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A NYSTRÖM METHOD FOR A CLASS OF INTEGRAL EQUATIONS ON THE REAL LINE WITH APPLICATIONS TO SCATTERING BY DIFFRACTION GRATINGS AND ROUGH SURFACES

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ABSTRACT. We propose a Nyström/product integration method for a class of second kind integral equations on the real line which arise in problems of two-dimensional scalar and elastic wave scattering by unbounded surfaces. Stability and convergence of the method is established with convergence rates dependent on the smoothness of components of the kernel. The method is applied to the problem of acoustic scattering by a sound soft one-dimensional surface which is the graph of a function f, and superalgebraic convergence is established in the case when f is infinitely smooth. Numerical results are presented illustrating this behavior for the case when f is periodic (the diffraction grating case). The Nyström method for this problem is stable and convergent uniformly with respect to the period of the grating, in contrast to standard integral equation methods for diffraction gratings which fail at a countable set of grating periods.

1. Introduction. The most general form of a Fredholm integral equation of the second kind on the real line is

$$x(s) = y(s) + \int_{-\infty}^{+\infty} k(s,t)x(t)\,dt, \quad s \in \mathbf{R}, \tag{1.1}$$

where the kernel k and the righthand side y are given and the unknown function x is to be determined. In this paper we consider the case when the kernel $k(s,t)$ takes the form

$$k(s,t) = a^*(s,t)\ln|s-t| + b^*(s,t), \tag{1.2}$$

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where $a^*, b^* \in C^n(\mathbf{R}^2)$ for some $n \in \mathbf{N}$, and $a^*(s,t)$, $b^*(s,t)$ decay like $|s-t|^{-q}$ as $|s-t| \to \infty$ for some $q > 1$. We propose and analyze a Nyström or product integration method for equations of the form (1.1), establishing its stability and proving convergence results which guarantee superalgebraic convergence in the case when $a^*, b^* \in C^\infty(\mathbf{R}^2)$.

Our motivation is that kernels of the class (1.2) arise when one reformulates two-dimensional problems of time harmonic wave scattering by unbounded obstacles as second kind boundary integral equations. In particular, we have in mind problems, in scalar wave scattering and elastodynamics, where one or more components of the boundary (or one or more interfaces in transmission problems) are the graph of a bounded smooth function $f : \mathbf{R} \to \mathbf{R}$. We discuss the application of the method to a simple problem of this type in Section 4. Specifically, we consider the problem of scattering of time harmonic acoustic waves by an infinite rough sound-soft surface; that is, we solve the Dirichlet boundary value problem for the Helmholtz equation in a non-locally perturbed half-plane.

Nyström/product integration methods are long established for solving Fredholm integral equations of the second kind (see [**5**], [**19**] and the references cited therein). Indeed, the method we propose is based in quite large part on a Nyström method, described in [**19**], suitable for second kind integral equations on finite intervals with logarithmically singular periodic kernels. Such integral equations arise, for example, from the boundary integral equation formulation of 2D problems in potential theory if the boundary is a smooth closed curve. The modification of the method and analysis in [**19**] to apply to the second kind integral equation (1.1) on the whole real line requires some subtlety, however. In particular, since the integral operator is no longer necessarily compact, collectively compact operator theory is not sufficient for a stability analysis. To prove stability we modify results in this theory so that compactness is no longer required.

For descriptions and analyses of related Nyström/product integration methods for second kind integral equations on the half-line, in the case when the integral operator is a convolution operator or a compact perturbation of a convolution operator, see [**4**], [**7**], [**2**], [**15**].

We start in Section 2 by making more explicit the assumptions on the kernel function. The mapping properties of the integral operator K that follow from these assumptions are established, and we use these to derive regularity results for (1.1). In Section 3 we describe our Nyström method which uses equally spaced quadrature points on the real line and reduces (1.1) to the solution of an infinite set of linear equations. In the general case this infinite system requires truncation prior to solution. We point out that a finite linear system is obtained in the case of a periodic kernel and righthand side. In Section 3 we also present our stability and convergence results, which show that the method is superalgebraically convergent if the original equation is well posed and a^* and b^* are smooth.

The application to rough surface scattering is carried out in Section 4. By using the Green's function for a half-plane with Dirichlet boundary conditions and representing the scattered field as a combined single- and double-layer potential, we obtain a novel integral equation equivalent to the scattering problem. Our method is applicable to this equation and is superalgebraically convergent if the scattering surface is smooth. In the engineering literature integral equation methods are widely used for rough surface scattering problems, with discretization based on collocation or Galerkin methods with piecewise polynomial approximation on a uniform or quasiuniform mesh (e.g., [**21**], [**26**], [**17**]). These methods achieve an algebraic convergence rate for smooth surfaces. We hope that the simplicity of implementation and superalgebraic convergence rate of the discretization scheme we propose will make it an attractive alternative. In Section 5 we present numerical results, in the case when the boundary is periodic, forming a diffraction grating, demonstrating that the claimed convergence rates are achieved.

Throughout the paper we pay attention to obtaining stability and error estimates which are uniform with respect to the kernel function k. As a consequence, the final error estimates in Section 4, for scattering by rough surfaces and diffraction gratings, depend on the maximum surface amplitude and slope (and on bounds on higher derivatives) but are otherwise independent of the surface shape. In particular, for the diffraction grating problem, the numerical scheme we propose is shown to be uniformly stable and convergent with respect to the grating period. Such a result does not hold for conventional integral equation formulations of diffraction grating problems; in fact, the standard integral equation formulations (e.g., [**6**], [**23**], [**18**]), which use the free field Green's function rather than the half-plane Dirichlet Green's function we propose, fail to hold for a countable set of values of the grating period.

Throughout the paper we will use the following notations. Let K denote the integral operator defined by

$$K\psi(s) = \int_{-\infty}^{+\infty} k(s,t)\psi(t)\,dt, \quad s \in \mathbf{R}. \tag{1.3}$$

$BC^n(\mathbf{R}^m)$ is the Banach space of all functions whose derivatives up to order n are bounded and continuous on $\mathbf{R}^m$. We abbreviate the norm $\|\cdot\|_{BC^n(\mathbf{R}^m)}$ on $BC^n(\mathbf{R}^m)$ by $\|\cdot\|_\infty$ in the case $n = 0$. For $a \in BC^1(\mathbf{R}^2)$, we denote by $\partial_j a$, $j = 1, 2$, the partial derivatives $\partial_1 a(s,t) = \partial a(s,t)/\partial s$, $\partial_2 a(s,t) = \partial a(s,t)/\partial t$, respectively. By $C^{n,\alpha}(\mathbf{R})$ we denote the usual Hölder space of those functions $\phi \in BC^n(\mathbf{R})$ for which $\phi^{(n)}$ satisfies a uniform Hölder condition of index α, a Banach space under the norm

$$\|\psi\|_{C^{n,\alpha}(\mathbf{R})} := \|\psi\|_{BC^n(\mathbf{R})} + [\psi^{(n)}]_{\alpha;\mathbf{R}},$$

where

$$[\psi]_{\alpha;\mathbf{R}} := \sup_{\substack{s,t\in\mathbf{R}\\ s\neq t}} \frac{|\psi(s)-\psi(t)|}{|s-t|^\alpha}.$$

We define $C^n_{0,\pi}(\mathbf{R}) := \{\psi \in BC^n(\mathbf{R}) : \psi(s) = 0, |s| \geq \pi\}$ and $C^n_{0,\pi}(\mathbf{R}^2) := \{a \in BC^n(\mathbf{R}^2) : a(s,t) = 0, |s-t| \geq \pi\}$ and note that $C^n_{0,\pi}(\mathbf{R})$ and $C^n_{0,\pi}(\mathbf{R}^2)$ are closed subspaces of $BC^n(\mathbf{R})$ and $BC^n(\mathbf{R}^2)$, respectively. We introduce the further nonstandard notation

$$\begin{aligned} BC^n_p(\mathbf{R}) := \{\psi \in BC^n(\mathbf{R}) : {}& \|\psi\|_{BC^n_p(\mathbf{R})} \\ &:= \sup_{m=0,1,\dots,n} \|w_p \psi^{(m)}\|_\infty < \infty\}, \end{aligned}$$

where $w_p(t) = (1+|t|)^p$, and

$$\begin{aligned} BC^n_p(\mathbf{R}^2) := \Big\{a \in BC^n(\mathbf{R}^2) : {}& \|a\|_{BC^n_p(\mathbf{R}^2)} \\ &:= \sup_{\substack{j,k=0,\dots,n\\ j+k\leq n}} \|\tilde{w}_p \partial_1^j \partial_2^k a\|_\infty < \infty\Big\} \end{aligned}$$

where $\tilde{w}_p(s,t) := w_p(s-t)$.

Throughout, $\mathbf{N}_0$ will denote $\mathbf{N} \cup \{0\}$, and $\chi \in C_0^\infty(\mathbf{R})$ will denote a fixed 'cut-off' function, satisfying that $0 \le \chi(s) \le 1$, $s \in \mathbf{R}$, $\chi(s) = 0$, $|s| \ge \pi$, $\chi(s) = 1$, $|s| \le 1$, $\chi(-s) = \chi(s)$, $s \in \mathbf{R}$. In the numerical examples in Section 5, we use the specific choice of χ given by equation (5.8) below.

2. Regularity results. The integral equation (1.1) can be written in operator notation as

$$x = y + Kx. \tag{2.1}$$

In this section we consider mapping properties of the operator K from which we obtain results on the regularity of the solution of (2.1) needed in the later convergence analysis.

It is well known (e.g., [**16**]) that, if the kernel k satisfies the following two properties **A** and **B**, then K maps $BC(\mathbf{R})$ to $BC(\mathbf{R})$ and is bounded with norm $\|K\| = \sup_{s\in\mathbf{R}} \int_{-\infty}^{\infty} |k(s,t)|\,dt$.

A. $\sup_{s\in\mathbf{R}} \int_{-\infty}^{+\infty} |k(s,t)|\,dt < \infty$.

B. For all $s \in \mathbf{R}: \int_{-\infty}^{+\infty} |k(s,t) - k(s',t)|\,dt \to 0 \quad \text{as } s' \to s$.

Throughout we will assume, motivated as indicated in the introduction by applications to scattering by rough surfaces and diffraction gratings, that k takes the form specified in one of the following three conditions $\mathbf{C_n}$, $\mathbf{C'_n}$ and $\mathbf{C''_n}$. We will show shortly that these three conditions are equivalent and that they imply **A** and **B** and thus imply that K is bounded as an operator on $BC(\mathbf{R})$ (though not necessarily compact).

We will use all three conditions and their equivalence extensively throughout the paper. It is simplest, in specific applications (see Section 4) to show that $\mathbf{C}_n$ is satisfied. The form $\mathbf{C''_n}$ for the kernel will be needed to implement the Nyström method. The representation $\mathbf{C'_n}$ will prove particularly convenient in this section for deducing regularity properties. The proof of equivalence, in particular that $\mathbf{C_n} \Longrightarrow \mathbf{C''_n}$, is constructive, and we will use this construction when implementing the Nyström method in Section 4.

$\mathbf{C_n}$.
$$k(s,t) = a^*(s,t)\ln|s-t| + b^*(s,t), \quad s,t \in \mathbf{R},\ s \neq t,$$

where $a^*, b^* \in C^n(\mathbf{R}^2)$, and constants $C > 0$ and $p > 1$ exist such that, for all j, $l \in \mathbf{N}_0$, with $j + l \leq n$, we have

(2.2)
$$\left| \frac{\partial^{j+l} a^*(s,t)}{\partial s^j \partial t^l} \right| \leq C, \quad s,t \in \mathbf{R},\ |s-t| \leq \pi,$$

(2.3)
$$\left| \frac{\partial^{j+l} b^*(s,t)}{\partial s^j \partial t^l} \right| \leq C, \quad s,t \in \mathbf{R},\ |s-t| \leq \pi$$

and

(2.4)
$$\left| \frac{\partial^{j+l} k(s,t)}{\partial s^j \partial t^l} \right| \leq C(1+|s-t|)^{-p}, \quad s,t \in \mathbf{R},\ |s-t| \geq \pi.$$

$\mathbf{C'_n}$.
$$k(s,t) = a(s,t)\ln|s-t| + b(s,t), \quad s,t \in \mathbf{R},\ s \neq t,$$

where $a \in C^n_{0,\pi}(\mathbf{R}^2)$ and $b \in BC^n_p(\mathbf{R}^2)$, for some $p > 1$.

$\mathbf{C''_n}$.
$$k(s,t) = \frac{1}{2\pi} A(s,t) \ln\left(4\sin^2\left(\frac{s-t}{2}\right)\right) + B(s,t),$$
$$s,t \in \mathbf{R},\ s \neq t,$$

where $A \in C^n_{0,\pi}(\mathbf{R}^2)$ and $B \in BC^n_p(\mathbf{R}^2)$ for some $p > 1$.

Theorem 2.1. *For $n \in \mathbf{N}_0$, $\mathbf{C_n}$, $\mathbf{C'_n}$ and $\mathbf{C''_n}$ are equivalent. Moreover, there exists a constant $c > 1$ depending only on n and p such that, if k satisfies $\mathbf{C_n}$, then the functions a, b in $\mathbf{C'_n}$ and A, B in $\mathbf{C''_n}$ can be chosen to satisfy*

(2.5)
$$\begin{aligned} C &\leq c(\|a\|_{BC^n(\mathbf{R}^2)} + \|b\|_{BC^n_p(\mathbf{R}^2)}) \\ &\leq c^2(\|A\|_{BC^n(\mathbf{R}^2)} + \|B\|_{BC^n_p(\mathbf{R}^2)}) \leq c^3 C. \end{aligned}$$

Proof. $\mathbf{C_n'} \Longrightarrow \mathbf{C_n}$. Set $a^* := a$, $b^* := b$ and $C := \|a\|_{BC^n(\mathbf{R}^2)} + \|b\|_{BC_p^n(\mathbf{R}^2)}$. Then (2.2) and (2.3) follow immediately; for (2.4), observe that a^* and all its derivatives vanish for $|s-t| \geq \pi$ and $b^* \in BC_p^n(\mathbf{R})$.

$\mathbf{C_n} \Longrightarrow \mathbf{C_n''}$. Set

$$A(s,t) := \pi a^*(s,t)\chi(s-t) \tag{2.6}$$

and

$$\begin{aligned} B(s,t) := a^*(s,t)\Bigg[&\ln|s-t|(1-\chi(s-t)) \\ &- \chi(s-t)\ln\left(\frac{\sin((s-t)/2)}{(s-t)/2}\right)\Bigg] + b^*(s,t). \end{aligned} \tag{2.7}$$

An easy calculation yields the representation of k in $\mathbf{C_n''}$. As $a^* \in C^n(\mathbf{R}^2)$ and $\chi \in C^\infty_{0,\pi}(\mathbf{R})$, $A \in C^n_{0,\pi}(\mathbf{R}^2)$ follows. Furthermore, $\ln(\sin t/t) \in C^\infty(-\pi,\pi)$ and $0 \notin \operatorname{supp}\{\ln(t)(1-\chi(t))\}$, so (2.2), (2.3) and (2.4) imply $B \in BC_p^n(\mathbf{R}^2)$. Moreover, by straightforward calculations, we obtain the estimate

$$\|A\|_{BC^n(\mathbf{R}^2)} + \|B\|_{BC_p^n(\mathbf{R}^2)} \leq c_1 C,$$

where c_1 is a constant only depending on n, p and the cut-off function χ.

$\mathbf{C_n''} \Longrightarrow \mathbf{C_n'}$. Set

$$\gamma(x) := \begin{cases} \ln\left(\dfrac{\sin x}{x}\right) & x \neq 0, \\ 0 & x = 0, \end{cases}$$

and

$$\begin{aligned} a(s,t) &:= \frac{1}{\pi}A(s,t), \\ b(s,t) &:= \frac{1}{\pi}\gamma\left(\frac{s-t}{2}\right)A(s,t) + B(s,t). \end{aligned}$$

Again, the representation of k as in $\mathbf{C_n'}$ is obvious; so is the fact that $a \in C^n_{0,\pi}(\mathbf{R}^2)$. From $A \in C^n_{0,\pi}(\mathbf{R}^2)$, $B \in BC_p^n(\mathbf{R}^2)$ and the definition of γ, we also conclude that $b \in BC_p^n(\mathbf{R}^2)$. Again, by straightforward calculations, we obtain the estimate

$$\|a\|_{BC^n(\mathbf{R}^2)} + \|b\|_{BC_p^n(\mathbf{R}^2)} \leq c_2(\|A\|_{BC^n(\mathbf{R}^2)} + \|B\|_{BC_p^n(\mathbf{R}^2)}),$$

where c_2 is a constant dependent only on n and p. $\square$

By applications of the dominated convergence theorem to show **B**, we easily establish the following result.

Theorem 2.2. *If k satisfies $\mathbf{C}_0'$, then k also satisfies* **A** *and* **B**. *Consequently, K is bounded as an operator from $BC(\mathbf{R})$ to $BC(\mathbf{R})$. In fact,*

$$\|K\| \leq C(\|a\|_{BC(\mathbf{R}^2)} + \|b\|_{BC_p^0(\mathbf{R}^2)}),$$

where $C > 0$ is a constant only dependent on p.

Let

$$\omega(h) := \sup\left\{ \int_{-\infty}^{+\infty} |k(s_1,t) - k(s_2,t)|\, dt : s_1, s_2 \in \mathbf{R},\ |s_1 - s_2| \leq h \right\},$$

and note that assumption **B** is certainly satisfied if $\omega(h) \to 0$ as $h \to 0$. The following sharper bound on the behavior of $\omega(h)$ as $h \to 0$ is the basis of the regularity results in this section.

Theorem 2.3. *If k satisfies $\mathbf{C}_1'$, then*

$$\omega(h) \leq C(\|a\|_{BC^1(\mathbf{R}^2)} + \|b\|_{BC_p^1(\mathbf{R}^2)})h(1 + |\ln h|), \quad 0 < h \leq 1,$$

where $C > 0$ is a constant only dependent on p.

Proof. Let $s_1, s_2 \in \mathbf{R}$, $h = |s_1 - s_2| \le 1$, $\bar{s} = (s_1 + s_2)/2$. Note that

$$\begin{aligned}\int_{-\infty}^{+\infty} |k(s_1,t) - k(s_2,t)|\,dt \\ &\le \int_{-\infty}^{+\infty} |b(s_1,t) - b(s_2,t)|\,dt \\ &\quad + \int_{\bar{s}-\pi-(1/2)}^{\bar{s}+\pi+(1/2)} |a(s_1,t) - a(s_2,t)||\ln|s_1 - t||\,dt \\ &\quad + \int_{\bar{s}-\pi-(1/2)}^{\bar{s}+\pi+(1/2)} |a(s_2,t)||\ln|s_1 - t| - \ln|s_2 - t||\,dt.\end{aligned}$$

By the mean value theorem, there holds

$$\int_{-\infty}^{+\infty} |b(s_1,t) - b(s_2,t)|\,dt \le Ch\|b\|_{BC_p^1(\mathbf{R}^2)} \tag{2.8}$$

and

$$\begin{aligned}\int_{\bar{s}-\pi-(1/2)}^{\bar{s}+\pi+(1/2)} |a(s_1,t) - a(s_2,t)||\ln|s_1 - t||\,dt \\ &\le h\|a\|_{BC^1(\mathbf{R}^2)} \int_{\bar{s}-\pi-(1/2)}^{\bar{s}+\pi+(1/2)} |\ln|s_1 - t||\,dt \\ &\le Ch\|a\|_{BC^1(\mathbf{R}^2)}.\end{aligned} \tag{2.9}$$

Finally, there holds

$$\begin{aligned}\int_{\bar{s}-\pi-(1/2)}^{\bar{s}+\pi+(1/2)} |a(s_2,t)||\ln|s_1 - t| - \ln|s_2 - t||\,dt \\ &\le \|a\|_{BC(\mathbf{R}^2)} \int_{\bar{s}-\pi-(1/2)}^{\bar{s}+\pi+(1/2)} |\ln|s_1 - t| - \ln|s_2 - t||\,dt, \\ &\le \|a\|_{BC(\mathbf{R}^2)} Ch(1 + |\ln h|).\end{aligned}$$

This bound, combined with (2.8) and (2.9), yields the assertion. □

To derive further mapping properties for the integral operator K, we introduce for $b \in BC_p^0(\mathbf{R}^2)$, the operator M^b defined by

$$M^b\phi(s) := \int_{-\infty}^{+\infty} b(s,t)\phi(t)\,dt, \quad s \in \mathbf{R},$$

and, for $a \in C^0_{0,\pi}(\mathbf{R}^2)$, the operator L^a defined by

$$L^a\phi(s) := \int_{-\infty}^{+\infty} a(s,t) \ln |s-t| \phi(t)\, dt, \quad s \in \mathbf{R}.$$

We will proceed by establishing certain mapping properties of these operators which are stated in the following theorem.

Theorem 2.4. *For $n \in \mathbf{N}$,*

(a) *If $b \in BC^n_p(\mathbf{R}^2)$, then M^b maps $BC(\mathbf{R})$ to $BC^n(\mathbf{R})$ and is bounded with*

$$\|M^b\| \le C\|b\|_{BC^n_p}(\mathbf{R}^2),$$

where the constant $C > 0$ only depends on n and p.

(b) *If $a \in C^n_{0,\pi}(\mathbf{R}^2)$, then L^a maps $BC^{n-1}(\mathbf{R})$ to $C^{n-1,\alpha}(\mathbf{R})$, $\alpha \in (0,1)$, and is bounded with*

$$\|L^a\| \le C\|a\|_{BC^n(\mathbf{R}^2)},$$

where the constant $C > 0$ only depends on n and α.

(c) *If $a \in C^n_{0,\pi}(\mathbf{R}^2)$, then L^a maps $C^{n-1,\alpha}(\mathbf{R})$ to $BC^n(\mathbf{R})$, $\alpha \in (0,1)$, and is bounded with*

$$\|L^a\| \le C\|a\|_{BC^n(\mathbf{R}^2)},$$

where the constant $C > 0$ only depends on n and α.

To prove this theorem, the following three technical results concerning the operator L^a are needed.

Lemma 2.5. *If $a \in C^1_{0,\pi}(\mathbf{R}^2)$, then L^a maps $BC(\mathbf{R})$ to $C^{0,\alpha}(\mathbf{R})$, $\alpha \in (0,1)$, and is bounded with*

$$\|L^a\| \le C\|a\|_{BC^1(\mathbf{R}^2)}, \tag{2.10}$$

for some constant $C > 0$ depending only on α.

Proof. Setting $k(s,t) := a(s,t)\ln|s-t|$ and applying Theorem 2.3 yields that, for $|s_1 - s_2| \le 1$,

$$|L^a\phi(s_1) - L^a\phi(s_2)| \le \|\phi\|_\infty \omega(|s_1-s_2|) \\ \le C|s_1-s_2|\|a\|_{BC^1(\mathbf{R}^2)}\left(1+\left|\ln\frac{1}{|s_1-s_2|}\right|\right)\|\phi\|_\infty.$$

For $\alpha \in (0,1)$ and $|s_1-s_2| \le 1$, $|s_1-s_2|\ln(1/|s_1-s_2|) \le (1/(1-\alpha))|s_1-s_2|^\alpha$ holds and we have

$$(2.11) \qquad |L^a\phi(s_1) - L^a\phi(s_2)| \le C\|\phi\|_\infty\|a\|_{BC^1(\mathbf{R}^2)}|s_1 - s_2|^\alpha.$$

Now, by Theorem 2.2, $\|L^a\phi\|_\infty \le C\|a\|_{BC(\mathbf{R}^2)}\|\phi\|_\infty$. Together with (2.11), this yields that $L^a\phi \in C^{0,\alpha}(\mathbf{R})$ and the bound (2.10). □

Lemma 2.6. *If $a \in C^N_{0,\pi}(\mathbf{R}^2)$ for some $N \in \mathbf{N}$, then, for $n = 1, 2, \dots, N$ and $\phi \in BC^n(\mathbf{R})$, there holds $L^a\phi \in BC^n(\mathbf{R})$. Moreover, there exist functions $a^n_j \in C^{N+j-n}_{0,\pi}(\mathbf{R}^2)$, $j = 0, \dots, n$, satisfying*

$$\frac{d^n}{ds^n}(L^a\phi(s)) = \sum_{j=0}^{n}(L^{a^n_j}\phi^{(j)})(s), \quad s \in \mathbf{R},$$

for all $\phi \in BC^n(\mathbf{R})$. Moreover, for some constant $C > 0$ depending only on N,

$$(2.12) \qquad \|a^n_j\|_{BC^{j+N-n}(\mathbf{R}^2)} \le C\|a\|_{BC^N(\mathbf{R}^2)}, \quad j = 0, \dots, n.$$

Proof. We prove this lemma by induction. In the case $n = 1$, we have for $s \in \mathbf{R}$ and any $h \in (-1,1) \setminus \{0\}$, that

$$\frac{1}{h}(L^a\phi(s+h) - L^a\phi(s)) \\ = \int_{s-\pi-1}^{s+\pi+1}\frac{1}{h}(a(s+h,t) - a(s,t))\ln|s-t|\phi(t)\,dt \\ + \int_{s-\pi-1}^{s+\pi+1}\frac{1}{h}(a(s+h,t+h)\phi(t+h) - a(s+h,t)\phi(t))\ln|s-t|\,dt.$$

Letting $h \to 0$, we conclude by the dominated convergence theorem that

$$\frac{d}{ds}(L^a \phi(s)) = L^{a_0^1}\phi(s) + L^{a_1^1}\phi'(s), \tag{2.13}$$

where $a_0^1 := \partial_1 a + \partial_2 a$ and $a_1^1 := a$ satisfy (2.12) with $n = 1$.

Assume now that $1 \le n < N$, $\phi \in BC^{n+1}(\mathbf{R})$ and that the assertion of the lemma holds for n. Then there exist $a_j^n \in C_{0,\pi}^{N+j-n}(\mathbf{R}^2)$, $j = 0, \dots, n$, such that

$$\frac{d^n}{ds^n}(L^a \phi(s)) = \sum_{j=0}^{n} (L^{a_j^n}\phi^{(j)})(s).$$

Now set

$$a_j^{n+1} := \begin{cases} a_n^n & j = n+1, \\ \partial_1 a_j^n + \partial_2 a_j^n + a_{j-1}^n & 1 \le j \le n, \\ \partial_1 a_0^n + \partial_2 a_0^n & j = 0. \end{cases}$$

Then $a_j^{n+1} \in C_{0,\pi}^{N+j-n-1}(\mathbf{R}^2)$ and, by the same argument as for the case $n = 1$, notably by (2.13), there holds

$$\frac{d^{n+1}}{ds^{n+1}}(L^a \phi(s)) = \sum_{j=0}^{n+1} (L^{a_j^{n+1}}\phi^{(j)})(s).$$

The estimate (2.12) for n replaced by $n+1$ follows from the definition of the functions a_j^{n+1} and the inductive assumption. $\square$

Lemma 2.7. *If $a \in C_{0,\pi}^1(\mathbf{R}^2)$, then L^a maps $C^{0,\alpha}(\mathbf{R})$ to $BC^1(\mathbf{R})$, $\alpha \in (0,1)$, and is bounded with*

$$\|L^a\| \le C\|a\|_{BC^1(\mathbf{R}^2)}, \tag{2.14}$$

for some constant $C > 0$ depending only on α.

Proof. Choose $q > 2\pi$ and set $J := [-q, q]$. Further, let $\chi^* \in C^\infty(\mathbf{R})$ with $\operatorname{supp}\chi^* \subset J$ and $\chi^* \equiv 1$ on $[\pi - q, q - \pi]$. Defining $\tilde{a}(s,t) := \chi^*(t)a(s,t)$ and $\psi(s) := 1$, we have $L^{\tilde{a}}\psi \in BC^1(\mathbf{R})$ by Lemma 2.6 and hence, by Lemma 1 (iii) in [**25**], we obtain $L^{\tilde{a}}\phi \in C^1(J)$. However, as $L^{\tilde{a}}\phi(s) = L^a\phi(s)$ for $|s| \leq q - 2\pi$ and, since q was chosen arbitrarily, there also holds $L^a\phi \in C^1(\mathbf{R})$. From Lemma 1 in [**25**], we also obtain the representation

$$\begin{aligned}\frac{d}{ds}(L^a\phi)(s) \\ &= \int_{-\infty}^{+\infty} \frac{\partial}{\partial s}\{a(s,t)\ln|s-t|\}(\phi(t)-\phi(s))\,dt + \phi(s)\frac{d}{ds}L^a\psi(s).\end{aligned}$$

Since $a \in C^1_{0,\pi}(\mathbf{R}^2)$ and $\phi \in C^{0,\alpha}(\mathbf{R})$ we easily obtain that $L^a\phi \in BC^1(\mathbf{R})$ and the bound (2.14) by applications of Theorem 2.2, Lemma 2.6 and elementary estimates. $\square$

Proof of Theorem 2.4. With regard to the operator M^b, we may interchange differentiation and integration so that, for $\phi \in BC(\mathbf{R})$ and $m = 0, \dots, n$, we obtain

$$\frac{d^m}{ds^m}M^b\phi(s) = \int_{-\infty}^{+\infty} \frac{\partial^m}{\partial s^m}b(s,t)\phi(t)\,dt.$$

By applying Theorem 2.2, we conclude the proof of (a).

Lemma 2.5 and Lemma 2.6 immediately imply (b).

For the case $n = 1$, (c) is the assertion of Lemma 2.7. For $n > 1$, by Lemma 2.6 we have

$$\frac{d^{n-1}}{ds^{n-1}}(L^a\phi(s)) = \sum_{j=0}^{n-1}(L^{a_j^{n-1}}\phi^{(j)})(s), \quad s \in \mathbf{R}, \tag{2.15}$$

where, for $j = 0, \dots, n-1$, $a_j^{n-1} \in C^{j+1}_{0,\pi}(\mathbf{R}^2)$ with $\|a_j^{n-1}\|_{BC^{j+1}(\mathbf{R}^2)} \leq C\|a\|_{BC^n(\mathbf{R}^2)}$. Applying Lemma 2.7 to the righthand side of (2.15), it follows that

$$\frac{d^{n-1}}{ds^{n-1}}(L^a\phi(s)) \in BC^1(\mathbf{R}),$$

with

$$\|(L^a\phi)^{(n-1)}\|_{BC^1(\mathbf{R})} \leq C\|a\|_{BC^n(\mathbf{R}^2)}\|\phi\|_{C^{n-1,\alpha}(\mathbf{R})}.$$

Thus $L^a\phi \in BC^n(\mathbf{R})$ and the bound $\|L^a\| \leq C\|a\|_{BC^n(\mathbf{R}^2)}$ follows. $\square$

Corollary 2.8. *If k satisfies $\mathbf{C_n'}$ for some $n \in \mathbf{N}$, then for all $\alpha \in (0,1)$ and $m = 0, \dots, n-1$, the operator K maps $BC^m(\mathbf{R})$ to $C^{m,\alpha}(\mathbf{R})$ and $C^{m,\alpha}(\mathbf{R})$ to $BC^{m+1}(\mathbf{R})$, respectively. Further, each of these mappings is bounded and the operator norms satisfy*

$$\|K\| \le C(\|a\|_{BC^n(\mathbf{R}^2)} + \|b\|_{BC_p^n(\mathbf{R}^2)}),$$

where the constant $C > 0$ depends only on α, n and p.

To complete this section we apply the above mapping properties to establish the regularity of the solution of (2.1) when K satisfies $\mathbf{C_n}$ and y is sufficiently smooth.

Theorem 2.9. *If k satisfies $\mathbf{C_n'}$ for some $n \in \mathbf{N}$ and $x \in BC(\mathbf{R})$ satisfies the integral equation (2.1) and $y \in BC^n(\mathbf{R})$, then $x \in BC^n(\mathbf{R})$ and, for some constant $C > 0$ dependent only on n and p,*

$$\|x\|_{BC^n(\mathbf{R})} \le C(\|y\|_{BC^n(\mathbf{R})} + (\|a\|_{BC^n(\mathbf{R}^2)} + \|b\|_{BC_p^n(\mathbf{R}^2)})\|x\|_\infty).$$

Proof. This follows easily by induction using Corollary 2.8 and equation (2.1). $\square$

3. The Nyström method. We first describe the Nyström/product integration method we propose and then analyze its stability and convergence.

For $s \in \mathbf{R}$, define the periodic extension operator $E_s : BC(\mathbf{R}) \to L_\infty(\mathbf{R})$ implicitly by

$$E_s\phi(t) = \phi(t), \quad s - \pi \le t < s + \pi,$$

and

$$E_s\phi(t + 2\pi) = E_s\phi(t), \quad t \in \mathbf{R}.$$

Throughout this section we continue to assume that (as a minimum) assumption $\mathbf{C_0''}$ holds, so that $K = K^A + K^B$, where

$$\begin{aligned} K^A\phi(s) &:= \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(s,t)\ln\left(4\sin^2\left(\frac{s-t}{2}\right)\right)\phi(t)\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \ln\left(4\sin^2\left(\frac{s-t}{2}\right)\right)E_s(A(s,\cdot)\phi)(t)\,dt \end{aligned}$$

and

$$K^B\phi(s) := \int_{-\infty}^{+\infty} B(s,t)\phi(t)\,dt.$$

Note that $\mathbf{C_0''}$ implies that $A \in C^n_{0,\pi}(\mathbf{R}^2)$, so that $A(s,t) = 0$, $|s-t| \geq \pi$. We will use this fact extensively in this section.

In the case when $f \in BC^n_p(\mathbf{R})$ for some $p > 1$ and $n \in \mathbf{N}_0$, we can approximate

$$If := \int_{-\infty}^{+\infty} f(t)\,dt$$

by the trapezoidal rule approximation

$$I_h f := h \sum_{j\in\mathbf{Z}} f(jh),$$

and we will see later that this approximation is very rapidly convergent as $h \to 0$ if n is large. It makes sense therefore to approximate $K^B\phi$ using the trapezoidal rule, by $K^B_N\phi$, where the operator K^B_N is defined by

$$K^B_N\phi(s) = I_{\pi/N}(B(s,\cdot)\phi) = \frac{\pi}{N}\sum_{j\in\mathbf{Z}} B(s,t_j)\phi(t_j), \quad s \in \mathbf{R},$$

where $t_j = j\pi/N$, $j \in \mathbf{Z}$. We choose $h = \pi/N$ for compatibility with our approximation of the operator K^A.

We turn now to the approximation of the operator K^A. Define the integral operator $Q : BC(\mathbf{R}) \to BC(\mathbf{R})$ by

$$Q\phi(s) := \frac{1}{2\pi}\int_0^{2\pi} \ln\left(4\sin^2\left(\frac{s-t}{2}\right)\right)\phi(t)\,dt, \quad s \in \mathbf{R}.$$

For $N \in \mathbf{N}$, let $Q_N\phi := Q\phi_N$, where ϕ_N is the unique trigonometric polynomial of the form $\phi_N(t) = \sum_{j=0}^N \alpha_j \cos jt + \sum_{j=1}^{N-1} \beta_j \sin jt$ which interpolates ϕ at t_j, $j = 0, \dots, 2N-1$. Then (see [**19**, p. 177]) we have that

$$Q_N\phi(t) = Q\phi_N(t) = \sum_{j=0}^{2N-1} R_j^{(N)}(t)\phi(t_j), \quad t \in \mathbf{R},$$

where

$$(3.1)\qquad R_j^{(N)}(t) := -\frac{1}{N}\left\{\sum_{m=1}^{N-1}\frac{1}{m}\cos m(t-t_j) + \frac{1}{2N}\cos N(t-t_j)\right\}.$$

We note from this formula that

$$R_j^{(N)}(t_k) = R_{j-k}^{(N)}, \quad j,k\in\mathbf{Z},$$

where

(3.2)

$$R_j^{(N)} := R_j^{(N)}(0) = -\frac{1}{N}\left\{\sum_{m=1}^{N-1}\frac{1}{m}\cos mt_j + \frac{1}{2N}\cos Nt_j\right\}, \quad j\in\mathbf{Z}.$$

Remark 3.1. Note that $Q_N\phi = Q\phi$ for any trigonometric polynomial ϕ of order N. For if $\phi(t) = \sum_{j=0}^{N} a_j\cos jt + \sum_{j=1}^{N-1} b_j \sin jt$, there holds $\phi_N = \phi$, so $Q\phi = Q\phi_N = Q_N\phi$. Also, $\phi(t) = \sin(Nt)$ implies $Q\phi = Q_N\phi = 0$.

We will approximate $K^A\phi(s)$ which we may write as $Q(E_s(A(s,\cdot)\phi))(s)$ by $K_N^A\phi(s)$ where, for $s\in\mathbf{R}$,

$$\begin{aligned}K_N^A\phi(s) &:= Q_N(E_s(A(s,\cdot)\phi))(s)\\ &= \sum_{j=0}^{2N-1} R_j^{(N)}(s)(E_s(A(s,\cdot)\phi))(t_j)\\ &= \sum_{j\in\mathbf{Z}} R_j^{(N)}(s)A(s,t_j)\phi(t_j).\end{aligned}$$

Finally then, our Nyström method is to approximate $K = K^A + K^B$ by $K_N = K_N^A + K_N^B$, and we have that

$$K_N\phi(s) = \sum_{j\in\mathbf{Z}}\alpha_j^{(N)}(s)\phi(t_j), \quad s\in\mathbf{R},$$

where

$$\alpha_j^{(N)}(s) := R_j^{(N)}(s)A(s,t_j) + \frac{\pi}{N}B(s,t_j), \quad s\in\mathbf{R}.$$

The Nyström method approximation, $x_N \in BC(\mathbf{R})$, is defined by the equation

$$x_N = y + K_N x_N, \tag{3.3}$$

or, explicitly,

$$x_N(s) = y(s) + \sum_{j\in\mathbf{Z}} \alpha_j^{(N)}(s) x_N(t_j), \quad s \in \mathbf{R}. \tag{3.4}$$

To obtain $x_N^j := x_N(t_j)$, $j \in \mathbf{Z}$, we set $s = t_k$ to obtain the infinite set of linear equations

$$x_N^k = y^k + \sum_{j\in\mathbf{Z}} a_{kj} x_N^j, \quad k \in \mathbf{Z}, \tag{3.5}$$

where

$$\begin{aligned} y^k &:= y(t_k), \\ a_{kj} &:= \alpha_j^{(N)}(t_k) = R_{j-k}^{(N)} A(t_k, t_j) + \frac{\pi}{N} B(t_k, t_j), \quad j, k \in \mathbf{Z}. \end{aligned} \tag{3.6}$$

In general the solution of the infinite linear system (3.5) cannot be computed exactly. We will analyze the stability and convergence of truncating (3.5) to a finite linear system in a future publication; see [**8**] for this analysis carried out for a related class of integral equations on the real line. But in certain cases (3.5) reduces to a finite linear system. One such case is when the following additional assumption on k is satisfied and $y \in C_L(\mathbf{R})$, the space of L-periodic continuous functions on $\mathbf{R}$.

D. There exists $L > 0$ such that

$$k(s + L, t + L) = k(s, t), \quad s, t \in \mathbf{R}.$$

If Assumptions **A**, **B** and **D** hold, then K is a compact operator on $C_L(\mathbf{R})$ but, in general, not on $BC(\mathbf{R})$. (Consider, for example, the case when $k(s, t) = \kappa(s - t)$ with $\kappa \in L_1(\mathbf{R})$, i.e., K is a convolution operator on the real line.) However, the solvability of (1.1) in $BC(\mathbf{R})$ has been addressed in Theorem 2.10 in [**10**]. As a corollary of this theorem and of the compactness of K on $C_L(\mathbf{R})$, the following result holds.

Theorem 3.2. *Suppose that assumptions* **A**, **B** *and* **D** *are satisfied and the homogeneous integral equation $x = Kx$ has no nontrivial solution in $BC(\mathbf{R})$. Then the integral equation* (1.1) *has exactly one solution $x \in BC(\mathbf{R})$ for every $y \in BC(\mathbf{R})$ and, for some constant $C > 0$ independent of y,*

$$\|x\|_\infty \le C\|y\|_\infty. \tag{3.7}$$

If also $y \in C_L(\mathbf{R})$, then $x \in C_L(\mathbf{R})$.

Now suppose that assumptions $\mathbf{C}_0$ and **D** are satisfied, that $x = Kx$ has no nontrivial solution, and that $y \in C_L(\mathbf{R})$. Then, by Theorems 2.2 and 3.2, $x \in C_L(\mathbf{R})$. If also $L = (M\pi/N)$ for some $M \in \mathbf{N}$, then $x_N^{\mathbf{j}+M} = x_N^j$, $y^{j+M} = y^j$, $a_{k+M,j+M} = a_{k,j}$. Then (3.6) reduces to the finite linear system

$$x_N^k = y^k + \sum_{j=1}^{M} \tilde{a}_{kj} x_N^j, \quad k = 1, \dots, M, \tag{3.8}$$

where

$$\tilde{a}_{kj} = \sum_{n \in \mathbf{Z}} a_{k,j+nM}.$$

3.1. Stability analysis. We turn next to establishing stability of the Nyström method we have proposed. Our stability analysis depends crucially on the following error estimates which establish convergence of $K_N\phi$ to $K\phi$, uniformly on bounded and uniformly equicontinuous sets.

Theorem 3.3. *If k satisfies assumption $\mathbf{C_1''}$ and $\phi \in BC(\mathbf{R})$ is uniformly continuous, then*

$$\|K\phi - K_N\phi\|_\infty \longrightarrow 0$$

as $N \to \infty$. Further, if $S \subset BC(\mathbf{R})$ is bounded and uniformly equicontinuous on $\mathbf{R}$, then, for every $\varepsilon > 0$ and $\beta > 0$, there exists $N_0 \in \mathbf{N}$ dependent only on ε, β and S such that

$$\|K\phi - K_N\phi\|_\infty \le \varepsilon, \tag{3.9}$$

for all $N > N_0$, $\phi \in S$ and $k \in T := \{k : k$ satisfies $\mathbf{C_1''}$ with $\|A\|_{BC^1(\mathbf{R}^2)} + \|B\|_{BC_p^1(\mathbf{R}^2)} \le \beta\}$.

Proof. Clearly it is sufficient to prove that for every $\varepsilon > 0$, $\beta > 0$, there exists $N_0 \in \mathbf{N}$ such that the estimate (3.9) holds for all $N > N_0$, $\phi \in S$ and $k \in T$. Throughout, let C denote a generic constant dependent only on p and fix $\beta > 0$. For $\delta > 0$, let

$$\Omega(\delta) := \sup\{|f(s) - f(t)| : f \in S,\ s, t \in \mathbf{R},\ |s - t| \le \delta\},$$

and note that, since S is uniformly equicontinuous, $\Omega(\delta) \to 0$ as $\delta \to 0$. Similarly, let

$$M := \sup_{f \in S} \|f\|_\infty.$$

For $k \in T$, we have $K = K^A + K^B$ with $A \in C^1_{0,\pi}(\mathbf{R}^2)$, $B \in BC^1_p(\mathbf{R}^2)$ and $\|A\|_{BC^1(\mathbf{R}^2)} + \|B\|_{BC_p^1(\mathbf{R}^2)} \le \beta$.

a) First of all, consider $K^B\phi - K_N^B\phi$. For $\phi \in C(\mathbf{R})$ and $N \in \mathbf{N}$, let $P_N\phi$ denote the piecewise constant approximation to ϕ given by

$$P_N\phi(s) = \phi(jh), \quad \left(j - \frac{1}{2}\right)h < s \le \left(j + \frac{1}{2}\right)h, \quad j \in \mathbf{Z},$$

where $h = (\pi/N)$. Then, provided $\sum_{j \in \mathbf{Z}} \phi(jh)$ exists, $I_{\pi/N}(\phi) = I(P_N\phi)$ and, for $\phi, \psi \in C(\mathbf{R})$, there holds $P_N(\phi\psi) = P_N\phi P_N\psi$. Thus

$$\begin{aligned}|K^B\phi(s) - K_N^B\phi(s)| &= |I(B(s,\cdot)\phi) - I_{\pi/N}(B(s,\cdot)\phi)| \\ &= |I(B(s,\cdot)\phi - P_N(B(s,\cdot))P_N\phi)| \\ &\le \|B(s,\cdot)\|_1 \|\phi - P_N\phi\|_\infty \\ &\quad + \|P_N\phi\|_\infty \|B(s,\cdot) - P_N(B(s,\cdot))\|_1,\end{aligned}$$

where $\|f\|_1 := \int_{-\infty}^{+\infty} |f(t)|\, dt$. Now

$$\|B(s,\cdot)\|_1 \le C\|B\|_{BC_p^1(\mathbf{R}^2)} \le C\beta,$$

and

$$\|\phi - P_N\phi\|_\infty \le \Omega\left(\frac{\pi}{2N}\right), \qquad \|P_N\phi\|_\infty \le \|\phi\|_\infty \le M.$$

Also, for $(j-(1/2))h < t \le (j+(1/2))h$, $j \in \mathbf{Z}$ and some ρ_j between t and $t_j = j\pi/N$,

$$\begin{aligned} |B(s,t) - P_N(B(s,t))| &= |B(s,t) - B(s,t_j)| \\ &\le |t-t_j| \left|\frac{\partial B}{\partial t}(s,\rho_j)\right| \\ &\le \frac{\pi}{2N}\|B\|_{BC_p^1(\mathbf{R}^2)}(1+|s-\rho_j|)^{-p}. \end{aligned}$$

Thus, for $N \in \mathbf{N}$,

$$\begin{aligned} \|B(s,\cdot) - P_N(B(s,\cdot))\|_1 &= \sum_{j\in\mathbf{Z}} \int_{(j-(1/2))h}^{(j+(1/2))h} |B(s,t) - P_N(B(s,t))|\,dt \\ &\le \frac{\pi}{N}\sum_{j\in\mathbf{Z}} \frac{\pi}{2N}\|B\|_{BC_p^1(\mathbf{R}^2)}(1+|s-\rho_j|)^{-p} \\ &\le \frac{C}{N}\beta. \end{aligned}$$

Thus we have

$$\|K^B\phi - K_N^B\phi\|_\infty \le C\beta\left(\Omega\left(\frac{\pi}{2N}\right) + \frac{M}{N}\right),$$

so that for all $\varepsilon > 0$, there exists $N_0 \in \mathbf{N}$ such that $\|K^B\phi - K_N^B\phi\|_\infty \le \varepsilon$ for all $N \ge N_0$, $\phi \in S$, $k \in T$.

b) Now consider $\|K^A\phi - K_N^A\phi\|_\infty \le \sup_{s\in\mathbf{R}} \|(Q-Q_N)(E_s(A(s,\cdot)\phi))\|_\infty$. We show first that $E := \{E_s(A(s,\cdot)\phi) : \phi \in S, s \in \mathbf{R}, A \in C^1_{0,\pi(\mathbf{R}^2)}, \|A\|_{BC^1(\mathbf{R}^2)} \le \beta\}$ is bounded and uniformly equicontinuous on $\mathbf{R}$. For $\phi \in S$,

$$\|E_s(A(s,\cdot)\phi)\|_\infty \le M\|A(s,\cdot)\|_\infty \le M\beta.$$

Thus E is bounded.

Further, after some calculation, we see that for all $s, t_1, t_2 \in \mathbf{R}$,

$$\begin{aligned}|E_s(A(s,\cdot)\phi)(t_1) - E_s(A(s,\cdot)\phi)(t_2)| \\ \leq 2\beta(M|t_1 - t_2| + \Omega(|t_1 - t_2|)),\end{aligned}$$

so that E is also uniformly equicontinuous. Thus $\{E_s(A(s,\cdot)\phi)|_{[0,2\pi]} : \phi \in S,\, s \in \mathbf{R},\, k \in T\}$ is a bounded and equicontinuous subset of $C[0, 2\pi]$, and thus a compact subset by the Arzéla-Ascoli theorem. It is shown in [**19**, Theorem 12.13] that the set $\{Q_N\}$ is collectively compact and pointwise convergent to Q. It follows that $\{Q_N\}$ is uniformly convergent on the compact set E (see [**19**, Corollary 10.4]). Therefore,

$$\begin{aligned}\|K^A\phi - K_N^A\phi\|_\infty &\leq \sup_{s\in\mathbf{R}} \|(Q - Q_N)E_s(A(s,\cdot)\phi)\|_\infty \\ &\leq \sup_{f\in E} \|(Q - Q_N)f\|_\infty \longrightarrow 0\end{aligned}$$

as $N \to \infty$, so that for all $\varepsilon > 0$, there exists $N_0 \in \mathbf{N}$ such that $\|K^A\phi - K_N^A\phi\|_\infty \leq \varepsilon$ for all $N \geq N_0$, $\phi \in S$, $k \in T$. □

Theorem 3.4. *Suppose that $S \subset BC(\mathbf{R})$ is bounded and that $\beta > 0$, and let $T := \{k : k$ satisfies $\mathbf{C_1''}$ with $\|A\|_{BC^1(\mathbf{R}^2)} + \|B\|_{BC_p^1(\mathbf{R}^2)} \leq \beta\}$. Then $\{K_N\phi : N \in \mathbf{N},\, \phi \in S,\, k \in T\}$ is bounded and uniformly equicontinuous.*

Proof. Let $N \in \mathbf{N}$. With P_N defined as in the previous proof, we have that

$$\begin{aligned}K_N\phi(s) &= K_N^A\phi(s) + K_N^B\phi(s) \\ &= Q_N(E_s(A(s,\cdot)\phi)) + IP_N(B(s,\cdot)\phi).\end{aligned}$$

Moreover, since $\{Q_N\}$ is collectively compact [**19**, Theorem 12.13], we conclude by the Arzéla-Ascoli theorem that for any bounded set $U \subset BC(\mathbf{R})$ the set $\cup_{N\in\mathbf{N}} Q_N U$ is bounded and uniformly equicontinuous on $[0, 2\pi]$ and hence on $\mathbf{R}$, as $Q_N\phi$ is 2π-periodic for all $\phi \in BC(\mathbf{R})$, $N \in \mathbf{N}$. By an argument similar to that employed in the proof of Theorem 3.3, we see that $\{E_s(A(s,\cdot)\phi) : \phi \in S,\, s \in \mathbf{R},\, k \in T\}$ is bounded. Thus we have that $\{Q_N E_s(A(s,\cdot)\phi) : \phi \in S,\, s \in \mathbf{R},\, k \in T\}$ is bounded and uniformly equicontinuous and, consequently, the same statement applies to $\{K_N^A\phi : N \in \mathbf{N},\, \phi \in S,\, k \in T\}$.

Since S is bounded with $M := \sup_{f\in S} \|f\|_\infty$ and since $k \in T$, we have

$$\begin{aligned}\|IP_N(B(s,\cdot)\phi)\|_\infty &= \|I(P_N(B(s,\cdot))P_N\phi)\|_\infty \\ &\le CM\beta.\end{aligned}$$

For $0 \le s_2 - s_1 \le 1$ and some $\rho_j \in [s_1, s_2]$, $j \in \mathbf{Z}$, we have

$$\begin{aligned}|K_N^B\phi(s_1) - K_N^B\phi(s_2)| &= \left|\frac{\pi}{N}\sum_{j\in\mathbf{Z}} B(s_1,t_j)\phi(t_j) - B(s_2,t_j)\phi(t_j)\right| \\ &\le \frac{\pi}{N}\|\phi\|_\infty \sum_{j\in\mathbf{Z}} |B(s_1,t_j) - B(s_2,t_j)| \\ &\le \frac{\pi}{N}\|\phi\|_\infty \sum_{j\in\mathbf{Z}} |s_1 - s_2|\beta(1+|\rho_j - t_j|)^{-p} \\ &\le CM\beta|s_1 - s_2|.\end{aligned}$$

Therefore also $\{K_N^B\phi : N \in \mathbf{N},\ \phi \in S,\ k \in T\}$ is bounded and uniformly equicontinuous. $\square$

Theorem 3.5. *If k satisfies $\mathbf{C_1''}$, then*

$$\|(K - K_N)K_N\|_\infty \longrightarrow 0$$

and

$$\|(K - K_N)K\|_\infty \longrightarrow 0 \quad \text{as } N \longrightarrow \infty.$$

Moreover, for every $\beta > 0$, this convergence is uniform in k for $k \in T := \{k : k$ satisfies $\mathbf{C_1''}$ with $\|A\|_{BC^1(\mathbf{R}^2)} + \|B\|_{BC^1_p(\mathbf{R}^2)} \le \beta\}$.

Proof. Define the set

$$S := \{K_N\phi : N \in \mathbf{N},\ \phi \in BC(\mathbf{R}),\ \|\phi\|_\infty = 1,\ k \in T\}.$$

By Theorem 3.4, S is bounded and uniformly equicontinuous. Thus we know from Theorem 3.3 that there exists $N_0 = N_0(\varepsilon, \beta)$ such that for $\phi \in BC(\mathbf{R})$ with $\|\phi\|_\infty = 1$ and $k \in T$,

$$\|(K - K_N)K_N\phi\|_\infty \le \varepsilon, \quad N \ge N_0.$$

This proves the assertion for $(K-K_N)K_N$. Next consider $\tilde{S} := \{K\phi : \phi \in BC(\mathbf{R}),\ \|\phi\|_\infty = 1,\ k \in T\}$. By Corollary 2.8, $\tilde{S}$ is a bounded subset of $C^{0,\alpha}(\mathbf{R})$ and hence is bounded and equicontinuous in $BC(\mathbf{R})$. The same argument above can be repeated to prove the assertion for $(K-K_N)K$. $\square$

To make use of the above results we state in the following theorem and corollary two general results on operator approximation in Banach spaces. These results, of some interest in their own right, are modifications of results usually seen as part of collectively compact operator theory [**3**], [**19**]. In particular, Theorem 3.6 up to (3.11) is well known, see [**3**, Theorem 1.10] or [**19**, Theorem 10.8]. If $\mathcal{L}$ in Theorem 3.6 is compact, in particular if $\mathcal{L}$ is of finite rank, then the remaining part of Theorem 3.6 is superfluous, for $I-\mathcal{L}$ injective implies $(I-\mathcal{L})(X) = X$. But our interest is in cases where the operators involved are bounded but not compact. Theorem 3.6 and Corollary 3.7 are results which can be applied in this case.

Theorem 3.6. *Let X be a Banach space and $B(X)$ the set of bounded linear operators on X. Suppose that $\mathcal{K}, \mathcal{L}, (I-\mathcal{K})^{-1} \in B(X)$ and that*

$$\Delta := \|(I-\mathcal{K})^{-1}(\mathcal{L}-\mathcal{K})\mathcal{L}\| < 1. \tag{3.10}$$

Then $I-\mathcal{L}$ is injective so that there exists $(I-\mathcal{L})^{-1}$ as an operator on $(I-\mathcal{L})(X)$ and

$$\|(I-\mathcal{L})^{-1}\| \le (1 + \|(I-\mathcal{K})^{-1}\|\|\mathcal{L}\|)(1-\Delta)^{-1}. \tag{3.11}$$

Moreover, if also

$$\|(\mathcal{L}-\mathcal{K})(I-\mathcal{K})^{-1}\mathcal{L}\| < 1 \tag{3.12}$$

then $(I-\mathcal{L})(X) = X$ so that $(I-\mathcal{L})^{-1} \in B(X)$.

Proof. That $I-\mathcal{L}$ is injective and (3.11) holds follows exactly as in the proof of [**19**, Theorem 10.8]. In the case where $\mathcal{L}$ is compact, it follows automatically that also $(I-\mathcal{L})(X) = X$. In the general case, if (3.12) holds, we find that $(I+(I-\mathcal{K})^{-1}\mathcal{L})(I-(\mathcal{L}-\mathcal{K})(I-\mathcal{K})^{-1}\mathcal{L})^{-1}$ is a right inverse for $I - \mathcal{L}$ and hence $(I - \mathcal{L})(X) = X$ and $(I - \mathcal{L})^{-1} \in B(X)$. $\square$

We remark that $(I - \mathcal{K})^{-1} = I + \mathcal{K}(I - \mathcal{K})^{-1}$, so that

$$\|(\mathcal{L} - \mathcal{K})(I - \mathcal{K})^{-1}\mathcal{L}\| \leq \|(\mathcal{L} - \mathcal{K})\mathcal{L}\| + \|(\mathcal{L} - \mathcal{K})\mathcal{K}\| \|(I - \mathcal{K})^{-1}\| \|\mathcal{L}\|. \tag{3.13}$$

Bearing in mind this observation, we obtain the following corollary of Theorem 3.6.

Corollary 3.7. *Let X be an arbitrary Banach space and suppose that $\mathcal{K}$, $(I - \mathcal{K})^{-1} \in B(X)$, $\mathcal{K}_N \in B(X)$, $N = 1, 2, \ldots$, and that*

$$\left.\begin{aligned} &\|\mathcal{K}_N\| = O(1), \\ &\|(\mathcal{K}_N - \mathcal{K})\mathcal{K}_N\| \longrightarrow 0, \\ &\|(\mathcal{K}_N - \mathcal{K})\mathcal{K}\| \longrightarrow 0, \end{aligned}\right\} \text{ as } N \longrightarrow \infty. \tag{3.14}$$

Then for all N sufficiently large such that

$$\|(I - \mathcal{K})^{-1}(\mathcal{K}_N - \mathcal{K})\mathcal{K}_N\| < 1$$

and

$$\|(\mathcal{K}_N - \mathcal{K})(I - \mathcal{K})^{-1}\mathcal{K}_N\| < 1,$$

it holds that

$$(I - \mathcal{K}_N)^{-1} \in B(X)$$

with

$$\|(I - \mathcal{K}_N)^{-1}\| \leq \frac{1 + \|(I - \mathcal{K})^{-1}\| \|\mathcal{K}_N\|}{1 - \|(I - \mathcal{K})^{-1}(\mathcal{K}_N - \mathcal{K})\mathcal{K}\|}.$$

To apply these results to our integral equation, we will need the following assumption of well-posedness:

E. For every $y \in BC(\mathbf{R})$, the integral equation (1.1) has exactly one solution $x \in BC(\mathbf{R})$.

Assumption $\mathbf{C_1}$ (see Theorem 2.2) implies that $I - K : BC(\mathbf{R}) \to BC(\mathbf{R})$ and is bounded. If also assumption $\mathbf{E}$ is satisfied, then $I - K : BC(\mathbf{R}) \to BC(\mathbf{R})$ is also bijective and so $(I-K)^{-1} : BC(\mathbf{R}) \to BC(\mathbf{R})$ exists and, by the Banach theorem, is bounded. Thus, by Theorem 2.1, Theorem 3.5, and Corollary 3.7, and bearing in mind (3.13), we have the following result.

Theorem 3.8. *If assumptions* $\mathbf{C_1''}$ *and* $\mathbf{E}$ *are satisfied, then there exist* $\tilde{N} \in \mathbf{N}$ *and* $C > 0$ *such that, for all* $N \geq \tilde{N}$*, there holds* $(I - K_N)^{-1} \in B(BC(\mathbf{R}))$ *and*

$$\|(I-K_N)^{-1}\|_\infty \leq C. \tag{3.15}$$

For all $N \geq \tilde{N}$ *equation* (3.3) *has a unique solution,* x_N*, and*

$$\|x_N\|_\infty \leq C\|y\|_\infty.$$

Further, given any $\beta > 0$*, the constants* $\tilde{N}$ *and* C *can be chosen independently of* k*, for* $k \in T := \{k : k$ *satisfies* $\mathbf{C_1''}$ *and* $\mathbf{E}$ *with* $\|A\|_{BC^1(\mathbf{R}^2)} + \|B\|_{BC^1_p(\mathbf{R}^2)} \leq \beta$, $\|(I-K)^{-1}\| \leq \beta\}$.

3.2. Convergence analysis. The first element in our convergence analysis is the following error estimate for the trapezoidal rule.

Lemma 3.9. *Let* $m, N \in \mathbf{N}$, $g \in C^m[a,b]$ *with* $g^{(j)}(a) = g^{(j)}(b) = 0$, $j = 0, 1, \ldots, m-1$, *and define* $h := (b-a)/N$, $s_j := a + jh$, $j = 0, 1, \ldots, N$. *Then*

$$e_N := \left| \int_a^b g(s)\, ds - h \sum_{j=1}^{N-1} g(s_j) \right| \leq C \|g\|_{C^m[a,b]} h^m,$$

where the constant $C > 0$ *depends only on* a, b *and* m.

Proof. For $m = 1, 2$, this is a standard result. If $m \geq 3$ is an odd integer, the result follows from the standard Euler-Maclaurin expansion [**13**, p. 108] which gives, under the assumptions of the lemma, that

$$e_N = \left| h^m \int_a^b P_m\left(\frac{s-a}{h}\right) g^{(m)}(s)\, ds \right| \tag{3.16}$$

with $P_m \in C(\mathbf{R})$ given by

$$P_m(s) = \sum_{n=1}^{\infty} \frac{2\sin 2\pi ns}{(2\pi n)^m}. \tag{3.17}$$

Integrating (3.16) by parts we see that if $g \in C^m[a,b]$ and m is even, then (3.16) holds with P_m given by (3.17) but with the sin replaced by cos. $\square$

Lemma 3.10. *If, for some $p > 1$ and $m \in \mathbf{N}$, $f \in BC_p^m(\mathbf{R})$, then*

$$|If - I_h f| \le C\|f\|_{BC_p^m(\mathbf{R})} h^m, \quad h > 0,$$

where the constant $C > 0$ depends only on m and p.

Proof. Let $\phi \in C^\infty(\mathbf{R})$ be such that $\phi(s) = -(1/2)$, $s \le -(1/2)$, $\phi(s) = (1/2)$, $s \ge (1/2)$. Let $\psi_0(s) := \phi(s) - \phi(s-1)$ and let $\psi_j(s) := \psi_0(s-j)$, $j \in \mathbf{Z}$. Then $\psi_j \in C_0^\infty(\mathbf{R})$, with $\operatorname{supp}\psi_j = [-(1/2)+j, (3/2)+j]$ and

$$\sum_{j\in\mathbf{Z}} \psi_j(s) = 1, \quad s \in \mathbf{R},$$

so that we have a partition of unity. Let, for $h > 0$, $j \in \mathbf{Z}$, $e_j(h) := I(\psi_j f) - I_h(\psi_j f)$. Then

$$If - I_h f = \sum_{j\in\mathbf{Z}} e_j(h)$$

and, by Lemma 3.9,

$$|e_j(h)| \le C_m h^m \|f\|_{BC_p^m(\mathbf{R})} \max_{t\in[-(1/2)+j,(3/2)+j]} (1+|t|)^{-p},$$

where C_m depends only on m and on $\|\psi_j\|_{C^m[-(1/2)+j,(3/2)+j]} = \|\psi_0\|_{C^m[-(1/2),(3/2)]}$. Thus,

$$\begin{aligned}
&|If - I_h f| \\
&\quad \le \sum_{j\in\mathbf{Z}} |e_j(h)| \le C_m \|f\|_{BC_p^m(\mathbf{R})} h^m \sum_{j\in\mathbf{Z}} \max_{t\in[-(1/2)+j,(3/2)+j]} (1+|t|)^{-p} \\
&\quad \le C\|f\|_{BC_p^m(\mathbf{R})} h^m,
\end{aligned}$$

where $C > 0$ depends only on m and p. $\square$

Lemma 3.11. *If k satisfies assumption $\mathbf{C''_m}$ for some $m \in \mathbf{N}$ and $\phi \in BC^m(\mathbf{R})$, then*

$$\|K^B\phi - K^B_N\phi\| \le C\|B\|_{BC^m_p(\mathbf{R}^2)}\|\phi\|_{BC^m(\mathbf{R})}N^{-m}, \tag{3.18}$$

for some constant $C > 0$ dependent only on p and m.

Proof.

$$\begin{aligned}\|K^B\phi - K^B_N\phi\| &= \sup_{s\in\mathbf{R}} |I(B(s,\cdot)\phi) - I_{\pi/N}(B(s,\cdot)\phi| \\ &\le C\left(\frac{\pi}{N}\right)^m \|B(s,\cdot)\phi\|_{BC^m_p(\mathbf{R})},\end{aligned}$$

by Lemma 3.10. Thus the bound (3.18) holds. $\square$

Lemma 3.12. *If k satisfies assumption $\mathbf{C''_m}$ for some $m \in \mathbf{N}$ and $\phi \in BC^m(\mathbf{R})$, then*

$$\|K^A\phi - K^A_N\phi\| \le C\|A\|_{BC^m(\mathbf{R}^2)}\|\phi\|_{BC^m(\mathbf{R})}N^{-m},$$

for some constant $C > 0$ dependent only on m.

Proof. We have

$$(K^A\phi - K^A_N\phi)(s) = (Q - Q_N)((E_s(A(s,\cdot)\phi))(s).$$

Let p be the trigonometric polynomial of order N which is the best approximation to $E_s(A(s,\cdot)\phi)$ with respect to the norm $\|\cdot\|_\infty$. Then, see Remark 3.1, $Qp = Q_Np$. Thus,

$$\begin{aligned}|K^A\phi(s) - K^A_N\phi(s)| &\le \|(Q - Q_N)(E_s(A(s,\cdot)\phi) - p)\|_\infty \\ &\le (\|Q\| + \|Q_N\|)\|E_s(A(s,\cdot)\phi) - p\|_\infty \\ &\le \tilde{C}\|E_s(A(s,\cdot)\phi) - p\|_\infty,\end{aligned}$$

where $\tilde{C} > 0$ is a constant independent of N, since Q is bounded and Q_N is bounded uniformly in N [**19**, Theorem 12.13]. Now $A \in C^m_{0,\pi}(\mathbf{R}^2)$, $\phi \in BC^m(\mathbf{R})$, so that $E_s(A(s,\cdot)\phi) \in BC^m(\mathbf{R})$ with

$$\|E_s(A(s,\cdot)\phi)\|_{BC^m(\mathbf{R})} \leq C\|A\|_{BC^m(\mathbf{R}^2)}\|\phi\|_{BC^m(\mathbf{R})}.$$

It follows from Theorem 13.6 in [**28**] that

$$\|K^A\phi - K^A_N\phi\| \leq C\|A\|_{BC^m(\mathbf{R}^2)}\|\phi\|_{BC^m(\mathbf{R})}N^{-m}. \qquad \square$$

Combining the numerical quadrature estimate in Lemmas 3.11 and 3.12 with the stability result of Theorem 3.8, we obtain the following main convergence result.

Theorem 3.13. *If assumptions $\mathbf{C_1''}$ and $\mathbf{E}$ are satisfied, then there exist $\tilde{N} \in \mathbf{N}$ and $C_1, C_2 > 0$ such that, for $N \geq \tilde{N}$, a uniquely determined numerical solution x_N exists and satisfies*

$$\|x - x_N\|_\infty \leq C_1\|(K - K_N)x\|_\infty.$$

If also assumption $\mathbf{C_m''}$ is satisfied for some $m \in \mathbf{N}$ and $y \in BC^m(\mathbf{R})$, then

$$\|x - x_N\|_\infty \leq C_2 N^{-m}\|y\|_{BC^m(\mathbf{R})}$$

for $N \geq \tilde{N}$. Moreover, given any $\beta > 0$, the constants $\tilde{N}$, C_1 and C_2 can be chosen independently of k for $k \in T := \{k :$ k satisfies $\mathbf{C_m''}$ and $\mathbf{E}$ with $\|A\|_{BC^m(\mathbf{R}^2)} + \|B\|_{BC^m_p(\mathbf{R}^2)} \leq \beta, \|(I-K)^{-1}\| \leq \beta\}$.

Proof. That, for some $\tilde{N} \in \mathbf{N}$, $(I - K_N)^{-1} \in BC(\mathbf{R})$ and is bounded by (3.15), follows from Theorem 3.8. Further,

$$x_N - K_N x_N = y \Longleftrightarrow x_N = (I - K_N)^{-1}y.$$

Thus x_N exists and is unique for $N \geq \tilde{N}$. Also for $N \geq \tilde{N}$,

$$x - x_N = (I - K_N)^{-1}(K - K_N)x,$$

so that, by (3.15),

$$\|x - x_N\|_\infty \leq C_1 \|(K - K_N)x\|_\infty.$$

From Theorem 2.9 we have that $x \in BC^m(\mathbf{R})$ since $y \in BC^m(\mathbf{R})$. Applying Lemmas 3.11 and 3.12,

$$\begin{aligned}\|Kx - K_N x\|_\infty &\leq \|K^A x - K_N^A x\|_\infty + \|K^B x - K_N^B x\|_\infty \\ &\leq C(\|A\|_{BC^m(\mathbf{R}^2)} + \|B\|_{BC_p^m(\mathbf{R}^2)})\|x\|_{BC^m(\mathbf{R})} N^{-m}.\end{aligned}$$

Since $\|x\|_{BC^m(\mathbf{R})}$ can be bounded in terms of $\|y\|_{BC^m(\mathbf{R})}$ and $\|x\|_\infty$ by Theorem 2.9, and $\|x\|_\infty \leq \|(I-K)^{-1}\| \|y\|_\infty$, we obtain the required result. □

4. Application to scattering by rough surfaces. In this section we will consider the application of the Nyström method to a problem of the scattering of time-harmonic waves by unbounded rough surfaces. The propagation of time-harmonic acoustic waves with wave number κ in a domain Ω is governed by the reduced wave equation or Helmholtz equation,

$$\Delta u + \kappa^2 u = 0 \quad \text{in } \Omega.$$

We will consider here domains of the form $\Omega := \{\mathbf{x} \in \mathbf{R}^2 : x_2 > f(x_1)\}$ where we assume $f \in BC^{n+2}(\mathbf{R})$ for some $n \in \mathbf{N}_0$, and the existence of constants c_1, c_2 with $0 < c_1 \leq f(s) \leq c_2$ for all $s \in \mathbf{R}$. Set $\Gamma := \partial\Omega$.

In the following, let Φ denote the free field Green's function for the Helmholtz equation,

$$\Phi(\mathbf{x}, \mathbf{y}) := \frac{i}{4} H_0^{(1)}(\kappa|\mathbf{x} - \mathbf{y}|), \quad \mathbf{x}, \mathbf{y} \in \mathbf{R}^2, \quad \mathbf{x} \neq \mathbf{y},$$

where $H_0^{(1)}$ denotes the Hankel function of the first kind of order zero. Use shall also be made of the notations $U_h := \{\mathbf{x} \in \mathbf{R}^2 : x_2 > h\}$ and $\Gamma_h := \{\mathbf{x} \in \mathbf{R}^2 : x_2 = h\}$.

We suppose that a field u^i is incident on the boundary Γ and that u^i is a bounded solution to the Helmholtz equation in a neighborhood of Γ and seek to find the scattered field u^s as the solution to the following scattering problem. We consider the sound soft case where the total field $u = u^i + u^s$ vanishes on Γ.

Problem 4.1. *Given the incident field u^i, find the scattered field $u^s \in C^2(\Omega) \cap C(\overline{\Omega})$ that satisfies*

1. $\Delta u^s + \kappa^2 u^s = 0$ *in* Ω,

2. $u^s = -u^i$ *on* Γ,

3. *the* upwards propagating radiation condition [**11**] *in* Ω *that, for some* $h > \sup f$ *and some* $\phi \in L^\infty(\Gamma_h)$,

$$u^s(\mathbf{x}) = 2\int_{\Gamma_h} \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial y_2}\phi(\mathbf{y})\, ds(\mathbf{y}), \quad \mathbf{x} \in U_h,$$

holds,

4. *for every* $a > 0$, u^s *is bounded in the horizontal strip* $\Omega \setminus U_a$.

Remark 4.2. Uniqueness of solution for Problem 4.1 was shown in [**12**] and existence of solution in [**12**], [**9**], [**24**].

In reformulating Problem 4.1 as a boundary integral equation, we will make use of Green's function for the Helmholtz equation in a half-plane with Dirichlet boundary conditions. This function is given by

$$G(\mathbf{x},\mathbf{y}) := \Phi(\mathbf{x},\mathbf{y}) - \Phi(\mathbf{x},\mathbf{y}'), \quad \mathbf{x},\mathbf{y} \in \overline{U_0}, \quad \mathbf{x} \neq \mathbf{y},$$

where $\mathbf{y}' := (y_1, -y_2)^\top$. We will make the following Brakhage-Werner type ansatz for the scattered field:

$$u^s(\mathbf{x}) = \int_\Gamma \left(\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} - i\,\eta\, G(\mathbf{x},\mathbf{y})\right)\psi(\mathbf{y})\, ds(\mathbf{y}), \quad \mathbf{x} \in \Omega,$$

with some density $\psi \in BC(\Gamma)$, where $\eta > 0$ is a fixed constant and $\mathbf{n}(\mathbf{y})$ denotes the unit normal to Γ at $\mathbf{y}$ directed into Ω.

A scattered field of this type is a solution to Problem 4.1 if and only if the density ψ satisfies the boundary integral equation

$$\psi + D\psi - i\,\eta S\psi = -2u^i \quad \text{on } \Gamma, \tag{4.1}$$

where $D, S : BC(\Gamma) \to BC(\Gamma)$ are the boundary integral operators given by

$$D\psi(\mathbf{x}) := 2\int_\Gamma \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})}\psi(\mathbf{y})\, ds(\mathbf{y}), \quad \mathbf{x} \in \Gamma,$$

$$S\psi(\mathbf{x}) := 2\int_\Gamma G(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\, ds(\mathbf{y}), \quad \mathbf{x} \in \Gamma.$$

Using a parameterization of Γ as $\{(s, f(s))^\top : s \in \mathbf{R}\}$ and setting $\phi(s) := \psi(s, f(s))$ and $g(s) := -2u^i(s, f(s))$, $s \in \mathbf{R}$, we find (4.1) to be equivalent to

$$\phi - K\phi = g, \tag{4.2}$$

where

$$K\phi(s) := \int_{-\infty}^{\infty} k(s,t)\phi(t)\,dt, \quad s \in \mathbf{R},$$

and

$$k(s,t) := 2\left(i\,\eta G(\mathbf{x},\mathbf{y}) - \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})}\right)\Bigg|_{\substack{\mathbf{x}=(s,f(s))\\ \mathbf{y}=(t,f(t))}} \sqrt{1+f'(t)^2},$$
$$s, t \in \mathbf{R}, \quad s \neq t.$$

Now define, for $c_1, M > 0$,

$$B_{c_1,M} := \{f \in BC^{n+2}(\mathbf{R}) : c_1 \leq \inf f,\ \|f\|_{BC^{n+2}(\mathbf{R})} \leq M\}.$$

Theorem 4.3. *For $f \in BC^{n+2}(\mathbf{R})$, $n \in \mathbf{N}_0$, k satisfies condition $\mathbf{C_n}$. Moreover, given any c_1, $M > 0$, condition $\mathbf{C}_n$ is satisfied with the same constant C for all $f \in B_{c_1,M}$.*

Proof. Using the notation $\mathbf{x} := (s, f(s))^\top$ and $\mathbf{y} := (t, f(t))^\top$, we set

$$a^*(s,t) := -\frac{1}{\pi}\Bigg(i\,\eta J_0(\kappa|\mathbf{x}-\mathbf{y}|) - \kappa J_1(\kappa|\mathbf{x}-\mathbf{y}|)\frac{\mathbf{n}(\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\Bigg)\sqrt{1+f'(t)^2}$$

and

$$b^*(s,t) := k(s,t) - a^*(s,t)\ln(|s-t|)$$

for $s, t \in \mathbf{R}$, $s \neq t$, where J_j denotes the Bessel function of the first kind of order j, $j = 0, 1$. As $f \in BC^{n+2}(\mathbf{R})$, there holds $(\mathbf{n}(y)\cdot(\mathbf{x}-\mathbf{y})/|\mathbf{x}-\mathbf{y}|^2) \in BC^n(\mathbf{R}^2)$ [**5**, Section 7.1.3], and the norm of this function is uniformly bounded for all f with $\|f\|_{BC^{n+2}(\mathbf{R})} \leq M$.

Thus, from $H_n^{(1)} = J_n + iY_n$ and using ascending series expansions of Bessel functions (see [**1**, Equations 9.1.10 and 9.1.11]) and the regularity of f, we conclude that $a^*, b^* \in BC^n(\mathbf{R}^2)$. Further, for some constant $C > 0$ dependent on M and c_1,

$$\left| \frac{\partial^{j+l}}{\partial s^j \partial t^l} b^*(s,t) \right| \leq C \tag{4.3}$$

for all $j + l \leq n$ and all $s, t \in \mathbf{R}$, $|s - t| \leq \pi$, and the same estimate holds for $a^*(s,t)$.

The kernel function k can also be written as

$$k(s,t) := 2i\,\eta G(\mathbf{x},\mathbf{y})\sqrt{1 + f'(t)^2} + 2\mathrm{grad}_{\mathbf{y}} G(\mathbf{x},\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}).$$

In [**24**] it is shown that for $|\mathbf{x} - \mathbf{y}| \geq \varepsilon > 0$, there holds

$$|G(\mathbf{x},\mathbf{y})| \leq C \frac{(1 + x_2)(1 + y_2)}{|\mathbf{x} - \mathbf{y}|^{3/2}},$$

and from regularity estimates for solutions to elliptic partial differential equations [**14**, Theorem 3.9], it can be seen that such estimates in fact hold for partial derivatives of $G(\mathbf{x},\mathbf{y})$ of any order. On the other hand, as $f \in BC^{n+2}(\mathbf{R})$ with $\|f\|_{BC^{n+2}(\mathbf{R})} \leq M$, we have that $\sqrt{1 + f'^2}, \mathbf{n}((., f(.))^T) \in BC^n(\mathbf{R})$ and are bounded in norm by a constant only dependent on M. Combining these bounds we thus conclude that

$$\left| \frac{\partial^{j+l}}{\partial s^j \partial t^l} k(s,t) \right| \leq \frac{C}{(1 + |s - t|)^{3/2}}, \quad j + l \leq n$$

for $s, t \in \mathbf{R}$, $|s - t| \geq \pi$, where C is a constant dependent on M. $\square$

Theorem 4.4. *The kernel function k satisfies assumptions* **A**, **B** *and* **E**. *Further, given any c_1, $M > 0$, there exists $\beta > 0$ such that $\|(I - K)^{-1}\| \leq \beta$ for all $f \in B_{c_1,M}$.*

Proof. That **A** and **B** are satisfied follows from Theorem 4.3 and Remark 2.2. That assumption **E** is satisfied and the uniform bound on $(I - K)^{-1}$ is the content of Theorem 3.2 in [**27**]. $\square$

By the two previous theorems, we can apply the theoretical Nyström method to the integral equation (4.2) and, by Theorems 2.1, 3.8 and 3.13, the following result is proved:

Theorem 4.5. *Suppose* $n \in \mathbf{N}_0$ *and* c_1, $M > 0$. *Then there exists* $\tilde{N} \in \mathbf{N}$ *such that, provided* $N \geq \tilde{N}$ *and* $f \in B_{c_1,M}$, *the equation*

$$(I - K_N)\phi_N = g \tag{4.4}$$

has a unique solution ϕ_N. *Further, there exists a constant* $C > 0$ *such that, provided* $f \in B_{c_1,M}$, *it holds that* $g \in BC^n(\mathbf{R})$ *and*

$$\|\phi - \phi_N\|_\infty \leq C\|g\|_{BC^n(\mathbf{R})}N^{-n}, \quad N \geq \tilde{N}. \tag{4.5}$$

Remark 4.6. In the next section we consider the case when the incident wave is a plane wave. Then

$$u^i(\mathbf{x}) = e^{i\kappa \mathbf{x}\cdot\hat{\theta}}, \tag{4.6}$$

where $\hat{\theta} := (\cos\theta, -\sin\theta)^\top$ is the direction of the plane wave and $\theta \in (0, \pi)$ specifies its angle of incidence. In this case there exists a constant $c > 0$, depending only on κ and n, such that

$$\|g\|_{BC^n(\mathbf{R})} \leq c\|f\|_{BC^n(\mathbf{R})}.$$

As a consequence, (4.5) simplifies to

$$\|\phi - \phi_N\|_\infty \leq CN^{-n}, \quad N \geq \tilde{N},$$

where the value of $C > 0$ depends only on κ, n, c_1 and M.

Remark 4.7. In the case when $f \in C^\infty(\mathbf{R})$, Theorem 4.5 predicts that ϕ_N is superalgebraically convergent, i.e., that

$$\|\phi - \phi_N\|_\infty = o(N^{-n}) \quad \text{as } N \to \infty$$

for every $n \in \mathbf{N}$.

5. Scattering by periodic surfaces. In general ϕ_N cannot be computed exactly; to obtain an approximation to ϕ_N the infinite system of equations (4.4) has to be truncated. As was pointed out in Section 3, exact computation of ϕ_N is however possible if the kernel function and the righthand side of equation (4.2) show some periodicity. For the scattering problem 4.1 this is the case if we assume f to be L-periodic and the incident wave to be a plane wave, so that u^i is given by (4.6). By multiplying (4.2) by $e^{-i\kappa s\cos\theta}$ and setting

$$\begin{aligned}\tilde{\phi}(s) &:= e^{-i\kappa s\cos\theta}\phi(s),\\ \tilde{K}\tilde{\phi}(s) &:= \int_{-\infty}^{\infty} e^{i\kappa(t-s)\cos\theta}k(s,t)\tilde{\phi}(t)\,dt,\\ \tilde{g}(s) &:= e^{-i\kappa s\cos\theta}g(s) = -2e^{-i\kappa s\cos\theta}u^i(s,f(s)),\end{aligned}$$

the modified integral equation

$$(I-\tilde{K})\tilde{\phi} = \tilde{g} \tag{5.7}$$

is obtained, which satisfies the periodicity assumption **D** of Section 3.

The implementation of the numerical scheme also relies on a representation of k as in assumption $\mathbf{C}''_{\mathbf{n}}$. Recalling the proof of Theorem 2.1, we find that

$$k(s,t) = \frac{1}{2\pi}A(s,t)\ln\left(4\sin^2\left(\frac{s-t}{2}\right)\right) + B(s,t),$$

where A and B are given by (2.6) and (2.7), respectively. We choose the cut-off function χ in these equations to be

$$\chi(t) := \begin{cases} 1 & |t|\le 1,\\ \left(1+\exp\left(\dfrac{\pi-1}{\pi-|t|} - \dfrac{1-\pi}{1-|t|}\right)\right)^{-1} & 1<|t|<\pi,\\ 0 & \pi<|t|.\end{cases} \tag{5.8}$$

Now, setting

$$\tilde{k}(s,t) := \left(\frac{1}{2\pi}A(s,t)\ln\left(4\sin^2\left(\frac{s-t}{2}\right)\right) + B(s,t)\right)e^{i\kappa(t-s)\cos\theta},$$

so that $\tilde{K}\tilde{\phi}(s) = \int_{-\infty}^{\infty} \tilde{k}(s,t)\tilde{\phi}(t)\,dt$, Theorems 4.3, 4.4 and 4.5 also hold with k replaced by $\tilde{k}$ and, by Remark 4.6, we deduce that the Nyström method approximation, $\tilde{\phi}_N$, satisfies that

$$\|\tilde{\phi} - \tilde{\phi}_N\|_\infty \leq CN^{-n}, \quad N \geq \tilde{N}, \tag{5.9}$$

where the constants $C > 0$ and $\tilde{N} \in \mathbf{N}$ depend on κ, c_1, M and n, but not on θ or on the particular choice of $f \in B_{c_1,M}$, and hence not on the period of the diffraction grating, L.

By choosing the unit of length measurement appropriately, we can ensure that L is a multiple of π/N. Then, recalling that $t_k := k\pi/N$ and setting $x_N^k := \phi_N(t_k)$, $y^k := g(t_k)$, we have to solve the linear system (3.8) with $M = LN/\pi$. The entries of the matrix are given by

$$\tilde{a}_{kj} = \sum_{n\in\mathbf{Z}} a_{k,j+nM} \tag{5.10}$$

and, recalling (3.2),

$$a_{kj} := (R_{j-k}^{(N)} A(t_k,t_j) + \frac{\pi}{N} B(t_k,t_j))e^{i\kappa(t_j-t_k)\cos\theta}.$$

Noting that $a_{kj} = \pi\tilde{k}(t_k,t_j)/N$ for $|k-j| > N$, to compute $\tilde{a}_{kj}$ we have to evaluate sums of the form

$$k_+^{\mathrm{sum}}(s,t) = \sum_{d=0}^{\infty} \tilde{k}(s,t+dL) \quad \text{for } t-s > \pi,$$
$$k_-^{\mathrm{sum}}(s,t) = \sum_{d=0}^{\infty} \tilde{k}(s,t-dL) \quad \text{for } s-t < \pi.$$

Each of these was rewritten as a Laplace-type integral (cf. [**20**, Equation 2.36]), via integral representations of Hankel functions ([**22**, formulae 2.13.52 and 2.13.60]). Then each integrand was expressed as the sum of a simple term containing the simple pole singularity nearest the positive real axis, evaluated exactly in terms of the complementary error function, and a complex but smooth remainder. The integral corresponding to this latter part of the integrand was evaluated by a 40 point Gauß-Laguerre rule.

5.1. Results for a flat surface. In the case of a flat surface, the integral equation (5.7) is a convolution equation on the real line, and its solution can be computed exactly. Setting $f(s) = h$ for some $h > 0$, there follows

$$\tilde{g}(s) = -2e^{-i\kappa s\cos\theta} e^{i\kappa(s\cos\theta - h\sin\theta)} = -2e^{-i\kappa h\sin\theta},$$

and

$$\tilde{k}(s,t) = \tilde{\kappa}(s-t)e^{-i\kappa(s-t)\cos\theta},$$

where

$$\begin{aligned}\tilde{\kappa}(z) := {} & \frac{\eta}{2}\left(H_0^{(1)}(\kappa\sqrt{z^2+4h^2}) - H_0^{(1)}(\kappa|z|)\right)\\ & + \frac{i}{2}\frac{\partial H_0^{(1)}}{\partial y_2}(\kappa\sqrt{z^2+(h+y_2)^2})\bigg|_{y_2=h}.\end{aligned}$$

It follows that ϕ is a constant, in fact

$$\phi = \tilde{\phi} = \frac{\tilde{g}}{1 - \int_{-\infty}^{\infty}\tilde{\kappa}(u)e^{i\kappa u\cos\theta}\,du} \tag{5.11}$$

and, computing the Fourier transform of $\tilde{\kappa}$, we find that

$$\int_{-\infty}^{\infty}\tilde{\kappa}(u)e^{i\kappa u\cos\theta}\,du = -\frac{\eta}{\kappa\sin\theta}(1 - e^{2i\kappa h\sin\theta}) - e^{2i\kappa h\sin\theta}.$$

In Table 1 results are presented showing the differences between the exact solution given by (5.11) and the approximations obtained by the Nyström method, solving (3.8) for various values of N, with $\tilde{a}_{kj}$ given by (5.10) and taking $L = 2\pi$. The error tabulated is

$$e_N := \max_{j\in\mathbf{Z}}|\tilde{\phi}(t_j) - \tilde{\phi}_N(t_j)|,$$

and the estimated order of convergence (EOC) is defined as

$$\mathrm{EOC} := \frac{\ln(e_N/e_{2N})}{\ln 2}. \tag{5.12}$$

The error, e_N, satisfies the bound (5.9), and the superalgebraic convergence rate predicted in Remark 4.7 can clearly be observed until

TABLE 1. Error and estimated order of convergence against N for (a) $\eta = \kappa = \sqrt{2}$, $h = 1$, $\theta = \pi/2$ and (b) $\eta = \kappa = 2\sqrt{2}$, $h = 5/2$, $\theta = \pi/6$.

N	e_N	EOC	N	e_N	EOC
2	1.329e-2		2	3.510e-1	
		4.71			3.47
4	5.072e-4		4	3.177e-2	
		5.18			9.72
8	1.398e-5		8	3.779e-5	
		6.72			6.00
16	1.326e-7		16	5.909e-7	
		6.58			9.34
32	1.385e-9		32	9.103e-10	
		9.61			9.08
64	1.769e-12		64	1.679e-12	
		4.73			2.72
128	6.664e-14		128	2.549e-13	
	(a)			(b)	

rounding errors and the errors inherent in the Gauß-Laguerre rule become significant.

5.2. Results for a sinusoidal surface. We next consider a configuration treated previously by other authors (see [**6**] and references contained therein). We assume the surface to be sinusoidal, given by

$$f(s) = h + \varepsilon \sin(s),$$

with $h > \varepsilon$. For this application we will assume that $\kappa = 1.25/0.546$ and $L = 2\pi$. These values correspond to the physical problem of scattering of a plane wave of wavelength 0.546 μm by a sinusoidal diffraction grating with period 1.25 μm and height $(1.25\varepsilon/\pi)\mu$m.

For $x_2 > \max f$, we can write the scattered field as a Rayleigh series,

$$u^s(\mathbf{x}) = \sum_{n \in \mathbf{Z}} u_n e^{i\alpha_n x_1 + i\beta_n x_2},$$

where $\alpha_n := n + \kappa \cos\theta$ and $\beta_n := \sqrt{\kappa^2 - \alpha_n^2}$ with $\operatorname{Im} \beta_n \geq 0$ and $u_n \in \mathbf{C}$, $n \in \mathbf{Z}$. Only a finite number of terms in this expansion

TABLE 2. Comparison of results for $\eta=\sqrt{2}$, $h=3$ and $\theta=\pi/2$.

ε	Rayleigh coefficients	our results $N=8$	$N=16$	v. d. Berg's results
	$\lvert u_0\rvert^2$	0.42284	0.42285	0.42229
0.375	$\lvert u_{-1}\rvert^2=\lvert u_1\rvert^2$	0.01294	0.01294	0.01282
	$\lvert u_{-2}\rvert^2=\lvert u_2\rvert^2$	0.56909	0.56907	0.56988
	$\beta_0^{-1}\sum_{n=-2}^{2}\beta_n\lvert u_n\rvert^2$	1.00000	1.00000	1.00002
	$\lvert u_0\rvert^2$	0.33677	0.33678	0.3352
0.500	$\lvert u_{-1}\rvert^2=\lvert u_1\rvert^2$	0.18897	0.18896	0.1888
	$\lvert u_{-2}\rvert^2=\lvert u_2\rvert^2$	0.33214	0.33213	0.3311
	$\beta_0^{-1}\sum_{n=-2}^{2}\beta_n\lvert u_n\rvert^2$	1.00002	1.00000	0.9971
	$\lvert u_0\rvert^2$	0.34570	0.34573	0.3443
0.700	$\lvert u_{-1}\rvert^2=\lvert u_1\rvert^2$	0.10551	0.10556	0.1049
	$\lvert u_{-2}\rvert^2=\lvert u_2\rvert^2$	0.47715	0.47709	0.4772
	$\beta_0^{-1}\sum_{n=-2}^{2}\beta_n\lvert u_n\rvert^2$	0.99994	1.00000	0.9975

represent plane waves propagating away from the diffraction grating. For these there holds (see [**6**])

$$\sum_{\alpha_n^2\le\kappa^2}\beta_n|u_n|^2=\beta_0.$$

Using the Rayleigh series expansion [**20**] for the sums of Hankel functions appearing in the kernel $\tilde{k}$,

$$\sum_{n\in\mathbf{Z}}e^{i\alpha n2\pi}H_0^{(1)}(\kappa|\mathbf{x}-\mathbf{y}-n\mathbf{p}|)=\frac{1}{\pi}\sum_{n\in\mathbf{Z}}\frac{1}{\beta_n}e^{i\alpha_n(x_1-y_1)+i\beta_n|x_2-y_2|},$$

where $\mathbf{p}=(2\pi,0)^\top$, we obtain the formula

$$\begin{aligned}u_n&=\int_0^{2\pi}\left(1-f'(t)\frac{\alpha_n}{\beta_n}+\frac{\eta}{\beta_n}\sqrt{1+f'(t)^2}\right)e^{-int-i\beta_nf(t)}\tilde{\phi}(t)\,dt\\&\quad+\int_0^{2\pi}\left(1+f'(t)\frac{\alpha_n}{\beta_n}-\frac{\eta}{\beta_n}\sqrt{1+f'(t)^2}\right)e^{-int+i\beta_nf(t)}\tilde{\phi}(t)\,dt\end{aligned}$$

TABLE 3. Estimated error and estimated order of convergence against N for $\eta=\sqrt{2}$, $h=3$, $\varepsilon=0.7$ and (a) $\theta=\pi/2$ and (b) $\theta=\pi/6$.

N	e_n	EOC	N	e_N	EOC
2	1.933e+0		2	8.038e-1	
		1.45			0.11
4	7.098e-1		4	7.429e-1	
		6.43			6.18
8	8.213e-3		8	1.028e-2	
		10.07			11.07
16	7.646e-6		16	4.773e-6	
		10.33			9.64
32	5.958e-9		32	5.974e-9	
		9.39			9.91
64	8.887e-12		64	6.191e-12	
		5.67			4.89
128	1.420e-13		128	2.095e-13	
	(a)			(b)	

for the Rayleigh coefficients u_n. Table 2 compares results obtained with the Nyström method to those obtained in [**6**] and shows that good accuracy is reached for even modest values of N. Table 3 lists the estimated error

$$e_N := \max_{j\in\mathbf{Z}} |\tilde{\phi}_{512}(t_j) - \tilde{\phi}_N(t_j)|,$$

and the estimated order of convergence given by (5.12). Again the claimed superalgebraic convergence rate is clearly demonstrated.

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REFERENCES

1. M. Abramowitz and I. Stegun, *Handbook on mathematical functions*, Dover, Washington, 1964.

2. S. Amini and I.H. Sloan, *Collocation methods for second kind integral equations with non-compact operators*, J. Integral Equations Appl. **2** (1988), 1–30.

3. P.M. Anselone, *Collectively compact operator theory*, Prentice-Hall, New York, 1971.

4. P.M. Anselone and I.H. Sloan, *Numerical solutions of integral equations on the half-line*, J. Integral Equations Appl. **1** (1988), 203–225.

5. K.E. Atkinson, *The numerical solution of integral equations of the second kind*, Cambridge University Press, Cambridge, 1997.

6. P.M. van den Berg, *Diffraction theory of a reflection grating*, Appl. Sci. Res. **24** (1971), 261–293.

7. G.A. Chandler and I.G. Graham, *The convergence of Nyström methods for Wiener-Hopf equations*, Numer. Math. **52** (1988), 345–364.

8. S.N. Chandler-Wilde, M. Rahman and C.R. Ross, *A fast two-grid and finite section method for a class of integral equations on the real line with application to an acoustic scattering problem in the half-plane*, submitted.

9. S.N. Chandler-Wilde, C.R. Ross and B. Zhang, *Scattering by infinite one-dimensional rough surfaces*, Proc. Roy. Soc. London Ser. A **455** (1999), 3767–3787.

10. S.N. Chandler-Wilde and B. Zhang, *On the solvability of a class of second kind integral equations on unbounded domains*, J. Math. Anal. Appl. **214** (1997), 482–502.

11. ———, *Electromagnetic scattering by an inhomogeneous conducting or dielectric layer on a perfectly conducting plate*, Proc. Roy. Soc. London Ser. A **454** (1998), 519–542.

12. ———, *A uniqueness result for scattering by infinite rough surfaces*, SIAM J. Appl. Math. **58** (1998), 1774–1790.

13. P.D. Davis and P. Rabinowitz, *Methods of numerical integration*, Academic Press, New York, 1975.

14. D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer, Berlin, 1983.

15. I.G. Graham and W. Mendes, *Nyström product integration for Wiener-Hopf equations with applications to radiative transfer*, IMA J. Numer. Anal. **9** (1989), 261–284.

16. K. Jörgens, *Linear integral operators*, Pitman, London, 1982.

17. D.A. Kapp and D.S. Brown, *A new numerical method for rough surface scattering calculations*, IEEE Trans. Antennas and Propagation **44** (1996), 711–721.

18. A. Kirsch, *Diffraction by periodic structures*, in *Inverse problems in mathematical physics* (L. Pävärinta and E. Somersalo, eds.), Springer, Berlin, 1993, 87–102.

19. R. Kress, *Linear integral equations*, Springer, Berlin, 1989.

20. C.M. Linton, *The Green's function for the two-dimensional Helmholtz equation in periodic domains*, J. Engrg. Math. **33** (1998), 377–402.

21. D. Maystre, M. Saillard and J. Ingers, *Scattering by one- or two-dimensional randomly rough surfaces*, Waves Random Media, **3** (1991), s143–s155.

22. T. Oberhettinger and L. Badii, *Tables of Laplace transforms*, Springer, Berlin, 1973.

23. R. Petit, ed., *Electromagnetic theory of gratings*, Springer, Berlin, 1980.

24. C.R. Ross, *Direct and inverse scattering by rough surfaces*, Ph.D. Thesis, Brunel University, 1996.

25. C. Schneider, *Regularity of the solution to a class of weakly singular Fredholm integral equations of the second kind*, Integral Equations Operator Theory **2** (1979), 62–68.

26. L. Tsang, C.H. Chan, K. Pak and H. Sangani, *Monte-Carlo simulations of large-scale problems of random rough surface scattering and applications to grazing incidence with the bmia/canonical grid method*, IEEE Trans. Antennas and Propagation **43** (1995), 851–859.

27. B. Zhang and S.N. Chandler-Wilde, *An integral equation approach for rough surface scattering*, in *Proc. of the Fourth Internat. Conf. on Mathematical and Numerical Aspects of Wave Propagation* (J. De Santo, ed.), SIAM, 1998.

28. A. Zygmund, *Trigonometric series*, Cambridge University Press, Cambridge, 1959.

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