JGSP 33 (2014) 47–78

JOURNAL OF
Geometry and Symmetry in Physics
ISSN 1312-5192

A DECOUPLED SOLUTION TO THE GENERALIZED EULER DECOMPOSITION PROBLEM IN $\mathbb{R}^3$ AND $\mathbb{R}^{2,1}$

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Abstract. In this article we suggest a new method, partially based on earlier works of Wohlhart [15], Mladenova and Mladenov [11], Brezov et al [3], that resolves the generalized Euler decomposition problem (about arbitrary axes) using a system of quadratic equations. The main contribution made here is that we manage to decouple this system and express the solutions independently in a compact covariant form. We apply the same technique to the Lorentz group in $2+1$ dimensions and discuss certain complications related to the presence of isotropic directions in $\mathbb{R}^{2,1}$.

Contents

doi: 10.7546/jgsp-33-2014-47-78

1. Introduction

The problem of generalizing the classical Euler ZXZ and Bryan ZYX decomposition settings to non-orthogonal axes has been on the table for decades now. Real solutions are guaranteed as long as the second axis is normal to the other two (the so-called *Davenport condition* [6, 14]) and in the generic case one needs a non-negative discriminant relation to be satisfied [11, 13]. The standard procedure is to solve a coupled system of quadratic equations for the scalar (angular) parameters and then sort out the actual solutions from the fake ones as has been done in [11,15], or use Rodrigues' formula to isolate the symmetric and skew-symmetric parts of the rotations, thus obtaining the angles as proper quadrant inverse tangents [13]. Unfortunately, one ends up with somewhat ambiguous expressions in the former case and overwhelmingly complicated formulae in the latter. Exploiting the *vector-parameter* representation briefly explained in Section 2, which reveals some linear-fractional relations between the parameters in the decomposition, we overcome these problems in [3,5]. All those methods, however, have the disadvantage of using the solution to one of the equations as a parameter in the other two. Here we manage to decouple the system and express all parameters independently in terms of several matrix entries and determinants (or angles). As can be expected, the method works also for the Lorentz group $\mathsf{SO}(2,1)$ with a few modifications. The formulae obtained in Section 4 significantly improve our previous results on the problem [4, 5]. Moreover, we discover a light cone singularity: when all three axes are normal to some null direction in $\mathbb{R}^{2,1}$, the solutions are either infinitely many or none, as shown in Section 4.3. Along with the many examples[1], we provide a list of decomposition configurations applicable in special relativity [4,9] and scattering theory [2, 4]. The method has been naturally adapted to moving frames and quaternion (respectively split quaternion) decompositions in sections 5 and 6.

2. Quaternions and Vector-Parameters

We choose a basis in $\mathfrak{su}(2)$ in the form

$$\mathbf{i} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{k} = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{1}$$

and introduce the set of unit quaternions as

$$\zeta = \zeta_0 + \zeta_1\mathbf{i} + \zeta_2\mathbf{j} + \zeta_3\mathbf{k}, \qquad |\zeta|^2 = 1, \qquad \zeta_\mu \in \mathbb{R}$$

[1] we also obtain a peculiar relation between the Euler and Bryan angles in this context

with norm given by

$$|\,\zeta\,|^2 = \frac{1}{2}\mathrm{tr}(\zeta\bar{\zeta}) = \det(\zeta) = \sum_{\mu=0}^{3}\zeta_\mu^2 \tag{2}$$

where $\bar{\zeta} = \zeta_0 - \zeta_1\mathbf{i} - \zeta_2\mathbf{j} - \zeta_3\mathbf{k}$ stands for the *conjugate quaternion*. Next, we associate with each vector $\mathbf{x}\in\mathbb{R}^3$ a skew-hermitian matrix by the rule (cf [8, 12])

$$\mathbf{x} \to \Psi = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$$

where x_i are the Cartesian coordinates of $\mathbf{x}$ in the default basis and let $\mathsf{SU}(2)$ act in its Lie algebra via the adjoint representation $\mathrm{Ad}_\zeta:\ \Psi \to \zeta\,\Psi\,\bar{\zeta}$, which can be viewed as a norm-preserving automorphism of $\mathbb{R}^3$. It is straightforward to obtain the corresponding orthogonal transformation for the Cartesian coordinates of three-dimensional vectors in the form

$$\mathcal{R}(\zeta) = (\zeta_0^2 - \boldsymbol{\zeta}^2)\mathcal{I} + 2\boldsymbol{\zeta}\otimes\boldsymbol{\zeta}^t + 2\zeta_0\boldsymbol{\zeta}^\times \tag{3}$$

where $\boldsymbol{\zeta} = \Im(\zeta) \in \mathbb{R}^3$ stands for the *imaginary* (or *vector*) part of the quaternion $\zeta = (\zeta_0, \boldsymbol{\zeta})$ and $\zeta_0 = \Re(\zeta)$ is referred to as its *real* (*scalar*) part, $\mathcal{I}$ denotes the identity matrix in $\mathbb{R}^3$ and $\boldsymbol{\zeta}^\times$ - the skew-symmetric transformation, associated with $\boldsymbol{\zeta}$ via *Hodge duality*, i.e., $\boldsymbol{\zeta}^\times\mathbf{x} = \boldsymbol{\zeta}\times\mathbf{x}$ for $\mathbf{x}\in\mathbb{R}^3$. The famous *Rodrigues' formula*

$$\mathcal{R}(\varphi, \mathbf{n}) = \cos\varphi\,\mathcal{I} + (1-\cos\varphi)\,\mathbf{n}\otimes\mathbf{n}^t + \sin\varphi\,\mathbf{n}^\times \tag{4}$$

follows directly with the substitution $\zeta_0 = \cos\dfrac{\varphi}{2}$, $\boldsymbol{\zeta} = \sin\dfrac{\varphi}{2}\,\mathbf{n}$, where $\mathbf{n}$ is a vector with unit Euclidean norm, i.e., $(\mathbf{n},\mathbf{n}) = 1$. Alternatively, we may project $\zeta \to \mathbf{c} = \dfrac{\boldsymbol{\zeta}}{\zeta_0} = \tau\mathbf{n}$ with $\tau = \tan\dfrac{\varphi}{2}$ (the *scalar parameter*) and thus express the matrix entries in (3) as rational functions of the *vector-parameter* $\mathbf{c}$ in the form

$$\mathcal{R}(\mathbf{c}) = \frac{(1-\mathbf{c}^2)\,\mathcal{I} + 2\,\mathbf{c}\otimes\mathbf{c}^t + 2\,\mathbf{c}^\times}{1+\mathbf{c}^2}\cdot \tag{5}$$

The inverse relation is given by the formula

$$\mathbf{c}^\times = \frac{\mathcal{R} - \mathcal{R}^t}{1 + \mathrm{tr}\mathcal{R}}\cdot$$

This projection, on the other hand, can be lifted back to the two-sheeted cover as

$$\zeta_0^\pm = \pm(1+\mathbf{c}^2)^{-\frac{1}{2}}, \qquad \boldsymbol{\zeta}^\pm = \zeta_0^\pm\mathbf{c}. \tag{6}$$

Quaternion multiplication then gives the composition law for the vector-parameters of two successive rotations $\mathcal{R}(\langle \mathbf{c}_2, \mathbf{c}_1 \rangle) = \mathcal{R}(\mathbf{c}_2)\,\mathcal{R}(\mathbf{c}_1)$ in the form

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1}{1 - (\mathbf{c}_2, \mathbf{c}_1)} \tag{7}$$

and in the case of three transformations, i.e., $\mathbf{c} = \langle \mathbf{c}_3, \langle \mathbf{c}_2, \mathbf{c}_1 \rangle \rangle$ we have (see [7])

$$\mathbf{c} = \frac{\mathbf{c}_3 + \mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_3 \times \mathbf{c}_2 + \mathbf{c}_3 \times \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1 + (\mathbf{c}_3 \times \mathbf{c}_2) \times \mathbf{c}_1 - (\mathbf{c}_3, \mathbf{c}_2)\,\mathbf{c}_1}{1 - (\mathbf{c}_3, \mathbf{c}_2) - (\mathbf{c}_3, \mathbf{c}_1) - (\mathbf{c}_2, \mathbf{c}_1) - (\mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1)}$$

where $(\mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1) = (\mathbf{c}_3 \times \mathbf{c}_2, \mathbf{c}_1)$. The latter constitutes a representation with

$$\langle \mathbf{c}_3, \langle \mathbf{c}_2, \mathbf{c}_1 \rangle \rangle = \langle \langle \mathbf{c}_3, \mathbf{c}_2 \rangle, \mathbf{c}_1 \rangle, \qquad \langle\, \mathbf{c},\, 0 \,\rangle = \langle\, 0,\, \mathbf{c} \,\rangle = \mathbf{c}, \qquad \langle\, \mathbf{c},\, -\mathbf{c} \,\rangle = 0.$$

Among the advantages of vector parametrization are exact rational expressions for the rotation matrix entries (5), more efficient composition of group elements (7) as well as more accurate description of the orthogonal group's topology $\mathsf{SO}(3) \cong \mathbb{RP}^3$.

Similarly, in $\mathfrak{sl}(2, \mathbb{R})$ one has the *split quaternion* basis [1,4]

$$\tilde{\mathbf{i}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tilde{\mathbf{j}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \tilde{\mathbf{k}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{8}$$

which can be mapped to $\mathfrak{su}(1,1)$ via the isomorphism $\mathcal{H}^2 \to \mathbb{D} :\ z \to \mathrm{i}\,\dfrac{z - \mathrm{i}}{z + \mathrm{i}}$ as

$$\tilde{\mathbf{i}} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tilde{\mathbf{j}} \to \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \tilde{\mathbf{k}} \to \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\cdot \tag{9}$$

Expansion in the above bases allows for an explicit isometry $\mathbb{R}^{2,1} \to \mathfrak{sl}(2, \mathbb{R})$: $\mathbf{x} \to \Psi = x_1 \tilde{\mathbf{i}} + x_2 \tilde{\mathbf{j}} + x_3 \tilde{\mathbf{k}},\ \ \mathbf{x} \cdot \mathbf{x} = -\det\Psi$ and the projection onto $\mathsf{SO}^+(2,1)$ is given by the adjoint action of the group of unit split quaternions $\mathsf{SL}(2,\mathbb{R}) \cong \mathsf{SU}(1,1)$ in its Lie algebra $\mathrm{Ad}_{\zeta} :\ \Psi \to \zeta\,\Psi\,\bar{\zeta}$, which is a norm-preserving automorphism. Using the familiar notation $\zeta = (\zeta_0, \boldsymbol{\zeta})$, $\bar{\zeta} = (\zeta_0, -\boldsymbol{\zeta})$, $\boldsymbol{\zeta} \in \mathbb{R}^{2,1}$ we write the pseudo-orthogonal matrix transforming the Cartesian coordinates of $\mathbf{x}$ as

$$\mathcal{R}_h(\zeta) = (\zeta_0^2 + \boldsymbol{\zeta}^2)\mathcal{I} - 2\,\boldsymbol{\zeta} \otimes \eta\,\boldsymbol{\zeta} + 2\,\zeta_0\,\boldsymbol{\zeta}^{\curlywedge} \tag{10}$$

where $\eta = \mathrm{diag}(1,1,-1)$ is the flat metric in $\mathbb{R}^{2,1}$, $(\boldsymbol{\zeta} \otimes \eta\,\boldsymbol{\zeta})^i_j = \eta_{jk}\,\zeta^i\zeta^k$ (summation over repeated indices is assumed) and $\boldsymbol{\zeta}^{\curlywedge} = \eta\,\boldsymbol{\zeta}^{\times}$, so that we also denote $\boldsymbol{\zeta} \curlywedge \boldsymbol{\xi} = \boldsymbol{\zeta}^{\curlywedge}\,\boldsymbol{\xi}$. Furthermore, we may introduce the *hyperbolic vector-parameter* in the usual manner $\mathbf{c} = \dfrac{\boldsymbol{\zeta}}{\zeta_0}$ and write (10) as

$$\mathcal{R}_h(\mathbf{c}) = \frac{(1 + \mathbf{c}^2)\mathcal{I} - 2\,\mathbf{c} \otimes \eta\,\mathbf{c} + 2\,\mathbf{c}^{\curlywedge}}{1 - \mathbf{c}^2} \tag{11}$$

which allows for expressing $\mathbf{c}$ in terms of the matrix entries of $\mathcal{R}_h$ as

$$\mathbf{c}^{\curlywedge} = \frac{\mathcal{R}_h - \eta\mathcal{R}_h^t\eta}{1 + \mathrm{tr}\mathcal{R}_h}.$$

The inverse transformation is given by

$$\zeta_0^{\pm} = \pm(1 - \mathbf{c}^2)^{-\frac{1}{2}}, \qquad \boldsymbol{\zeta}^{\pm} = \zeta_0^{\pm}\mathbf{c} \tag{12}$$

where the two signs correspond to different sheets of the spin cover.
From the multiplication rule of split quaternions we easily derive the composition law of hyperbolic vector parameters in the form

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \curlywedge \mathbf{c}_1}{1 + \mathbf{c}_2 \cdot \mathbf{c}_1} \tag{13}$$

and for the case of three transformations $\mathbf{c} = \langle \mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1 \rangle$ we have (cf [4])

$$\mathbf{c} = \frac{\mathbf{c}_3 + \mathbf{c}_2 + \mathbf{c}_1 + (\mathbf{c}_3 \cdot \mathbf{c}_2)\,\mathbf{c}_1 + \mathbf{c}_3 \curlywedge \mathbf{c}_2 + \mathbf{c}_3 \curlywedge \mathbf{c}_1 + \mathbf{c}_2 \curlywedge \mathbf{c}_1 + (\mathbf{c}_3 \curlywedge \mathbf{c}_2) \curlywedge \mathbf{c}_1}{1 + \mathbf{c}_3 \cdot \mathbf{c}_2 + \mathbf{c}_3 \cdot \mathbf{c}_1 + \mathbf{c}_2 \cdot \mathbf{c}_1 + (\mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1)}.$$

Moreover, this construction constitutes a representation of $\mathsf{SO}(2,1)$ as well, with the advantages already discussed. Here we have several analogues of *Rodrigues*' rotation formula (4) depending on the geometric type of the invariant axis (cf [4,5])

1. *Hyperbolic:* $\mathrm{Tr}\,\mathcal{R}_h(\zeta) > 3,\ \boldsymbol{\zeta}^2 = \zeta_0^2 - 1 > 0$ (*space-like*) $\Rightarrow\ \tau = \mathrm{th}\,\dfrac{\varphi}{2}$

$$\mathcal{R}_h(\mathbf{n}, \varphi) = \mathrm{ch}\,\varphi\,\mathcal{I} + (1 - \mathrm{ch}\,\varphi)\,\mathbf{n} \otimes \eta\,\mathbf{n} + \mathrm{sh}\,\varphi\,\mathbf{n}^{\curlywedge}. \tag{14}$$

2. *Elliptic:* $\mathrm{Tr}\,\mathcal{R}_h(\zeta) < 3,\ \boldsymbol{\zeta}^2 < 0$ (*time-like*) $\Rightarrow\ \tau = \tan\dfrac{\varphi}{2}$

$$\mathcal{R}_h(\mathbf{n}, \varphi) = \cos\varphi\,\mathcal{I} + (\cos\varphi - 1)\,\mathbf{n} \otimes \eta\,\mathbf{n} + \sin\varphi\,\mathbf{n}^{\curlywedge}. \tag{15}$$

3. *Parabolic:* $\mathrm{Tr}\,\mathcal{R}_h(\zeta) = 3,\ \boldsymbol{\zeta}^2 = 0$ (*isotropic*) $\Rightarrow\ \tau = \dfrac{\varphi}{2}$

$$\mathcal{R}_h(\mathbf{n}, \varphi) = \mathcal{I} + \varphi\,\mathbf{n}^{\curlywedge} - \frac{\varphi^2}{2}\,\mathbf{n} \otimes \eta\,\mathbf{n}. \tag{16}$$

4. *Non-Orthochronous*[2]: $\mathcal{R}_{33} < 0,\ \boldsymbol{\zeta}^2 = \zeta_0^2 + 1\ \Rightarrow\ \tau = \coth\dfrac{\varphi}{2}$

$$\mathcal{R}_h(\mathbf{n}, \varphi) = -\mathrm{ch}\,\varphi\,\mathcal{I} + (1 + \mathrm{ch}\,\varphi)\,\mathbf{n} \otimes \eta\,\mathbf{n} - \mathrm{sh}\,\varphi\,\mathbf{n}^{\curlywedge}. \tag{17}$$

[2]this case is not in the proper Lorentz group $\mathsf{SO}^+(2,1)$ and is obtained via analytic continuation.

3. The Decomposition Setting

We consider the generalized Euler decomposition

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c}_3)\mathcal{R}(\mathbf{c}_2)\mathcal{R}(\mathbf{c}_1) \tag{18}$$

with respect to arbitrary axes[3], determined by the unit vectors $\hat{\mathbf{c}}_k$. We denote the corresponding vector-parameters with $\mathbf{c}_k = \tau_k \hat{\mathbf{c}}_k$, where $\tau_k = \tan\dfrac{\varphi_k}{2}$ are the so-called *scalar parameters* and φ_k - the generalized Euler angles of rotation about $\hat{\mathbf{c}}_k$. Considering appropriately chosen matrix entries of $\mathcal{R}(\mathbf{c})$ in the basis[4] $\{\mathbf{c}_k\}$, obtained with the aid of formula (18) and taking into account that $\hat{\mathbf{c}}_k$ is an invariant vector for $\mathcal{R}(\mathbf{c}_k)$, we come to a system of quadratic equations for the parameters τ_k in the form (see [3] for more details on the derivation)

$$\begin{aligned}
&(r_{32} + g_{32} - 2g_{12}r_{31})\,\tau_1^2 - 2\tilde{\omega}\tau_1 + r_{32} - g_{32} = 0\\
&(r_{31} + g_{31} - 2g_{12}g_{23})\,\tau_2^2 + 2\omega\tau_2 + r_{31} - g_{31} = 0\\
&(r_{21} + g_{21} - 2g_{23}r_{31})\,\tau_3^2 - 2\tilde{\omega}\tau_3 + r_{21} - g_{21} = 0
\end{aligned} \tag{19}$$

where we make use of the notation

$$g_{ij} = (\hat{\mathbf{c}}_i, \hat{\mathbf{c}}_j)\,, \qquad r_{ij} = (\hat{\mathbf{c}}_i, \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_j)\,, \qquad \omega = (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3) \tag{20}$$

and $\tilde{\omega} = (\mathcal{R}(\mathbf{c}_2)\,\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3)$. Note that since the second equation determines in the generic case two solutions for τ_2, $\tilde{\omega}$ is actually a double-valued function

$$\tilde{\omega}^{\pm} = \left(\mathcal{R}(\tau_2^{\pm}\hat{\mathbf{c}}_2)\,\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3\right), \qquad \tilde{\omega}^{-} + \tilde{\omega}^{+} = 0. \tag{21}$$

We also introduce the discriminants of the above equations (modulo a factor of 4)

$$\Delta_1 = \begin{vmatrix} 1 & g_{12} & r_{31}\\ g_{21} & 1 & r_{32}\\ r_{31} & r_{32} & 1 \end{vmatrix}, \qquad \Delta = \begin{vmatrix} 1 & g_{12} & r_{31}\\ g_{21} & 1 & g_{23}\\ r_{31} & g_{32} & 1 \end{vmatrix}, \qquad \Delta_3 = \begin{vmatrix} 1 & r_{21} & r_{31}\\ r_{21} & 1 & g_{23}\\ r_{31} & g_{32} & 1 \end{vmatrix} \tag{22}$$

and let $\Delta_2 = \det g$ be the Gram determinant for the vector system $\{\hat{\mathbf{c}}_k\}$. Denoting

$$\omega_1 = \left(\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \mathcal{R}^t(\mathbf{c})\,\hat{\mathbf{c}}_3\right), \qquad \omega_2 = \omega, \qquad \omega_3 = (\mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3) \tag{23}$$

we obviously have $\Delta_k = \omega_k^2$ and $\Delta = \tilde{\omega}^2$, which explains the property (21). Hence, $\Delta_k \geq 0$ and the necessary and sufficient condition for the existence of real, or rather $\mathbb{RP}^1$ solutions (with ∞ added) of (19) can be written in the compact form

$$\Delta \geq 0. \tag{24}$$

[3]provided that $\hat{\mathbf{c}}_2$ is not parallel to any of the other two, so successive rotations are independent.

[4]strictly speaking, it is not always a basis, since the axes are allowed to be coplanar.

Since the parameters τ_k in (19) satisfy certain linear-fractional relations as well (cf [5]), it is clear that we typically have two solutions, rather than eight. Taking into account some orientation arguments justified below, we obtain the formulae

$$\tau_1^{\pm} = \frac{-\omega_1 \pm \sqrt{\Delta}}{r_{32} + g_{32} - 2g_{12}r_{31}}, \qquad \tau_2^{\pm} = \frac{-\omega_2 \pm \sqrt{\Delta}}{r_{31} + g_{31} - 2g_{12}r_{23}} \qquad (25)$$

$$\tau_3^{\pm} = \frac{-\omega_3 \pm \sqrt{\Delta}}{r_{21} + g_{21} - 2g_{23}r_{31}}$$

which can be simplified even further in the generic case $r_{ij} \neq g_{ij}$ for $i > j$. Namely, applying the identity $\Delta_k - \Delta = (\omega_k - \sqrt{\Delta})(\omega_k + \sqrt{\Delta})$ one easily obtains

$$\tau_1^{\pm} = \frac{r_{32} - g_{32}}{\omega_1 \pm \sqrt{\Delta}}, \qquad \tau_2^{\pm} = \frac{g_{31} - r_{31}}{\omega_2 \pm \sqrt{\Delta}}, \qquad \tau_3^{\pm} = \frac{r_{21} - g_{21}}{\omega_3 \pm \sqrt{\Delta}}. \qquad (26)$$

On the other hand, if any of the relations

$$r_{ij} = g_{ij}, \qquad i > j \qquad (27)$$

holds, beside the trivial solution for the corresponding parameter[5] $\varepsilon_{ijk}\tau_k = 0$, there is one more, which may be retrieved from (25) in the form

$$\tau_1 = \frac{\omega_1}{g_{12}r_{31} - g_{23}}, \qquad \tau_2 = \frac{\omega_2}{g_{12}g_{23} - g_{13}}, \qquad \tau_3 = \frac{\omega_3}{g_{23}r_{31} - g_{12}} \qquad (28)$$

that becomes a decomposition of the identity transformation if all three of the above relations take place simultaneously. In this case we have also $\omega_k = \omega$ and therefore

$$\tau_1 = \frac{\omega}{g_{12}g_{31} - g_{23}}, \qquad \tau_2 = \frac{\omega}{g_{12}g_{23} - g_{13}}, \qquad \tau_3 = \frac{\omega}{g_{23}g_{31} - g_{12}}. \qquad (29)$$

3.1. Half-Turns

The explicit form of (19) and (26) allows for a straightforward investigation of the cases when a half-turn is present in (18), i.e., when some of the parameters become infinite ($\varphi_k = \pi \Leftrightarrow \tau_k = \infty$). Namely, from (26) we easily obtain

$$\Delta_k = \Delta, \qquad r_{ij} \neq g_{ij}, \quad i > j \quad \Rightarrow \quad \varepsilon_{ijk}\tau_k^{\pm} = \infty, \quad \frac{g_{ij} - r_{ij}}{2\omega_k} \qquad (30)$$

and obviously encounter a double root at infinity for $\Delta = 0$.
Note that the compound rotation may be a half-turn itself $\mathcal{O}(\mathbf{n}) = 2\mathbf{n}\otimes\mathbf{n}^t - \mathcal{I}$, in which case nothing changes about the above formulae, except that we may simplify further by substituting $r_{ij} = 2\upsilon_i\upsilon_j - g_{ij}$, where $\upsilon_i = (\mathbf{n}, \hat{\mathbf{c}}_i)$. Half-turns are even easier to explore in the two-axes decomposition setting considered below (cf [3,5]).

[5] here and below ε_{ijk} denotes the Levi-Civita symbol and δ_{ij} - the Kronecker one.

3.2. The Case of Two Axes

When the condition (27) is satisfied, the trivial solution $\varepsilon_{ijk}\tau_k = 0$ yields a decomposition with respect to only two axes. Alternatively, one may consider a setting with only two gimbals to rotate about and try to obtain the decomposition

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c}_2)\mathcal{R}(\mathbf{c}_1). \tag{31}$$

Proceeding as in the case of three axes, we find the necessary and sufficient condition to be $r_{21} = g_{21}$ and from the quadratic equations for τ_k, obtain

$$\tau_1 = \pm\sqrt{\frac{1 - r_{22}}{1 + r_{22} - 2r_{21}^2}}\,, \qquad \tau_2 = \pm\sqrt{\frac{1 - r_{11}}{1 + r_{11} - 2r_{21}^2}}\,.$$

Actually, the solution in this case is only one and can be expressed by means of

$$\mathring{\omega}_1 = \left(\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \mathcal{R}^t(\mathbf{c})\,\hat{\mathbf{c}}_2\right), \qquad \mathring{\omega}_2 = \left(\mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2\right) \tag{32}$$

in the compact form

$$\tau_1 = \frac{r_{22} - 1}{\mathring{\omega}_1}\,, \qquad \tau_2 = \frac{r_{11} - 1}{\mathring{\omega}_2}\,. \tag{33}$$

The above numerators are strictly negative for any non-trivial axis configuration. Therefore, the condition for a half-turn $\varphi_k = \pi \Leftrightarrow \mathring{\omega}_k = 0$ is given simply by

$$1 + r_{22} = 2g_{12}^2 \Leftrightarrow \tau_1 = \infty, \qquad 1 + r_{11} = 2g_{12}^2 \Leftrightarrow \tau_2 = \infty \tag{34}$$

which generalizes a classical result asserting that each rotation is a composition of two reflections with respect to axes (or planes), which make an angle, equal to half the angle of compound rotation. In this case we may denote $\theta_{ij} = \measuredangle(\hat{\mathbf{c}}_i, \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_j)$, so θ_{kk} is the angle by which $\mathcal{R}(\mathbf{c})$ rotates $\hat{\mathbf{c}}_k$ and let $\gamma_{ij} = \measuredangle(\hat{\mathbf{c}}_i, \hat{\mathbf{c}}_j)$ be the acute angle between the two axes. Then, (34) yields $\cos\theta_{kk} = \cos 2\gamma_{12}$ meaning that

$$\theta_{22} = \pm 2\gamma_{12} \Leftrightarrow \varphi_1 = \pi, \qquad \theta_{11} = \pm 2\gamma_{12} \Leftrightarrow \varphi_2 = \pi. \tag{35}$$

In particular, if the above relations are both satisfied, the axes lie in the plane of rotation (normal to $\mathbf{n}$) and if in addition they are mutually perpendicular, it follows that $\theta_{11} = \theta_{22} = \pi$, so the compound transformation is a half-turn itself.

Note that all solutions obtained so far can easily be expressed solely in terms of sine and cosine functions of certain known angles. For example, in (26) one may substitute $g_{ij} = \cos\gamma_{ij}$ and $r_{ij} = \cos\theta_{ij}$, as well as $\omega_1 = \sin\gamma_{12}\cos\tilde{\beta}_{123}$, $\omega_2 = \sin\gamma_{12}\cos\beta_0$ and $\omega_3 = \sin\gamma_{23}\cos\beta_{123}$, where $\tilde{\beta}_{ijk} = \measuredangle\left(\hat{\mathbf{c}}_i \times \hat{\mathbf{c}}_j, \mathcal{R}^t(\mathbf{c})\,\hat{\mathbf{c}}_k\right)$, $\beta_{ijk} = \measuredangle(\mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_i, \hat{\mathbf{c}}_j \times \hat{\mathbf{c}}_k)$ and $\beta_0 = \measuredangle(\hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3)$. Similarly, in (33) we have $\mathring{\omega}_1 = \sin\gamma_{12}\cos\tilde{\beta}_{122}$, $\mathring{\omega}_2 = \sin\gamma_{12}\cos\beta_{112}$, so τ_k depend only on relative angles.

3.3. Signs and Orientation

So far we avoided discussing our motivation for the particular choice of signs for the determinants in (26) and (33), slipping out with the vague argument of "orientation", that is not even a real argument. In this paragraph, we finally pay our debt to the reader, thus making the whole method rigorous. We start with the simpler case of two axes and use an observation, pointed out by Davenport [6] and later exploited (although not very efficiently) in this context by Piovan and Bullo [13]. Namely, if we consider a generic rotation in $\mathbb{R}^3$ given by $\mathcal{R}(\tau\mathbf{n}) : \mathbf{x} \to \tilde{\mathbf{x}}$, where $\mathcal{R}(\tau\mathbf{n}) = \mathcal{R}(\mathbf{n}, \varphi)$ is determined by (4) or (5), taking consecutive scalar products with $\mathbf{x}$ and $\mathbf{n}\times\mathbf{x}$ allows for expressing trigonometric functions of the angle φ as

$$\cos\varphi = \frac{(\mathbf{x}, \tilde{\mathbf{x}}) - (\mathbf{n}, \mathbf{x})}{\mathbf{x}^2 - (\mathbf{n}, \mathbf{x})}, \qquad \sin\varphi = \frac{(\mathbf{n}, \mathbf{x}, \tilde{\mathbf{x}})}{\mathbf{x}^2 - (\mathbf{n}, \mathbf{x})}, \qquad \tau = \frac{\mathbf{x}^2 - (\mathbf{n}, \mathbf{x})}{(\mathbf{n}, \mathbf{x}, \tilde{\mathbf{x}})}\cdot \tag{36}$$

Applying the last formula above for the decomposition setting (31), where we have

$$\mathcal{R}(\tau_2\hat{\mathbf{c}}_2) : \ \hat{\mathbf{c}}_1 \longrightarrow \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1, \qquad \mathcal{R}^t(\tau_1\hat{\mathbf{c}}_1) : \ \hat{\mathbf{c}}_2 \longrightarrow \mathcal{R}^t(\mathbf{c})\,\hat{\mathbf{c}}_2$$

and taking into account that all vectors are unit and $\mathcal{R}^t(\tau_1\mathbf{c}_1) = \mathcal{R}(-\tau_1\hat{\mathbf{c}}_1)$, we obtain exactly (33) and thus prove that the choice of signs made in (32) is correct. As for the generic case of three axes (18), one may consider a situation, in which $r_{31} = g_{31}$ and $\Delta = 0$. Although (26) fails to determine the value of τ_2 for this case, formula (28) yields a double vanishing root as long as the co-factor $g_{12}g_{23} - g_{13}$ is non-zero. Then, (33) predicts a single solution for τ_1 and τ_3 in the form

$$\tau_1 = \frac{r_{32} - g_{32}}{\omega_1}, \qquad \tau_3 = \frac{r_{21} - g_{21}}{\omega_3}\cdot$$

Since the above is a two-gimbal decomposition, the choice of a third axis is arbitrary. In particular, we may set $\hat{\mathbf{c}}_2 = \hat{\mathbf{c}}_3$ for the first equation and $\hat{\mathbf{c}}_2 = \hat{\mathbf{c}}_1$ for the second one, thus obtaining with the aid of (23) the two expressions

$$\tau_1 = \frac{r_{33} - 1}{(\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_3, \mathcal{R}^t(\mathbf{c})\,\hat{\mathbf{c}}_3)}, \qquad \tau_3 = \frac{r_{11} - 1}{(\mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_1, \hat{\mathbf{c}}_3)} \tag{37}$$

which agree with formula (33). Although we proved the signs correct only in one very specific case, since the solutions are continuous maps between two connected compacts $\mathcal{E} : \ (\mathbf{c}, \{\hat{\mathbf{c}}_k\}) \in \mathbb{RP}^3 \times (\mathbb{S}^2)^3 \longrightarrow \{\varphi_k\} \in \mathbb{T}^3$ on the whole semi-axis $\Delta \geq 0$, no sign jumps are allowed for the determinants ω_k.

3.4. Gimbal Lock

There is a singularity of $\mathcal{E}$, known as *gimbal lock*, at which τ_1 and τ_3 cannot be determined from (19) due to zero coefficients. It is characterized by the condition

$$\hat{\mathbf{c}}_3 = \pm\mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1 \tag{38}$$

that makes (24) equivalent to $r_{21} = g_{21}$, since it yields $\Delta = -(r_{21} - g_{21})^2$. Note that with the aid of group conjugation

$$\mathcal{R}(\mathbf{c})\,\mathcal{R}(\tilde{\mathbf{c}})\,\mathcal{R}^{-1}(\mathbf{c}) = \mathcal{R}(\mathcal{R}(\mathbf{c})\,\tilde{\mathbf{c}}) \tag{39}$$

the decomposition (18) can be expressed in this case also as

$$\mathcal{R}(\tau_2\hat{\mathbf{c}}_2)\mathcal{R}(\tau_1\hat{\mathbf{c}}_1) = \mathcal{R}(\mp\tau_3\mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1)\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c})\mathcal{R}(\mp\tau_3\hat{\mathbf{c}}_1)$$

which can easily be written in the form

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}(\tau_2\hat{\mathbf{c}}_2)\mathcal{R}(\langle\tau_1\hat{\mathbf{c}}_1, \pm\tau_3\hat{\mathbf{c}}_1\rangle) = \mathcal{R}(\tau_2\hat{\mathbf{c}}_2)\mathcal{R}(\tilde{\tau}_1\hat{\mathbf{c}}_1).$$

This yields a two-axes decomposition, so (7) and (33) provide the solution

$$\tau_2 = \frac{r_{11} - 1}{\mathring{\omega}_2}\,, \qquad \tilde{\tau}_1 = \frac{\tau_1 \pm \tau_3}{1 \mp \tau_1\tau_3} = \frac{r_{22} - 1}{\mathring{\omega}_1} \tag{40}$$

which may also be written in terms of the generalized Euler angles as

$$\varphi_2 = 2\arctan\frac{r_{11} - 1}{\mathring{\omega}_2}, \qquad \varphi_1 \pm \varphi_3 = 2\arctan\frac{r_{22} - 1}{\mathring{\omega}_1}\cdot$$

A well-known example is the gimbal lock one encounters in the classical ZXZ Euler setting decomposing a half-turn about the OY axis. Note that if we consider a similar situation in the two-gimbal case $\hat{\mathbf{c}}_2 = \pm\mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1$, the condition $r_{21} = g_{21}$ specifies that only the trivial decomposition, for which $\hat{\mathbf{c}}_1 \sim \hat{\mathbf{c}}_2 \sim \mathbf{n}$, is possible.

This concludes our construction. For an illustration of the algorithm see Fig. 1 on the next page.

3.5. Two Familiar Examples

First, we consider the classical Euler ZXZ setting, in which $g_{12} = g_{23} = 0$, $g_{31} = 1$, $r_{32} = \mathcal{R}_{31}$, $r_{31} = \mathcal{R}_{33}$ and $r_{21} = \mathcal{R}_{13}$, as well as

$$\Delta = 1 - \mathcal{R}_{33}^2, \qquad \omega_1 = \mathcal{R}_{32}, \qquad \omega_2 = 0, \qquad \omega_3 = -\mathcal{R}_{23}$$

so the solutions are given, according to (26), in the form

$$\tau_1^{\pm} = \frac{\mathcal{R}_{31}}{\mathcal{R}_{32} \pm \sqrt{1 - \mathcal{R}_{33}^2}}, \qquad \tau_2^{\pm} = \pm\sqrt{\frac{1 - \mathcal{R}_{33}}{1 + \mathcal{R}_{33}}}, \qquad \tau_3^{\pm} = -\frac{\mathcal{R}_{13}}{\mathcal{R}_{23} \mp \sqrt{1 - \mathcal{R}_{33}^2}}\cdot$$

Another famous example is Bryan's XYZ decomposition with $g_{ij} = \delta_{ij}$, $r_{ij} = \mathcal{R}_{ij}$

$$\Delta = 1 - \mathcal{R}_{31}^2, \qquad \omega_1 = \mathcal{R}_{33}, \qquad \omega_2 = 1, \qquad \omega_3 = \mathcal{R}_{11}$$

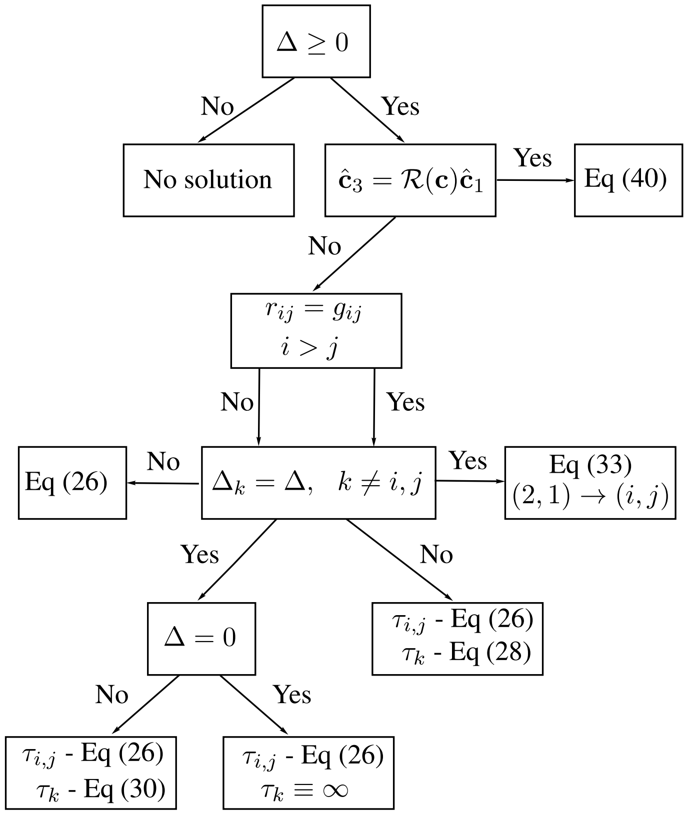

Figure 1. Decomposition flowchart in the Euclidean case.

that yields the solution in the form

$$\tau_1^{\pm} = \frac{\mathcal{R}_{32}}{\mathcal{R}_{33} \pm \sqrt{1 - \mathcal{R}_{31}^2}}, \qquad \tau_2^{\pm} = -\frac{\mathcal{R}_{31}}{1 \pm \sqrt{1 - \mathcal{R}_{31}^2}}, \qquad \tau_3^{\pm} = \frac{\mathcal{R}_{21}}{\mathcal{R}_{11} \pm \sqrt{1 - \mathcal{R}_{31}^2}}.$$

The corresponding angles of rotation can be obtained directly as $\varphi_k^{\pm} = 2 \arctan \tau_k^{\pm}$. On the other hand, in the gimbal lock setting we have $r_{11} = r_{22} = \mathcal{R}_{33} = -1$, $\mathring{\omega}_1 = -\mathcal{R}_{12} = \mp(\hat{\mathbf{c}}_1, \mathbf{c}_1, \hat{\mathbf{c}}_2) = 0$ and $\mathring{\omega}_2 = -\mathcal{R}_{23} = 0$ in the Euler case, leading to

$$\varphi_2 = \pi, \qquad \varphi_1 - \varphi_3 = \pi.$$

Similarly, for the Bryan decomposition considered above the singularity condition (38) yields $r_{11} = r_{22} = 0$, $\mathring{\omega}_1 = \mathcal{R}_{23}$, $\mathring{\omega}_2 = \mathcal{R}_{31} = \pm 1$, so we have

$$\varphi_2 = \mp \pi/2, \qquad \varphi_1 \pm \varphi_3 = -2 \arctan(\mathcal{R}_{23})^{-1}.$$

Denoting the Euler angles by $\{\phi, \vartheta, \psi\}$ and the Bryan ones - by $\{\tilde{\phi}, \tilde{\vartheta}, \tilde{\psi}\}$, we have two equivalent representations of the compound rotation $\mathcal{R}(\phi, \vartheta, \psi) = \mathcal{R}(\tilde{\phi}, \tilde{\vartheta}, \tilde{\psi})$, the former written explicitly as

$$
\begin{aligned}
&\mathcal{R}(\phi,\vartheta,\psi)=\mathcal{R}(\hat{\mathbf{e}}_z,\psi)\,\mathcal{R}(\hat{\mathbf{e}}_x,\vartheta)\,\mathcal{R}(\hat{\mathbf{e}}_z,\phi)\\
&=\begin{pmatrix}
\cos\psi\cos\phi-\sin\psi\cos\vartheta\sin\phi & -\cos\psi\sin\phi-\sin\psi\cos\vartheta\cos\phi & \sin\psi\sin\vartheta\\
\sin\psi\cos\phi+\cos\psi\cos\vartheta\sin\phi & \cos\psi\cos\vartheta\cos\phi-\sin\psi\sin\phi & -\cos\psi\sin\vartheta\\
\sin\vartheta\sin\phi & \sin\vartheta\cos\phi & \cos\vartheta
\end{pmatrix}
\end{aligned}
$$

where $\{\hat{\mathbf{e}}_x,\hat{\mathbf{e}}_y,\hat{\mathbf{e}}_z\}$ stand for the unit vectors along the coordinate axes and the three rotations in the decomposition are constructed with the aid of formula (4).

Similarly, in the Bryan setting the compound matrix is given as

$$
\begin{aligned}
&\mathcal{R}(\tilde{\phi},\tilde{\vartheta},\tilde{\psi})=\mathcal{R}(\hat{\mathbf{e}}_z,\tilde{\psi})\,\mathcal{R}(\hat{\mathbf{e}}_y,\tilde{\vartheta})\,\mathcal{R}(\hat{\mathbf{e}}_x,\tilde{\phi})\\
&=\begin{pmatrix}
\cos\psi\cos\tilde{\vartheta} & \cos\tilde{\psi}\sin\tilde{\vartheta}\sin\tilde{\phi}-\sin\tilde{\psi}\cos\tilde{\phi} & \sin\tilde{\psi}\sin\tilde{\phi}+\cos\tilde{\psi}\sin\tilde{\vartheta}\cos\tilde{\phi}\\
\sin\tilde{\psi}\cos\tilde{\vartheta} & \sin\tilde{\psi}\sin\tilde{\vartheta}\sin\tilde{\phi}-\cos\tilde{\psi}\cos\tilde{\phi} & \sin\tilde{\psi}\sin\tilde{\vartheta}\cos\tilde{\phi}-\cos\tilde{\psi}\sin\tilde{\phi}\\
-\sin\tilde{\vartheta} & \cos\tilde{\vartheta}\sin\tilde{\phi} & \cos\tilde{\vartheta}\cos\tilde{\phi}
\end{pmatrix}\cdot
\end{aligned}
$$

Then, we may derive the explicit relation between the two parameterizations by expressing the matrix entries of $\mathcal{R}$ in one of these two sets of parameters and use the decomposition formulae for the other. In this way, it is straightforward to obtain

$$
\begin{aligned}
\tilde{\phi}^{\pm} &= 2\arctan\frac{\cos\phi\sin\vartheta}{\cos\vartheta\pm\sqrt{1-\sin^2\!\phi\sin^2\!\vartheta}}\\
\tilde{\vartheta}^{\pm} &= -2\arctan\frac{\sin\phi\sin\vartheta}{1\pm\sqrt{1-\sin^2\!\phi\sin^2\!\vartheta}}\\
\tilde{\psi}^{\pm} &= 2\arctan\frac{\cos\phi\sin\psi+\sin\phi\cos\vartheta\cos\psi}{\cos\phi\cos\psi-\sin\phi\cos\vartheta\sin\psi\pm\sqrt{1-\sin^2\!\phi\sin^2\!\vartheta}}\cdot
\end{aligned}
\tag{41}
$$

In the inverse direction we have respectively

$$
\begin{aligned}
\phi^{\pm} &= -2\arctan\frac{\sin\tilde{\vartheta}}{\sin\tilde{\phi}\cos\tilde{\vartheta}\pm\sqrt{1-\cos^2\!\tilde{\phi}\cos^2\!\tilde{\vartheta}}}\\
\vartheta^{\pm} &= \pm 2\arctan\sqrt{\frac{1-\cos\tilde{\phi}\cos\tilde{\vartheta}}{1+\cos\tilde{\phi}\cos\tilde{\vartheta}}}\\
\psi^{\pm} &= 2\arctan\frac{\cos\tilde{\phi}\sin\tilde{\vartheta}\cos\tilde{\psi}+\sin\tilde{\phi}\sin\tilde{\psi}}{\sin\tilde{\phi}\cos\tilde{\psi}-\cos\tilde{\phi}\sin\tilde{\vartheta}\sin\tilde{\psi}\pm\sqrt{1-\cos^2\tilde{\phi}\cos^2\tilde{\vartheta}}}\cdot
\end{aligned}
\tag{42}
$$

Note that substituting different solutions (corresponding to the same compound matrix) for the angles in the righthand sides of the above equations still yields the same result, as should be expected due to formula (21). From (42) and (43) one may also derive interesting relations for a fixed value of certain parameter. For example, a symmetric first factor in the Bryan decomposition $\tilde{\phi} \in \{0, \pi\}$ yields for the Euler case $\phi = \mp\pi/2$, $\vartheta \in \pm\{\tilde{\vartheta}, \pi - \tilde{\vartheta}\}$ and $\psi = \tilde{\psi} \pm \pi/2$. Similarly, $\vartheta = \pm\pi/2$ in the latter leads to $\tilde{\phi} = \pm\pi/2$, $\tilde{\vartheta} \in \mp\{\phi, \pi - \phi\}$ as well as $\tilde{\psi} = 2\,(\psi \mp \phi) \pm \pi$. Both relations can be inverted for the constant parameter, i.e., if ϕ, respectively $\tilde{\phi}$ is a right angle, the above provide adequate expressions for the remaining parameters.

4. The Hyperbolic Case

In this section we study decompositions of the Lorentz group $\mathrm{SO}(2,1)$ in the form

$$\mathcal{R}_h(\mathbf{c}) = \mathcal{R}_h(\mathbf{c}_3)\mathcal{R}_h(\mathbf{c}_2)\mathcal{R}_h(\mathbf{c}_1) \tag{43}$$

where we use the same notation $\mathbf{c} = \tau\,\mathbf{n}$ and $\mathbf{c}_k = \tau_k\,\hat{\mathbf{c}}_k$ for the corresponding vector-parameters and $\mathbf{n}$, $\hat{\mathbf{c}}_k$ are the *quasi-unit* vectors with magnitude ± 1 or 0. It is convenient to introduce the coefficients $\epsilon = \mathbf{n} \cdot \mathbf{n} = \pm 1$ in the space-like, respectively time-like case and $\epsilon = 0$ in the isotropic one, where the normalization is arbitrary, so we may choose for example the Euclidean one setting $(\mathbf{n}, \mathbf{n}) = 1$. Similar considerations hold for the coefficients $\epsilon_k = \hat{\mathbf{c}}_k \cdot \hat{\mathbf{c}}_k$. Let us also denote

$$r_{ij} = \hat{\mathbf{c}}_i \cdot \mathcal{R}_h\hat{\mathbf{c}}_j, \qquad g_{ij} = \hat{\mathbf{c}}_i \cdot \hat{\mathbf{c}}_j, \qquad \omega = \hat{\mathbf{c}}_1 \cdot \hat{\mathbf{c}}_2 \curlywedge \hat{\mathbf{c}}_3 \tag{44}$$

and point out that the hyperbolic triple product of vectors coincides with the Euclidean one $\omega = \hat{\mathbf{c}}_1 \cdot \hat{\mathbf{c}}_2 \curlywedge \hat{\mathbf{c}}_3 = (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3)$ and can also be expressed as determinant. Next, we proceed just as in the previous section, considering different entries r_{ij}, calculated with the aid of (11) and the property that $\hat{\mathbf{c}}_k$ is an invariant eigenvector of $\mathcal{R}(\mathbf{c}_k)$. This leads to the system of quadratic equations for the parameters τ_k

$$\begin{aligned} &\big(\epsilon_1(r_{32} + g_{32}) - 2g_{12}r_{31}\big)\,\tau_1^2 + 2\tilde{\omega}\tau_1 + g_{32} - r_{32} = 0 \\ &\big(\epsilon_2(r_{31} + g_{31}) - 2g_{12}g_{23}\big)\,\tau_2^2 - 2\omega\tau_2 + g_{31} - r_{31} = 0 \\ &\big(\epsilon_3(r_{21} + g_{21}) - 2g_{23}r_{31}\big)\,\tau_3^2 + 2\tilde{\omega}\tau_3 + g_{21} - r_{21} = 0 \end{aligned} \tag{45}$$

where $\tilde{\omega}^{\pm} = (\mathcal{R}_h(\tau_2^{\pm}\hat{\mathbf{c}}_2)\,\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3)$ is again double-valued with $\tilde{\omega}^- + \tilde{\omega}^+ = 0$. Next, we introduce the discriminants of the equations (45) in the form

$$\Delta_1 = -\begin{vmatrix} \epsilon_1 & g_{12} & r_{31} \\ g_{21} & \epsilon_2 & r_{32} \\ r_{31} & r_{32} & \epsilon_3 \end{vmatrix}, \quad \Delta = -\begin{vmatrix} \epsilon_1 & g_{12} & r_{31} \\ g_{21} & \epsilon_2 & g_{23} \\ r_{31} & g_{32} & \epsilon_3 \end{vmatrix}, \quad \Delta_3 = -\begin{vmatrix} \epsilon_1 & r_{21} & r_{31} \\ r_{21} & \epsilon_2 & g_{23} \\ r_{31} & g_{32} & \epsilon_3 \end{vmatrix}\cdot \tag{46}$$

We also let $\Delta_2 = \det g$ be the Gram determinant of the ordered vector system $\{\hat{\mathbf{c}}_i\}$ and point out that $\tilde{\omega}^2 = \Delta$. The solutions in the regular case are thus given as

$$\tau_1^{\pm} = \frac{\omega_1 \pm \sqrt{\Delta}}{\epsilon_1(r_{32}+g_{32}) - 2g_{12}r_{31}}, \qquad \tau_2^{\pm} = \frac{\omega_2 \pm \sqrt{\Delta}}{\epsilon_2(r_{31}+g_{31}) - 2g_{12}g_{23}} \tag{47}$$

$$\tau_3^{\pm} = \frac{\omega_3 \pm \sqrt{\Delta}}{\epsilon_3(r_{21}+g_{21}) - 2g_{23}r_{31}}$$

where we make use of the notation

$$\omega_1 = \left(\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \mathcal{R}_h^{-1}(\mathbf{c})\,\hat{\mathbf{c}}_3\right), \qquad \omega_2 = \omega, \qquad \omega_3 = \left(\mathcal{R}_h(\mathbf{c})\,\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3\right). \tag{48}$$

Just as in the Euclidean setting, we may simplify further as long as $r_{ij} \neq g_{ij}, \ i > j$ and express the above solutions in the form

$$\tau_1^{\pm} = \frac{r_{32} - g_{32}}{\omega_1 \mp \sqrt{\Delta}}, \qquad \tau_2^{\pm} = \frac{g_{31} - r_{31}}{\omega_2 \mp \sqrt{\Delta}}, \qquad \tau_3^{\pm} = \frac{r_{21} - g_{21}}{\omega_3 \mp \sqrt{\Delta}}\cdot \tag{49}$$

On the other hand, if any of the relations $r_{ij} = g_{ij}, \ i > j$ takes place, formula (49) shows that $\varepsilon_{ijk}\tau_k = 0$, but there is one more solution revealed by formula (47) as

$$\tau_1 = \frac{\omega_1}{\epsilon_1 g_{23} - g_{12}r_{31}}, \qquad \tau_2 = \frac{\omega_2}{\epsilon_2 g_{31} - g_{12}g_{23}}, \qquad \tau_3 = \frac{\omega_3}{\epsilon_3 g_{12} - g_{23}r_{31}}\cdot \tag{50}$$

In particular, when $\mathcal{R} \equiv \mathcal{I}$ all three conditions in (27) are satisfied and $\omega_k = \omega$, in which case the decomposition is determined by (see also [5])

$$\tau_1 = \frac{\omega}{\epsilon_1 g_{23} - g_{12}g_{31}}, \qquad \tau_2 = \frac{\omega}{\epsilon_2 g_{31} - g_{12}g_{23}}, \qquad \tau_3 = \frac{\omega}{\epsilon_3 g_{12} - g_{23}g_{31}}\cdot \tag{51}$$

4.1. Two-Axes Decompositions

Let us now investigate the much simpler case of two axes

$$\mathcal{R}_h(\mathbf{c}) = \mathcal{R}_h(\mathbf{c}_2)\mathcal{R}_h(\mathbf{c}_1) \tag{52}$$

for which the necessary and sufficient condition is easily seen to be also $r_{21} = g_{21}$, the scalar products this time being calculated with respect to the Lorentz metric η. From the expressions (11) for r_{11} and r_{22} one easily obtains the magnitudes of τ_k

$$\tau_1 = \pm\sqrt{\frac{r_{22} - \epsilon_1}{\epsilon_2(r_{22} + \epsilon_1) - 2r_{21}^2}}, \qquad \tau_2 = \pm\sqrt{\frac{r_{11} - \epsilon_2}{\epsilon_1(r_{11} + \epsilon_2) - 2r_{21}^2}}\,.$$

In order to determine the above signs, we exploit the Davenport technique, considered in the preceding section, this time for the pseudo-rotation $\mathcal{R}_h(\tau\mathbf{n}) : \mathbf{x} \to \tilde{\mathbf{x}}$, where $\mathbf{x}, \tilde{\mathbf{x}} \in \mathbb{R}^{2,1}$. Although there are four analogues of Rodrigues' formula in this case (14) - (17), one could work directly with (11) in order to obtain

$$\tau = \frac{\mathbf{x}^2 - \mathbf{x} \cdot \tilde{\mathbf{x}}}{\mathbf{n} \cdot \mathbf{x} \curlywedge \tilde{\mathbf{x}}} \tag{53}$$

which can be applied to the transformations

$$\mathcal{R}_h(\tau_2 \hat{\mathbf{c}}_2) : \ \hat{\mathbf{c}}_1 \longrightarrow \mathcal{R}_h(\mathbf{c})\, \hat{\mathbf{c}}_1, \qquad \mathcal{R}_h^{-1}(\tau_1 \hat{\mathbf{c}}_1) : \ \hat{\mathbf{c}}_2 \longrightarrow \mathcal{R}_h^{-1}(\mathbf{c})\, \hat{\mathbf{c}}_2.$$

In this way, we see that the solutions are uniquely determined by the expressions

$$\tau_1 = \frac{r_{22} - \epsilon_2}{\mathring{\omega}_1}, \qquad \tau_2 = \frac{r_{11} - \epsilon_1}{\mathring{\omega}_2} \tag{54}$$

where the denominators are defined just as in the Euclidean case

$$\mathring{\omega}_1 = \left(\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \mathcal{R}_h^{-1}(\mathbf{c})\, \hat{\mathbf{c}}_2\right), \qquad \mathring{\omega}_2 = \left(\mathcal{R}_h(\mathbf{c})\, \hat{\mathbf{c}}_1, \hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2\right). \tag{55}$$

Note that the numerators in (54) are strictly positive, so the scalar parameters have the signs of the determinants $\mathring{\omega}_k$. Moreover, the same argument we used in the previous section shows that the signs of ω_k are properly chosen. Considering the case $r_{31} = g_{31}$ and $\hat{\mathbf{c}}_2 = \alpha\, \hat{\mathbf{c}}_1 + \beta\, \hat{\mathbf{c}}_3$, i.e., $\Delta = \omega = 0$, we make sure that only the two-gimbal decomposition (54) takes place. On the other hand, (49) are correct as well (even when the expression for τ_2 is undetermined). Then, since the second axis is irrelevant, we may set $\alpha = 0$ for the first expression in (49) and $\beta = 0$ for the second one, respectively. This yields a coincidence between (49) and (54) (the latter is also written for $\hat{\mathbf{c}}_1$ and $\hat{\mathbf{c}}_3$) for this particular configuration. Since ω_k is a continuous function of $\mathbf{c}$ and $\hat{\mathbf{c}}_k$, and so are the matrix entries in (11) on each connected component of $\mathsf{SO}(2,1)$, the corresponding signs are chosen correctly.

4.2. Half-Turns, Time-Reversing Boosts and Locked Gimbals

Just as in the Euclidean case, (45) and (49) provide the solutions involving infinite parameters directly in the form (with a a double root at infinity if $\Delta = 0$)

$$\Delta_k = \Delta, \qquad r_{ij} \neq g_{ij}, \quad i > j \quad \Longrightarrow \quad \varepsilon_{ijk} \tau_k^{\pm} = \infty, \quad \frac{g_{ij} - r_{ij}}{2\omega_k} \cdot \tag{56}$$

We also note that the hyperbolic analogue of an Euclidean half-turn is somewhat ambiguous. By the formula $\mathcal{O}_h(\mathbf{n}) = 2\,\epsilon\, \mathbf{n} \otimes \mathbf{n}^t - \mathcal{I}$, i.e., $r_{ij} = 2\epsilon v_i v_j - g_{ij}$ with $v_i = \mathbf{n} \cdot \hat{\mathbf{c}}_i$, we understand different things for $\epsilon = -1$ (time-like direction), in which case we are dealing with an actual half-turn, and $\epsilon = 1$ (space-like direction), related to a non-proper Lorentz transformation (time-reversing boost). Finally, for

$\epsilon = 0$ we obtain a divergent matrix in (11). Therefore $\tau \to \infty$ will not be allowed for null directions in the same way we have the restriction $|\tau| \neq 1$ for space-like ones, since (11) is ill-defined in this case as well, so our parameters need to satisfy

$$\epsilon = 0 \ \Rightarrow \ |\tau| \neq \infty, \qquad \epsilon = 1 \ \Rightarrow \ |\tau| \neq 1. \tag{57}$$

Since the non-negative discriminant condition $\Delta \geq 0$ cannot guarantee that the letter are satisfied, we consider it only necessary and generically insufficient. There are two more counterexamples considered bellow - the degenerate gimbal lock setting and the light cone singularity. Both these configurations almost certainly satisfy the discriminant condition, but allow solutions only on a zero measure set.

Degenerate Solutions. The hyperbolic case of gimbal lock is quite similar to the Euclidean one. In particular, it is defined by the same condition

$$\hat{\mathbf{c}}_3 = \pm \mathcal{R}_h(\mathbf{c})\,\hat{\mathbf{c}}_1 \tag{58}$$

and we may resort to the same technique, using the identity (39) in order to obtain

$$\mathcal{R}_h(\mathbf{c}) = \mathcal{R}_h(\tau_2\hat{\mathbf{c}}_2)\mathcal{R}_h(\langle \tau_1\hat{\mathbf{c}}_1, \pm\tau_3\hat{\mathbf{c}}_1\rangle) = \mathcal{R}_h(\tau_2\hat{\mathbf{c}}_2)\mathcal{R}_h(\tilde{\tau}_1\hat{\mathbf{c}}_1).$$

This leads to a decomposition with respect to two axes and formula (54) yields

$$\tau_2 = \frac{r_{11} - \epsilon_1}{\mathring{\omega}_2}\,, \qquad \tilde{\tau}_1 = \frac{\tau_1 \pm \tau_3}{1 \pm \epsilon_1\tau_1\tau_3} = \frac{r_{22} - \epsilon_2}{\mathring{\omega}_1} \tag{59}$$

as long as the relation $r_{21} = g_{21}$ holds. Note that the latter is not guaranteed by $\Delta \geq 0$, since in this case we always have $\Delta = \epsilon_1(r_{21} - g_{21})^2 \geq 0$ for space-like and null $\hat{\mathbf{c}}_1$, so it should be considered as a replacement for the discriminant condition in the gimbal lock setting. If $\hat{\mathbf{c}}_1$ is time-like, on the other hand, it follows naturally. Note, on the other hand, that gimbal lock in this setting is only possible if $\epsilon_1 = \epsilon_3$.

4.3. Light Cone Singularities

Although the analogy between $\mathbb{R}^3$ and $\mathbb{R}^{2,1}$ has been close so far, there are crucial differences. Above all, the discriminant condition $\Delta \geq 0$ in the latter case is a necessary, but generally not sufficient for the existence of (43), as we already discussed. Here we investigate another degenerate setting that is typical only for the hyperbolic case - a singularity, present when all axes lie in the normal complement of a null vector in $\mathbb{R}^{2,1}$. First, we prove the following

Lemma 1. *Let* $\mathbf{c}_0 \in \mathbb{R}^{2,1}$ *be a null vector* ($\mathbf{c}_0^2 = 0$) *and* $\mathbf{c}_0^\perp$ *denote its orthogonal complement with respect to* η. *Then* $\mathbf{c}_0^\perp$ *is closed under the composition* (13) *and* *for each* $\mathbf{c} \in \mathbf{c}_0^\perp$ *the transformation* $\mathcal{R}_h(\mathbf{c})$ *preserves* $\mathbf{c}_0^\perp$*, and in particular, the* $\mathbf{c}_0$*-direction, i.e.,*

$$\mathbf{c}, \tilde{\mathbf{c}} \in \mathbf{c}_0^\perp \quad \Rightarrow \quad \langle \mathbf{c}, \tilde{\mathbf{c}} \rangle \in \mathbf{c}_0^\perp, \qquad \mathcal{R}_h(\mathbf{c}) : \mathbf{c}_0^\perp \to \mathbf{c}_0^\perp, \qquad \mathcal{R}_h(\mathbf{c})\, \mathbf{c}_0 \sim \mathbf{c}_0.$$

Proof: First, we note that $\mathbf{c}_0^\perp$ is two-dimensional, so one has $\tilde{\mathbf{c}} \in \operatorname{span}\{\mathbf{c}, \mathbf{c}_0\}$. Next, if we take into account that $\mathbf{c}_0 \in \mathbf{c}_0^\perp$ and $\mathbf{c} \curlywedge \tilde{\mathbf{c}} \sim \mathbf{c}_0$, the first part follows directly from formula (13) and the second one - from (11). Finally, $\operatorname{span}\{\mathbf{c}_0\}$ is the only null direction in $\mathbf{c}_0^\perp$ and $\mathcal{R}_h(\mathbf{c})$ is norm-preserving, hence the last relation. ∎

There is a light-cone singularity for (43), as shown by the following

Proposition 2. *Let* $\{\hat{\mathbf{c}}_k\} \in \mathbf{c}_0^\perp$ *for some null vector* $\mathbf{c}_0 \in \mathbb{R}^{2,1}$*. Then,* $\mathcal{R}_h(\mathbf{c})$ *has the representation* (43) *or* (52) *if and only if* $\mathbf{c} \in \mathbf{c}_0^\perp$ *and in the case of three axes the solutions form a degenerate one-parameter set.*

Proof: Lemma 1 shows that $\mathbf{c}_0^\perp$ is closed under the composition $\langle \cdot, \cdot \rangle$, which proves the necessity of the condition $\mathbf{c} \in \mathbf{c}_0^\perp$. As we show in a while, it also implies that for arbitrary (non-coincident) axes $\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2 \in \mathbf{c}_0^\perp$ there exist $\tau_1, \tau_2 \in \mathbb{RP}^1$, such that $\mathcal{R}_h(\mathbf{c}) = \mathcal{R}_h(\tau_2\hat{\mathbf{c}}_2)\mathcal{R}_h(\tau_1\hat{\mathbf{c}}_1)$. The proof is constructive and actually provides the solution. First, we see that for $\mathbf{c} \in \mathbf{c}_0^\perp$ all coefficients in (45) vanish, so the method described above fails on any tangent plane[6] to the light cone in $\mathbb{R}^{2,1}$. In particular, the three axes are coplanar, so we have $\omega = 0$ and $\omega_k = \tilde{\omega} = 0$, since $\mathbf{c} \in \mathbf{c}_0^\perp$ implies $\mathcal{R}_h(\mathbf{c}) : \mathbf{c}_0^\perp \to \mathbf{c}_0^\perp$, which also yields $\mathring{\omega}_k = 0$ in (54). Moreover, it allows for expanding $\mathbf{c}_j = \alpha_j \mathbf{c} + \beta_j \mathbf{c}_0$, so that we have $\mathcal{R}_h(\mathbf{c})\, \mathbf{c}_j = \alpha_j \mathbf{c} + \tilde{\beta}_j \mathbf{c}_0$, where the second term does not contribute to the scalar products in $\mathbf{c}_0^\perp$, hence $r_{ij} = g_{ij}$, and in the case of two axes, $r_{kk} = \epsilon_k$. Next, since the condition $r_{21} = g_{21}$ is always satisfied as long as $\mathbf{c} \in \mathbf{c}_0^\perp$, we may use (13) to write $\mathbf{c}_2 = \langle \mathbf{c}, -\mathbf{c}_1 \rangle$ and $\mathbf{c}_1 = \langle -\mathbf{c}_2, \mathbf{c} \rangle$ with the usual notation $\mathbf{c}_k = \tau_k \hat{\mathbf{c}}_k$. We multiply the first equation by $\hat{\mathbf{c}}_1^\curlywedge$ and the second one - by $\hat{\mathbf{c}}_2^\curlywedge$ (which effectively projects on the null direction) and then consider Euclidean scalar product with $\mathbf{c}_0$. Denoting $\mathbf{x}^\circ = (\mathbf{x}, \mathbf{c}_0)$ for arbitrary $\mathbf{x} \in \mathbb{R}^{2,1}$, we obtain the solution in the form

$$\tau_1 = \frac{(\hat{\mathbf{c}}_2 \curlywedge \mathbf{n})^\circ \tau}{\upsilon_2 \hat{\mathbf{c}}_1^\circ \tau - g_{12} \mathbf{n}^\circ \tau - (\hat{\mathbf{c}}_1 \curlywedge \hat{\mathbf{c}}_2)^\circ}, \qquad \tau_2 = \frac{(\hat{\mathbf{c}}_1 \curlywedge \mathbf{n})^\circ \tau}{(\hat{\mathbf{c}}_1 \curlywedge \hat{\mathbf{c}}_2)^\circ + g_{12} \mathbf{n}^\circ \tau - \upsilon_1 \hat{\mathbf{c}}_2^\circ \tau}. \tag{60}$$

Lemma 1 also guarantees that the numerators are non-zero, unless $\mathbf{n}$ is collinear with one of the axes, in which case the decomposition is trivial. The denominators,

[6] $\mathbf{c}_0^\perp$ may be considered as a projective plane (with $\pm\hat{\mathbf{c}}_k$ identified), or a sphere (with ∞ added).

on the other hand, are allowed to vanish in the space-like case, since ∞ is admissible value for the scalar parameter. However, it can be seen from the composition law (13) that the above solutions are regular in the sense (57) as long as $\mathcal{R}_h(\mathbf{c})$ is.

Finally, we consider the case of three axes (43), which can be described the vector-parameter composition $\mathbf{c} = \langle \mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1 \rangle$, where $\mathbf{c}, \mathbf{c}_k \in \mathbf{c}_0^{\perp}$. Denoting $\mathbf{c}^* = \langle \mathbf{c}_2, \mathbf{c}_1 \rangle$, we point out that by the result just proved, for any pair of fixed direction $\hat{\mathbf{c}}_3, \hat{\mathbf{c}}^*$ there are appropriate scalar parameters τ_3, τ^*, such that $\mathbf{c} = \langle \tau_3 \hat{\mathbf{c}}_3, \tau^* \hat{\mathbf{c}}^* \rangle$. On the other hand, by the same argument we have $\mathbf{c}^* = \tau^* \hat{\mathbf{c}}^* = \langle \tau_2 \hat{\mathbf{c}}_2, \tau_1 \hat{\mathbf{c}}_1 \rangle$ and since the direction of $\hat{\mathbf{c}}^*$ can be arbitrary (in $\mathbf{c}_0^{\perp}$), the solutions are infinitely many. ■

Apart from the above described method for obtaining the infinite set of solutions we may write explicit formulae for τ_k using a technique developed in [4, 5]. Namely, we express the decomposition (43) in a vector parameter form as $\mathbf{c}_1 = \langle -\mathbf{c}_2, -\mathbf{c}_1, \mathbf{c} \rangle$ or $\mathbf{c}_3 = \langle \mathbf{c}, -\mathbf{c}_1, -\mathbf{c}_2 \rangle$. Then, multiplying the first equality on the left by $\hat{\mathbf{c}}_1^{\curlywedge}$ and the second one - by $\hat{\mathbf{c}}_3^{\curlywedge}$, we eliminate one of the unknown scalar parameters and project on the null direction $\mathbf{c}_0$. Therefore, considering Euclidean dot product with $\mathbf{c}_0$ transforms the vector equations into scalar ones without loss of information and thus yields a pair of linear-fractional expressions for τ_1 and τ_3 in terms of the undetermined parameter τ_2 in the form

$$\tau_1 = \frac{(\sigma_{32} + (\upsilon_3 \hat{\mathbf{c}}_2^{\circ} - g_{23}\mathbf{n}^{\circ})\tau)\,\tau_2 - \rho_3 \tau}{(g_{13}\hat{\mathbf{c}}_2^{\circ} - g_{23}\hat{\mathbf{c}}_1^{\circ} + (\sigma_{13}\upsilon_2 - \sigma_{23}\upsilon_1 + g_{12}\rho_3)\tau)\,\tau_2 - (\upsilon_3 \hat{\mathbf{c}}_1^{\circ} - g_{13}\mathbf{n}^{\circ})\tau + \sigma_{13}}$$

$$\tau_3 = \frac{(\sigma_{12} - (\upsilon_1 \hat{\mathbf{c}}_2^{\circ} - g_{12}\mathbf{n}^{\circ})\tau)\,\tau_2 - \rho_1 \tau}{(g_{12}\hat{\mathbf{c}}_3^{\circ} - g_{13}\hat{\mathbf{c}}_2^{\circ} + (\sigma_{12}\upsilon_3 - \sigma_{13}\upsilon_2 + g_{23}\rho_1)\tau)\,\tau_2 + (\upsilon_1 \hat{\mathbf{c}}_3^{\circ} - g_{13}\mathbf{n}^{\circ})\tau + \sigma_{31}} \tag{61}$$

where we denote

$$\sigma_{ij} = (\hat{\mathbf{c}}_i \curlywedge \hat{\mathbf{c}}_j)^{\circ} = (\hat{\mathbf{c}}_i \curlywedge \hat{\mathbf{c}}_j, \mathbf{c}_0), \qquad \rho_k = (\hat{\mathbf{c}}_k \curlywedge \mathbf{n})^{\circ} = (\hat{\mathbf{c}}_3 \curlywedge \mathbf{n}, \mathbf{c}_0).$$

Clearly, similar (invertible) relations hold for each pair τ_i, τ_j (see [5] for details), so one may equivalently set τ_1 or τ_3 as a free parameter and express the other two.

We note one more peculiar property of null planes. Namely, for arbitrary normalized $\hat{\mathbf{c}}_i, \hat{\mathbf{c}}_j \in \mathbf{c}_0^{\perp}$ the scalar product $\hat{\mathbf{c}}_i \cdot \hat{\mathbf{c}}_j$ is equal to either 0 (if at least one of the vectors is null) or ± 1 (if they are both space-like[7]). By the hyperbolic polar change $(\hat{\mathbf{c}}_i \cdot \hat{\mathbf{c}}_j)^2 = \hat{\mathbf{c}}_i^2\, \hat{\mathbf{c}}_j^2 + (\hat{\mathbf{c}}_i \curlywedge \hat{\mathbf{c}}_j)^2 = 1$, since $\hat{\mathbf{c}}_i \curlywedge \hat{\mathbf{c}}_j \sim \mathbf{c}_0$. Moreover, from the property that $\mathbf{n} \curlywedge \mathbf{c}_0 = \lambda\, \mathbf{c}_0$ for some $\lambda \in \mathbb{R}$, which yields $\mathbf{n} \curlywedge (\mathbf{n} \curlywedge \mathbf{c}_0) = \lambda^2 \mathbf{c}_0 = \mathbf{n}^2 \mathbf{c}_0 = \mathbf{c}_0$, we see that for each unit space-like vector $\mathbf{n} \in \mathbf{c}_0^{\perp}$ we have $\mathbf{n} \curlywedge \mathbf{c}_0 = \pm \mathbf{c}_0$. Then, by induction $(\mathbf{n}^{\curlywedge})^n \mathbf{c}_0 = \pm \mathbf{c}_0$ and the sign is positive for $n = 2k$.

[7]there are no time-like directions in $\mathbf{c}_0^{\perp}$ - only space-like ones and $\mathbf{c}_0$ itself, since a null and a time-like vector cannot be normal to each other and neither can be two non-parallel isotropic ones.

4.4. Configurations of Axes

In order to obtain all configurations of axes in $\mathbb{R}^{2,1}$ that guarantee the decomposition (43) for arbitrary pseudo-rotation $\mathcal{R}(\mathbf{c})$, we may adapt the *Davenport condition* $\hat{\mathbf{c}}_2 \perp \hat{\mathbf{c}}_{1,3}$ ($g_{12} = g_{23} = 0$) from the Euclidean case and set $\epsilon_2 = 1$, while ϵ_1 and ϵ_3 can be -1 and 0, -1 and 1, or 0 and 1 (as long as $g_{13} \neq 0$). More generally

$$\epsilon_1 = \epsilon_2 = 1, \quad \epsilon_3 \leq 0, \quad g_{12} = 0 \quad \Rightarrow \quad \Delta = g_{23}^2 + r_{31}^2 - \epsilon_3 \geq 0$$

$$\epsilon_2 = \epsilon_3 = 1, \quad \epsilon_1 \leq 0, \quad g_{23} = 0 \quad \Rightarrow \quad \Delta = g_{12}^2 + r_{31}^2 - \epsilon_1 \geq 0.$$

The coefficients ϵ_k, on the other hand, allow for providing more configurations, reliable away from the gimbal lock setting, for example

- $\hat{\mathbf{c}}_2$ is space-like and normal to $\hat{\mathbf{c}}_1$ or $\hat{\mathbf{c}}_3$, which is null and $\{\hat{\mathbf{c}}_k\}$ is a basis

$$\epsilon_1 = g_{12} = 0 \quad \text{or} \quad \epsilon_3 = g_{23} = 0, \qquad \omega \neq 0 \quad \Rightarrow \quad \Delta = r_{31}^2 \geq 0.$$

- $\hat{\mathbf{c}}_2$ is time-like and normal to both $\hat{\mathbf{c}}_1$ and $\hat{\mathbf{c}}_3$, which are space-like

$$\epsilon_2 = -1, \quad \epsilon_1 = \epsilon_3 = 1, \quad g_{12} = g_{23} = 0 \quad \Rightarrow \quad \Delta = 1 - r_{31}^2 \geq 0.$$

- $\hat{\mathbf{c}}_2$ - time-like or null, with equal in absolute value (non-zero) projections on $\hat{\mathbf{c}}_1$ and $\hat{\mathbf{c}}_3$, which are both space-like

$$\begin{aligned} g_{12} &= \pm g_{23} \neq 0, \qquad \epsilon_2 \leq 0, \qquad \epsilon_1 = \epsilon_3 = 1 \\ \Rightarrow \quad \Delta &= \epsilon_2(r_{31}^2 - 1) + 2g_{12}^2(1 \mp r_{31}) \geq 0. \end{aligned}$$

At the end of this section we provide a Lorentz analogue of the Bryan decomposition about the axes ZYX (the first one is bound to be time-like, so that the condition $\Delta \geq 0$ is sufficient even in the gimbal lock setting). We have in this case

$$\Delta = 1 + \mathcal{R}_{13}^2, \qquad \omega_1 = -\mathcal{R}_{11}, \qquad \omega_2 = -1, \qquad \omega_3 = -\mathcal{R}_{33}$$

where $\mathcal{R}_{ij}$ are the matrix entries in the XYZ reference frame[8] and thus

$$\tilde{\tau}_1^{\pm} = -\frac{\mathcal{R}_{12}}{\mathcal{R}_{11} \pm \sqrt{1 + \mathcal{R}_{13}^2}}, \quad \tilde{\tau}_2^{\pm} = \frac{\mathcal{R}_{13}}{1 \pm \sqrt{1 + \mathcal{R}_{13}^2}}, \quad \tilde{\tau}_3^{\pm} = -\frac{\mathcal{R}_{23}}{\mathcal{R}_{33} \pm \sqrt{1 + \mathcal{R}_{13}^2}}.$$

Similarly, one may consider decomposition about ZYW, where the axis OW is determined by the null vector $\hat{\mathbf{c}}_3 = (1, 0, 1)^t = \hat{\mathbf{e}}_x + \hat{\mathbf{e}}_z$, so we have

$$r_{32} = \mathcal{R}_{12} - \mathcal{R}_{32}, \qquad r_{31} = \mathcal{R}_{13} - \mathcal{R}_{33}, \qquad r_{21} = \mathcal{R}_{23}, \qquad g_{31} = -1$$

[8] we write $\mathcal{R}_{ij}$ instead of $\mathcal{R}^i_j$ in order to avoid confusion with powers.

$$\Delta = (\mathcal{R}_{13} - \mathcal{R}_{33})^2, \qquad \omega_1 = \mathcal{R}_{31} - \mathcal{R}_{11}, \qquad \omega_2 = -1, \qquad \omega_3 = \mathcal{R}_{13} - \mathcal{R}_{33}$$

and thus the unique solution satisfying (57) is given in the form

$$\tau_1 = \frac{\mathcal{R}_{12} - \mathcal{R}_{32}}{\mathcal{R}_{31} + \mathcal{R}_{13} - \mathcal{R}_{11} - \mathcal{R}_{33}}, \qquad \tau_2 = \frac{1 + \mathcal{R}_{13} - \mathcal{R}_{33}}{1 - \mathcal{R}_{13} + \mathcal{R}_{33}}, \qquad \tau_3 = \frac{\mathcal{R}_{23}}{2(\mathcal{R}_{13} - \mathcal{R}_{33})}.$$

Note that when lifted back to the spin cover, the above corresponds to the well-known *Iwasawa decomposition* of $\mathsf{SL}(2,\mathbb{R}) = NAK$ (cf [1]). Working in the standard basis (8), with the aid of (12) and some basic trigonometry one obtains

$$\frac{1}{\sqrt{1-\mathbf{c}^2}}\begin{pmatrix} 1+c_2 & c_1+c_3 \\ c_1-c_3 & 1-c_2 \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \exp\frac{\beta}{2} & 0 \\ 0 & \exp -\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

where we exploit the notation $\theta = 2\arctan\tau_1$, $\beta = 2\,\mathrm{arcth}\,\tau_2$ and $\lambda = 2\tau_3$. Denoting the Bryan parameters with $\tilde{\phi} = 2\arctan\tilde{\tau}_1$ for the angle of rotation about OZ and respectively $\tilde{\vartheta} = 2\,\mathrm{arcth}\,\tilde{\tau}_2$, $\tilde{\psi} = 2\,\mathrm{arcth}\,\tilde{\tau}_3$ for the *rapidities* of the two boosts, we may use either (11) or the representations (14)-(17) for the factors in each decomposition of the compound transformation $\mathcal{R}_h(\theta,\beta,\lambda) = \mathcal{R}_h(\tilde{\phi},\tilde{\vartheta},\tilde{\psi})$. For the Bryan ZYX setting this yields

$$\mathcal{R}_h = \begin{pmatrix} \cos\tilde{\phi}\,\mathrm{ch}\,\tilde{\vartheta} & -\sin\tilde{\phi}\,\mathrm{ch}\,\tilde{\vartheta} & \mathrm{sh}\,\tilde{\vartheta} \\ \sin\tilde{\phi}\,\mathrm{ch}\,\tilde{\psi} - \cos\tilde{\phi}\,\mathrm{sh}\,\tilde{\vartheta}\,\mathrm{sh}\,\tilde{\psi} & \cos\tilde{\phi}\,\mathrm{ch}\,\tilde{\psi} + \sin\tilde{\phi}\,\mathrm{sh}\,\tilde{\vartheta}\,\mathrm{sh}\,\tilde{\psi} & -\mathrm{ch}\,\tilde{\vartheta}\,\mathrm{sh}\,\tilde{\psi} \\ \cos\tilde{\phi}\,\mathrm{sh}\,\tilde{\vartheta}\,\mathrm{sh}\,\tilde{\psi} - \sin\tilde{\phi}\,\mathrm{sh}\,\tilde{\psi} & -\cos\tilde{\phi}\,\mathrm{sh}\,\tilde{\psi} - \sin\tilde{\phi}\,\mathrm{sh}\,\tilde{\vartheta}\,\mathrm{ch}\,\tilde{\psi} & \mathrm{ch}\,\tilde{\vartheta}\,\mathrm{ch}\,\tilde{\psi} \end{pmatrix}$$

while in the Iwasawa case we have

$$\mathcal{R}_h(\theta,\beta,\lambda) = \begin{pmatrix} \mu\cos\theta - \lambda\sin\theta & -\mu\sin\theta - \lambda\cos\theta & \mathrm{e}^{\beta} - \mu \\ \lambda\mathrm{e}^{-\beta}\cos\theta + \sin\theta & \cos\theta - \lambda\mathrm{e}^{-\beta}\sin\theta & -\lambda\mathrm{e}^{-\beta} \\ (\mu - \mathrm{e}^{-\beta})\cos\theta - \lambda\sin\theta & (\mathrm{e}^{-\beta} - \mu)\sin\theta - \lambda\cos\theta & 2\,\mathrm{ch}\beta - \mu \end{pmatrix}$$

with the notation $\mu(\lambda,\beta) = \mathrm{ch}\,\beta - \dfrac{\lambda^2}{2}\,\mathrm{e}^{-\beta}$.

We may substitute the matrix entries of $\mathcal{R}_h(\tilde{\phi},\tilde{\vartheta},\tilde{\psi})$ in the expressions obtained above for the Iwasawa decomposition or, alternatively, use the entries of $\mathcal{R}_h(\theta,\beta,\lambda)$ in the formulae defining the Bryan scalar parameters just as in the Euclidean case. Thus, we derive the relation between the two parameterizations in the form

$$\theta = 2\arctan\frac{\sin\tilde{\phi}(\mathrm{ch}\,\tilde{\vartheta} - \mathrm{sh}\,\tilde{\vartheta}\,\mathrm{ch}\,\tilde{\psi}) - \cos\tilde{\phi}\,\mathrm{sh}\,\tilde{\psi}}{\cos\tilde{\phi}(\mathrm{ch}\,\tilde{\vartheta} - \mathrm{sh}\,\tilde{\vartheta}\,\mathrm{ch}\,\tilde{\psi}) + \sin\tilde{\phi}\,\mathrm{sh}\,\tilde{\psi} + \mathrm{ch}\,\tilde{\vartheta}\,\mathrm{ch}\,\tilde{\psi} - \mathrm{sh}\,\tilde{\vartheta}} \tag{62}$$

$$\beta = 2\,\mathrm{arcth}\frac{1 + \mathrm{sh}\,\tilde{\vartheta} - \mathrm{ch}\,\tilde{\vartheta}\,\mathrm{ch}\,\tilde{\psi}}{1 - \mathrm{sh}\,\tilde{\vartheta} + \mathrm{ch}\,\tilde{\vartheta}\,\mathrm{ch}\,\tilde{\psi}}, \qquad \lambda = \frac{\mathrm{sh}\,\tilde{\psi}}{\mathrm{ch}\,\tilde{\psi} - \mathrm{th}\,\tilde{\vartheta}}.$$

In the reverse direction it is

$$\tilde{\phi}^{\pm} = 2 \arctan \frac{2\lambda e^{\beta} \cos\theta + (e^{2\beta} + 1 - \lambda^2) \sin\theta}{(e^{2\beta} + 1 - \lambda^2) \cos\theta - 2\lambda e^{\beta} \sin\theta \mp \sqrt{D}} \tag{63}$$

$$\tilde{\vartheta}^{\pm} = 2 \operatorname{arcth} \frac{\lambda^2 + e^{2\beta} - 1}{2e^{\beta} \pm \sqrt{D}}, \qquad \tilde{\psi}^{\pm} = 2 \operatorname{arcth} \frac{2\,\lambda}{\lambda^2 + e^{2\beta} + 1 \pm \sqrt{D}}$$

where we make use of the notation $D = \lambda^4 + 2\lambda^2(e^{2\beta} - 1) + (e^{2\beta} + 1)^2$. Note also that in the Iwasawa decomposition we only have one regular solution, so the other one, which is easily seen to occur at $\operatorname{th} \tilde{\vartheta} = \operatorname{ch} \tilde{\psi}$, will be divergent.

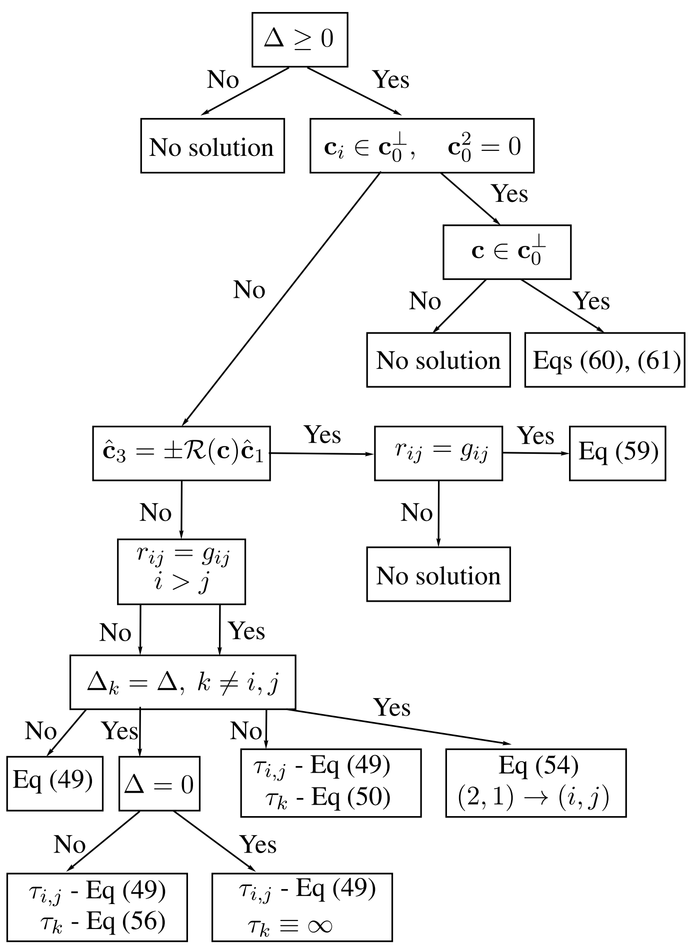

Figure 2. Decomposition flowchart in the hyperbolic case.

Moreover, the above representations provide a convenient tool to study the way certain parameters in one decomposition affect those in the other. For example, it is straightforward to see that in the case $\lambda = 0$ both decompositions coincide, i.e., $\tilde{\psi} = 0$, $\tilde{\vartheta} = \beta$ and $\tilde{\phi} = \theta$. Similarly, we have

$$\tilde{\phi}{=}0 \;\to\; \theta = -2\tan^{-1}\left(\mathrm{e}^{\tilde{\vartheta}}\,\mathrm{th}\,\frac{\tilde{\psi}}{2}\right), \qquad \tilde{\phi}{=}\pi \;\to\; \theta = 2\tan^{-1}\left(\mathrm{e}^{-\tilde{\vartheta}}\coth\frac{\tilde{\psi}}{2}\right)$$

and finally, for $\tilde{\vartheta} = 0$ the correspondence is

$$\theta = 2\tan^{-1}\frac{\sin\tilde{\phi} - \cos\tilde{\phi}\,\mathrm{sh}\,\tilde{\psi}}{\cos\tilde{\phi} + \sin\tilde{\phi}\,\mathrm{sh}\,\tilde{\psi} + \mathrm{ch}\,\tilde{\psi}}, \qquad \beta = -2\,\mathrm{th}^{-1}\left(\mathrm{th}^2\frac{\tilde{\psi}}{2}\right), \qquad \lambda = \mathrm{th}\,\tilde{\psi}\,.$$

The algorithm for a generic pseudo-rotation is illustrated in Fig. 2 on previous page.

5. Transition to Moving Frames

So far we obtained all possible decompositions of three-dimensional rotations and pseudo-rotations with respect to fixed axes. For the applications, however, it is often preferable to consider axes, attached to the moving object. Both in classical mechanics and relativity it is more natural to work in the "frame at rest" of a moving particle or a rotating rigid body. As shown in [5], the decompositions in the static $\{\mathbf{c}_k\}$ and the dynamic $\{\mathbf{c}'_k\}$ systems of axes are related via the formula[9]

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c}'_3)\mathcal{R}(\mathbf{c}'_2)\mathcal{R}(\mathbf{c}'_1) = \mathcal{R}(\mathbf{c}_1)\mathcal{R}(\mathbf{c}_2)\mathcal{R}(\mathbf{c}_3) \tag{64}$$

and in the case of two axes $\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c}'_2)\mathcal{R}(\mathbf{c}'_1) = \mathcal{R}(\mathbf{c}_1)\mathcal{R}(\mathbf{c}_2)$, respectively. Note that the norm-preserving property of the operators involved yields $\tau'_k = \tau_k$. Moreover, one may think of the system $\{\mathbf{c}_k\}$ as representing $\{\mathbf{c}'_k\}$ before the transformation has begun. Then, as it undergoes a series of rotations, we have

$$\hat{\mathbf{c}}'_1 = \hat{\mathbf{c}}_1, \qquad \hat{\mathbf{c}}'_2 = \mathcal{R}(\mathbf{c}'_1)\,\hat{\mathbf{c}}_2, \qquad \hat{\mathbf{c}}'_3 = \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_3 \tag{65}$$

which yields, with the notation $g'_{ij} = (\hat{\mathbf{c}}'_i, \hat{\mathbf{c}}'_j)$ and $r'_{ij} = (\hat{\mathbf{c}}'_i, \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}'_j)$, the relations

$$g'_{12} = g_{12}, \qquad g'_{23} = g_{23}, \qquad g'_{13} = r_{13}, \qquad r'_{31} = g_{31}.$$

As a straightforward consequence we derive the relations between the volume element in one of the systems and the discriminant in the other $D = \omega^2$ and $\Delta = {\omega'}^2$,

[9] although the notation is borrowed from the Euclidean case, all results here apply also to $\mathbb{R}^{2,1}$.

which guarantees, at least in the Euclidean case, that the decomposition is justified in $\{\mathbf{c}'_k\}$ as long as it is in $\{\mathbf{c}_k\}$ and vice versa. In the hyperbolic one we easily see that the gimbal lock condition in one of the systems yields zero volume element in the other and the light cone singularity is either present in both systems, or in none. Next, we multiply (64) with $\mathcal{R}(-\mathbf{c}'_3)$ on the left, which leads to

$$\langle -\mathbf{c}'_3, \mathbf{c}\,\rangle = \langle \mathbf{c}'_2, \mathbf{c}'_1\,\rangle \quad \Rightarrow \quad \langle -\mathcal{R}(\mathbf{c})\,\mathbf{c}_3, \mathbf{c}\,\rangle = \langle \mathcal{R}(\mathbf{c}_1)\,\mathbf{c}_2, \mathbf{c}_1\,\rangle$$

in accordance with (65). Then, applying formula (39), that may also be written as $\langle\,\mathbf{c},\,\tilde{\mathbf{c}}\,\rangle = \langle \mathcal{R}(\mathbf{c})\,\tilde{\mathbf{c}},\,\mathbf{c}\,\rangle = \langle\,\tilde{\mathbf{c}}, \mathcal{R}(-\tilde{\mathbf{c}})\,\mathbf{c}\,\rangle$, we obtain the desired result

$$\langle \mathbf{c}, -\mathbf{c}_3\rangle = \langle \mathbf{c}_1, \mathbf{c}_2\rangle \quad \Rightarrow \quad \mathbf{c} = \langle \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\rangle.$$

Similarly, in the case of two axes we have $\mathbf{c} = \langle \mathbf{c}'_2, \mathbf{c}'_1\rangle = \langle \mathbf{c}'_1, \mathcal{R}(-\mathbf{c}'_1)\,\mathbf{c}'_2\rangle = \langle \mathbf{c}_1, \mathbf{c}_2\rangle$. The above result can be generalized to an arbitrary number of factors by induction. In fact, it applies to arbitrary groups and parameterizations. The proof is a combination of the braiding property of groups (easily obtained by induction)

$$\mathcal{R}_1\mathcal{R}_2\ldots\mathcal{R}_n = \mathcal{R}'_n\mathcal{R}'_{n-1}\ldots\mathcal{R}'_1, \qquad \mathcal{R}'_k = \mathcal{R}_1\mathcal{R}_2\ldots\mathcal{R}_{k-1}\mathcal{R}_k\mathcal{R}^{-1}_{k-1}\ldots\mathcal{R}^{-1}_2\mathcal{R}^{-1}_1$$

and conjugation $f[g(x)] = g \circ f(x) \circ g^{-1}$, applied to the parameters $\mathbf{c}_k$. Thus, if the transformations $\mathcal{R}_k$ constitute a representation of some group G, we have

$$\mathcal{R}(\mathbf{c}_1)\mathcal{R}(\mathbf{c}_2)\ldots\mathcal{R}(\mathbf{c}_n) = \mathcal{R}(\mathbf{c}'_n)\mathcal{R}(\mathbf{c}'_{n-1})\ldots\mathcal{R}(\mathbf{c}'_1), \qquad \mathbf{c}'_k = \mathcal{R}_1\mathcal{R}_2\ldots\mathcal{R}_{k-1}\,\mathbf{c}_k.$$

6. Quaternion and Split Quaternion Decompositions

As explained in the beginning, the vector-parameter technique is based on projecting a quaternion (or split quaternion) construction from the spin covering group and this makes it, apart from its other merits, very appropriate for emphasizing the relation between the corresponding projective group and its spin cover. In particular, lifting up our solutions is almost straightforward using the correspondence (6) in the Euclidean and (12) in the hyperbolic case. Let us now consider a decomposition problem that is initially formulated for the spin cover, starting with the compact case. Namely, we are given a generic unit quaternion $\zeta \in \mathsf{SU}(2)$ and three purely imaginary ones $\hat{\xi}_k \in \mathfrak{su}(2) \cap \mathbb{S}^3 \cong \mathbb{S}^2$ to determine the axes of rotation. In the spin representation we rely on the Killing form induced metric

$$(\zeta, \xi) = \frac{1}{2}\,\mathrm{tr}\,(\zeta\bar{\xi}), \qquad \zeta, \xi \in \mathsf{SU}(2) \tag{66}$$

and the formula (6). Used together with (5), these two yield

$$r_{ij} - g_{ij} = 2\left[(\zeta, \hat{\xi}_i)(\zeta, \hat{\xi}_j) + (\zeta_0^2 - 1)(\hat{\xi}_i, \hat{\xi}_j) - \zeta_0(\zeta, \Im(\hat{\xi}_i\,\hat{\xi}_j))\right] \tag{67}$$

where $\zeta_0 = \Re(\zeta) = \dfrac{1}{2}\operatorname{tr}\zeta$ and we may denote $\kappa_k = \varepsilon_{ijk}(g_{ij} - r_{ij})$ for $i > j$. The correspondence between conjugations in $\mathfrak{su}(2)$ and rotations in $\mathbb{R}^3$ then yields

$$\omega_1 = (\Im(\hat{\xi}_1\,\hat{\xi}_2), \bar{\zeta}\,\hat{\xi}_3\,\zeta), \qquad \omega_2 = (\Im(\hat{\xi}_1\,\hat{\xi}_2), \hat{\xi}_3), \qquad \omega_3 = (\zeta\,\hat{\xi}_1\,\bar{\zeta}, \Im(\hat{\xi}_2\,\hat{\xi}_3))$$

and finally, the discriminant of the second equation in (19) can be written as

$$\Delta = \omega_2^2 - \kappa_2(2\mathcal{G}^{31} + \kappa_2), \qquad \mathcal{G}^{31} = g_{12}g_{23} - g_{31} = (\Im(\hat{\xi}_1\,\hat{\xi}_2), \Im(\hat{\xi}_2\,\hat{\xi}_3)).$$

Then, the condition $\Delta \geq 0$ is still relevant and the solutions (26) are given as

$$\tau_i^{\pm} = \frac{\kappa_i}{\omega_i \pm \sqrt{\Delta}}. \tag{68}$$

In the case of two axes, as long as $g_{21} = (\hat{\xi}_2, \hat{\xi}_1) = (\hat{\xi}_2, \zeta\,\hat{\xi}_1\,\bar{\zeta}) = r_{21}$ is satisfied, formula (33) provides the unique solution with

$$\frac{r_{kk} - 1}{2} = \zeta_0^2 + (\zeta, \hat{\xi}_k)^2 - 1, \quad \mathring{\omega}_1 = (\Im(\hat{\xi}_1\,\hat{\xi}_2), \bar{\zeta}\,\hat{\xi}_2\,\zeta), \quad \mathring{\omega}_2 = (\zeta\,\hat{\xi}_1\,\bar{\zeta}, \Im(\hat{\xi}_1\,\hat{\xi}_2))$$

that we also use in (40) for the gimbal lock setting $\hat{\xi}_3 = \pm\zeta\,\hat{\xi}_1\,\bar{\zeta}$.
Once we have determined the scalar parameters, the quaternions ξ_k in the decomposition $\zeta = \xi_3\,\xi_2\,\xi_1$ are obtained with the aid of formula (6) as[10]

$$\xi_k = \pm\frac{1}{\sqrt{1 + \tau_k^2}}\left(\sigma_0 + \tau_k\hat{\xi}_k\right), \qquad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{69}$$

In the hyperbolic case we consider a generic unit split quaternion $\zeta \in \mathsf{SL}(2, \mathbb{R})$ and three purely imaginary ones[11] $\hat{\xi}_k \in \mathfrak{su}(2, \mathbb{R})$, then define the scalar product as

$$\zeta \cdot \xi = -\frac{1}{2}\operatorname{tr}(\zeta\bar{\xi}), \qquad \zeta, \xi \in \mathsf{SL}(2, \mathbb{R}). \tag{70}$$

From the relation (12) and formula (11) we easily obtain

$$r_{ij} - g_{ij} = -2\left[(\zeta \cdot \hat{\xi}_i)(\zeta \cdot \hat{\xi}_j) + (1 - \zeta_0^2)\,\hat{\xi}_i \cdot \hat{\xi}_j + \zeta_0\,\zeta \cdot \Im(\hat{\xi}_i\,\hat{\xi}_j)\right]. \tag{71}$$

Moreover, the correspondence between pseudo-rotations in $\mathbb{R}^{2,1}$ and conjugations in $\mathfrak{su}(2, \mathbb{R})$,i.e., the adjoint action of $\mathsf{Spin}(2, 1) \cong \mathsf{SL}(2, \mathbb{R})$, yields

$$\omega_1 = \Im(\hat{\xi}_1\,\hat{\xi}_2) \cdot \bar{\zeta}\,\hat{\xi}_3\,\zeta, \qquad \omega_2 = \Im(\hat{\xi}_1\,\hat{\xi}_2) \cdot \hat{\xi}_3, \qquad \omega_3 = \zeta\,\hat{\xi}_1\,\bar{\zeta} \cdot \Im(\hat{\xi}_2\,\hat{\xi}_3)$$

[10] each separate ξ_k may be taken with either "+" or "−", but the three signs must agree, e.g., if we choose "+" for ξ_1 and ξ_2, the third one is determined by $\xi_3 = \zeta\,\bar{\xi}_1\,\bar{\xi}_2$.

[11] we assume that the $\mathbb{R}^{2,1}$ images of $\hat{\xi}_k$ are quasi-unit in the sense specified above.

and for the discriminant of the second equation in (45) we have

$$\Delta = \omega_2^2 - \kappa_2(2\,\mathcal{G}^{31} - \epsilon_2\kappa_2), \qquad \mathcal{G}^{31} = \Im(\hat{\xi}_1\,\hat{\xi}_2)\cdot\Im(\hat{\xi}_2\,\hat{\xi}_3).$$

With the notation $\kappa_k = \varepsilon_{ijk}(g_{ij} - r_{ij})$ for $i > j$ we write the solutions (49) as

$$\tau_i^{\pm} = \frac{\kappa_i}{\omega_i \mp \sqrt{\Delta}} \tag{72}$$

and the factors ξ_k in the decomposition $\zeta = \xi_3\,\xi_2\,\xi_1$ are given by (12) in the form

$$\xi_k^{\pm} = \pm\frac{1}{\sqrt{1-\epsilon_k\tau_k^2}}\left(\sigma_0 + \tau_k\hat{\xi}_k\right) \tag{73}$$

with $\epsilon_k = -\det\hat{\zeta}_k$ in accordance with our previous definitions. Just as in the Euclidean case, we may choose arbitrary signs for two of the three spinors, while the third one is fixed, e.g., $\xi_3 = \zeta\,\bar{\xi}_1\,\bar{\xi}_2$. For this reason, one has four times more solutions in the spin cover compared to the corresponding projective group.

In the case of two axes the scalar parameters are given by formula (54) with

$$r_{kk} - \epsilon_k = 2(\zeta_0^2 - 1)\epsilon_k - 2(\zeta\cdot\hat{\xi}_k)^2, \quad \mathring{\omega}_1 = \Im(\hat{\xi}_1\,\hat{\xi}_2)\cdot\bar{\zeta}\,\hat{\xi}_2\,\zeta, \quad \mathring{\omega}_3 = \zeta\,\hat{\xi}_1\,\bar{\zeta}\cdot\Im(\hat{\xi}_1\,\hat{\xi}_2)$$

as long as the condition $\hat{\xi}_2\cdot\hat{\xi}_1 = \hat{\xi}_2\cdot\zeta\,\hat{\xi}_1\,\bar{\zeta}$ is satisfied. The latter is also relevant in the gimbal lock setting $\hat{\xi}_3 = \pm\zeta\,\hat{\xi}_1\,\bar{\zeta}$, in which the solutions are provided by (59). The singular solutions (61) may also be obtained in this way by introducing Wick rotation. Namely, if we expand the null direction as $\mathbf{c}_0 \to c_{01}\,\tilde{\mathbf{i}} + c_{02}\,\tilde{\mathbf{j}} - c_{03}\,\tilde{\mathbf{k}}$, calculating the Euclidean scalar products σ_{ij}, ρ_k, $\hat{\mathbf{c}}_k^{\circ}$ and $\mathbf{n}^{\circ}$ is straightforward.

Back to the Classics. We consider once more the classical Euler decomposition, this time for the spin covering group. Applying formula (6) in the basis (1) yields

$$\frac{1}{\sqrt{1+\mathbf{c}^2}}\begin{pmatrix} 1+\mathrm{i}c_1 & c_2+\mathrm{i}c_3 \\ \mathrm{i}c_3-c_2 & 1-\mathrm{i}c_1\end{pmatrix} = \begin{pmatrix} \cos\frac{\psi}{2} & \mathrm{i}\sin\frac{\psi}{2} \\ \mathrm{i}\sin\frac{\psi}{2} & \cos\frac{\psi}{2}\end{pmatrix}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{\vartheta}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{\vartheta}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\phi}{2} & \mathrm{i}\sin\frac{\phi}{2} \\ \mathrm{i}\sin\frac{\phi}{2} & \cos\frac{\phi}{2}\end{pmatrix}$$

for the Euler case and for the Bryan one we have similarly

$$\frac{1}{\sqrt{1+\mathbf{c}^2}}\begin{pmatrix} 1+\mathrm{i}c_1 & c_2+\mathrm{i}c_3 \\ \mathrm{i}c_3-c_2 & 1-\mathrm{i}c_1\end{pmatrix} = \begin{pmatrix} \cos\frac{\tilde{\psi}}{2} & \mathrm{i}\sin\frac{\tilde{\psi}}{2} \\ \mathrm{i}\sin\frac{\tilde{\psi}}{2} & \cos\frac{\tilde{\psi}}{2}\end{pmatrix}\begin{pmatrix} \cos\frac{\tilde{\vartheta}}{2} & \sin\frac{\tilde{\vartheta}}{2} \\ -\sin\frac{\tilde{\vartheta}}{2} & \cos\frac{\tilde{\vartheta}}{2}\end{pmatrix}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{\tilde{\phi}}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{\tilde{\phi}}{2}}\end{pmatrix}.$$

Using the standard notation for the components of ζ and formula (71) we obtain

$$\kappa_1 = 2\left(\zeta_1\zeta_3 - \zeta_0\zeta_2\right), \qquad \kappa_2 = 2\left(1 - \zeta_3^2 - \zeta_0^2\right), \qquad \kappa_3 = 2\left(\zeta_1\zeta_3 + \zeta_0\zeta_2\right)$$

as well as $\Delta = \kappa_2(2-\kappa_2) = 4(1-\zeta_3^2-\zeta_0^2)(\zeta_3^2+\zeta_0^2)$ and finally

$$\omega_1 = 2\left(\zeta_2\zeta_3+\zeta_0\zeta_1\right), \qquad \omega_2 = 0, \qquad \omega_3 = 2\left(\zeta_0\zeta_1-\zeta_2\zeta_3\right).$$

The solution is then retrieved by substituting the above expressions in (68), namely

$$\tau_1^\pm = \frac{\zeta_1\zeta_3-\zeta_0\zeta_2}{\zeta_2\zeta_3+\zeta_0\zeta_1\pm\sqrt{D_s}}, \quad \tau_2^\pm = \pm\sqrt{\frac{\zeta_1^2+\zeta_2^2}{\zeta_3^2+\zeta_0^2}}, \quad \tau_3^\pm = \frac{\zeta_1\zeta_3+\zeta_0\zeta_2}{\zeta_0\zeta_1-\zeta_2\zeta_3\pm\sqrt{D_s}}$$

with $D_s = (1-\zeta_3^2-\zeta_0^2)(\zeta_3^2+\zeta_0^2)$ and in terms of the vector-parameter components

$$\tau_1^\pm = \frac{c_1c_3-c_2}{c_2c_3+c_1\pm\sqrt{D_v}}, \qquad \tau_2^\pm = \pm\sqrt{\frac{c_1^2+c_2^2}{c_3^2+1}}, \qquad \tau_3^\pm = \frac{c_1c_3+c_2}{c_1-c_2c_3\pm\sqrt{D_v}}$$

where $D_v = (c_1^2+c_2^2)(c_3^2+1)$. Similarly, in the Bryan XYZ setting we have

$$\kappa_1 = 2\left(\zeta_2\zeta_3+\zeta_0\zeta_1\right), \qquad \kappa_2 = 2\left(\zeta_0\zeta_2-\zeta_1\zeta_3\right), \qquad \kappa_3 = 2\left(\zeta_1\zeta_2+\zeta_0\zeta_3\right)$$

the discriminant is given by $\Delta = 1-\kappa_2^2$ and ω_k can be written as

$$\omega_1 = 2\left(\zeta_3^2+\zeta_0^2\right)-1, \qquad \omega_2 = 1, \qquad \omega_3 = 2\left(\zeta_1^2+\zeta_0^2\right)-1.$$

Hence, our method yields for the scalar parameters

$$\tau_1^\pm = \frac{2\left(\zeta_2\zeta_3+\zeta_0\zeta_1\right)}{2\left(\zeta_3^2+\zeta_0^2\right)-1\pm\sqrt{D_s}}, \quad \tau_2^\pm = \frac{2\left(\zeta_0\zeta_2-\zeta_1\zeta_3\right)}{1\pm\sqrt{D_s}}, \quad \tau_3^\pm = \frac{2\left(\zeta_1\zeta_2+\zeta_0\zeta_3\right)}{2\left(\zeta_1^2+\zeta_0^2\right)-1\pm\sqrt{D_s}}$$

where $D_s = 1-4\left(\zeta_1\zeta_3-\zeta_0\zeta_2\right)^2$, or in terms of $\mathbf{c}$ and $D_v = (1+\mathbf{c}^2)^2-4\left(c_1c_3-c_2\right)^2$

$$\tau_1^\pm = \frac{2\left(c_2c_3+c_1\right)}{1-c_1^2-c_2^2+c_3^2\pm\sqrt{D_v}}, \quad \tau_2^\pm = \frac{2\left(c_2-c_1c_3\right)}{1+\mathbf{c}^2\pm\sqrt{D_v}}, \quad \tau_3^\pm = \frac{2\left(c_1c_2+c_3\right)}{1+c_1^2-c_2^2-c_3^2\pm\sqrt{D_v}}.$$

Alternatively, we may work directly with the matrix entries $\mathcal{R}_{ij}$ expressed in terms of the corresponding quaternion parameters by means of formula (3), namely as

$$\mathcal{R}(\zeta) = \begin{pmatrix} 1-2(\zeta_2^2+\zeta_3^2) & 2(\zeta_1\,\zeta_2-\zeta_0\,\zeta_3) & 2(\zeta_1\,\zeta_3+\zeta_0\,\zeta_2) \\ 2(\zeta_1\,\zeta_2+\zeta_0\,\zeta_3) & 1-2(\zeta_1^2+\zeta_3^2) & 2(\zeta_2\,\zeta_3-\zeta_0\,\zeta_1) \\ 2(\zeta_1\,\zeta_3-\zeta_0\,\zeta_2) & 2(\zeta_2\,\zeta_3+\zeta_0\,\zeta_1) & 1-2(\zeta_1^2+\zeta_2^2) \end{pmatrix}.$$

Next, we derive the quaternion representations via scalar parameters for the Euler and Bryan decompositions, denoting the former with τ_k and the latter - with $\tilde{\tau}_k$. The vector-parameter composition law (7) yields in the Euler case

$$\mathbf{c} = \frac{1}{1-\tau_1\tau_3}\left(\tau_2(1+\tau_1\tau_3),\ \tau_2(\tau_3-\tau_1),\ \tau_3+\tau_1\right)^t$$

which gives, according to (6) and with the notation $E(\tau_1,\tau_2,\tau_3)=\prod_{k=1}^{3}\sqrt{1+\tau_k^2}$

$$\zeta=\frac{1-\tau_1\tau_3}{E(\tau_1,\tau_2,\tau_3)}+\frac{\tau_2(1+\tau_1\tau_3)}{E(\tau_1,\tau_2,\tau_3)}\,\mathbf{i}+\frac{\tau_2(\tau_3-\tau_1)}{E(\tau_1,\tau_2,\tau_3)}\,\mathbf{j}+\frac{\tau_3+\tau_1}{E(\tau_1,\tau_2,\tau_3)}\,\mathbf{k}.$$

Similarly, in the Bryan setting we have

$$\mathbf{c}=\frac{1}{1+\tilde\tau_1\tilde\tau_2\tilde\tau_3}\left(\tilde\tau_1-\tilde\tau_2\tilde\tau_3,\ \tilde\tau_2+\tilde\tau_1\tilde\tau_3,\ \tilde\tau_3-\tilde\tau_1\tilde\tau_3\right)^t$$

and thus the quaternion representation takes the form

$$\zeta=\frac{1+\tilde\tau_1\tilde\tau_2\tilde\tau_3}{E(\tilde\tau_1,\tilde\tau_2,\tilde\tau_3)}+\frac{\tilde\tau_1-\tilde\tau_2\tilde\tau_3}{E(\tilde\tau_1,\tilde\tau_2,\tilde\tau_3)}\,\mathbf{i}+\frac{\tilde\tau_2+\tilde\tau_1\tilde\tau_3}{E(\tilde\tau_1,\tilde\tau_2,\tilde\tau_3)}\,\mathbf{j}+\frac{\tilde\tau_3-\tilde\tau_1\tilde\tau_3}{E(\tilde\tau_1,\tilde\tau_2,\tilde\tau_3)}\,\mathbf{k}.$$

Using the Euler representation of the quaternion ζ and the solution (68) for the Bryan case, we may express the parameters $\tilde\tau_k$ as functions of the τ_k's in the form

$$\begin{aligned}\tilde\tau_1^{\pm}&=\frac{2\tau_2(1-\tau_1^2)}{(1-\tau_2^2)(1+\tau_1^2)\pm\sqrt{D}},\qquad \tilde\tau_2^{\pm}=\frac{-4\tau_1\tau_2}{(1+\tau_1^2)(1+\tau_2^2)\pm\sqrt{D}}\\ \tilde\tau_3^{\pm}&=\frac{2\tau_1(1-\tau_2^2)(1-\tau_3^2)+2\tau_3(1-\tau_1^2)(1+\tau_2^2)}{(1-\tau_1^2)(1+\tau_2^2)(1-\tau_3^2)-4\tau_1\tau_3(1-\tau_2^2)\pm(1+\tau_3^2)\sqrt{D}}\end{aligned}\tag{74}$$

with $D=(1+\tau_1^2)^2(1+\tau_2^2)^2-(4\tau_1\tau_2)^2$ and in the inverse direction[12]

$$\begin{aligned}\tau_1^{\pm}&=-\frac{\tilde\tau_2(1+\tilde\tau_1^2)}{\tilde\tau_1(1-\tilde\tau_2^2)\pm\sqrt{(\tilde\tau_1^2+\tilde\tau_2^2)(1+\tilde\tau_1^2\tilde\tau_2^2)}},\qquad \tau_2^{\pm}=\pm\sqrt{\frac{\tilde\tau_1^2+\tilde\tau_2^2}{1+\tilde\tau_1^2\tilde\tau_2^2}}\\ \tau_3^{\pm}&=\frac{\tilde\tau_2(1-\tilde\tau_1^2)(1-\tilde\tau_3^2)+2\tilde\tau_1\tilde\tau_3(1+\tilde\tau_2^2)}{\tilde\tau_1(1+\tilde\tau_2^2)(1-\tilde\tau_3^2)-2\tilde\tau_2\tilde\tau_3(1-\tilde\tau_1^2)\pm(1+\tilde\tau_3^2)\sqrt{(\tilde\tau_1^2+\tilde\tau_2^2)(1+\tilde\tau_1^2\tilde\tau_2^2)}}.\end{aligned}\tag{75}$$

Next, we obtain the relations between the solutions for the two examples considered above in the hyperbolic case using similar technique, i.e., express the parameters $\tilde\tau_k$ in the ZYX Bryan setting in terms of those for the Iwasawa decomposition denoted by τ_k and vice versa. Applying the method to the former case we obtain

$$\kappa_1=-2\left(\zeta_1\zeta_2+\zeta_0\zeta_3\right),\qquad \kappa_2=-2\left(\zeta_0\zeta_2+\zeta_1\zeta_3\right),\qquad \kappa_3=2\left(\zeta_2\zeta_3-\zeta_0\zeta_1\right)$$

as well as $\omega_1=2\left(\zeta_1^2-\zeta_0^2\right)+1$, $\omega_2=-1$, $\omega_3=1-2\left(\zeta_3^2+\zeta_0^2\right)$ and $\Delta=1+\kappa_2^2$, where we use the standard notation $\zeta_1=(\tilde{\mathbf{i}},\,\zeta)$, $\zeta_2=(\tilde{\mathbf{j}},\,\zeta)$ and $\zeta_3=-(\tilde{\mathbf{k}},\,\zeta)$.

[12] these results may also be derived from (42) and (43) using the Euler trigonometric substitution.

With this result it is straightforward to derive expressions for the scalar parameters in the decomposition in terms of the corresponding split quaternion components

$$\tilde{\tau}_1^{\pm}=\frac{2\,(\zeta_1\zeta_2+\zeta_0\zeta_3)}{2\big(\zeta_0^2-\zeta_1^2\big)-1\pm\sqrt{D_s}},\quad \tilde{\tau}_2^{\pm}=\frac{2\,(\zeta_0\zeta_2+\zeta_1\zeta_3)}{1\pm\sqrt{D_s}},\quad \tilde{\tau}_3^{\pm}=\frac{2\,(\zeta_0\zeta_1-\zeta_2\zeta_3)}{2\big(\zeta_0^2+\zeta_3^2\big)-1\pm\sqrt{D_s}}$$

where $D_s=1+4\,(\zeta_0\zeta_2+\zeta_1\zeta_3)^2$ and the above gives, by projection $\zeta\to\mathbf{c}$

$$\tilde{\tau}_1^{\pm}=\frac{2\,(c_1c_2+c_3)}{1-c_1^2+c_2^2-c_3^2\pm\sqrt{D_v}},\quad \tilde{\tau}_2^{\pm}=\frac{2\,(c_2+c_1c_3)}{1-\mathbf{c}^2\pm\sqrt{D_v}},\quad \tilde{\tau}_3^{\pm}=\frac{2\,(c_1-c_2c_3)}{1+c_1^2+c_2^2+c_3^2\pm\sqrt{D_v}}$$

with the notation $D_v=(1-\mathbf{c}^2)^2+4\,(c_2+c_1c_3)^2$.

Similarly, for the Iwasawa decomposition considered above, our method gives

$$\kappa_1=2(\zeta_1-\zeta_3)(\zeta_0-\zeta_2),\quad \kappa_2=2(\zeta_1^2+\zeta_2^2-\zeta_1\zeta_3-\zeta_0\zeta_2),\quad \kappa_3=2(\zeta_2\zeta_3-\zeta_0\zeta_1)$$

along with $\Delta=(1+\kappa_2)^2$, $\omega_1=2(\zeta_0\zeta_2-\zeta_1\zeta_3-\zeta_2^2+\zeta_3^2)-1$, $\omega_2=-1$ and $\omega_3=-\kappa_2-1$. Hence, the scalar parameters in this case can be written also as

$$\tau_1=\frac{\zeta_1-\zeta_3}{\zeta_2-\zeta_0},\qquad \tau_2=\frac{\zeta_1\zeta_3+\zeta_0\zeta_2-\zeta_1^2-\zeta_2^2}{\zeta_3^2+\zeta_0^2-\zeta_1\zeta_3-\zeta_0\zeta_2},\qquad \tau_3=\frac{\zeta_2\zeta_3-\zeta_0\zeta_1}{2(\zeta_1\zeta_3+\zeta_0\zeta_2-\zeta_3^2-\zeta_0^2)+1}$$

and in vector-parameter notation they take the form

$$\tau_1=\frac{c_1-c_3}{c_2-1},\qquad \tau_2=\frac{c_1c_3+c_2-c_1^2-c_2^2}{c_3^2-c_1c_3-c_2+1},\qquad \tau_3=\frac{c_2c_3-c_1}{2(c_1c_3+c_2)-c_1^2-c_2^2-c_3^2-1}.$$

These same results may also be obtained from the matrix entries $\mathcal{R}_{ij}$ expressed in terms of the corresponding split quaternion parameters by (10), namely as

$$\mathcal{R}_h(\zeta)=\begin{pmatrix}1+2(\zeta_2^2-\zeta_3^2) & -2(\zeta_1\,\zeta_2+\zeta_0\,\zeta_3) & 2(\zeta_1\,\zeta_3+\zeta_0\,\zeta_2)\\ 2(\zeta_0\,\zeta_3-\zeta_1\,\zeta_2) & 1+2(\zeta_1^2-\zeta_3^2) & 2(\zeta_2\,\zeta_3-\zeta_0\,\zeta_1)\\ 2(\zeta_0\,\zeta_2-\zeta_1\,\zeta_3) & -2(\zeta_2\,\zeta_3+\zeta_0\,\zeta_1) & 1+2(\zeta_1^2+\zeta_2^2)\end{pmatrix}.$$

Both here and in the Euclidean case this alternative approach (which is actually easier) may serve as a test for the consistency of our methods.

Denoting τ_k the scalar parameters in the Iwasawa decomposition and $\tilde{\tau}_k$ - the ones in the Bryan case and using the composition law (13) we obtain for the latter

$$\mathbf{c}=\frac{1}{1+\tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_3}\left(\tilde{\tau}_3+\tilde{\tau}_1\tilde{\tau}_2,\ \tilde{\tau}_2-\tilde{\tau}_1\tilde{\tau}_3,\ \tilde{\tau}_1-\tilde{\tau}_2\tilde{\tau}_3\right)^t$$

which can be lifted up to the spin cover as

$$\zeta=\frac{1+\tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_3}{H_1(\tilde{\tau}_1,\tilde{\tau}_2,\tilde{\tau}_3)}+\frac{\tilde{\tau}_3+\tilde{\tau}_1\tilde{\tau}_2}{H_1(\tilde{\tau}_1,\tilde{\tau}_2,\tilde{\tau}_3)}\,\tilde{\mathbf{i}}+\frac{\tilde{\tau}_2-\tilde{\tau}_1\tilde{\tau}_3}{H_1(\tilde{\tau}_1,\tilde{\tau}_2,\tilde{\tau}_3)}\,\tilde{\mathbf{j}}+\frac{\tilde{\tau}_1-\tilde{\tau}_2\tilde{\tau}_3}{H_1(\tilde{\tau}_1,\tilde{\tau}_2,\tilde{\tau}_3)}\,\tilde{\mathbf{k}}.$$

with the notation $H_1(\tilde\tau_1,\tilde\tau_2,\tilde\tau_3)=\sqrt{(1+\tilde\tau_1^2)(1-\tilde\tau_2^2)(1-\tilde\tau_3^2)}$.
Similarly, in the Iwasawa setting we have the representation

$$\mathbf{c}=\frac{1}{1-\tau_1\tau_3(1-\tau_2)}\left(\tau_1\tau_2+\tau_3(1-\tau_2),\ \tau_2-\tau_1\tau_3(1-\tau_2),\ \tau_1+\tau_3(1-\tau_2)\right)^t$$

or in split quaternion terms

$$\zeta=\frac{1-\tau_1\tau_3+\tau_1\tau_2\tau_3}{H_2(\tau_1,\tau_2,\tau_3)}+\frac{\tau_1\tau_2+\tau_3(1-\tau_2)}{H_2(\tau_1,\tau_2,\tau_3)}\tilde{\mathbf{i}}+\frac{\tau_2-\tau_1\tau_3(1-\tau_2)}{H_2(\tau_1,\tau_2,\tau_3)}\tilde{\mathbf{j}}+\frac{\tau_1+\tau_3(1-\tau_2)}{H_2(\tau_1,\tau_2,\tau_3)}\tilde{\mathbf{k}}$$

where we denote $H_2(\tau_1,\tau_2,\tau_3)=\sqrt{(1+\tau_1^2)(1-\tau_2^2)}$.
Proceeding exactly as in the Euclidean case, we derive a set of explicit relations between the scalar parameters of the hyperbolic Bryan and Iwasawa decompositions in the form

$$\begin{gathered}\tilde\tau_1^{\pm}=2\frac{\tau_3(1-\tau_1^2)(1-\tau_2^2)+\tau_1\left(1+\tau_2^2-2\tau_3^2(1-\tau_2)^2\right)}{(1-\tau_1^2)\left(1+\tau_2^2-2\tau_3^2(1-\tau_2)^2\right)-4\tau_1\tau_3(1-\tau_2^2)\pm(1+\tau_1^2)\sqrt{\tilde D}}\\ \tilde\tau_2^{\pm}=2\frac{\tau_2+\tau_3^2(1-\tau_2)^2}{1-\tau_2^2\pm\sqrt{\tilde D}},\qquad \tilde\tau_3^{\pm}=\frac{2\tau_3(1-\tau_2)^2}{2\tau_3^2(1-\tau_2)^2+1+\tau_2^2\pm\sqrt{\tilde D}}\end{gathered}\tag{76}$$

with the notation $\tilde D=(1-\tau_2^2)^2+4\left(\tau_2+\tau_3^2(1-\tau_2)^2\right)^2$ and in the reverse direction

$$\begin{gathered}\tau_1=-\frac{\tilde\tau_2(\tilde\tau_3+\tilde\tau_1)+\tilde\tau_3-\tilde\tau_1}{\tilde\tau_1\tilde\tau_3(1+\tilde\tau_2)+1-\tilde\tau_2},\qquad \tau_2=\frac{\tilde\tau_2(1-\tilde\tau_3^2)-\tilde\tau_2^2-\tilde\tau_3^2}{1+\tilde\tau_2^2\tilde\tau_3^2-\tilde\tau_2(1-\tilde\tau_3^2)}\\ \tau_3=\frac{\tilde\tau_3(1+\tilde\tau_2^2)}{(1+\tilde\tau_2^2)(1+\tilde\tau_3^2)-2\tilde\tau_2(1-\tilde\tau_3^2)}.\end{gathered}\tag{77}$$

This technique may be used to obtain such relations between each pair of decompositions, both Euclidean and hyperbolic, as long as they are well-defined.

7. Numerical Examples

We start with a purely numerical example in the spirit of [15]. The unit vectors $\hat{\mathbf{c}}_k$ and $\mathbf{n}$ are given in spherical coordinates (measuring the azimuthal angle from the equator) as $\mathbf{x}(\theta,\vartheta)=(\cos\theta\cos\vartheta,\cos\theta\sin\vartheta,\sin\theta)^t$. Let us choose, for instance $\hat{\mathbf{c}}_1(22.62°,67.38°)$, $\hat{\mathbf{c}}_2(46.4°,43.6°)$, $\hat{\mathbf{c}}_3(61.93°,28.07°)$ and $\mathbf{n}(36.87°, 53.13°)$ with $\varphi=33°$. Applying the algorithm described in Section 3 we obtain

$$\{\varphi_k^+\}=\{52.81°,-78.05°,66.67°\},\qquad \{\varphi_k^-\}=\{9.47°,32.35°,-8.69°\}.$$

A typical example for a gimbal lock is the *Euler ZXZ* decomposition of a half-turn about the OY axis. Formula (40) yields $\{\varphi_k\}=\{\vartheta,\ \pi,\ \vartheta-\pi\}$ with $\vartheta\in\mathbb{R}$ and the corresponding matrix decomposition is given by

$$\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}=\begin{pmatrix}-\cos\vartheta&\sin\vartheta&0\\-\sin\vartheta&-\cos\vartheta&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}\cos\vartheta&-\sin\vartheta&0\\\sin\vartheta&\cos\vartheta&0\\0&0&1\end{pmatrix}.$$

In particular, the two-gimbal decompositions obtained for $\vartheta=0$ and $\vartheta=\pi$ present the compound half-turn as a product of two reflections in the XZ-plane. Alternatively, one may decompose into a half-turn $\varphi_2=\pi$ and two quarter-turns at $\vartheta=\pi/2$. For the Lorentz case we choose a decomposition that takes place entirely on the light cone: $\hat{\mathbf{c}}_1=(5,12,13)^t$, $\hat{\mathbf{c}}_2=(21,20,29)^t$, $\hat{\mathbf{c}}_3=(15,8,17)^t$ and $\mathbf{n}=(3,4,5)^t$ with $\tau=11/7$. Our method easily provides the exact solutions in the form

$$\{\tau_k^+\}=\{-43/36,13/84,-17/27\},\qquad \{\tau_k^-\}=\{-11/18,11/42,11/27\}.$$

Let us also consider the more exotic example of a massless particle with relativistic momentum $\hat{\mathbf{p}}=(3,4,5)^t$, boosted by $\mathcal{R}_h(\mathbf{c})$, where $\mathbf{c}=(5/2,5/2,7/2)^t$. We decompose with respect to the axes $\hat{\mathbf{c}}_1=\hat{\mathbf{p}}=(3,4,5)^t$, $\hat{\mathbf{c}}_2=(0,5/3,4/3)^t$ and $\hat{\mathbf{c}}_3=(5/4,0,3/4)^t$. Formula (61) yields the one-parameter solution

$$\tau_1=\left(\frac{5}{12}\right)\frac{2s+3}{s+1},\qquad \tau_2=s,\qquad \tau_3=\frac{2s+1}{s+2},\qquad s\in\mathbb{R}/\{\pm1\}$$

where $s\neq\pm1$ due to (57). Normalization in the null direction $\hat{\mathbf{c}}_1$ is arbitrary, since the light cone is scale invariant and if we choose $(\hat{\mathbf{c}}_1,\hat{\mathbf{c}}_1)=1$ for example, i.e., multiply $\hat{\mathbf{c}}_1$ by $\sqrt{2}/10$, the pre-factor in the expression for τ_1 becomes $25\sqrt{2}/12$.

Our last example is the Iwasawa decomposition of the split quaternion

$$\zeta=\begin{pmatrix}1&1\\2&3\end{pmatrix}=2+\frac{3}{2}\,\tilde{\mathbf{i}}-\tilde{\mathbf{j}}-\frac{1}{2}\,\tilde{\mathbf{k}}.$$

By the results in the previous section we obtain $\tau_k=\{-2/3,-6/7,5/13\}$ and thus

$$\xi_1=\frac{1}{\sqrt{13}}\begin{pmatrix}3&-2\\2&3\end{pmatrix},\qquad \xi_1=\frac{1}{\sqrt{13}}\begin{pmatrix}1&0\\0&13\end{pmatrix},\qquad \xi_1=\begin{pmatrix}1&\frac{5}{13}\\0&1\end{pmatrix}.$$

Final Remarks

The method provided here can easily be generalized to other low-dimensional Lie groups (cf [8]), such as $\mathsf{SO}(4)$, $\mathsf{SO}(3,1)$, $\mathsf{SO}(2,2)$ and $\mathsf{SO}^*(4)$, for which vector parametrization is still available in some form [7]. Moreover, the compact solutions we obtain make it appropriate also for the study of continuous transformations in rigid body mechanics [10,12]. On the other hand, as shown in Section 6, the correspondence between (split) quaternions and vector-parameters lifts up all results to the spin covering groups $\mathsf{SU}(2)$ and $\mathsf{SU}(1,1) \cong \mathsf{SL}(2,\mathbb{R})$, which play central role both in classical and quantum mechanics in the description of symplectic maps, deformable media, qubit systems, scattering and many other areas.

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