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CHANNEL LINEAR WEINGARTEN SURFACES

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Abstract. We demonstrate that every non-tubular channel linear Weingarten surface in Euclidean space is a surface of revolution, hence parallel to a catenoid or a rotational surface of non-zero constant Gauss curvature. We provide explicit parametrizations and deduce existence of complete hyperbolic linear Weingarten surfaces.

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1. Introduction

This note is based on the second author's Bachelor thesis, the purpose of which was to understand the classification of cyclic linear Weingarten surfaces from [6]. In particular, we obtained a very simple proof for Corollary 3.6 of [6], using explicit parametrizations in terms of Jacobi elliptic functions based on [11]. This method will also be applicable to the results of [7]. Moreover, in our attempt to derive a more conceptual proof for the key step [6, Theorem 2.1] in the classification of cyclic linear Weingarten surfaces of [6, Theorem 1.1], we derived the classification of channel linear Weingarten surfaces Theorem 3 below – that, in fact, had already been obtained in [4], using different methods based on [3]. As a consequence of our Theorem 3 we obtain a partial but rather explicit classification, of channel linear Weingarten surfaces as special cyclic linear Weingarten surfaces, in Section 3 of this note.

This paper does not contain substantial new results, but merely employs various well known methods and results in order to elucidate the main classification result of [6]. We feel, however, that its publication may serve the mathematical community by recalling these methods and by demonstrating how they beautifully serve to classify channel linear Weingarten surfaces. In fact, basic scholarly work suggests that the classical and rather direct methods we employ have fallen into oblivion and require revivification.

2. Parallel Linear Weingarten Surfaces

It is well known that the parallel surfaces $\mathbf{x}^t = \mathbf{x} + t\mathbf{n}$ of a linear Weingarten surface $\mathbf{x}$ in the Euclidean space are linear Weingarten surfaces, cf [10, §3.4.3] and [5, §2.7.2]

Lemma 1. *If the Gauss and mean curvatures K and H, respectively, of a surface $\mathbf{x} : \Sigma^2 \to \mathbb{R}^3$ satisfy a linear Weingarten condition*[1]

$$0 = aK + 2bH + c \tag{1}$$

then the Gauss and mean curvatures of its parallel surfaces $\mathbf{x}^t = \mathbf{x} + t\mathbf{n}$ satisfy

$$0 = (a + 2tb + t^2c)K^t + 2(b + tc)H^t + c. \tag{2}$$

Note that surfaces of constant mean or constant Gauss curvature ($a = 0$ or $b = 0$, respectively) yield examples of linear Weingarten surfaces, with parallel linear Weingarten surfaces that do generally not have a constant curvature.

A simple and fairly efficient proof of Lemma 1 is obtained by employing the Cayley-Hamilton identity for the fundamental forms of $\mathbf{x}^t = \mathbf{x} + t\mathbf{n}$

$$\mathrm{III}^t = \mathrm{III}, \qquad \mathrm{II}^t = \mathrm{II} - t\mathrm{III} \qquad \text{and} \qquad \mathrm{I}^t = \mathrm{I} - 2t\mathrm{II} + t^2\mathrm{III}$$

the Cayley-Hamilton identity $0 = \mathrm{III} - 2H\mathrm{II} + K\mathrm{I}$ of the initial surface $t = 0$ yields

$$0 = (1 - 2Ht + Kt^2)\mathrm{III}^t - 2(H - tK)\mathrm{II}^t + K\mathrm{I}^t.$$

Hence $K^t = \frac{K}{1-2tH+t^2K}$ and $H^t = \frac{H-tK}{1-2tH+t^2K}$, so that the claim follows from the linear Weingarten condition for the initial surface

$$(a + 2tb + t^2c)K + 2(b + tc)(H - tK) + c(1 - 2Ht + Kt^2) = 0.$$

Note that the linear Weingarten condition (1) factorizes when its discriminant vanishes

$$0 = \Delta := -ac + b^2 \qquad \Rightarrow \qquad 0 = (a\kappa_1 + b)(a\kappa_2 + b)$$

[1] We call a surface "linear Weingarten" if its Gauss and mean curvature satisfy an affine relation - another notion of "linear Weingarten" is, for example, adopted in [8], where the principal curvatures are assumed to satisfy a linear relation.

so that one of the principal curvatures κ_i is constant, $\kappa_i \equiv -\frac{b}{a}$, cf [1, §2.V] or [2]. In this case the linear Weingarten surface $\mathbf{x}$ is either developable, if $b = 0$, or it is a tube of radius $\frac{a}{b}$ about its focal curve $\mathbf{x} - \frac{a}{b}\mathbf{n}$. In either case, we will refer to the linear Weingarten surface $\mathbf{x}$ as *tubular*. Note that we will also consider totally umbilic surfaces as *tubular linear Weingarten surfaces*.

If, on the other hand, $\mathbf{x}$ is a non-tubular linear Weingarten surface (i.e., $\Delta \neq 0$), then equation (2) shows that its parallel family $(\mathbf{x} + t\mathbf{n})_{t\in\mathbb{R}}$ contains a constant curvature representative.

In particular, if $c = 0$ then $b \neq 0$ and $a + 2tb = 0$ yields $H^t = 0$, a minimal surface. If $c \neq 0$ then $b + tc = 0$ yields a surface of constant Gauss curvature $K^t = \frac{c^2}{\Delta}$. If, moreover, $\Delta > 0$ then we recover Bonnet's theorem on parallel surfaces of constant curvature from (2) with $a + 2tb + t^2c = 0$ or $t = \frac{-b\pm\sqrt{\Delta}}{c}$, cf [5, §2.7.4].

Thus we obtain three classes of parallel families of non-tubular linear Weingarten surfaces

Corollary 2. *Let* $\mathbf{x} : \Sigma^2 \to \mathbb{R}^3$ *be a non-tubular linear Weingarten surface*

$$0 = aK + 2bH + c \qquad with \qquad \Delta := b^2 - ac \neq 0.$$

If $c = 0$, $\mathbf{x}$ *is parallel to a minimal surface. If* $c \neq 0$, $\mathbf{x}$ *is parallel to a surface of constant Gauss curvature* $K \neq 0$, *which has two parallel constant mean curvature surfaces when* $K > 0$, *i.e., when* $\Delta > 0$.

3. Channel Linear Weingarten Surfaces

If a linear Weingarten surface has a family of circular curvature lines, then it is called a *channel* linear Weingarten surface[2]. In this section, we show that such a surface is either tubular or a surface of revolution, cf [4, Theorem 1] or [6, Theorem 1.1].

Following [6] we parametrize a surface foliated by circles: if $u \mapsto \mathbf{z}(u)$ denotes the centre curve of the foliating circles and $r = r(u)$ their radii[3] then

$$(u, v) \mapsto \mathbf{x}(u, v) = \mathbf{z}(u) + r(u)\boldsymbol{\nu}(u, v) \tag{3}$$

[2] A channel surface can be characterized as the envelope of a one-parameter family of spheres or, equivalently (in conformal geometry), as a surface with a family of circular curvature lines.

[3] Here we exclude surfaces where the family of circular curvature lines contains lines such as on developable surfaces.

where we choose standard parametrizations $v \mapsto \boldsymbol{\nu}(u,v)$ of the unit circles, such that $\boldsymbol{\nu}_v \perp \boldsymbol{\nu}$ and, with unit normals $\boldsymbol{\tau} = \boldsymbol{\tau}(u)$ of the planes of the foliating circles, we obtain an orthonormal frame field $(\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\nu}_v)$ for the surface. Assuming that u parametrizes the circle planes by means of a unit speed orthogonal trajectory[4], $\boldsymbol{\tau}$ becomes the tangent field of this orthogonal trajectory. To facilitate our analysis we require that $u \mapsto \boldsymbol{\nu}(u,0)$, hence $u \mapsto \boldsymbol{\nu}(u,v)$ for every v, be parallel along this orthogonal trajectory, that is

$$\boldsymbol{\nu}_u = -\kappa\boldsymbol{\tau}.$$

Then $\boldsymbol{\tau}' = \kappa\boldsymbol{\nu} + \lambda\boldsymbol{\nu}_v$, where κ and λ are the curvatures of the trajectory with respect to the (parallel) normal fields $\boldsymbol{\nu}$ and $\boldsymbol{\nu}_v$ for fixed v, and

$$0 = \boldsymbol{\tau}'_v = \kappa_v\boldsymbol{\nu} + \kappa\boldsymbol{\nu}_v + \lambda_v\boldsymbol{\nu}_v + \lambda\boldsymbol{\nu}_{vv}$$

since $\boldsymbol{\nu}_{vv} + \boldsymbol{\nu} = 0$, yields

$$\lambda = \kappa_v \qquad \text{and} \qquad \kappa + \kappa_{vv} = 0.$$

As $\boldsymbol{\nu}_v \parallel \mathbf{x}_v$ is tangential, the Gauss map $\mathbf{n}$ of the channel surface $\mathbf{x}$ is given by

$$\mathbf{n} = \boldsymbol{\tau}\cos\alpha + \boldsymbol{\nu}\sin\alpha$$

where $\alpha = \alpha(u)$ denotes the intersection angle of the surface with the planes of the foliating circles – which is constant along the intersection lines $v \mapsto \mathbf{x}(u,v)$ by Joachimsthal's theorem, since these lines are curvature lines on the surface. Note that this also proves one direction of the equivalence noted in Footnote 2. As a consequence, since $\mathbf{x}_u \perp \mathbf{n}$

$$\mathbf{x}_u = w(-\boldsymbol{\tau}\sin\alpha + \boldsymbol{\nu}\cos\alpha) + w_v\boldsymbol{\nu}_v\cos\alpha$$

where $(u,v) \mapsto w(u,v)$ denotes a suitable function, and the form of the $\boldsymbol{\nu}_v$-coefficient is obtained by using that $0 = \mathbf{z}_{uv} = (\mathbf{x} - r\boldsymbol{\nu})_{uv}$ as well as

$$r' = (w_{vv} + w)\cos\alpha \qquad \text{and} \qquad \kappa_v = \tfrac{1}{r}w_v\sin\alpha. \tag{4}$$

Hence

$$\kappa = -\tfrac{1}{r}w_{vv}\sin\alpha \tag{5}$$

since $\kappa_{vv} + \kappa = 0$. With $\mathbf{x}_v = r\boldsymbol{\nu}_v$ and

$$\mathbf{n}_u = (\alpha' + \kappa)(-\boldsymbol{\tau}\sin\alpha + \boldsymbol{\nu}\cos\alpha) + \kappa_v\boldsymbol{\nu}_v\cos\alpha \qquad \text{and} \qquad \mathbf{n}_v = \boldsymbol{\nu}_v\sin\alpha$$

[4]Such an orthogonal trajectory does not always exist, as the example of a surface of revolution with closed profile curve shows.

it is straightforward to determine the Gauss and mean curvatures of $\mathbf{x}$

$$K = \frac{\alpha' + \kappa}{w}\frac{\sin\alpha}{r} \qquad \text{and} \qquad H = -\tfrac{1}{2}\Big(\frac{\alpha' + \kappa}{w} + \frac{\sin\alpha}{r}\Big).$$

Hence the linear Weingarten condition $0 = aK + 2bH + c$ for our channel surface $\mathbf{x}$ reads

$$0 = (\alpha' + \kappa)(a\sin\alpha - br) - w(b\sin\alpha - cr). \tag{6}$$

Differentiating the linear Weingarten equation (6) twice and using (5) we deduce

$$0 = w_{vv}(a\sin^2\alpha - 2br\sin\alpha + cr^2)$$

as a necessary condition for a channel surface (3) to be linear Weingarten and this yields two cases to analyze.

If $a\sin^2\alpha - 2br\sin\alpha + cr^2 \equiv 0$ on an open interval then the channel surface $\mathbf{x}$ has a constant principal curvature, $\frac{1}{r}\sin\alpha \equiv \text{const}$. That is, $\mathbf{x}$ is tubular on that interval.

If $a\sin^2\alpha - 2br\sin\alpha + cr^2$ has at most isolated zeroes then $w_{vv} \equiv 0$. Thus, by (5), $\kappa \equiv 0$, which shows that the orthogonal trajectory of the circle planes is a straight line, so that the circle planes are parallel. Consequently, using (4), this yields $w_v \equiv 0$ and $r' = w\cos\alpha$, and, with (3)

$$\mathbf{z}' = (\mathbf{x} - r\boldsymbol{\nu})_u = -w\boldsymbol{\tau}\sin\alpha$$

so that the curve of circle centres is a straight line and, in particular, an orthogonal trajectory of the circle planes. Thus the channel surface $\mathbf{x}$ is a surface of revolution with its centre curve $\mathbf{z}$ taking values on the axis, cf [1, §2.III] and [4, Theorem 1].

Theorem 3. *A non-tubular linear Weingarten surface with circular curvature lines is a surface of revolution.*

4. Linear Weingarten Surfaces of Revolution

With Corollary 2, Theorem 3 yields a classification result for channel linear Weingarten surfaces:

Corollary 4. *A channel linear Weingarten surface is either tubular* (*tube about a space curve, developable or totally umbilic*) *or parallel surface of a minimal* ($H = 0$) *or a constant Gauss curvature* $K \neq 0$ *surface of revolution.*

It is well known that the catenoid is the only minimal surface of revolution in Euclidean three-space, cf [1, §2.VIII] or [9, §78], so the following classification of surfaces of revolution with constant Gauss curvature completes our classification of channel linear Weingarten surfaces, cf [11, Section 6.7]

Theorem 5. *Every constant Gauss curvature $K \neq 0$ surface of revolution is homothetic to*

$$(s, \vartheta) \quad \mapsto \quad x(s, \vartheta) := (r(s)\cos\vartheta, r(s)\sin\vartheta, h(s))$$

with one of the following profile curves

$$r(s) = p\,\mathrm{cn}_p(s), \qquad \begin{cases} h(s) = (E_p \circ \mathrm{am}_p)(s), & K = +1 \\ h(s) = (E_p \circ \mathrm{am}_p)(s) - s, & K = -1 \end{cases}$$

$$r(s) = \tfrac{1}{\cosh(s)}, \qquad \begin{cases} h(s) = \tanh(s), & K = +1, \textit{ sphere} \\ h(s) = \tanh(s) - s, & K = -1, \textit{ pseudosphere} \end{cases}$$

$$r(s) = \tfrac{1}{p}\,\mathrm{dn}_p(\tfrac{s}{p}), \qquad \begin{cases} h(s) = \tfrac{1}{p}(E_p \circ \mathrm{am}_p)(\tfrac{s}{p}) - \tfrac{1-p^2}{p^2}s, & K = +1 \\ h(s) = \tfrac{1}{p}(E_p \circ \mathrm{am}_p)(\tfrac{s}{p}) - \tfrac{1}{p^2}s, & K = -1. \end{cases}$$

Here $\mathrm{sn}_p = \sin \circ \mathrm{am}_p$, $\mathrm{cn}_p = \cos \circ \mathrm{am}_p$ and dn_p denote Jacobi elliptic functions, with the Jacobi amplitude function am_p, and $E_p(\phi) = \int_0^\phi \sqrt{1 - p^2 \sin^2\varphi}\,\mathrm{d}\varphi$ is the incomplete elliptic integral of the second kind, with elliptic modulus $p \in (0, 1)$.

It is straightforward to verify that the surfaces listed in Theorem 5 do indeed have constant Gauss curvatures $K = \pm 1$. The Gauss curvature of a surface of revolution is given by

$$K = \frac{r'h'' - r''h'}{\sqrt{r'^2 + h'^2}^3}\frac{h'}{r\sqrt{r'^2 + h'^2}} = -\frac{1}{2rr'}\Big(\frac{r'^2}{r'^2 + h'^2}\Big)'$$

in terms of its profile curve $s \mapsto (r(s), h(s))$ and hence the surface has constant Gauss curvature if and only if there is a constant $C \in \mathbb{R}$ so that

$$r'^2 = (r'^2 + h'^2)(C - Kr^2).$$

Note that

$$0 \le C - Kr^2 = \tfrac{r'^2}{r'^2 + h'^2} \le 1.$$

Given a surface of revolution with constant Gauss curvature K this differential equations for its profile curve can be further simplified to an elliptic differential equation

$$r'^2 = ((1 - C) + Kr^2)(C - Kr^2) \tag{7}$$

for the radius function $s \mapsto r(s)$ alone by adjusting the parametrization of the profile curve so that

$$r'^2 + h'^2 = (1 - C) + Kr^2$$

since $(1 - C) + Kr^2 \geq 0$. Hence, without loss of generality

$$h' = (1 - C) + Kr^2. \tag{8}$$

To analyze the solutions of the differential equations (7) and (8) that yield the occurring profile curves we distinguish the cases $K = \pm 1$.

If $K = +1$ then $0 \leq r^2 \leq C \leq 1 + r^2$ and (7) reads

$$y'^2 = \begin{cases} (1 - y^2)(q^2 + p^2 y^2) & \text{with} \quad p = \sqrt{C}, \ \ q = \sqrt{1 - C} \text{ and } y = \frac{r}{p} \text{ if } C < 1 \\ y^2 - 1 & \text{with} \quad y = \frac{1}{r} \text{ if } C = 1 \\ \frac{1}{p^2}(1 - y^2)(y^2 - q^2) & \text{with} \quad p = \frac{1}{\sqrt{C}}, \ \ q = \sqrt{\frac{C-1}{C}} \text{ and } y = pr \text{ if } C > 1. \end{cases}$$

The solutions of these differential equations are, up to parameter shift

$$y(s) = \mathrm{cn}_p(s), \qquad y(s) = \cosh s \qquad \text{and} \qquad y(s) = \mathrm{dn}_p(\frac{s}{p})$$

respectively. Integration of (8) then yields the respective three profile curves, as claimed in Theorem 5, up to translation or reflection.

If $K = -1$ then $0 \leq r^2 \leq 1 - C \leq 1 + r^2$ and swapping the roles of $1 - C$ and C leads to the same elliptic differential equations for r as in the $K = +1$ cases, up to a sign choice and (8) then differs from the same equation in the $K = +1$ case by an additive 1. Hence we obtain the profile curves as claimed in Theorem 5, up to translation and reflection again.

This completes the proof of Theorem 5.

The explicit parametrizations of Theorem 5 now enable us to verify the results of [7] and, in particular, to confirm Corollary 3.6 of [6] in a simple way

Corollary 6. *There is a one-parameter family of complete, immersed, rotational linear Weingarten surfaces of hyperbolic type, that is, with $\Delta < 0$.*

These surfaces have a translational period.

As the discriminant of the linear Weingarten equation is an invariant of a parallel family of linear Weingarten surfaces, $\Delta^t = \Delta$ by (2), we consider the parallel surfaces of a pseudospherical surface of revolution, where $\Delta = -1$, for

$$\boldsymbol{\xi} = (r, h) \quad \text{with} \quad r(s) = \frac{1}{p}\,\mathrm{dn}_p(\frac{s}{p}) \quad \text{and} \quad h(s) = \frac{1}{p}(E_p \circ \mathrm{am}_p)(\frac{s}{p}) - \frac{1}{p^2}s$$

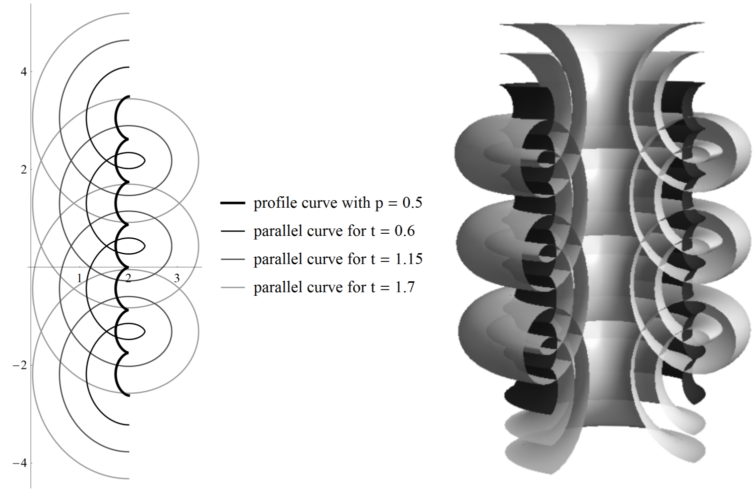

Figure 1. Parallel hyperbolic linear Weingarten surfaces of revolution.

we obtain $\boldsymbol{\xi}'(s) = -\operatorname{sn}_p(\frac{s}{p})\boldsymbol{\tau}(s)$ with $\boldsymbol{\tau}(s) = (\operatorname{cn}_p, \operatorname{sn}_p)(\frac{s}{p})$. Hence the parallel curves

$$\boldsymbol{\xi}^t = \boldsymbol{\xi} + t\boldsymbol{\nu} \qquad \text{with} \qquad \boldsymbol{\nu}(s) = (-\operatorname{sn}_p, \operatorname{cn}_p)(\frac{s}{p})$$

hit the axis of rotation where

$$0 = r^t(s) = (\frac{1}{p}\operatorname{dn}_p - t\operatorname{sn}_p)(\frac{s}{p}) \qquad \Leftrightarrow \qquad t = \frac{1}{p}\frac{\operatorname{dn}_p}{\operatorname{sn}_p}(\frac{s}{p}) \in (-\infty, -\frac{q}{p}] \cup [\frac{q}{p}, \infty)$$

with the elliptic co-modulus $q = \sqrt{1-p^2}$, and they develop singularities where

$$0 = (\boldsymbol{\xi}^t)'(s) = -(\operatorname{sn}_p + \frac{t}{p}\operatorname{dn}_p)(\frac{s}{p})\boldsymbol{\tau}(s) \qquad \Leftrightarrow \qquad t = -p\frac{\operatorname{sn}_p}{\operatorname{dn}_p}(\frac{s}{p}) \in [-\frac{p}{q}, \frac{p}{q}].$$

Consequently, if the elliptic modulus is chosen so that

$$\frac{p}{q} < \frac{q}{p} \qquad \Leftrightarrow \qquad p^2 < q^2 = 1 - p^2 \qquad \Leftrightarrow \qquad p < \frac{1}{\sqrt{2}}$$

then all $t \in (\frac{p}{q}, \frac{q}{p}) \neq \emptyset$ yield a family of immersed parallel linear Weingarten surfaces of revolution with profile curves $\boldsymbol{\xi}^t$, cf Fig. 1. Clearly these surfaces are complete, and they have a translational period since r and h are periodic.

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