta+\gamma} + (n-i)^{\beta+\gamma}\right) \leq 2\sum_{i=1}^{\infty} i^{\beta+\gamma} = c;$$

If $\beta \geq 0$ and $\gamma < -1$ then

$$\sum_{i=1}^{n-1} i^{\beta}(n-i)^{\gamma} \leq n^{\beta} \sum_{i=1}^{n-1}(n-i)^{\gamma} \leq n^{\beta} \sum_{j=1}^{\infty} j^{\gamma} = cn^{\beta};$$

If $\beta+\gamma \geq -1$ and $\gamma > -1$ then

$$\begin{aligned} \sum_{i=1}^{n-1} i^{\beta}(n-i)^{\gamma} &\leq \sum_{i=1}^{[n/2]} i^{\beta}\left(\frac{n}{2}\right)^{\gamma} + \sum_{i=[n/2]+1}^{n-1} \max\left\{\left(\frac{n}{2}\right)^{\beta}, n^{\beta}\right\}(n-i)^{\gamma} \\ &\leq \left(\frac{n}{2}\right)^{\gamma} \frac{n^{\beta+1}}{\beta+1} + \max\left\{1, \frac{1}{2^{\beta}}\right\} n^{\beta} \frac{n^{\gamma+1}}{\gamma+1} \\ &\leq cn^{\beta+\gamma+1}, \end{aligned}$$

where $[n/2]$ denotes the integer part of $n/2$. □

Now we are ready to prove a superconvergence result of method $\{(3),(5)\}$ for the values of v_N at the collocation points.

Lemma 2. *Assume* (2) *with $k = m+1$ and that the parameters η_j are chosen so that the quadrature approximation $\int_0^1 \varphi(s)\,ds \approx$*

$\sum_{j=1}^{m} A_j\varphi(\eta_j)$, *with appropriate weights* $\{A_j\}$, *is exact for all polynomials of degree* m. *Then there exists* $N_0 \in \mathbf{N}$ *such that for all* $N \geq N_0$ *the error estimate*

$$(12) \quad |y'(t_{nj}) - v_N(t_{nj})| \leq c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leq r < \dfrac{m+1}{2-\nu}, \\ N^{-r(2-\nu)}(1+\log N) & \text{for } r = \dfrac{m+1}{2-\nu}, \\ N^{-m-1} & \text{for } r > \dfrac{m+1}{2-\nu} \end{cases}$$

holds, where $c = c(r,\nu,m)$ *is a positive constant which is independent of* N.

Proof. Note, that we can write equation (1) in the form

$$(13) \quad y'(t) = f_1(t) + p(t)\int_0^t y'(s)\,ds + \int_0^t K_1(t,s)y'(s)\,ds, \quad t \in [0,T],$$

where

$$f_1(t) = q(t) + y_0 p(t) + y_0 \int_0^t K(t,s)\,ds, \quad t \in [0,T]$$

and

$$K_1(t,s) = \int_s^t K(t,\tau)\,d\tau, \quad 0 \leq s < t \leq T.$$

It is easy to check that if $K \in \mathcal{W}^{m+1,\nu}(\Delta_T)$, then $K_1 \in \mathcal{W}^{m+2,\nu-1}(\Delta_T)$. Let P_m be defined by (7), with $k = m$ and $t'_{nj} = t_{nj}$, then the collocation equations (5) can be written as

$$v_n = P_m f_1 + P_m S_1 v_n,$$

where

$$(S_1 x)(t) = \int_0^t \tilde{K}_1(t,s)x(s)\,ds,$$
$$\tilde{K}(t,s) = p(t) + K_1(t,s).$$

Denote $w = v_n - P_m y'$. Since

$$P_m y' = P_m f_1 + P_m S_1 y'$$

it follows that

$$w = P_m S_1 w + P_m S_1 (y' - P_m y').$$

As $I - S_1$ is invertible in $C[0,T]$, $I - P_m S_1$ is also invertible for sufficiently large N. Therefore

$$w = (I - P_m S_1)^{-1} P_m S_1 (y' - P_m y')$$

and

$$\begin{aligned} |y'(t_{nj}) - v_n(t_{nj})| &\le \|w\|_\infty \le c\|S_1(y' - P_m y')\|_\infty \\ &\le c \max_{t\in[0,T]} \left| \int_0^t \tilde{K}(t,s)\big(y'(s) - (P_m y')(s)\big)\, ds \right|. \end{aligned}$$

Fix t and let $n \in \{0, \dots, N-1\}$, $t \in [t_n, t_{n+1}]$. If $n = 0$ then, according to (8)

(14)

$$\begin{aligned} \left| \int_0^t \tilde{K}(t,s)\big(y'(s) - (P_m y')(s)\big)\, ds \right| &\le c t_1 \|y' - P_m y'\|_{[0,t_1]} \le c' t_1 t_1^{1-\nu} \\ &= c' t_1^{2-\nu} \le c'' N^{-r(2-\nu)}. \end{aligned}$$

If $n = 1, \dots, N-1$, then we have

$$\begin{aligned} &\left| \int_0^t \tilde{K}(t,s)\big(y'(s) - (P_m y')(s)\big)\, ds \right| \\ &\qquad \le \int_0^{t_1} \left| \tilde{K}(t,s)\big(y'(s) - (P_m y')(s)\big) \right| ds \\ &\qquad\quad + \left| \sum_{i=2}^{n-1} \int_{t_{i-1}}^{t_i} \tilde{K}(t,s)\big(y'(s) - (P_m y')(s)\big)\, ds \right| \\ &\qquad\quad + \int_{t_{n-1}}^{t} \left| \tilde{K}(t,s)\big(y'(s) - (P_m y')(s)\big) \right| ds. \end{aligned}$$

It is quite easy to estimate the first and last terms using the inequality (8):

$$\int_0^{t_1} \big|\tilde{K}(t,s)\big(y'(s)-(P_m y')(s)\big)\big|\,ds \le ct_1\|y'-P_m y'\|_{[0,t_1]} \le c'N^{-r(2-\nu)} \tag{15}$$

and

$$\begin{aligned}
&\int_{t_{n-1}}^{t} \big|\tilde{K}(t,s)\big(y'(s)-(P_m y')(s)\big)\big|\,ds \\
&= \int_{t_{n-1}}^{t_n} \big|\tilde{K}(t,s)\big(y'(s)-(P_m y')(s)\big)\big|\,ds \\
&\quad + \int_{t_n}^{t} \big|\tilde{K}(t,s)\big(y'(s)-(P_m y')(s)\big)\big|\,ds \\
&\le ch_n\|y'-P_m y'\|_{[t_{n-1},t_n]} + ch_{n+1}\|y'-P_m y'\|_{[t_n,t_{n+1}]} \\
&\le ch_n^{m+1}\begin{cases} 1 & \text{if } m<1-\nu \\ t_{n-1}^{1-\nu-m} & \text{if } m>1-\nu \end{cases} + ch_{n+1}^{m+1}\begin{cases} 1 & \text{if } m<1-\nu \\ t_n^{1-\nu-m} & \text{if } m>1-\nu \end{cases} \\
&\le c\begin{cases} N^{-r(2-\nu)} & \text{for } 1\le r\le \dfrac{m+1}{2-\nu}, \\[2mm] N^{-m-1} & \text{for } r>\dfrac{m+1}{2-\nu}. \end{cases}
\end{aligned} \tag{16}$$

The estimation of the remaining term requires more work.

$$\begin{aligned}
&\left|\sum_{i=2}^{n-1}\int_{t_{i-1}}^{t_i} \tilde{K}(t,s)\big(y'(s)-(P_m y')(s)\big)\,ds\right| \\
&\qquad \le \left|\sum_{i=2}^{n-1}\int_{t_{i-1}}^{t_i} \big(K_1(t,s)-K_1(t,t_i)\big)\big(y'(s)-(P_m y')(s)\big)\,ds\right| \\
&\qquad\quad + \left|\sum_{i=2}^{n-1}\int_{t_{i-1}}^{t_i} \tilde{K}(t,t_i)\big(y'(s)-(P_m y')(s)\big)\,ds\right|.
\end{aligned}$$

Using the facts that $K_1 \in \mathcal{W}^{m+2,\nu-1}(\Delta_T)$, $y' \in C^{m+1,\nu}(0,T]$ and (8) we get

$$\begin{aligned}
&\Big|\sum_{i=2}^{n-1}\int_{t_{i-1}}^{t_i}\big(K_1(t,s)-K_1(t,t_i)\big)\big(y'(s)-(P_my')(s)\big)\,ds\Big| \\
&\le c\sum_{i=2}^{n-1}h_i^2\left\{\begin{matrix}1 & \text{if } \nu<0\\ |t_i-t|^{-\nu} & \text{if } \nu>0\end{matrix}\right\}\|y'-P_my'\|_{C[t_{i-1},t_i]} \\
&\le c\sum_{i=2}^{n-1}h_i^2\left\{\begin{matrix}1 & \text{if } \nu<0\\ (n-i)^{-\nu}h_i^{-\nu} & \text{if } \nu>0\end{matrix}\right\}h^m\left\{\begin{matrix}1 & \text{if } m<1-\nu\\ t_i^{1-\nu-m} & \text{if } m>1-\nu\end{matrix}\right\} \\
&\le c\sum_{i=2}^{n-1}h_i^{m+2}\begin{cases}1 & \text{if } \nu<0 \text{ and } m<1-\nu,\\ t_i^{1-\nu-m} & \text{if } \nu<0 \text{ and } m>1-\nu,\\ (n-i)^{-\nu}h_i^{-\nu}t_i^{1-\nu-m} & \text{if } \nu>0 \text{ and } m>1-\nu.\end{cases}
\end{aligned}$$

1) In case $\nu<0$ and $m<1-\nu$:

$$c\sum_{i=2}^{n-1}h_i^{m+2}\le cT^{m+2}r^{m+2}N^{-r(m+2)}\sum_{i=2}^{n-1}i^{(r-1)(m+2)}\le cN^{-m-1};$$

2) In case $\nu<0$ and $m>1-\nu$:

$$\begin{aligned}
c\sum_{i=2}^{n-1}h_i^{m+2}t_i^{1-\nu-m}&\le cT^{3-\nu}r^{m+2}N^{-r(3-\nu)}\sum_{i=2}^{n-1}i^{r(3-\nu)-m-2}\\
&\le cN^{-r(3-\nu)}\sum_{i=2}^{n-1}i^{r(3-\nu)-m-2}\\
&\le c\begin{cases}N^{-r(3-\nu)} & \text{for } 1\le r<\dfrac{m+1}{3-\nu},\\ N^{-r(3-\nu)}(1+\log N) & \text{for } r=\dfrac{m+1}{3-\nu},\\ N^{-m-1} & \text{for } r>\dfrac{m+1}{3-\nu};\end{cases}
\end{aligned}$$

3) In case $\nu > 0$ and $m > 1-\nu$:

$$
\begin{aligned}
&c\sum_{i=2}^{n-1} h_i^{m+2-\nu}(n-i)^{-\nu} t_i^{1-\nu-m} \\
&\qquad\qquad \le cN^{-r(3-2\nu)} \sum_{i=2}^{n-1} i^{r(3-2\nu)-m-2+\nu}(n-i)^{-\nu} \\
&\qquad\qquad \le c\begin{cases} N^{-r(3-2\nu)} & \text{for } r < \dfrac{m+1}{3-2\nu}, \\ N^{-m-1} & \text{for } r \ge \dfrac{m+1}{3-2\nu}. \end{cases}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(17)\quad &\left|\sum_{i=2}^{n-1}\int_{t_{i-1}}^{t_i} \big(K_1(t,s)-K_1(t,t_i)\big)\big(y'(s)-(P_m y')(s)\big)\,ds\right| \\
&\le c\begin{cases} N^{-r(3-\nu)} & \text{if } \ \nu<0 \text{ and } 1\le r < \dfrac{m+1}{3-\nu} \\ N^{-r(3-\nu)}(1+\log N) & \text{if } \ \nu<0 \text{ and } r = \dfrac{m+1}{3-\nu} \\ N^{-m-1} & \text{if } \ \nu<0 \text{ and } r > \dfrac{m+1}{3-\nu}, \\ N^{-r(3-2\nu)} & \text{if } \ \nu>0 \text{ and } 1\le r < \dfrac{m+1}{3-2\nu} \\ N^{-m-1} & \text{if } \ \nu>0 \text{ and } r \ge \dfrac{m+1}{3-2\nu}. \end{cases}
\end{aligned}
$$

We choose $0 \le \eta'_1 < \cdots < \eta'_{m+1} \le 1$ such that $\{\eta_1,\dots,\eta_m\} \subset \{\eta'_1,\dots,\eta'_{m+1}\}$ and P_{m+1} is given by equality (7), where $k=m+1$.

Note that

$$
\begin{aligned}
&\int_{t_{i-1}}^{t_i} \big((P_{m+1}y')(s)-(P_m y')(s)\big)\,ds \\
&\qquad\qquad = \sum_{j=1}^{m} h_i A_j\big((P_{m+1}y')(t_{ij})-(P_m y')(t_{ij})\big) = 0;
\end{aligned}
$$

therefore,

$$\Big|\sum_{i=2}^{n-1}\int_{t_{i-1}}^{t_i}\tilde{K}(t,t_i)\big((P_{m+1}y')(s)-(P_my')(s)\big)ds\Big|=0.$$

Hence

$$\begin{aligned}\Big|\sum_{i=2}^{n-1}\int_{t_{i-1}}^{t_i}\tilde{K}(t,t_i)\big(y'(s)-(P_my')(s)\big)\,ds\Big|\\ &=\Big|\sum_{i=2}^{n-1}\int_{t_{i-1}}^{t_i}\tilde{K}(t,t_i)\big(y'(s)-(P_{m+1}y')(s)\big)\,ds\Big|\\ &\le c\int_0^T\big|y'(s)-(P_{m+1}y')(s)\big|\,ds \qquad (18)\\ &\le c\begin{cases}N^{-r(2-\nu)} & \text{for } 1\le r<\dfrac{m+1}{2-\nu},\\ N^{-r(2-\nu)}(1+\log N) & \text{for } r=\dfrac{m+1}{2-\nu},\\ N^{-m-1} & \text{for } r>\dfrac{m+1}{2-\nu}.\end{cases}\end{aligned}$$

By combining the estimates (17) and (18) it follows that

$$\begin{aligned}(19)\quad \Big|\sum_{i=2}^{n-1}\int_{t_{i-1}}^{t_i}K_1(t,s)\big(y'(s)-(P_my')(s)\big)\,ds\Big|\\ &\le c\begin{cases}N^{-r(2-\nu)} & \text{for } 1\le r<\dfrac{m+1}{2-\nu},\\ N^{-r(2-\nu)}(1+\log N) & \text{for } r=\dfrac{m+1}{2-\nu},\\ N^{-m-1} & \text{for } r>\dfrac{m+1}{2-\nu}.\end{cases}\end{aligned}$$

Finally, using the estimates (15), (16) and (19) we obtain the estimate

$$\left|\int_0^t K_1(t,s)\big(y'(s)-(P_m y')(s)\big)\,ds\right| \le c\begin{cases} N^{-r(2-\nu)} & \text{for } 1\le r<\dfrac{m+1}{2-\nu},\\ N^{-r(2-\nu)}(1+\log N) & \text{for } r=\dfrac{m+1}{2-\nu},\\ N^{-m-1} & \text{for } r>\dfrac{m+1}{2-\nu}. \end{cases} \tag{20}$$

Lemma 2 is proved. □

3. Proof of theorem 2. Fix $t\in[0,T]$, let $i\in\{0,\dots,N-1\}$ be such that $t\in[t_i,t_{i+1}]$. Using initial condition $y(0)=y_0$ and equation (3) we obtain

$$\begin{aligned} |y(t)-u_N(t)| &= \left|\int_0^t (y'(s)-v_N(s))\,ds\right| \\ &\le \left|\int_0^{t_i}(y'(s)-v_N(s))\,ds\right| + \left|\int_{t_i}^t (y'(s)-v_N(s))\,ds\right|. \end{aligned} \tag{21}$$

Consider the first term. Define

$$KV(x,0,t_i)=\sum_{j=1}^{i}\sum_{q=1}^{m} h_j A_q x(t_{jq}).$$

Then

$$\begin{aligned} \left|\int_0^{t_i}(y'(s)-v_N(s))\,ds\right| &= \left|\int_0^{t_i} y'(s)\,ds-\int_0^{t_i} v_N(s)\,ds\right| \\ &= \left|\int_0^{t_i} y'(s)\,ds - KV(v_N,0,t_i)\right| \\ &\le \left|\int_0^{t_i} y'(s)\,ds - KV(y',0,t_i)\right| \\ &\quad + \left|KV(y',0,t_i)-KV(v_N,0,t_i)\right|. \end{aligned} \tag{22}$$

We choose $0 \le \eta'_1 < \cdots < \eta'_{m+1} \le 1$ such that $\{\eta_1, \dots, \eta_m\} \subset \{\eta'_1, \dots, \eta'_{m+1}\}$ and P_{m+1} is given by equality (7), where $k = m+1$. Since

$$KV(y', 0, t_i) = \int_0^{t_i} (P_{m+1}y')(s)\, ds$$

we get

$$\begin{aligned}\left| \int_0^{t_i} y'(s)\, ds - KV(y', 0, t_i) \right| &= \left| \int_0^{t_i} y'(s)\, ds - \int_0^{t_i} (P_{m+1}y')(s)\, ds \right| \\ &= \left| \int_0^{t_i} (y'(s) - (P_{m+1}y')(s))\, ds \right| \\ &\le \int_0^{t_i} |y'(s) - (P_{m+1}y')(s)|\, ds.\end{aligned}$$

Since $y' \in C^{m+1,\nu}(0,T]$ we obtain from Lemma 1 following estimates:

in case $m < -\nu$

$$\int_0^{t_i} |y'(s) - (P_{m+1}y')(s)|\, ds \le cN^{-m-1} \quad \text{for } r \ge 1; \tag{23}$$

in case $m > -\nu$

$$\begin{aligned}&\int_0^{t_i} |y'(s) - (P_{m+1}y')(s)|\, ds \\ &\qquad \le c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \le r < \dfrac{m+1}{2-\nu}, \\ N^{-r(2-\nu)}(1+\log N) & \text{for } r = \dfrac{m+1}{2-\nu}, \\ N^{-m-1} & \text{for } r > \dfrac{m+1}{2-\nu}. \end{cases}\end{aligned} \tag{24}$$

As by Lemma 2 we have

$$\begin{aligned}&|KV(y', 0, t_i) - KV(v_N, 0, t_i)| \\ &\qquad \le \max_{n,j} |y'(t_{nj}) - v_N(t_{nj})| \\ &\qquad \le c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \le r < \dfrac{m+1}{2-\nu}, \\ N^{-r(2-\nu)}(1+\log N) & \text{for } r = \dfrac{m+1}{2-\nu}, \\ N^{-m-1} & \text{for } r > \dfrac{m+1}{2-\nu} \end{cases}\end{aligned}$$

it follows that

$$\left|\int_0^{t_i}(y'(s)-v_N(s))\,ds\right| \le c\begin{cases} N^{-r(2-\nu)} & \text{for } 1\le r<\dfrac{m+1}{2-\nu},\\ N^{-r(2-\nu)}(1+\log N) & \text{for } r=\dfrac{m+1}{2-\nu},\\ N^{-m-1} & \text{for } r>\dfrac{m+1}{2-\nu}.\end{cases}$$

It remains to estimate the last term in (21).

$$\left|\int_{t_i}^{t}(y'(s)-v_N(s))\,ds\right| \le \int_{t_i}^{t_{i+1}}|y'(s)-(P_my')(s)|\,ds + \int_{t_i}^{t_{i+1}}|(P_my')(s)-v_N(s)|\,ds.$$

Using (8) we get

$$\begin{aligned}\int_{t_i}^{t_{i+1}}|y'(s)-(P_my')(s)|\,ds &\le c h_{i+1}^{m+1}\begin{cases}1 & \text{if } m<1-\nu\\ t_{i+1}^{1-\nu-m} & \text{if } m>1-\nu\end{cases}\\ &\le c'\begin{cases}T^{m+1}r^{m+1}N^{-r(m+1)}(i+1)^{(r-1)(m+1)} & \text{if } m<1-\nu\\ T^{2-\nu}r^{m+1}N^{-r(2-\nu)}(i+1)^{r(2-\nu)-m-1} & \text{if } m>1-\nu\end{cases}\\ &\le c''\begin{cases}N^{-r(2-\nu)} & \text{for } 1\le r\le\dfrac{m+1}{2-\nu},\\ N^{-m-1} & \text{for } r>\dfrac{m+1}{2-\nu}.\end{cases}\end{aligned}$$

Since by Lemma 2

$$\begin{aligned}\int_{t_i}^{t_{i+1}} |(P_m y')(s) - v_N(s)|\, ds &\le h_{i+1}\|P_m y' - v_N\|_\infty \\ &\le c h_{i+1} \max_{j=1,\dots,m} |y'(t_{ij}) - v_N(t_{ij})| \\ &\le c' N^{-1} \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \le r < \dfrac{m+1}{2-\nu}, \\ N^{-r(2-\nu)}(1+\log N) & \text{for } r = \dfrac{m+1}{2-\nu}, \\ N^{-m-1} & \text{for } r > \dfrac{m+1}{2-\nu} \end{cases}\end{aligned}$$

it follows that

$$\left| \int_{t_i}^{t} (y'(s) - v_N(s))\, ds \right| \le c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \le r < \dfrac{m+1}{2-\nu}, \\ N^{-m-1} & \text{for } r \ge \dfrac{m+1}{2-\nu}. \end{cases}$$

Theorem 2 is proved. ◻

4. Numerical experiments. We consider the integro-differential equation

$$\begin{aligned} y'(t) &= -y(t) + (2-\nu)t^{1-\nu} + t^{2-\nu} \\ &\quad + t^{3-2\nu} \int_0^1 (1-x)^{-\nu} x^{2-\nu}\, dx - \int_0^t (t-s)^{-\nu} y(s)\, ds, \\ &\qquad t \in [0,1] \end{aligned} \tag{25}$$

with initial condition $y(0) = 0$.

First note that equation (25) is an equation of type (1), where $p(t) = -1$,

$$\begin{aligned} q(t) &= (2-\nu)t^{1-\nu} + t^{2-\nu} + t^{3-2\nu} \int_0^1 (1-x)^{-\nu} x^{2-\nu}\, dx, \\ K(t,s) &= -(t-s)^{-\nu}. \end{aligned}$$

It is easy to check that the assumptions of p, q and K hold with arbitrary $\nu \in (0,1)$ and arbitrary m. The exact solution of the initial value problem is $y(t) = t^{2-\nu}$.

Numerical results for this problem in case $m = 2$, when our approximate solution u_N is a piecewise quadratic polynomial, are presented for $\nu = 1/10$, $\nu = 1/2$ and $\nu = 9/10$ in Tables 1–4. In order to estimate the error $\|u_N - y\|_\infty$ and $\|v_N - y'\|_\infty$ we have used the points

$$\tau_{nj} = t_{n-1} + j\,\frac{t_n - t_{n-1}}{10}, \quad j = 1, \dots, 9; \quad n = 1, \dots, N,$$

corresponding error estimates are denoted by

$$\varepsilon_N = \{\max |u(\tau_{nj}) - y(\tau_{nj})| : j = 1, \dots, 9;\, n = 1, \dots, N\}$$

and

$$\varepsilon'_N = \{\max |u'(\tau_{nj}) - y'(\tau_{nj})| : j = 1, \dots, 9;\, n = 1, \dots, N\}.$$

To illustrate the fact that the superconvergence of the approximate solution in the supremum norm is the result of the superconvergence of the derivative v_N at the collocation points, see Lemma 2, the errors of v_N at collocation points, denoted by

$$\delta'_N = \{\max |u'(t_{nj}) - y'(t_{nj})| : j = 1, \dots, m;\, n = 1, \dots, N\}$$

are also presented. The ratios of the actual errors are presented in the columns with labels in the form $\varrho(x)$, where x is a real number corresponding to the ratios of the error estimates.

Tables 1 and 2 show the dependence of the convergence rates of the approximate solution and its derivative on the choice of collocation parameters η_1, η_2. In the case $\eta_1 = 1/4$, $\eta_2 = 3/4$ the corresponding quadrature formula is exact only for linear polynomials. As we can see, in this case the convergence rates agree well with the estimates of Theorem 1. Collocation parameters $\eta_1 = 1/4$, $\eta_2 = 5/6$ and $\eta_1 = 1/3$, $\eta_2 = 1$ correspond to quadrature formulas that are exact for all polynomials up to order 2. Numerical results show that the convergence rate is much better in these cases and agrees well with the estimate of Theorem 2. From the last column of the table we

TABLE 1. Dependence of the convergence rate on collocation parameters.

$\nu = 0.1$, $r = 1.8$	$\eta_1 = 1/4,\ \eta_2 = 3/4$		$\eta_1 = 1/4,\ \eta_2 = 5/6$		$\eta_1 = 1/3,\ \eta_2 = 1$		$\eta_1 = \frac{3-\sqrt{3}}{6},\ \eta_2 = \frac{3+\sqrt{3}}{6}$	
N	ε_N	$\varrho\,(4.0)$	ε_N	$\varrho\,(8.0)$	ε_N	$\varrho\,(8.0)$	ε_N	$\varrho\,(8.0)$
8	6.8E-5	4.6	2.1E-5	7.8	4.3E-5	7.3	1.5E-5	7.8
16	1.5E-5	4.5	2.6E-6	7.9	5.7E-6	7.5	1.9E-6	7.9
32	3.6E-6	4.3	3.3E-7	8.0	7.4E-7	7.7	2.4E-7	8.0
64	8.7E-7	4.1	4.1E-8	8.0	9.6E-8	7.8	3.1E-8	8.0
128	2.1E-7	4.1	5.2E-9	8.0	1.2E-8	7.8	3.8E-9	8.0
256	5.3E-8	4.0	6.5E-10	8.0	1.6E-9	7.9	4.8E-10	8.0
N	ε'_N	$\varrho\,(3.1)$	ε'_N	$\varrho\,(3.1)$	ε'_N	$\varrho\,(3.1)$	ε'_N	$\varrho\,(3.1)$
8	1.0E-3	3.0	1.1E-3	3.0	1.7E-3	3.0	8.3E-4	3.0
16	3.4E-4	3.1	3.6E-4	3.1	5.6E-4	3.1	2.7E-4	3.1
32	1.1E-4	3.1	1.2E-4	3.1	1.8E-4	3.1	8.9E-5	3.1
64	3.6E-5	3.1	3.8E-5	3.1	5.9E-5	3.1	2.9E-5	3.1
128	1.2E-5	3.1	1.2E-5	3.1	1.9E-5	3.1	9.4E-6	3.1
256	3.8E-6	3.1	4.0E-6	3.1	6.3E-6	3.1	3.1E-6	3.1
N	δ'_N	$\varrho\,(4.0)$	δ'_N	$\varrho\,(8.0)$	δ'_N	$\varrho\,(8.0)$	δ'_N	$\varrho\,(8.0)$
8	1.1E-4	4.1	2.7E-5	7.8	5.8E-5	7.3	1.7E-5	8.1
16	2.6E-5	4.1	3.4E-6	7.9	7.7E-6	7.5	2.1E-6	8.1
32	6.4E-6	4.1	4.3E-7	8.0	1.0E-6	7.7	2.6E-7	8.1
64	1.6E-6	4.0	5.4E-8	8.0	1.3E-7	7.8	3.2E-8	8.1
128	3.9E-7	4.0	6.8E-9	8.0	1.7E-8	7.8	4.0E-9	8.1
256	9.8E-8	4.0	8.5E-10	8.0	2.1E-9	7.9	4.9E-10	8.1

TABLE 1. (Cont'd.)

$\nu=0.5$ $r=2.5$	$\eta_1=1/4,\ \eta_2=3/4$		$\eta_1=1/4,\ \eta_2=5/6$		$\eta_1=1/3,\ \eta_2=1$		$\eta_1=\frac{3-\sqrt{3}}{6},\ \eta_2=\frac{3+\sqrt{3}}{6}$	
N	ε_N	$\varrho\,(4.0)$	ε_N	$\varrho\,(8.0)$	ε_N	$\varrho\,(8.0)$	ε_N	$\varrho\,(8.0)$
8	2.8E-4	5.0	1.2E-4	7.6	2.3E-4	6.8	8.9E-5	7.5
16	6.0E-5	4.6	1.5E-5	8.0	3.1E-5	7.4	1.1E-5	7.8
32	1.4E-5	4.3	1.8E-6	8.1	4.1E-6	7.7	1.4E-6	7.9
64	3.3E-6	4.2	2.2E-7	8.1	5.2E-7	7.8	1.8E-7	8.0
128	8.0E-7	4.1	2.8E-8	8.0	6.7E-8	7.9	2.2E-8	8.0
256	2.0E-7	4.0	3.5E-9	8.0	8.4E-9	7.9	2.8E-9	8.0
N	ε'_N	$\varrho\,(2.4)$	ε'_N	$\varrho\,(2.4)$	ε'_N	$\varrho\,(2.4)$	ε'_N	$\varrho\,(2.4)$
8	8.2E-3	2.4	8.6E-3	2.4	1.3E-2	2.4	6.8E-3	2.4
16	3.5E-3	2.4	3.6E-3	2.4	5.3E-3	2.4	2.9E-3	2.4
32	1.5E-3	2.4	1.5E-3	2.4	2.2E-3	2.4	1.2E-3	2.4
64	6.1E-4	2.4	6.4E-4	2.4	9.4E-4	2.4	5.0E-4	2.4
128	2.6E-4	2.4	2.7E-4	2.4	4.0E-4	2.4	2.1E-4	2.4
256	1.1E-4	2.4	1.1E-4	2.4	1.7E-4	2.4	8.9E-5	2.4
N	δ'_N	$\varrho\,(4.0)$	δ'_N	$\varrho\,(8.0)$	δ'_N	$\varrho\,(8.0)$	δ'_N	$\varrho\,(8.0)$
8	6.1E-4	4.5	2.0E-4	8.0	4.3E-4	7.0	1.2E-4	8.8
16	1.4E-4	4.3	2.4E-5	8.2	5.8E-5	7.5	1.4E-5	8.7
32	3.3E-5	4.2	3.0E-6	8.2	7.5E-6	7.7	1.7E-6	8.5
64	8.2E-6	4.1	3.6E-7	8.2	9.5E-7	7.8	2.0E-7	8.4
128	2.0E-6	4.0	4.5E-8	8.1	1.2E-7	7.9	2.4E-8	8.3
256	5.0E-7	4.0	5.5E-9	8.1	1.5E-8	7.9	3.0E-9	8.2

TABLE 2. Dependence of the convergence rate on collocation parameters.

$\nu = 0.9$, $r = 3.2$	$\eta_1 = 1/4$, $\eta_2 = 3/4$		$\eta_1 = 1/4$, $\eta_2 = 5/6$		$\eta_1 = 1/3$, $\eta_2 = 1$		$\eta_1 = \frac{3-\sqrt{3}}{6}$, $\eta_2 = \frac{3+\sqrt{3}}{6}$	
N	ε_N	$\varrho\,(4.0)$	ε_N	$\varrho\,(8.0)$	ε_N	$\varrho\,(8.0)$	ε_N	$\varrho\,(8.0)$
8	1.0E-4	4.5	7.0E-5	6.0	1.2E-4	4.9	5.3E-5	7.7
16	2.2E-5	4.6	9.5E-6	7.3	1.9E-5	6.4	6.3E-6	8.5
32	4.9E-6	4.4	1.2E-6	7.7	2.8E-6	6.9	7.8E-7	8.0
64	1.1E-6	4.2	1.6E-7	7.9	3.8E-7	7.3	9.9E-8	8.0
128	2.8E-7	4.1	1.9E-8	8.0	5.1E-8	7.5	1.2E-8	7.9
256	6.9E-8	4.1	2.4E-9	8.0	6.6E-9	7.7	1.6E-9	8.0
N	ε'_N	$\varrho\,(1.2)$	ε'_N	$\varrho\,(1.2)$	ε'_N	$\varrho\,(1.2)$	ε'_N	$\varrho\,(1.2)$
8	2.6E-2	1.2	2.7E-2	1.2	3.7E-2	1.2	2.2E-2	1.2
16	2.1E-2	1.2	2.2E-2	1.2	3.0E-2	1.2	1.7E-2	1.2
32	1.7E-2	1.2	1.7E-2	1.2	2.4E-2	1.2	1.4E-2	1.2
64	1.3E-2	1.2	1.4E-2	1.2	1.9E-2	1.2	1.1E-2	1.2
128	1.1E-2	1.2	1.1E-2	1.2	1.5E-2	1.2	9.0E-3	1.2
256	8.6E-3	1.2	8.9E-3	1.2	1.2E-2	1.2	7.2E-3	1.2
N	δ'_N	$\varrho\,(4.0)$	δ'_N	$\varrho\,(8.0)$	δ'_N	$\varrho\,(8.0)$	δ'_N	$\varrho\,(8.0)$
8	8.4E-4	4.4	5.5E-4	5.8	9.9E-4	4.8	4.2E-4	7.2
16	1.8E-4	4.6	7.4E-5	7.5	1.5E-4	6.5	5.1E-5	8.4
32	4.2E-5	4.4	9.3E-6	7.9	2.2E-5	7.0	5.9E-6	8.6
64	9.9E-6	4.2	1.2E-6	8.1	2.9E-6	7.4	6.8E-7	8.6
128	2.4E-6	4.1	1.4E-7	8.1	3.9E-7	7.6	8.0E-8	8.5
256	6.0E-7	4.0	1.7E-8	8.2	5.0E-8	7.7	9.4E-9	8.5

TABLE 3. Dependence of the convergence rate on the nonuniformity parameter r.

$\nu = 0.9$	$r = 1$		$r = \frac{m}{2-\nu} = 1.82$		$r = \frac{m+0.5}{2-\nu} = 2.27$		$r = \frac{m+1.2}{2-\nu} = 2.91$	
N	ε_N	$\varrho\,(2.1)$	ε_N	$\varrho\,(4.0)$	ε_N	$\varrho\,(5.7)$	ε_N	$\varrho\,(8.0)$
8	1.1E-3	1.8	2.0E-4	3.6	9.3E-5	4.1	5.4E-5	6.5
16	5.6E-4	1.9	5.0E-5	3.9	1.8E-5	5.3	7.1E-6	7.6
32	2.7E-4	2.0	1.3E-5	4.0	3.2E-6	5.6	8.7E-7	8.1
64	1.3E-4	2.1	3.2E-6	4.0	5.6E-7	5.6	1.0E-7	8.3
128	6.2E-5	2.1	7.9E-7	4.0	9.9E-8	5.7	1.2E-8	8.4
256	2.9E-5	2.1	2.0E-7	4.0	1.7E-8	5.7	1.5E-9	8.4
N	ε'_N	$\varrho\,(1.1)$	ε'_N	$\varrho\,(1.1)$	ε'_N	$\varrho\,(1.2)$	ε'_N	$\varrho\,(1.2)$
8	2.5E-2	1.0	2.8E-2	1.0	2.6E-2	1.1	2.3E-2	1.2
16	2.7E-2	0.9	2.5E-2	1.1	2.2E-2	1.2	1.9E-2	1.2
32	2.8E-2	1.0	2.2E-2	1.1	1.9E-2	1.2	1.5E-2	1.2
64	2.7E-2	1.0	2.0E-2	1.1	1.6E-2	1.2	1.3E-2	1.2
128	2.6E-2	1.1	1.8E-2	1.1	1.4E-2	1.2	1.0E-2	1.2
256	2.4E-2	1.1	1.5E-2	1.1	1.2E-2	1.2	8.4E-3	1.2
N	δ'_N	$\varrho\,(2.1)$	δ'_N	$\varrho\,(4.0)$	δ'_N	$\varrho\,(5.7)$	δ'_N	$\varrho\,(8.0)$
8	8.3E-3	1.9	1.3E-3	4.1	6.7E-4	4.6	4.0E-4	7.0
16	4.1E-3	2.0	3.2E-4	4.1	1.1E-4	6.0	5.0E-5	7.9
32	1.9E-3	2.2	7.4E-5	4.3	1.8E-5	6.1	5.9E-6	8.4
64	8.5E-4	2.2	1.7E-5	4.4	3.0E-6	6.1	6.8E-7	8.7
128	3.8E-4	2.2	3.9E-6	4.3	5.0E-7	6.0	7.8E-8	8.7
256	1.7E-4	2.3	9.5E-7	4.1	8.6E-8	5.9	8.9E-9	8.8

TABLE 4. Dependence of the convergence rate on the nonuniformity parameter r.

$\nu = 0.1$	$r = 1$		$r = \frac{m}{2-\nu} = 1.05$		$r = \frac{m+0.5}{2-\nu} = 1.32$		$r = \frac{m+1.2}{2-\nu} = 1.68$	
N	ε_N	$\varrho\,(3.7)$	ε_N	$\varrho\,(4.0)$	ε_N	$\varrho\,(5.7)$	ε_N	$\varrho\,(8.0)$
8	1.1E-4	3.6	8.7E-5	3.9	3.1E-5	5.5	1.4E-5	8.0
16	2.9E-5	3.7	2.2E-5	3.9	5.6E-6	5.6	1.7E-6	8.1
32	7.9E-6	3.7	5.6E-6	4.0	9.9E-7	5.6	2.1E-7	8.1
64	2.1E-6	3.7	1.4E-6	4.0	1.7E-7	5.6	2.6E-8	8.1
128	5.7E-7	3.7	3.5E-7	4.0	3.1E-8	5.7	3.2E-9	8.1
256	1.5E-7	3.7	8.7E-8	4.0	5.5E-9	5.7	4.0E-10	8.0
N	ε'_N	$\varrho\,(1.9)$	ε'_N	$\varrho\,(1.9)$	ε'_N	$\varrho\,(2.3)$	ε'_N	$\varrho\,(2.9)$
8	3.6E-3	1.8	3.3E-3	1.9	2.0E-3	2.2	1.0E-3	2.8
16	2.0E-3	1.8	1.7E-3	1.9	9.1E-4	2.2	3.6E-4	2.8
32	1.1E-3	1.8	9.1E-4	1.9	4.0E-4	2.3	1.3E-4	2.9
64	5.8E-4	1.9	4.7E-4	1.9	1.8E-4	2.3	4.5E-5	2.9
128	3.1E-4	1.9	2.5E-4	1.9	7.8E-5	2.3	1.6E-5	2.9
256	1.7E-4	1.9	1.3E-4	1.9	3.4E-5	2.3	5.5E-6	2.9
N	δ'_N	$\varrho\,(3.7)$	δ'_N	$\varrho\,(4.0)$	δ'_N	$\varrho\,(5.7)$	δ'_N	$\varrho\,(8.0)$
8	1.1E-4	3.7	9.0E-5	4.0	3.2E-5	5.6	1.5E-5	8.1
16	3.0E-5	3.7	2.2E-5	4.0	5.6E-6	5.7	1.9E-6	8.2
32	7.9E-6	3.7	5.6E-6	4.0	9.9E-7	5.7	2.3E-7	8.2
64	2.1E-6	3.7	1.4E-6	4.0	1.8E-7	5.7	2.8E-8	8.1
128	5.7E-7	3.7	3.5E-7	4.0	3.1E-8	5.7	3.5E-9	8.1
256	1.5E-7	3.7	8.8E-8	4.0	5.5E-9	5.7	4.3E-10	8.1

TABLE 4. (Cont'd.)

$\nu = 0.5$	$r = 1$		$r = \frac{m}{2-\nu} = 1.33$		$r = \frac{m+0.5}{2-\nu} = 1.67$		$r = \frac{m+1.2}{2-\nu} = 2.13$	
N	ε_N	$\varrho\,(2.8)$	ε_N	$\varrho\,(4.0)$	ε_N	ϱ	ε_N	$\varrho\,(8.0)$
8	1.0E-3	2.7	3.6E-4	3.9	1.3E-4	5.5	7.2E-5	7.7
16	3.6E-4	2.8	9.2E-5	4.0	2.4E-5	5.5	8.8E-6	8.2
32	1.3E-4	2.8	2.3E-5	4.0	4.3E-6	5.6	1.1E-6	8.3
64	4.6E-5	2.8	5.8E-6	4.0	7.6E-7	5.6	1.3E-7	8.3
128	1.6E-5	2.8	1.4E-6	4.0	1.3E-7	5.7	1.5E-8	8.3
256	5.8E-6	2.8	3.6E-7	4.0	2.4E-8	5.7	1.9E-9	8.2
N	ε'_N	$\varrho\,(1.4)$	ε'_N	$\varrho\,(1.6)$	ε'_N	$\varrho\,(1.8)$	ε'_N	$\varrho\,(2.1)$
8	3.1E-2	1.3	2.2E-2	1.5	1.6E-2	1.7	9.9E-3	2.1
16	2.2E-2	1.4	1.4E-2	1.6	9.0E-3	1.8	4.7E-3	2.1
32	1.6E-2	1.4	9.0E-3	1.6	5.1E-3	1.8	2.3E-3	2.1
64	1.1E-2	1.4	5.7E-3	1.6	2.9E-3	1.8	1.1E-3	2.1
128	8.1E-3	1.4	3.6E-3	1.6	1.6E-3	1.8	5.2E-4	2.1
256	5.7E-3	1.4	2.3E-3	1.6	9.0E-4	1.8	2.5E-4	2.1
N	δ'_N	$\varrho\,(2.8)$	δ'_N	$\varrho\,(4.0)$	δ'_N	$\varrho\,(5.7)$	δ'_N	$\varrho\,(8.0)$
8	1.3E-3	3.0	4.4E-4	4.3	1.7E-4	6.0	1.0E-4	8.4
16	4.4E-4	3.0	1.0E-4	4.2	2.8E-5	6.1	1.2E-5	8.6
32	1.5E-4	2.9	2.5E-5	4.2	4.7E-6	6.0	1.4E-6	8.6
64	5.1E-5	2.9	6.1E-6	4.1	8.0E-7	5.9	1.7E-7	8.5
128	1.7E-5	2.9	1.5E-6	4.1	1.4E-7	5.8	2.0E-8	8.5
256	6.1E-6	2.9	3.7E-7	4.0	2.4E-8	5.7	2.3E-9	8.4

can see that Gaussian parameters $\eta_1 = (3-\sqrt{3})/6$, $\eta_2 = (3+\sqrt{3})/6$ (that are exact for polynomials of order 3) do not give any further improvement in the convergence rate. In all of the cases presented we see that convergence rate of the derivative v_N at the collocation points is much higher than in the L^∞ norm and is the same as the convergence rate of the approximate solution u_N in the supremum norm.

Tables 3 and 4 show the dependence of the convergence rate on the nonuniformity parameter r, when Gaussian parameters $\eta_1 = (3-\sqrt{3})/6$, $\eta_2 = (3+\sqrt{3})/6$ are used. From Theorem 2 it follows that for uniform grid ($r = 1$) with $m = 2$ the ratios of consecutive values of ε_N ought to be approximately $2^{2-\nu}$ (3.7 for $\nu = 0.1$, 2.8 for $\nu = 0.5$, 2.1 for $\nu = 0.9$). For graded grids with $m = 2$ the ratios ought to be approximately $4(2^{r(2-\nu)})$ for $r = m/(2-\nu)$, 5.7 $(2^{r(2-\nu)})$ for $r = (m+0.5)/(2-\nu)$ and 8 (2^{m+1}) for $r = (m+1.2)/(2-\nu)$. From Tables 3 and 4 we can see that the numerical results are in good agreement with the theoretical estimates of Theorem 2.

REFERENCES

1. H. Brunner, A. Pedas and G. Vainikko, *Piecewise polynomial collocation methods for linear Volterra integro-differential equations with weakly singular kernels*, SIAM J. Numer. Anal. **39** (2001), 957–982.

2. ———, *A spline collocation method for linear Volterra integro-differential equations with weakly singular kernels*, BIT **41** (2001), 891–900.

3. Tao Tang, *Superconvergence of numerical solutions to weakly singular Volterra integro-differential equations*, Numer. Math. **61** (1992), 373–382.

4. G. Vainikko, *Multidimensional weakly singular integral equations*, Lecture Notes in Math., Vol. 1549, Springer-Verlag, Berlin, 1993.

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