



VECTOR PARAMETERS IN CLASSICAL HYPERBOLIC GEOMETRY

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Abstract. Here we use an extension of *Rodrigues’ vector parameter* construction for pseudo-rotations in order to obtain explicit formulae for the generalized *Euler* decomposition with arbitrary axes for the structure groups in the classical models of hyperbolic geometry. Although the construction is projected from the universal cover $SU(1, 1) \simeq SL(2, \mathbb{R})$, most attention is paid to the $2 + 1$ *Minkowski* space model, following the close analogy with the *Euclidean* case, and various decompositions of the restricted *Lorentz* group $SO^+(2, 1)$ are investigated in detail. At the end we propose some possible applications in *special relativity* and *scattering theory*.

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List of Symbols and Notations

$\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3, \hat{\mathbf{n}}$	normalized vectors in $\mathbb{R}^{2,1}$
$\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$	unit vectors along the coordinate axes
$\mathbf{e}_k, \tilde{\mathbf{e}}_k$	bases in $\mathfrak{su}(1, 1)$ and $\mathfrak{sl}(2, \mathbb{R})$
$z_k, \alpha, \beta, \alpha_k, \beta_k$	complex numbers
$\Re(z), \Im(z)$	real and imaginary part of a complex number
$\xi, \zeta, \tilde{\xi}, \tilde{\zeta}$	pseudo-quaternions
$\mathbf{c}, \tilde{\mathbf{c}}, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$	vector parameters of pseudo-rotations
$\mathbf{x} \otimes (\eta \mathbf{z})^t$	dyadic product of vectors in $\mathbb{R}^{2,1}$
\times, \wedge	cross product of vectors in \mathbb{R}^3 and $\mathbb{R}^{2,1}$
$(\mathbf{x}, \mathbf{z}), \mathbf{x} \cdot \mathbf{z}$	dot product of vectors in \mathbb{R}^3 and $\mathbb{R}^{2,1}$
ε_{ijk}	the <i>Levi-Civita</i> symbol
$\mathbf{c}^\times, \mathbf{c}^\wedge$	the <i>Hodge</i> dual in \mathbb{R}^3 and $\mathbb{R}^{2,1}$
\mathcal{I}, η	the identity matrix and the flat metric in $\mathbb{R}^{2,1}$
$\mathcal{A}(\mathbf{x}) = \mathbf{x}^\wedge$	an element of the <i>Lie</i> algebra $\mathfrak{so}(2, 1)$
$\mathcal{R}_h(\mathbf{c}), \mathcal{R}_h(\mathbf{c}_k)$	pseudo-rotations in $\mathbb{R}^{2,1}$
$\epsilon = \text{sgn}(\mathbf{c} \cdot \mathbf{c})$	sign of the pseudo-norm
$\tau, \tilde{\tau}, \tau_k$	scalar parameters
$\varphi, \theta, \phi, \gamma, \psi_k$	angles and rapidities
$\kappa_{ij} = \hat{\mathbf{c}}_i \cdot \hat{\mathbf{c}}_j, \epsilon_i = \kappa_{ii}$	elements of the <i>Gramm</i> matrix
$\sigma_{ij} = \hat{\mathbf{c}}_i \cdot \mathcal{R}_h(\mathbf{c}) \hat{\mathbf{c}}_j$	matrix entries of $\mathcal{R}_h(\mathbf{c})$
$\rho_k = \hat{\mathbf{n}} \cdot \hat{\mathbf{c}}_k, \varepsilon_{ijk} \tilde{\rho}^k = \hat{\mathbf{c}}_i \wedge \hat{\mathbf{c}}_j \cdot \hat{\mathbf{n}}$	some useful scalar products
$\omega = \hat{\mathbf{c}}_1 \cdot \hat{\mathbf{c}}_2 \wedge \hat{\mathbf{c}}_3$	a volume element

1. Introduction

Vector decompositions of arbitrary rotations in \mathbb{R}^3 into pairs or triplets of orthogonal transformations have been studied repeatedly since the time of *Euler*, who was the first to propose a parametrization of $SO(3)$ with three angles, later named after him. We can distinguish two powerful algebraic methods to deal with the more general case of nonorthogonal axes. The first one, explicitly relying on *Rodrigues'* formula, was proposed by Davenport [9] and recently exploited by Piovan and Bullo [27] in a more consistent framework. The second one is based on the *vector parameter* construction, proposed by *Rodrigues* [29] and developed in the works of *Gibbs* [11], who used to call it “*vector semitangent of version*” (now it is commonly known as *Gibbs'* or *Rodrigues'* vector) and *Fedorov* [10], while two of the present authors are among the first to utilize it in classical mechanics [23, 24] and certain quantum mechanical problems [19, 20, 22]. To our knowledge it was first exploited in the context of *Euler* decomposition by Wohlfart [34], but the idea was fully developed only recently in [5, 21], where analytic solutions based on this construction are presented.

Apart from its merits in terms of clarity and simplicity, vector parameter representation appears to provide most natural parametrization of $SO(3)$, as it is given by a projection from the universal cover, which has not been sufficiently emphasized by *Gibbs* and *Fedorov*. Here we use analogous construction to obtain the generalized *Euler* parameters for a decomposition of an arbitrary element of the *restricted Lorentz group*¹ $SO^+(2, 1)$ projecting from its spin cover $SU(1, 1)$. We also write the corresponding expressions in the $SL(2, \mathbb{R})$ case and thus provide explicit solutions for the classical models of hyperbolic geometry - the *Poincaré* disk Δ , the upper half-plane \mathcal{H}^2 and the flat three-dimensional *Minkowski* space $\mathbb{R}^{2,1}$.

In the following section we introduce *pseudo-quaternions* [8, 32] (also known as *split quaternions*) and construct the *hyperbolic vector parameter* with their help. In Section 3 we derive a simple method to obtain the *Euler* parameters in the decomposition of a generic pseudo-orthogonal transformation in $2 + 1$ dimensions with respect to two or three almost arbitrary axes, paying special attention to more specific cases, such as degenerate solutions and divergent parameters. Next we investigate into more detail how parabolic elements and pure rotations behave within the group and finally provide a relevant physical example in Section 5.

¹This is just the connected component of unity in $SO(2, 1)$, also called *orthochronous*, since its elements preserve the direction of time in $2 + 1$ dimensional *special relativity*.

2. Projective Construction

In this section we introduce the basic machinery needed for our work - *Rodrigues' formula* and the *vector parameter* representation in the hyperbolic case. The starting point is the universal cover $SU(1, 1)$, built in a quaternion fashion, in complete analogy with the *Euclidean* case. Then, following this analogy further, we project onto $SO^+(2, 1)$ where most of the work is being done. We also comment on the corresponding construction on $SL(2, \mathbb{R})$ using the well-known isomorphism of the *Lie* groups (see [12, 13]).

2.1. The Poincaré Disk and Pseudo-quaternions

We may choose a basis in $\mathfrak{su}(1, 1)$ in the form

$$\mathbf{e}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (1)$$

Then, adding the identity element

$$\mathbf{e}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we easily obtain a realization of the group $SU(1, 1)$ as a submanifold of \mathbb{R}^4 . More precisely, each vector $\zeta \in \mathbb{R}^4$ can be expanded as $\zeta = \zeta_i \mathbf{e}_i$, $i = 0 \dots 3$ in the above basis (summation over repeated indices is always assumed in the text, if not pointed otherwise). Using complex coordinates $\alpha = \zeta_0 + i\zeta_3$ and $\beta = \zeta_1 + i\zeta_2$ we may write

$$\zeta = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \quad (2)$$

which is invertible as long as $|\alpha|^2 - |\beta|^2 \neq 0$ (i.e., for any non-isotropic vector $\zeta \in \mathbb{R}^{2,2}$) and if we choose $|\alpha|^2 - |\beta|^2 = 1$, we obtain an element of $SU(1, 1)$ with inverse, equal to

$$\zeta^{-1} = \begin{pmatrix} \bar{\alpha} & -\beta \\ -\bar{\beta} & \alpha \end{pmatrix}.$$

In this way we easily identify $SU(1, 1)$ with the hyperboloid (or quasi-sphere)

$$\zeta_0^2 - \zeta_1^2 - \zeta_2^2 + \zeta_3^2 = 1. \quad (3)$$

For more details on the subject of quaternions and hypercomplex numbers we refer to [8, 17, 32] and [33] for some of their application in mechanics.

2.2. The Upper Half Plane Model

It is well known that $\mathrm{SL}(2, \mathbb{R})$ acts as a structure group in the upper half-plane $\mathcal{H}^2 = \{z \in \mathbb{C}; \Im z > 0\}$ via linear fractional transformations

$$\mathrm{SL}(2, \mathbb{R}) \ni A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \longrightarrow A(z) = \frac{az + b}{cz + d} \in \mathcal{H}^2$$

and since the action is only determined up to a scalar factor, we think of it as a representation of the projective group $\mathrm{PSL}(2, \mathbb{R})$, also called the *Möbius* group of \mathcal{H}^2 . Its Lie algebra is certainly the same - we may choose a basis in the space of 2×2 traceless real matrices $\mathfrak{sl}(2, \mathbb{R})$ in the form

$$\tilde{\mathbf{e}}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\mathbf{e}}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{\mathbf{e}}_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (4)$$

and use it to expand vectors in $\mathbb{R}^{2,1}$ as we did before. Moreover, we may add the identity element

$$\tilde{\mathbf{e}}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and thus represent the group itself. Actually, instead of rebuilding the whole construction from scratch, we give an explicit morphism between the two groups which will help us adapt the results obtained above without any trouble. We use the well-known geometric correspondence

$$\mathcal{U} : \mathcal{H}^2 \rightarrow \Delta, \quad z \rightarrow \mathcal{U}(z) := i \frac{z - i}{z + i} \quad (5)$$

that maps the upper half-plane to the unit disk and which can be extended to their groups of isometries. Its matrix

$$\mathcal{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \quad (6)$$

is unitary (with this particular normalization) and it is not hard to see that it maps the basis $\{\tilde{\mathbf{e}}_k\}$ to $\{\mathbf{e}_k\}$ and vice versa via conjugation

$$\mathbf{e}_k = \mathcal{U} \tilde{\mathbf{e}}_k \mathcal{U}^\dagger, \quad \tilde{\mathbf{e}}_k = \mathcal{U}^\dagger \mathbf{e}_k \mathcal{U} \quad (7)$$

which means that the *Lie* algebras are isomorphic and the same holds for the groups as well. Hence, all results for $\mathrm{SU}(1, 1)$ can be directly applied to $\mathrm{SL}(2, \mathbb{R})$ and vice versa. For a more advanced study on hyperbolic geometry we refer to [30, 31].

There are three classes of elements in $\mathrm{PSL}(2, \mathbb{R})$ and we have an elementary criterion to distinguish between them: for each matrix $\zeta \in \mathrm{SL}(2, \mathbb{R})$ the corresponding linear-fractional transformation is

1. *hyperbolic* if $|\mathrm{Tr} \zeta| > 2$
2. *elliptic* if $|\mathrm{Tr} \zeta| < 2$
3. *parabolic* if $|\mathrm{Tr} \zeta| = 2$

so if we expand the vector part of ζ over the basis specified in (4) we end up with

$$\zeta = \begin{pmatrix} \zeta_0 + \zeta_2 & \zeta_1 + \zeta_3 \\ \zeta_1 - \zeta_3 & \zeta_0 - \zeta_2 \end{pmatrix} \quad (8)$$

and by this construction it becomes evident that the above conditions concern only its scalar part. In the parabolic case for example, we have $\zeta_0 = 1$ and since $\det \zeta = 1$ this yields the equation of the *light cone*

$$\zeta_1^2 + \zeta_2^2 - \zeta_3^2 = 0.$$

Similarly hyperbolic transformations are associated with *space-like* vectors and the elliptic ones - with *time-like* vectors, the above quadratic form being positive in the former while negative in the latter case. It is easy to see for example, that *Möbius* transformations, corresponding to translations $z \rightarrow z + \alpha$

$$\mathcal{T}_p(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

are parabolic and direct comparison with (8) gives $\zeta_2 = 0$ and $\zeta_1 = \zeta_3 = \frac{\alpha}{2}$.

As for the elliptic and hyperbolic transformations, by the exponential map, we have two typical representatives in the form

$$\mathcal{T}_e(\varphi) = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}, \quad \mathcal{T}_h(\tau) = \begin{pmatrix} e^\tau & 0 \\ 0 & e^{-\tau} \end{pmatrix}.$$

Moreover, each element \mathcal{T} of $\mathrm{PSL}(2, \mathbb{R})$ can be written as (see [1])

$$\mathcal{T} = \mathcal{T}_e \mathcal{T}_h \mathcal{T}_p.$$

Note that if we restrict our considerations to integer coordinates on the light cone, the components of ζ form a *Pythagorean triple*, and thus we obtain a representation of the *Pythagorean triples* in the modular group (see [16] for details).

2.3. Rodrigues' Formula and Vector Parametrization

Any vector $\mathbf{x} \in \mathbb{R}^{2,1}$ expanded in the basis (1) can be mapped in the algebra $\mathfrak{su}(1, 1)$ as follows

$$\mathbf{x} \rightarrow \mathbf{X} = \begin{pmatrix} ix_3 & x_1 + ix_2 \\ x_1 - ix_2 & -ix_3 \end{pmatrix} \quad (9)$$

while its pseudo-Euclidean norm is given by

$$\mathbf{x} \cdot \mathbf{x} = -\det \mathbf{X} = x_1^2 + x_2^2 - x_3^2. \quad (10)$$

Then we may easily construct a norm preserving automorphism, using the adjoint action of the group in its algebra

$$\text{Ad}_\zeta : \mathbf{X} \rightarrow \zeta \mathbf{X} \zeta^{-1}.$$

The *Cartesian* coordinates of \mathbf{x} are then being transformed by a pseudo-orthogonal matrix $\mathcal{R}_h(\zeta)$ which is easily shown to have the form

$$\mathcal{R}_h(\zeta) = \begin{pmatrix} 1 - 2\zeta_1^2 & -2(\zeta_1\zeta_2 + \zeta_0\zeta_3) & 2(\zeta_1\zeta_3 + \zeta_0\zeta_2) \\ 2(\zeta_0\zeta_3 - \zeta_1\zeta_2) & 1 - 2\zeta_2^2 & 2(\zeta_2\zeta_3 - \zeta_0\zeta_1) \\ 2(\zeta_0\zeta_2 - \zeta_1\zeta_3) & -2(\zeta_2\zeta_3 + \zeta_0\zeta_1) & 1 - 2\zeta_3^2 \end{pmatrix}. \quad (11)$$

Then we may separate the *scalar* and *vector* part of the pseudo-quaternion

$$\zeta = (\zeta_0, \zeta), \quad \zeta \in \mathbb{R}^{2,1}$$

which allows to write down the above matrix invariantly

$$\mathcal{R}_h(\zeta) = \mathcal{I} - 2\zeta \otimes (\eta\zeta)^t + 2\zeta_0 \zeta^\wedge. \quad (12)$$

In the above formula \mathcal{I} is the 3×3 identity operator, $\eta = \text{diag}(1, 1, -1)$ is the flat metric tensor in $\mathbb{R}^{2,1}$, $\mathcal{P}_\zeta = \zeta \otimes (\eta\zeta)^t$, which can be explicitly written as $\mathcal{P}_{\zeta_j^i} = \eta_{jk}\zeta^i\zeta^k$, is a quasi-projector along ζ , obtained by the usual *tensor* or *dyadic* product of vectors and $\zeta^\wedge = \eta\zeta^\times$ (written in components as $\zeta^\wedge_j^i = \eta^{ik}\varepsilon_{klj}\zeta^l$) is the tensor, associated with ζ by the *Hodge duality*

$$\zeta^\wedge \xi = \zeta \wedge \xi = \eta \zeta \times \xi.$$

Note that a covariant definition of the tensor entities we work with allows for using \times instead of \wedge and $\zeta \otimes \zeta$ for $\zeta \otimes (\eta\zeta)^t$ - switching from upper to lower indices and vice versa then naturally involves the metric η . However, we chose to make our expressions as explicit as possible here and in order to avoid confusion, introduce

the unnecessary otherwise notation \wedge and $\zeta \otimes (\eta \zeta)^t$.

Now, since from (3) and (10) we have $\zeta_0^2 - \zeta^2 = 1$, where $\zeta^2 = \zeta \cdot \zeta$, as long as ζ is *space-like* ($\zeta^2 > 0$) we may introduce a unit vector

$$\hat{\mathbf{n}} \in \mathbb{R}^{2,1}, \quad \hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$$

to parameterize $\mathcal{R}_h(\zeta)$ by writing

$$\zeta_0 = \cosh \frac{\varphi}{2}, \quad \zeta = \sinh \frac{\varphi}{2} \hat{\mathbf{n}}, \quad \varphi \in \mathbb{R}$$

and then with the help of some basic hyperbolic trigonometry (12) becomes

$$\mathcal{R}_h(\hat{\mathbf{n}}, \varphi) = \cosh \varphi \mathcal{I} + (1 - \cosh \varphi) \hat{\mathbf{n}} \otimes (\eta \hat{\mathbf{n}})^t + \sinh \varphi \hat{\mathbf{n}}^\wedge \quad (13)$$

which we may refer to as the *hyperbolic Rodrigues' formula* and the pseudo-angular parameter φ is usually called *rapidity*. Furthermore, we may introduce the *hyperbolic vector parameter* $\mathbf{c} \in \mathbb{RP}^3$ in a projective manner

$$\mathbf{c} = \frac{\zeta}{\zeta_0} = \tanh \frac{\varphi}{2} \hat{\mathbf{n}} = \tau \hat{\mathbf{n}} \quad (14)$$

where the $\tau \in \mathbb{R}$ is referred to as the *scalar parameter* of the pseudo-rotation. Hyperbolic *Euler* substitution then leads to the alternative representation

$$\mathcal{R}_h(\mathbf{c}) = \frac{(1 + \mathbf{c}^2)\mathcal{I} - 2\mathbf{c} \otimes (\eta \mathbf{c})^t + 2\mathbf{c}^\wedge}{1 - \mathbf{c}^2}. \quad (15)$$

Note that our hyperbolic metric allows to have vectors with negative scalar squares, called *time-like vectors*. If ζ is such a vector ($\zeta^2 < 0$), it obviously cannot be proportional to a unit one (at least not with a real coefficient), but we may introduce a *quasi-unit* time-like vector

$$\hat{\mathbf{n}} \in \mathbb{R}^{2,1}, \quad \hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = -1$$

and then the natural parametrization is

$$\mathbf{c} = \frac{\zeta}{\zeta_0} = \tan \frac{\varphi}{2} \hat{\mathbf{n}} = \tau \hat{\mathbf{n}} \quad (16)$$

or

$$\zeta_0 = \cos \frac{\varphi}{2}, \quad \zeta = \sin \frac{\varphi}{2} \hat{\mathbf{n}}, \quad \varphi \in \mathbb{R}$$

which leads to another form of *Rodrigues' formula*, that is valid for time-like vectors

$$\mathcal{R}_h(\hat{\mathbf{n}}, \varphi) = \cos \varphi \mathcal{I} - (1 - \cos \varphi) \hat{\mathbf{n}} \otimes (\eta \hat{\mathbf{n}})^t + \sin \varphi \hat{\mathbf{n}}^\wedge. \quad (17)$$

The so-called *null* or *isotropic* case $\zeta^2 = 0$ is discussed separately in Section 4. Note that the vector parameter \mathbf{c} can be projected using either $SU(1, 1)$ or $SL(2, \mathbb{R})$ quaternion construction. In both cases we divide by $\zeta_0 = \frac{1}{2}\text{Tr } \zeta$ and the components are given respectively by

$$\mathbf{c} = \frac{1}{\text{Tr } \zeta} \begin{pmatrix} \mathfrak{z}_{12} + \mathfrak{z}_{21} \\ i\mathfrak{z}_{21} - i\mathfrak{z}_{12} \\ i\mathfrak{z}_{22} - i\mathfrak{z}_{11} \end{pmatrix}_{\zeta \in SU(1,1)} \quad \mathbf{c} = \frac{1}{\text{Tr } \zeta} \begin{pmatrix} \mathfrak{z}_{12} + \mathfrak{z}_{21} \\ \mathfrak{z}_{11} - \mathfrak{z}_{22} \\ \mathfrak{z}_{12} - \mathfrak{z}_{21} \end{pmatrix}_{\zeta \in SL(2, \mathbb{R})} \quad (18)$$

where \mathfrak{z}_{ij} stand for the components of the quaternion ζ , written as a 2×2 matrix. On the other hand, we may easily determine ζ from those of \mathbf{c} , using the fact that ζ is unit, which means $\zeta_0^2 - \zeta^2 = 1$ and thus

$$\mathbf{c}^2 = \frac{\zeta_0^2 - 1}{\zeta_0^2} \implies \zeta_0^{1,2} = \pm(1 - \mathbf{c}^2)^{-\frac{1}{2}}, \quad \zeta^{1,2} = \zeta_0^{1,2} \mathbf{c} \quad (19)$$

where the two signs correspond to the two sheets of the cover.

2.4. Composition Law for the Vector Parameters

Due to the multiplication table of the matrices in (1)

$$\mathbf{e}_1 \mathbf{e}_2 = -\mathbf{e}_3, \quad \mathbf{e}_2 \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \mathbf{e}_1 = \mathbf{e}_2, \quad \mathbf{e}_1^2 = \mathbf{e}_2^2 = -\mathbf{e}_3^2 = \mathbf{e}^0 \quad (20)$$

$$\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i = 0, \quad i, j = 1, 2, 3$$

we have a rather simple composition law for two pseudo-quaternions $\xi = (\xi_0, \boldsymbol{\xi})$ and $\zeta = (\zeta_0, \boldsymbol{\zeta})$, namely

$$\zeta \xi = (\zeta_0 \xi_0 + \boldsymbol{\zeta} \cdot \boldsymbol{\xi}, \zeta_0 \boldsymbol{\xi} + \xi_0 \boldsymbol{\zeta} + \boldsymbol{\zeta} \wedge \boldsymbol{\xi}). \quad (21)$$

In the spirit of projective geometry, we may think of ξ_1, ξ_2 and ξ_3 as the homogeneous coordinates of the \mathbb{RP}^3 - points $\mathbf{c}_1, \mathbf{c}_2$ and \mathbf{c}_3 respectively. In other words we have $^2(\xi_i, \boldsymbol{\xi}_i) \sim (1, \mathbf{c}_i)$ or $\mathbf{c}_i = \frac{\boldsymbol{\xi}_i}{\xi_i}$, so the composition of two vector parameters may be obtained by projection of the composition in $SU(1, 1)$, i.e., the ratio

²For simplicity of notations we use ξ_i instead of $(\xi_i)_0$ for the scalar component of ξ_i

between its vector and scalar part

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \wedge \mathbf{c}_1}{1 + \mathbf{c}_2 \cdot \mathbf{c}_1}. \quad (22)$$

Then we naturally have

$$\mathcal{R}_h(\mathbf{c}_2)\mathcal{R}_h(\mathbf{c}_1) = \mathcal{R}_h(\langle \mathbf{c}_2, \mathbf{c}_1 \rangle), \quad \mathcal{R}_h(-\mathbf{c}) = \mathcal{R}_h(\mathbf{c})^{-1}, \quad \mathcal{R}_h(0) = \mathcal{I}$$

as in the *Euclidean* case. From the above it becomes clear why we arrange the indices in the backwards direction - this is because operator composition takes place from right to left.

If we want to decompose into three pseudo-rotations, first we need to notice that the scalar part of the product $\xi_3 \xi_2 \xi_1$ is given by

$$(\xi_3 \xi_2 \xi_1)_0 = \overset{\circ}{\xi}_3 \overset{\circ}{\xi}_2 \overset{\circ}{\xi}_1 + \overset{\circ}{\xi}_3 \xi_2 \cdot \xi_1 + \overset{\circ}{\xi}_2 \xi_3 \cdot \xi_1 + \overset{\circ}{\xi}_1 \xi_3 \cdot \xi_2 + (\xi_3, \xi_2, \xi_1)$$

where

$$(\xi_3, \xi_2, \xi_1) = \xi_3 \wedge \xi_2 \cdot \xi_1$$

denotes the triple product, i.e., the volume, spanned by ξ_3, ξ_2 and ξ_1 . As for the vector part, we have respectively

$$\begin{aligned} \text{vec}(\xi_3 \xi_2 \xi_1) &= \overset{\circ}{\xi}_3 \overset{\circ}{\xi}_2 \xi_1 + \overset{\circ}{\xi}_3 \overset{\circ}{\xi}_1 \xi_2 + \overset{\circ}{\xi}_2 \overset{\circ}{\xi}_1 \xi_3 + (\xi_3 \cdot \xi_2) \xi_1 \\ &\quad + \overset{\circ}{\xi}_3 \xi_2 \wedge \xi_1 + \overset{\circ}{\xi}_2 \xi_3 \wedge \xi_1 + \overset{\circ}{\xi}_1 \xi_3 \wedge \xi_2 + (\xi_3 \wedge \xi_2) \wedge \xi_1 \end{aligned}$$

which can be projected to the composition $\mathbf{c} = \langle \mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1 \rangle$ given as

$$\mathbf{c} = \frac{\mathbf{c}_3 + \mathbf{c}_2 + \mathbf{c}_1 + (\mathbf{c}_3 \cdot \mathbf{c}_2) \mathbf{c}_1 + \mathbf{c}_3 \wedge \mathbf{c}_2 + \mathbf{c}_3 \wedge \mathbf{c}_1 + \mathbf{c}_2 \wedge \mathbf{c}_1 + (\mathbf{c}_3 \wedge \mathbf{c}_2) \wedge \mathbf{c}_1}{1 + \mathbf{c}_3 \cdot \mathbf{c}_2 + \mathbf{c}_3 \cdot \mathbf{c}_1 + \mathbf{c}_2 \cdot \mathbf{c}_1 + (\mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1)}.$$

Note that by construction we have for the double cross product signs, opposite to those in the *Euclidean* case

$$\begin{aligned} (\mathbf{c}_3 \wedge \mathbf{c}_2) \wedge \mathbf{c}_1 &= (\mathbf{c}_1 \cdot \mathbf{c}_2) \mathbf{c}_3 - (\mathbf{c}_1 \cdot \mathbf{c}_3) \mathbf{c}_2 \\ \mathbf{c}_3 \wedge (\mathbf{c}_2 \wedge \mathbf{c}_1) &= (\mathbf{c}_2 \cdot \mathbf{c}_3) \mathbf{c}_1 - (\mathbf{c}_1 \cdot \mathbf{c}_3) \mathbf{c}_2. \end{aligned} \quad (23)$$

The sign difference is explained by the fact that in the flat *Minkowski* space we have a non-trivial metric η to switch from upper to lower indices and vice versa. For example, the cross product of two vectors $c_k = (\mathbf{a} \times \mathbf{b})_k = \varepsilon_{jkl} a^j b^l$ is defined exactly as in the *Euclidean* case for the covariant (lower) indices, but the components of the expansion in the corresponding basis are $c^i = \eta^{ij} c_j$. Therefore, we use special notations for these operations to make the differences more apparent.

3. Generalized Euler Decomposition

Here we utilize the expressions derived in the previous section for obtaining the vector decomposition of a generic pseudo-orthogonal transformation into two or three transformations of the same kind. We focus mostly on the *Minkowski* space model due to the close analogy with the *Euclidean* case. The results for $SU(1, 1)$ and $SL(2, \mathbb{R})$ may be written using directly the explicit correspondence between pseudo-quaternions and vector parameters built in the previous section.

3.1. The Case of Three Axes

This is an analogue of the classical *Euler decomposition* for arbitrary axes in flat hyperbolic space

$$\mathcal{R}_h(\tau \hat{\mathbf{n}}) = \mathcal{R}_h(\tau_3 \hat{\mathbf{c}}_3) \mathcal{R}_h(\tau_2 \hat{\mathbf{c}}_2) \mathcal{R}_h(\tau_1 \hat{\mathbf{c}}_1) \quad (24)$$

where $\mathbf{c} = \tau \hat{\mathbf{n}}$ and $\mathbf{c}_k = \tau_k \hat{\mathbf{c}}_k$ are the vector parameters of the pseudo-rotations in the decomposition and $\hat{\mathbf{n}}, \hat{\mathbf{c}}_k$ - the corresponding *quasi-unit* vectors along the oriented axes. What we mean here by quasi-unit is the following: since the hyperbolic flat metric in *Minkowski* space allows non-zero vectors to have positive, negative and vanishing scalar square, it is not always possible to normalize to unit “length”. Instead, we have $\epsilon = \hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = \pm 1$ in the *space-like*, respectively *time-like* case and $\epsilon = 0$ in the *isotropic* one. Similarly, we use the notation $\epsilon_k = \hat{\mathbf{c}}_k \cdot \hat{\mathbf{c}}_k$.

In order to normalize an isotropic vector, we resort to a trick, known in *Physics* as *Wick rotation* - that is basically taking *Euclidean* rather than hyperbolic scalar product, or equivalently, multiplying the third component of the vector by a factor of $\sqrt{-1}$ (i.e., if $\epsilon = 0$, we write $\mathbf{c} = \tau \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is unit in the *Euclidean* metric $(\hat{\mathbf{n}}, \hat{\mathbf{n}}) = \hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2 = 1$).

Note that *Euler's* invariant axis theorem holds in the hyperbolic case as well (we comment on this fact into more detail in the next section) and from (13) it can be derived that $\hat{\mathbf{c}}_1$ and $\hat{\mathbf{c}}_3$ are eigenvectors of $\mathcal{R}_h(\tau_1 \hat{\mathbf{c}}_1)$ and $\mathcal{R}_h(\tau_3 \hat{\mathbf{c}}_3)$ respectively, with eigenvalue, equal to one. Thus we have

$$\hat{\mathbf{c}}_3 \cdot \mathcal{R}_h(\tau \hat{\mathbf{n}}) \hat{\mathbf{c}}_1 = \hat{\mathbf{c}}_3 \cdot \mathcal{R}_h(\tau_2 \hat{\mathbf{c}}_2) \hat{\mathbf{c}}_1 \quad (25)$$

and if we introduce the notations

$$\sigma_{ij} = \hat{\mathbf{c}}_i \cdot \mathcal{R}_h(\tau \hat{\mathbf{n}}) \hat{\mathbf{c}}_j, \quad \kappa_{ij} = \hat{\mathbf{c}}_i \cdot \hat{\mathbf{c}}_j, \quad \omega = \hat{\mathbf{c}}_1 \cdot \hat{\mathbf{c}}_2 \wedge \hat{\mathbf{c}}_3$$

and take into account (15), the above leads to a quadratic equation for the unknown middle scalar parameter τ_2

$$[\epsilon_2(\sigma_{31} + \kappa_{31}) - 2\kappa_{12}\kappa_{23}] \tau_2^2 - 2\omega\tau_2 + \kappa_{31} - \sigma_{31} = 0. \quad (26)$$

which has real roots whenever

$$\Delta = \omega^2 + [\epsilon_2(\sigma_{31} + \kappa_{31}) - 2\kappa_{12}\kappa_{23}] (\sigma_{31} - \kappa_{31}) \geq 0. \quad (27)$$

These solution, explicitly given by

$$\tau_2^\pm = \frac{\omega \pm \sqrt{\Delta}}{\epsilon_2(\sigma_{31} + \kappa_{31}) - 2\kappa_{12}\kappa_{23}} \quad (28)$$

will be used as parameters in the equations, determining τ_1 and τ_3 .

For the derivation of the latter, one may consider for example $\hat{\mathbf{c}}_1 \cdot \mathcal{R}_h(\tau \hat{\mathbf{n}}) \hat{\mathbf{c}}_1$ and $\hat{\mathbf{c}}_3 \cdot \mathcal{R}_h(\tau \hat{\mathbf{n}}) \hat{\mathbf{c}}_3$ in order to obtain the corresponding quadratic equations for the unknown scalar parameters with the help of (15).

However, this approach has the inconvenience of generating pseudo-solutions, that further need to be excluded, so we prefer to use another algorithm, which is linear. It is based on the vector parameter composition law, derived in the previous section. From (24) we may write

$$\mathbf{c}_1 = \langle -\mathbf{c}_2, -\mathbf{c}_3, \mathbf{c} \rangle, \quad \mathbf{c}_2 = \langle -\mathbf{c}_3, \mathbf{c}, -\mathbf{c}_1 \rangle, \quad \mathbf{c}_3 = \langle \mathbf{c}, -\mathbf{c}_1, -\mathbf{c}_2 \rangle \quad (29)$$

and consider two successive projections - first we premultiply the k^{th} equality above with $\hat{\mathbf{c}}_k^\perp$ and then take (hyperbolic) dot product with $\hat{\mathbf{n}}$. Note that since $\hat{\mathbf{n}}$ is not allowed to be collinear with any of the $\hat{\mathbf{c}}_k$'s (otherwise the decomposition would be trivial), the result is well defined and this trick allows for isolating one of the unknowns, thus obtaining explicit expressions for each pair of parameters as functions of the remaining one and the parameter of the compound pseudo-rotation. In particular we can express τ_1 and τ_3 in terms of τ and τ_2^\pm as

$$\tau_1^\pm = \frac{(\epsilon\kappa_{23} - \rho_2\rho_3)\tau\tau_2^\pm + \tilde{\rho}^1\tau_2^\pm}{(\rho_1\tilde{\rho}^1 + \rho_2\tilde{\rho}^2)\tau\tau_2^\pm + (\kappa_{23}\rho_1 - \kappa_{13}\rho_2)\tau_2^\pm + (\rho_1\rho_3 - \epsilon\kappa_{13})\tau + \tilde{\rho}^2} \quad (30)$$

and

$$\tau_3^\pm = \frac{(\epsilon\kappa_{12} - \rho_1\rho_2)\tau\tau_2^\pm + \tilde{\rho}^3\tau_2^\pm}{(\rho_2\tilde{\rho}^2 + \rho_3\tilde{\rho}^3)\tau\tau_2^\pm + (\kappa_{12}\rho_3 - \kappa_{13}\rho_2)\tau_2^\pm + (\rho_1\rho_3 - \epsilon\kappa_{13})\tau + \tilde{\rho}^2} \quad (31)$$

where we have used the notations

$$\rho_k = \hat{\mathbf{c}}_k \cdot \hat{\mathbf{n}}, \quad \varepsilon_{ijk} \tilde{\rho}^k = (\hat{\mathbf{c}}_i, \hat{\mathbf{c}}_j, \hat{\mathbf{n}})$$

or to be more explicit

$$\tilde{\rho}^1 = (\hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3, \hat{\mathbf{n}}), \quad \tilde{\rho}^2 = (\hat{\mathbf{c}}_3, \hat{\mathbf{c}}_1, \hat{\mathbf{n}}), \quad \tilde{\rho}^3 = (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{n}}).$$

We consider as an example the standard *Euler* decomposition (successive rotations about OZ , OX and OZ) of a $\text{SO}^+(2, 1)$ matrix in the form

$$\begin{aligned} & \begin{pmatrix} c\psi_3 & -s\psi_3 & 0 \\ s\psi_3 & c\psi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{ch}\psi_2 & -\text{sh}\psi_2 \\ 0 & -\text{sh}\psi_2 & \text{ch}\psi_2 \end{pmatrix} \begin{pmatrix} c\psi_1 & -s\psi_1 & 0 \\ s\psi_1 & c\psi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c\psi_3 c\psi_1 - s\psi_3 \text{ch}\psi_2 s\psi_1 & -c\psi_3 s\psi_1 - s\psi_3 \text{ch}\psi_2 c\psi_1 & s\psi_3 \text{sh}\psi_2 \\ s\psi_3 c\psi_1 + c\psi_3 \text{ch}\psi_2 s\psi_1 & c\psi_3 \text{ch}\psi_2 c\psi_1 - s\psi_3 s\psi_1 & -c\psi_3 \text{sh}\psi_2 \\ -\text{sh}\psi_2 s\psi_1 & -\text{sh}\psi_2 c\psi_1 & \text{ch}\psi_2 \end{pmatrix}. \end{aligned}$$

In front of the Greek letters we use here the abbreviations $s = \sin$, $c = \cos$, $\text{sh} = \sinh$ and $\text{ch} = \cosh$. Using the notations

$$\tau_2 = \tanh \frac{\psi_2}{2}, \quad \tau_{1,3} = \tan \frac{\psi_{1,3}}{2}$$

we easily obtain the (projective) compound vector parameter in the form

$$\mathbf{c} = \frac{1}{1 - \tau_1 \tau_3} \begin{pmatrix} \tau_2(1 + \tau_1 \tau_3) \\ \tau_2(\tau_3 - \tau_1) \\ \tau_1 + \tau_3 \end{pmatrix}$$

and hence by the sign of the expression

$$\mathbf{c}^2 \sim \tau_2^2(1 + \tau_1^2)(1 + \tau_3^2) - (\tau_1 + \tau_3)^2$$

determine whether it is time-like, space-like or null (respectively whether the transformation thus obtained is elliptic, hyperbolic or parabolic).

We may also decompose using (28), (30) and (31). Note that in this particular case we have $\kappa_{12} = \kappa_{23} = 0$, $\kappa_{13} = -1$, $\rho_1 = \rho_3 = \hat{n}_3 = -\hat{n}^3$, $\rho_2 = \hat{n}_1$, $\tilde{\rho}^3 = -\tilde{\rho}^1 = \hat{n}^2$, $\tilde{\rho}^2 = 0$, $\epsilon_1 = \epsilon_3 = -1$, $\epsilon_2 = 1$ and $\sigma_{31} = \mathcal{R}_{33} = -\mathcal{R}_3^3$, where \mathcal{R}_{ij} and \mathcal{R}_j^i are the components of \mathcal{R}_h in the standard basis and the sign change is due to the fact that we use the metric η to switch from lower to upper indices and vice versa. The above explicit form of the matrix gives the \mathcal{R}_j^i 's. With all this in

mind, we easily obtain

$$\tau_2^\pm = \pm \sqrt{\frac{\mathcal{R}_3^3 - 1}{\mathcal{R}_3^3 + 1}} \quad (32)$$

which gives $\tau_2^\pm = \pm \tanh \frac{\psi_2}{2}$ using elementary hyperbolic trigonometry and the fact that $\mathcal{R}_3^3 = \text{ch}\psi_2$. Furthermore we have

$$\tau_1^\pm = \frac{(\hat{n}^1 \hat{n}^3 \tau - \hat{n}^2) \tau_2^\pm}{(\hat{n}^1 + \hat{n}^2 \hat{n}^3 \tau) \tau_2^\pm + (\epsilon + (\hat{n}^3)^2) \tau}, \quad \tau_3^\pm = \frac{(\hat{n}^1 \hat{n}^3 \tau + \hat{n}^2) \tau_2^\pm}{(\hat{n}^1 - \hat{n}^2 \hat{n}^3 \tau) \tau_2^\pm + (\epsilon + (\hat{n}^3)^2) \tau}.$$

Alternatively, we may consider the classical *Bryan* decomposition about the axes OX , OY and OZ which can be written explicitly as

$$\begin{aligned} & \begin{pmatrix} c\psi_3 & -s\psi_3 & 0 \\ s\psi_3 & c\psi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \text{ch}\psi_2 & 0 & \text{sh}\psi_2 \\ 0 & 1 & 0 \\ \text{sh}\psi_2 & 0 & \text{ch}\psi_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{ch}\psi_1 & -\text{sh}\psi_1 \\ 0 & -\text{sh}\psi_1 & \text{ch}\psi_1 \end{pmatrix} \\ &= \begin{pmatrix} c\psi_3 \text{ch}\psi_2 & -s\psi_3 \text{ch}\psi_1 - c\psi_3 \text{sh}\psi_2 \text{sh}\psi_1 & s\psi_3 \text{sh}\psi_1 + c\psi_3 \text{sh}\psi_2 \text{ch}\psi_1 \\ s\psi_3 \text{ch}\psi_2 & c\psi_3 \text{ch}\psi_1 - s\psi_3 \text{sh}\psi_2 \text{sh}\psi_1 & s\psi_3 \text{sh}\psi_2 \text{ch}\psi_1 - c\psi_3 \text{sh}\psi_1 \\ \text{sh}\psi_2 & -\text{ch}\psi_2 \text{sh}\psi_1 & \text{ch}\psi_2 \text{ch}\psi_1 \end{pmatrix} \end{aligned}$$

and for the vector parameter we have

$$\mathbf{c} = \frac{1}{1 - \tau_1 \tau_2 \tau_3} \begin{pmatrix} \tau_1 - \tau_2 \tau_3 \\ \tau_2 + \tau_1 \tau_3 \\ \tau_3 + \tau_1 \tau_2 \end{pmatrix}.$$

If we denote $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)^t$ and $\tilde{\boldsymbol{\tau}} = (-\tau_2 \tau_3, \tau_1 \tau_3, \tau_1 \tau_2)^t$, the above is proportional to the vector sum $\boldsymbol{\tau} + \tilde{\boldsymbol{\tau}}$, so in order to determine the type of the composite vector (resp. transformation) we need the sign of the expression

$$\mathbf{c}^2 \sim (\boldsymbol{\tau} + \tilde{\boldsymbol{\tau}})^2 = \boldsymbol{\tau}^2 + \tilde{\boldsymbol{\tau}}^2 - 2\tau_1 \tau_2 \tau_3.$$

In this example we have $\kappa = \eta$, $\omega = 1$, $\rho_k = \hat{n}_k$, $\tilde{\rho}^k = \hat{n}^k$, $\epsilon_1 = \epsilon_2 = 1$, $\epsilon_3 = -1$ and $\sigma_{31} = \mathcal{R}_{31} = -\mathcal{R}_1^3 = -\text{sh}\psi_2$, so we easily obtain

$$\tau_2^\pm = -\frac{1 \pm \sqrt{1 + (\mathcal{R}_1^3)^2}}{\mathcal{R}_1^3} = -\frac{1 \pm \text{ch}\psi_2}{\text{sh}\psi_2} = \left\{ -\coth \frac{\psi_2}{2}, \tanh \frac{\psi_2}{2} \right\}$$

and for the other two scalar parameters

$$\tau_1^\pm = \frac{(\hat{n}^1 + \hat{n}^2 \hat{n}^3 \tau) \tau_2^\pm}{(\epsilon + (\hat{n}^3)^2) \tau \tau_2^\pm - \hat{n}^1 \hat{n}^3 \tau + \hat{n}^2}, \quad \tau_3^\pm = \frac{(\hat{n}^3 - \hat{n}^1 \hat{n}^2 \tau) \tau_2^\pm}{(\epsilon - (\hat{n}^1)^2) \tau \tau_2^\pm - \hat{n}^1 \hat{n}^3 \tau + \hat{n}^2}$$

which can easily be rewritten for the covariant components \mathcal{R}_{ij} and \hat{n}_j , as well as in terms of trigonometric functions of the ψ_i 's.

For a more detailed derivation of the analogous expressions in the *Euclidean* case we refer to [5].

To sum up, as it has been shown in [18], one may always decompose into three transformations with respect to non-collinear axes (actually, we demand only the second axis not to be collinear with the other two) among which one needs to be space-like in the generic case. Apart from that, these axes may be arbitrary, as long as the positive discriminant condition (27) is fulfilled.

3.2. The Case of Two Axes

We are also interested in deriving explicit formulae for a decomposition of the type

$$\mathcal{R}_h(\tau \hat{\mathbf{n}}) = \mathcal{R}_h(\tau_2 \hat{\mathbf{c}}_2) \mathcal{R}_h(\tau_1 \hat{\mathbf{c}}_1) \quad (33)$$

together with the necessary and sufficient conditions for its existence.

Following the same idea as before, we easily obtain

$$\sigma_{21} = \hat{\mathbf{c}}_2 \cdot \mathcal{R}_h(\tau \hat{\mathbf{n}}) \hat{\mathbf{c}}_1 = \hat{\mathbf{c}}_2 \cdot \hat{\mathbf{c}}_1 = \kappa_{21} \quad (34)$$

which plays the role of such condition.

As for the expressions for τ_1 and τ_2 we use the explicit form (22) of

$$\mathbf{c} = \langle \mathbf{c}_2, \mathbf{c}_1 \rangle$$

which we multiply on the left by $\hat{\mathbf{n}}^\lambda$ and then consider hyperbolic scalar products with $\hat{\mathbf{c}}_1$ and $\hat{\mathbf{c}}_2$ respectively to obtain (assuming $\tau_{1,2} \neq 0$)

$$\tau_1 = \frac{\tilde{\rho}^3}{\epsilon_1 \rho_2 - \kappa_{12} \rho_1}, \quad \tau_2 = \frac{\tilde{\rho}^3}{\epsilon_2 \rho_1 - \kappa_{12} \rho_2}. \quad (35)$$

In the *Euclidean* case of pure $\text{SO}(3)$ rotations we have similar expressions with $\epsilon_1 = \epsilon_2 = 1$ and $\tilde{\rho}_3 = -\tilde{\rho}^3$.

$$\tau_1 = \frac{\tilde{\rho}_3}{\kappa_{12} \rho_1 - \rho_2}, \quad \tau_2 = \frac{\tilde{\rho}_3}{\kappa_{12} \rho_2 - \rho_1} \quad (36)$$

and they ought to be considered instead of formula (29) in [5], which is mistaken for some reason (in the notations of [5] we have u, v instead of τ_1, τ_2).

3.3. Degenerate Solutions

There is one particular situation in which one cannot determine (30) and (31) independently, but only a certain combination of these two, due to a lack of sufficient information. This is an analogue of the classical *gimbal lock* - a phenomenon, well known in the applications.

The condition for such a degenerate type of solution is rather simple

$$\hat{\mathbf{c}}_3 = \pm \mathcal{R}_h(\mathbf{c}) \hat{\mathbf{c}}_1 \quad (37)$$

and in that case (24) takes the form

$$\mathcal{R}_h(\mathbf{c}) = \mathcal{R}_h(\pm \tau_3 \mathcal{R}_h(\mathbf{c}) \hat{\mathbf{c}}_1) \mathcal{R}_h(\tau_2 \hat{\mathbf{c}}_2) \mathcal{R}_h(\tau_1 \hat{\mathbf{c}}_1) \quad (38)$$

which can also be written as

$$\mathcal{R}_h(\tau_2 \hat{\mathbf{c}}_2) \mathcal{R}_h(\tau_1 \hat{\mathbf{c}}_1) = \mathcal{R}_h(\mp \tau_3 \mathcal{R}_h(\mathbf{c}) \hat{\mathbf{c}}_1) \mathcal{R}_h(\mathbf{c}) = \mathcal{R}_h(\mathbf{c}) \mathcal{R}_h(\mp \tau_3 \hat{\mathbf{c}}_1) \quad (39)$$

where for the last equality we use the well-known relation

$$\mathcal{R}(\mathbf{c}) \mathcal{R}(\mathbf{a}) \mathcal{R}(\mathbf{c})^{-1} = \mathcal{R}(\mathcal{R}(\mathbf{c}) \mathbf{a}) \quad (40)$$

that is equally valid for *Euclidean* and hyperbolic rotations.

Multiplying both sides of (39) on the right with $\mathcal{R}_h(\pm \tau_3 \hat{\mathbf{c}}_1)$ we end up with

$$\mathcal{R}_h(\mathbf{c}) = \mathcal{R}_h(\tau_2 \hat{\mathbf{c}}_2) \mathcal{R}_h(\langle \tau_1 \hat{\mathbf{c}}_1, \pm \tau_3 \hat{\mathbf{c}}_1 \rangle) = \mathcal{R}_h(\tau_2 \hat{\mathbf{c}}_2) \mathcal{R}_h(\tau'_1 \hat{\mathbf{c}}_1) \quad (41)$$

where

$$\tau'_1 = \frac{\tau_1 \pm \tau_3}{1 \pm \epsilon_1 \tau_1 \tau_3} \quad (42)$$

according to (22).

We see that the problem is reduced to decomposition into a pair of rotations and since (41) guarantees that the corresponding condition (34) is fulfilled, we may use (35) to obtain

$$\tau_2 = \frac{\tilde{\rho}^3}{\epsilon_2 \rho_1 - \kappa_{12} \rho_2}, \quad \frac{\tau_1 \pm \tau_3}{1 \pm \epsilon_1 \tau_1 \tau_3} = \frac{\tilde{\rho}^3}{\epsilon_1 \rho_2 - \kappa_{12} \rho_1}. \quad (43)$$

The first equation above determines τ_2 uniquely, but the second one fails to do so with the remaining parameters τ_1 and τ_3 . Instead, it provides a single relation for them, which allows each to be expressed from the other, or in other words we end up with a one-parameter degenerate solution, just as in the *Euclidean* case (see also [27]).

3.4. The Trivial Element

Here we consider a decomposition of the trivial element in $SO^+(2, 1)$ (the identity transformation) which cannot be assigned a particular direction $\hat{\mathbf{n}}$, since the scalar (and respectively the vector) parameter is identically zero. Therefore neither ρ nor $\tilde{\rho}$ is well defined and we cannot use the results, obtained above (formulae (28), (30) and (31)), so instead we take advantage of the fact that each of the pseudo-rotations in the decomposition can be expressed as a superposition of the remaining two

$$\mathbf{c}_1 = \langle -\mathbf{c}_2, -\mathbf{c}_3 \rangle, \quad \mathbf{c}_2 = \langle -\mathbf{c}_3, -\mathbf{c}_1 \rangle, \quad \mathbf{c}_3 = \langle -\mathbf{c}_1, -\mathbf{c}_2 \rangle \quad (44)$$

since $\langle \mathbf{a}, 0 \rangle = \langle 0, \mathbf{a} \rangle = \mathbf{a}$. Then we multiply the k^{th} equality in the system with $\hat{\mathbf{c}}_k^\lambda$ and consider (hyperbolic) dot products with $\hat{\mathbf{c}}_2$ for the first, respectively $\hat{\mathbf{c}}_3$ for the second and $\hat{\mathbf{c}}_1$ for the third relation to obtain

$$\begin{aligned} (\kappa_{12}\kappa_{23} - \epsilon_2\kappa_{13})\tau_2\tau_3 + \omega\tau_3 &= 0 \\ (\kappa_{13}\kappa_{23} - \epsilon_3\kappa_{12})\tau_1\tau_3 + \omega\tau_1 &= 0 \\ (\kappa_{12}\kappa_{13} - \epsilon_1\kappa_{23})\tau_1\tau_2 + \omega\tau_2 &= 0 \end{aligned} \quad (45)$$

which always has the trivial solution $\tau = 0$ and for $\omega \neq 0$ there is one more

$$\tau_1 = \frac{\omega}{\epsilon_1\kappa_{23} - \kappa_{12}\kappa_{13}}, \quad \tau_2 = \frac{\omega}{\epsilon_2\kappa_{13} - \kappa_{12}\kappa_{23}}, \quad \tau_3 = \frac{\omega}{\epsilon_3\kappa_{12} - \kappa_{13}\kappa_{23}}. \quad (46)$$

In the case $\omega = 0$, just as in its *Euclidean* analogue (see [5]), there is a non-trivial solution if and only if $\hat{\mathbf{c}}_1 = \pm\hat{\mathbf{c}}_3$. This solution, however, is not uniquely determined - we have $\tau_2^+ = \tau_2^- = 0$ and $\tau_1 = \mp\tau_3$, so that the two non-trivial transformations in the decomposition are mutually inverse, which provides a good example for a degenerate solution.

3.5. Divergencies

We note that the denominator in (22) is very likely to vanish in certain cases and there is nothing disturbing about that. Since we are dealing with projective quantities, after all, division by zero is well justified operation. In order to avoid the

inconvenience of dealing with vectors of infinite magnitude, however, one may simply lift up back to the universal cover. Diverging vector parameters correspond to pseudo-quaternions with vanishing scalar part $\zeta_0 = 0 \Rightarrow \zeta^2 = -1$, so they may appear only if the compound vector is time-like and thus the transformation associated with it - elliptic. In particular, if the invariant axis is collinear with OZ , we encounter a *half-turn* (rotation by an angle π). More generally, we may have a product of a pure rotation and a *Lorentz boost*. Here we consider different configurations of compositions that produce such a transformation. The condition for this to happen, as can be seen from (22), has the simple form

$$1 + \mathbf{c}_1 \cdot \mathbf{c}_2 = 0. \quad (47)$$

In the parabolic case for example, we may think of the vector parameters as generated by \mathbb{R}^2 that can be written as complex numbers

$$\mathbb{C} \ni z_k = x_k + iy_k \longrightarrow \mathbf{c}_k = (x_k, y_k, \sqrt{x_k^2 + y_k^2}) = (z_k, |z_k|), \quad k = 1, 2$$

and then the hyperbolic dot product of the \mathbf{c}_k 's is given by

$$\mathbf{c}_1 \cdot \mathbf{c}_2 = \Re(z_1 \bar{z}_2) - |z_1| |z_2| = |z_1| |z_2| (\cos \phi - 1), \quad \phi = \arg z_2 - \arg z_1$$

so in order to have (47) satisfied we need for the angle between the two vectors to obey the constraint

$$\cos \phi = 1 - \frac{1}{|z_1| |z_2|}.$$

In particular, we may have $|z_1| = |z_2| = 1$ and $\phi = \frac{\pi}{2}$, or let $|z_1|, |z_2| \rightarrow \infty$ which leads to $\phi \rightarrow 0$, once more leaving two out of three independent parameters. Another example would be to restrict to one-parameter submanifold by the following construction

$$\mathbf{c}_1 = (s, s + 1, \sqrt{2s(s + 1) + 1}), \quad \mathbf{c}_2 = (s + 1, s, \sqrt{2s(s + 1) + 1}), \quad s \in \mathbb{R}.$$

In the elliptic and hyperbolic cases, we deal with co-dimension one submanifolds since the restriction to the vector norm is given by an inequality, rather than equality. Compositions of pure boosts that result in elliptic transformations are interesting for the applications (see Section 5) and as for the case of rotations, they have been thoroughly studied in [5]. There we make use of *l'Hôpital's* rule to derive the scalar parameters of the generalized *Euler* decomposition, when the compound transformation happens to be a half-turn. Similarly, we may study (30) and (31) in

the limit $\tau \rightarrow \infty$ to obtain

$$\begin{aligned}\tau_1^\pm &= \frac{(\epsilon\kappa_{23} - \rho_2\rho_3)\tau_2^\pm}{(\rho_1\tilde{\rho}^1 + \rho_2\tilde{\rho}^2)\tau_2^\pm + \rho_1\rho_3 - \epsilon\kappa_{13}} \\ \tau_3^\pm &= \frac{(\epsilon\kappa_{12} - \rho_1\rho_2)\tau_2^\pm}{(\rho_2\tilde{\rho}^2 + \rho_3\tilde{\rho}^3)\tau_2^\pm + \rho_1\rho_3 - \epsilon\kappa_{13}}.\end{aligned}\tag{48}$$

It could happen in particular that $\tau_2 \rightarrow \infty$ as well (if $\sigma_{31} \neq \kappa_{31}$ and $\Delta = 0 \neq \omega$). Then we apply *l'Hôpital's* rule once more and end up with

$$\tau_1 = \frac{\epsilon\kappa_{23} - \rho_2\rho_3}{\rho_1\tilde{\rho}^1 + \rho_2\tilde{\rho}^2}, \quad \tau_3 = \frac{\epsilon\kappa_{12} - \rho_1\rho_2}{\rho_2\tilde{\rho}^2 + \rho_3\tilde{\rho}^3}.\tag{49}$$

This simple tricks works also for compositions of the type (22) - if either \mathbf{c}_1 or \mathbf{c}_2 has divergent pseudo-norm (i.e., $\tau_1 \rightarrow \infty$ or $\tau_2 \rightarrow \infty$), then it is straightforward to write

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle \rightarrow \frac{\mathbf{c}_1 + \mathbf{c}_2 \wedge \mathbf{c}_1}{\mathbf{c}_2 \cdot \mathbf{c}_1}, \quad \langle \mathbf{c}_2, \mathbf{c}_1 \rangle \rightarrow \frac{\mathbf{c}_2 + \mathbf{c}_2 \wedge \mathbf{c}_1}{\mathbf{c}_2 \cdot \mathbf{c}_1}$$

in the former and in the latter case respectively, so that the divergencies cancel out. On the other hand, if both scalar parameters tend to infinity simultaneously, then we simply have

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle \rightarrow \frac{\mathbf{c}_2 \wedge \mathbf{c}_1}{\mathbf{c}_2 \cdot \mathbf{c}_1}$$

similarly to the *Euclidean* case.

4. Null Vectors and Invariant Axes

In this section we discuss the hyperbolic analogue of *Euler's* fixed axis theorem and then focus on the behavior of parabolic and elliptic representatives of $\text{SO}^+(2, 1)$.

4.1. Euler's Fixed Axis Theorem

First we remind that in the *Euclidean* case (see e.g. [5, 26]) there is a theorem due to *Euler*, stating that a special orthogonal transformation in \mathbb{R}^{2k+1} has an invariant axis, determined by the eigenvector, corresponding to the eigenvalue one. The latter is unique except in the trivial case - the identity operator, for which the eigenspace of such vectors coincides with the imbedding space \mathbb{R}^{2k+1} .

One possible proof uses the fact that skew-symmetric matrices in odd dimensions are singular and thus have a zero eigenvalue. Since they constitute the *Lie algebra*

$\mathfrak{so}(2k + 1)$ of the corresponding group of rotations, for the latter we have a unit eigenvalue given by the exponential map, which is either simple in the generic case, or triple in the trivial one.

Similarly, we may construct the standard matrix representation of $\mathfrak{so}(2, 1)$ from the adjoint action of the generators \mathbf{e}_k (or $\tilde{\mathbf{e}}_k$), which allows for expressing a generic element $\mathcal{A} \in \mathfrak{so}(2, 1)$ as

$$\mathcal{A}(\mathbf{x}) = x^k \text{ad}(\mathbf{e}_k) = \mathbf{x}^\wedge = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}. \quad (50)$$

This gives for the eigenvalues $\lambda_1 = 0$ and $\lambda_{2,3} = \pm \sqrt{x_1^2 + x_2^2 - x_3^2} = \pm \sqrt{\mathbf{x} \cdot \mathbf{x}}$ and we may consider some specific cases:

1. For $x_3 = 0$ and $\mathbf{x} \in \mathbb{R}^2$, that is a particular example of a space-like vector (pointing outside the light cone), we have distinct real eigenvalues and eigenvectors. Moreover \mathcal{A} is symmetric in this case, so it can easily be put into diagonal form.
2. When $x_1 = x_2 = 0$, $x_3 \neq 0$ - the vector is time-like (inside the light cone) and we end up with a skew-symmetric matrix that generates a rotation. The eigenvalues are 0 and $\pm ix_3$ and the real eigenvector is oriented in the z direction (we end up with a pure rotation about OZ).
3. Finally, if \mathbf{x} is null ($\mathbf{x} \cdot \mathbf{x} = 0$), we end up with a triple eigenvalue $\lambda = 0$ and a triple eigenvector \mathbf{x} as already discussed.

The above present particular examples for generators of *hyperbolic*, *elliptic* and respectively *parabolic* transformations. More generally, similarly to $\text{SL}(2, \mathbb{R})$, we have an easy criterion to distinguish between these cases, namely \mathcal{R}_h is

1. *hyperbolic* if $\text{Tr}(\mathcal{R}_h) > 3$
2. *elliptic* if $\text{Tr}(\mathcal{R}_h) < 3$
3. *parabolic* if $\text{Tr}(\mathcal{R}_h) = 3$

which can be used to determine the type of its vector parameter: it is space-like in the first case, time-like in the second one and null or isotropic (also called *light-like*) in the third case.

After we have determined the type of the vector-parameter, it is not difficult to determine its direction and magnitude in a way, similar to the one, proposed in [26]

for the $SO(3)$ case. Actually, as long as $\varphi \neq 0$ and in the pure rotational case also $\varphi \neq \pi$, we can always determine \mathbf{c} , at least up to a scale factor, from the term \mathbf{c}^\wedge in (15). More precisely we have

$$\hat{\mathbf{n}} = \frac{1 - \epsilon\tau^2}{2\tau} \begin{pmatrix} -\mathcal{R}_{23} \\ \mathcal{R}_{13} \\ -\mathcal{R}_{12} \end{pmatrix} \quad (51)$$

where \mathcal{R}_{ij} denote the matrix entries of $\mathcal{R}_h(\tau\hat{\mathbf{n}})$ in the standard basis and the pre-factor on the right is equal to $(\sinh \varphi)^{-1}$, $(\sin \varphi)^{-1}$ or φ^{-1} for $\epsilon = \pm 1$ and $\epsilon = 0$ respectively. In the elliptic case this does not determine the angle completely, so we need also $\cos \varphi$ which is easy to find from $\text{Tr}(\mathcal{R}_h) = 2 \cos \varphi + 1$. One specific example is the so-called *half turn* $\mathcal{O}(\hat{\mathbf{n}}) = 2\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}^t - \mathcal{I}$.

With this technique we can always determine the vector parameter (or the angle and axis) for a $SO^+(2, 1)$ transformation and *vice versa*.

4.2. The Parabolic Case

In *Minkowski* space each null vector lies in its orthogonal complement - something that has no analogue in *Euclidean geometry*. When we use such vector in (15), the exponential map gives for the corresponding group element $\mathcal{R}_h(\mathbf{x}) = \exp \mathcal{A}(\mathbf{x})$ a triple eigenvalue $e^0 = 1$ although there is a single invariant axis, determined by $\hat{\mathbf{x}}$. We note that for a generic $\mathbf{x} \in \mathbb{R}^3$ we have

$$\mathcal{A}(\mathbf{x}) = \mathbf{x}^\wedge, \quad \mathcal{A}^2(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{x}) \mathcal{I} - \mathbf{x} \otimes (\eta \mathbf{x})^t, \quad \mathcal{A}^3(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{x}) \mathcal{A}(\mathbf{x}) \quad (52)$$

which leads back to different versions of *Rodrigues'* formula, based on the matrix exponent $\mathcal{R}_h(\mathbf{x}) = \exp \mathcal{A}(\mathbf{x})$ if we put $\mathbf{x} = \varphi \hat{\mathbf{n}}$. However, this works only if $\mathbf{x} \cdot \mathbf{x} \neq 0$ - for a null vector we end up with

$$\mathcal{R}_h(\mathbf{x}) = \exp \mathcal{A}(\mathbf{x}) = \mathcal{I} + \mathbf{x}^\wedge - \frac{1}{2} \mathbf{x} \otimes (\eta \mathbf{x})^t, \quad \mathbf{x} \cdot \mathbf{x} = 0 \quad (53)$$

since all higher order terms vanish.

According to our previous notations, however, we have $\mathbf{x} = \varphi \hat{\mathbf{n}}$ and thus

$$\mathcal{R}_h(\varphi \hat{\mathbf{n}}) = \exp \mathcal{A}(\varphi \hat{\mathbf{n}}) = \mathcal{I} + \varphi \hat{\mathbf{n}}^\wedge - \frac{\varphi^2}{2} \hat{\mathbf{n}} \otimes (\eta \hat{\mathbf{n}})^t, \quad \hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 0 \quad (54)$$

which can be used in an appropriate scalar product to obtain the analogue of (28) for the parabolic case

$$\varphi^\pm = \frac{-\omega \pm \sqrt{\Delta}}{\kappa_{12}\kappa_{23}} \quad (55)$$

with

$$\Delta = \omega^2 + 2(\kappa_{31} - \sigma_{31})\kappa_{12}\kappa_{23} \geq 0 \quad (56)$$

being the necessary and sufficient condition for the existence of real solutions.

We note that in this case the *Euler* substitution is not justified, since we are at a separatrix between elliptic and hyperbolic regions. That is why the parametrizations with the “angle” φ appears most natural - they leads to the *Rodrigues’* formula and we can write φ instead of $\sin \varphi$ ($\sinh \varphi$), 1 for $\cos \varphi$ (respectively $\cosh \varphi$) and corrected value $\frac{\varphi^2}{2}$ for $1 - \cos \varphi$ (or $\cosh \varphi - 1$).

From this point of view (54) is a substitute for (13) and (17) for the parabolic case. It also allows to use $\tau = \frac{\varphi}{2}$ so that (15) and (22) make sense as well.

On the other hand setting $\epsilon_2 = 0$, in (28) we easily derive the relation

$$\tau_2^\pm = \frac{-\omega \pm \sqrt{\Delta}}{2\kappa_{12}\kappa_{23}} \quad (57)$$

which is in complete accordance with the corresponding expressions obtained above.

Now let us see how composition of parabolic elements works on the level of vector parameters. Applying (22) with $\mathbf{c}_i = \frac{\varphi_i}{2} \hat{\mathbf{c}}_i$, $(\hat{\mathbf{c}}_i, \hat{\mathbf{c}}_i) = 1$ we end up with

$$\langle \mathbf{c}_2, \hat{\mathbf{c}}_1 \rangle = \frac{1}{2} \left(\varphi_1 \hat{\mathbf{c}}_1 + \varphi_2 \hat{\mathbf{c}}_2 + \frac{\varphi_1 \varphi_2}{2} \hat{\mathbf{c}}_2 \wedge \hat{\mathbf{c}}_1 \right) \quad (58)$$

in the case $\hat{\mathbf{c}}_1 \cdot \hat{\mathbf{c}}_2 = 0$. In particular for $\hat{\mathbf{c}}_1 = \hat{\mathbf{c}}_2$ the above reduces to

$$\varphi = \varphi_1 + \varphi_2.$$

On the other hand, one may consider the asymptotic behavior of the composition for small and large values of the parameters φ_i . In the first case the composition is reduced to a simple vector summation

$$\mathbf{c} \sim \mathbf{c}_1 + \mathbf{c}_2, \quad \varphi_1, \varphi_2 \rightarrow 0$$

while in the second one, we have a formula, familiar from the rotational case

$$\mathbf{c} \sim \frac{\mathbf{c}_2 \wedge \mathbf{c}_1}{\mathbf{c}_2 \cdot \mathbf{c}_1}, \quad \varphi_1, \varphi_2 \rightarrow \infty.$$

4.3. The Embedding $SO(2) \subset SO^+(2, 1)$

It is clear that $\mathfrak{so}(2) \subset \mathfrak{so}(2, 1)$ as the maximal compact subalgebra. The embedding is quite visible from the particular form of the matrix \mathcal{A} - one restricts to the skew-symmetric block by setting $x_1 = x_2 = 0$ and ends up with a generator of rotation about OZ . Obviously the same relation holds for the corresponding groups and one can make it explicit by considering the power series expansion for $\exp \mathcal{A}$ in this case - the upper-left 2×2 -block of \mathcal{A} when squared gives minus identity and thus, for this particular choice of vector-parameter, we end up with the classical *Rodrigues'* formula

$$\mathcal{R}(\hat{\mathbf{n}}, \varphi) = \cos \varphi \mathcal{I} + (1 - \cos \varphi) \hat{\mathbf{n}} \otimes \hat{\mathbf{n}}^t + \sin \varphi \hat{\mathbf{n}}^\times, \quad \hat{\mathbf{n}} = \hat{\mathbf{e}}_z \quad (59)$$

or, alternatively, one may use the basis of complex eigenvectors of \mathcal{A} to derive

$$\exp \mathcal{A} = \text{diag}(e^{i\varphi}, e^{-i\varphi}, 1), \quad \mathbf{x} \rightarrow (0, 0, \varphi)$$

which is an equivalent $U(1)$ representation.

For the physical applications considered in this paper, however, we are interested in the way ordinary $SO(2)$ rotations interact with pure *Lorentz boosts*. Suppose we want to compose two $SO^+(2, 1)$ transformations - one of the former and one of the latter type, so that we have $\hat{\mathbf{c}}_1 \in \mathbb{R}^2$, $\hat{\mathbf{c}}_2 = \hat{\mathbf{e}}_z$. Since the two vector parameters will be orthogonal in the hyperbolic (as well as in the *Euclidean*) metric, we have for the composition

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \wedge \mathbf{c}_1 \quad (60)$$

and since the last vector on the right has a vanishing third component, we may write $\mathbf{c}_2 \times \mathbf{c}_1$ instead of $\mathbf{c}_2 \wedge \mathbf{c}_1$. Furthermore we may decompose $\hat{\mathbf{c}}_1 = \cos \gamma \hat{\mathbf{e}}_x + \sin \gamma \hat{\mathbf{e}}_y$, where γ is the polar angle, representing the unit vector $\hat{\mathbf{c}}_1$ in the XOY -plane, and then use $\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_x = \hat{\mathbf{e}}_y$, $\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_y = -\hat{\mathbf{e}}_x$ to obtain

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \tau_1 (\cos \gamma - \tau_2 \sin \gamma) \hat{\mathbf{e}}_x + \tau_1 (\sin \gamma + \tau_2 \cos \gamma) \hat{\mathbf{e}}_y + \tau_2 \hat{\mathbf{e}}_z. \quad (61)$$

Alternatively, we may compose in the reverse order

$$\langle \mathbf{c}_1, \mathbf{c}_2 \rangle = \tau_1 (\cos \gamma + \tau_2 \sin \gamma) \hat{\mathbf{e}}_x + \tau_1 (\sin \gamma - \tau_2 \cos \gamma) \hat{\mathbf{e}}_y + \tau_2 \hat{\mathbf{e}}_z. \quad (62)$$

5. Wigner Rotation

A well-known fact from *special relativity* is that the composition of two non-collinear *Lorentz boosts* results in transformation that is no longer a pure boost,

but can be expressed as a product of a rotation and a boost

$$\mathcal{R}_h(\mathbf{c}_2)\mathcal{R}_h(\mathbf{c}_1) = \mathcal{R}_h(\tilde{\tau}\mathbf{e}_z)\mathcal{R}_h(\mathbf{c}) \quad (63)$$

where the first factor on the righthand side stands for the rotational contribution (a pure rotation in $SO^+(2, 1)$ is possible only about the third axis \mathbf{e}_z) and the second one is the hyperbolic (boost) contribution to the result.

Since by construction \mathbf{c}_1 and \mathbf{c}_2 have vanishing third components, they can be expressed with planar vectors, or more conveniently - complex numbers $z_1, z_2 \in \mathbb{C}$

$$z_k = x_k + iy_k \rightarrow \mathbf{c}_k = (x_k, y_k, 0)^t, \quad k = 1, 2. \quad (64)$$

From (61) we see that the third component of the composition $\langle \mathbf{c}_2, \mathbf{c}_1 \rangle$ is responsible for the rotational contribution to the compound transformation. Since \mathbf{c}_1 and \mathbf{c}_2 lie in the XOY -plane, this component is quite easy to obtain, using (22)

$$\tilde{\tau} = \langle \mathbf{c}_2, \mathbf{c}_1 \rangle_z = \frac{x_1y_2 - x_2y_1}{1 + x_1x_2 + y_1y_2} = -\frac{\Im(1 + z_1\bar{z}_2)}{\Re(1 + z_1\bar{z}_2)} \quad (65)$$

and therefore we have for the angle of that rotation $\theta = 2 \arctan \tilde{\tau}$ the simple expression

$$\theta = -2 \arg(1 + z_1\bar{z}_2). \quad (66)$$

The latter is known as *Wigner angle* and plays an important role in both *special relativity* and *quantum mechanics*. The simple and compact formula we obtain is very convenient for the interpretation of θ as a *Berry phase* [6, 25], or (negative) area of a hyperbolic triangle [2].

Note that if we reverse the order in the composition, only the sign of θ is changed. Moreover, once we have determined $\tilde{\tau}$, it is not difficult to retrieve the vector parameter \mathbf{c} of the boost in the decomposition

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \langle \tilde{\tau}\mathbf{e}_z, \mathbf{c} \rangle. \quad (67)$$

With the help of (61) and (62) it is almost straightforward to write

$$\mathbf{c}_\pm = \frac{\Re(1 + z_1\bar{z}_2)}{|1 + z_1\bar{z}_2|^2} \begin{pmatrix} \Re(z_{12}) \pm \tilde{\tau} \Im(z_{12}) \\ \Im(z_{12}) \mp \tilde{\tau} \Re(z_{12}) \\ 0 \end{pmatrix}, \quad z_{12} = z_1 + z_2 \quad (68)$$

where the two solutions \mathbf{c}_+ and \mathbf{c}_- correspond to the forward $\langle \mathbf{c}_2, \mathbf{c}_1 \rangle$ and backward $\langle \mathbf{c}_1, \mathbf{c}_2 \rangle$ composition respectively. Note that we may easily find the components of the corresponding pseudo-quaternion ζ with the help of (19)

$$\mathbf{c}_\pm^2 = \left| \frac{z_1 + z_2}{1 + z_1 \bar{z}_2} \right|^2 \implies \zeta_0 = (1 + \mathbf{c}_\pm^2)^{-\frac{1}{2}} = \frac{|1 + z_1 \bar{z}_2|}{\sqrt{|1 + z_1 \bar{z}_2|^2 - |z_1 + z_2|^2}} \quad (69)$$

$$\zeta_\pm = \frac{\Re(1 + z_1 \bar{z}_2)}{\sqrt{|1 + z_1 \bar{z}_2|^2 - |z_1 + z_2|^2}} \begin{pmatrix} \Re(z_1 + z_2) \pm \tilde{\tau} \Im(z_1 + z_2) \\ \Im(z_1 + z_2) \mp \tilde{\tau} \Re(z_1 + z_2) \\ 0 \end{pmatrix}$$

where we have written only the solution with positive scalar part, since ζ and $-\zeta$ participate symmetrically (the \pm subscript stands for the different permutations of the two boosts).

In scattering theory we encounter a $SU(1, 1)$ representation in the *monodromy matrix* formalism. More precisely, the monodromy matrix \mathcal{M} relates the left and right free particle asymptotic solutions (or rather one-jets of solutions) in the one dimensional scattering problem. When we write these solutions in a real basis (with \sin and \cos functions), this matrix is bound to be $SL(2, \mathbb{R})$, due to a conservation law for the phase volume. Then, we conclude that back in the standard complex basis, usually preferred in *quantum mechanics*, $\mathcal{M} \in SU(1, 1)$. Whether the process takes place at a point or over an interval, the monodromy matrix (as an alternative to the scattering matrix) describes the scattering potential and its entries are given by the corresponding *transition* and *reflection* coefficients. For simplicity of notations we are going to write it as

$$\mathcal{M} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(1, 1) \quad (70)$$

so if we have two scatterers on the same axis with potentials V_1 and V_2 , the compound monodromy matrix will be given by $\mathcal{M} = \mathcal{M}_2 \mathcal{M}_1$. It would be convenient to expand these matrices in the basis (1) built in Section 2, which is equivalent to associating to each of them a pseudo-quaternion: $\mathcal{M}_1 \rightarrow \xi$, $\mathcal{M}_2 \rightarrow \zeta$ and $\mathcal{M} \rightarrow \zeta \xi$, and then use the composition law (21) to find ζ explicitly. We may also use complex coordinates α_i, β_i to write \mathcal{M}_i in the form (70). Note that if we project this construction (dividing by the scalar part), we would obtain the composition for the vector-parameters (22) in the corresponding $SO^+(2, 1)$ representation.

What we are aiming for is to find the angle of *Wigner* rotation that appears as a result of such superposition of scatterers. Mathematically, this process is explained

with the fact that the compound monodromy \mathcal{M} can be represented as a composition of an elliptic and hyperbolic transformation

$$\mathcal{M} = \mathcal{M}_e \mathcal{M}_h, \quad \mathcal{M}_e \in \text{SO}(2, \mathbb{R}). \quad (71)$$

In particular we may also represent \mathcal{M}_e and \mathcal{M}_h with pseudo-quaternions, say $\tilde{\zeta}$ and $\tilde{\xi}$ respectively, where $\tilde{\xi}$ has vanishing fourth component while for $\tilde{\zeta}$ - the second and the third are zero. If we map these to $\text{SO}^+(2, 1)$ via projection, the composition for the vector parameters coincides with (61) so we may find the *Wigner* angle from the third component of the compound parameter as

$$\theta = 2 \arctan \langle \mathbf{c}_2, \mathbf{c}_1 \rangle_z \quad (72)$$

and since $\zeta \xi = \tilde{\zeta} \tilde{\xi}$, lifting up back to the universal cover gives simply

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\zeta_0 \boldsymbol{\xi} + \xi_0 \boldsymbol{\zeta} + \boldsymbol{\zeta} \wedge \boldsymbol{\xi}}{\zeta_0 \xi_0 + \boldsymbol{\zeta} \cdot \boldsymbol{\xi}}. \quad (73)$$

The normalized z -component is given by $\frac{\zeta_3}{\zeta_0}$, that is the ratio between the imaginary and the real part of the first matrix entry of the composition $\mathcal{M}_2 \mathcal{M}_1$ if we work in complex coordinates, i.e.,

$$\theta = 2 \arg(\alpha_2 \alpha_1 + \beta_2 \bar{\beta}_1), \quad \alpha_1, \alpha_2 \in \mathbb{R} \quad (74)$$

in a complete correspondence with (66) (in order to see this one needs only to introduce $z_k = \frac{\beta_k}{\alpha_k}$). The restriction on the α_k 's guarantees that the \mathcal{M}_k 's represent pure boosts. Moreover, one may easily find the remaining boost in the decomposition, substituting $z_k = \frac{\beta_k}{\alpha_k}$ in (69)

$$\zeta_0 = \frac{|\delta|}{\sqrt{|\delta|^2 - |\nu|^2}}, \quad \boldsymbol{\zeta}_{\pm} = \frac{\Re(\delta)}{|\alpha_1 \alpha_2| \sqrt{|\delta|^2 - |\nu|^2}} \begin{pmatrix} \Re(\nu) \pm \tilde{\tau} \Im(\nu) \\ \Im(\nu) \mp \tilde{\tau} \Re(\nu) \\ 0 \end{pmatrix} \quad (75)$$

where we use the short notations $\delta = \alpha_1 \alpha_2 + \beta_1 \bar{\beta}_2$ and $\nu = \alpha_1 \beta_2 + \beta_1 \alpha_2$.

Now to replace α_i and β_i with the physical quantities t_i and r_i - the transition and reflection coefficients (the latter is usually denoted by r_+ in the literature), defined

by the asymptotic behavior of the solutions

$$\begin{aligned}\Psi(k, x) &\sim e^{ikx} + r(k) e^{-ikx}, & x \rightarrow -\infty \\ \Psi(k, x) &\sim t(k) e^{ikx}, & x \rightarrow \infty.\end{aligned}$$

Using the well-known relations (see [3, 14, 15])

$$\alpha = \frac{1}{\bar{t}}, \quad \beta = -\frac{\bar{r}}{\bar{t}} \quad (76)$$

and restricting to $SU(1, 1)$ matrices with real diagonal entries $t_1, t_2 \in \mathbb{R}$ as they are the ones corresponding to pure boosts, we end up with the compact expression

$$\theta^\pm = \pm 2 \arg(1 + r_1 \bar{r}_2) \quad (77)$$

for the forward and the backward composition respectively, and the condition for zero *Wigner* rotation in both cases has the form $r_1 \bar{r}_2 \in \mathbb{R}$.

The matrix of the remaining boost in this representation can be derived from (69) with $z_k = -\bar{r}_k$. In pseudo-quaternion coordinates it has the form

$$\begin{aligned}\mathbf{c}_\pm^2 = \left| \frac{r_1 + r_2}{1 + r_1 \bar{r}_2} \right|^2 &\implies \zeta_0 = (1 + \mathbf{c}_\pm^2)^{-\frac{1}{2}} = \frac{|1 + r_1 \bar{r}_2|}{\sqrt{|1 + r_1 \bar{r}_2|^2 - |r_1 + r_2|^2}} \\ \zeta_\pm &= \frac{\Re(1 + r_1 \bar{r}_2)}{\sqrt{|1 + r_1 \bar{r}_2|^2 - |r_1 + r_2|^2}} \begin{pmatrix} -\Re(r_1 + r_2) \pm \tilde{\tau} \Im(r_1 + r_2) \\ \Im(r_1 + r_2) \pm \tilde{\tau} \Re(r_1 + r_2) \\ 0 \end{pmatrix}\end{aligned} \quad (78)$$

respectively for the forward and the backward composition of scatterers.

With a little effort we can relate the above results to the *Thomas precession* phenomenon, considering composition of infinitesimal vector parameters. However, we chose to ignore this opportunity for now and only refer to [4, 28] for different, but related treatment of the problem.

6. Concluding Remarks

In the present article we focus mainly on the generalized *Euler* parameters in the decomposition of three-dimensional pseudo-rotations. Using pseudo-quaternion and vector-parameter techniques, we manage to obtain quite simple and compact

explicit solutions, similar to the ones we give for the *Euclidean* (pure rotational) case [5].

Due to the well-known *Lie* algebra isomorphisms $\mathfrak{so}(2, 1) \simeq \mathfrak{su}(1, 1) \simeq \mathfrak{sl}(2, \mathbb{R})$, we are able to give analogous relations for the structure groups of the upper half-plane and the *Poincaré* disk - actually, projecting and lifting up to the universal cover is so easy for the formulae we obtain, that we did not bother to write all the expressions.

Finally, we provide a well-known physical example - calculation of the *Wigner* angle in *special relativity* and *scattering theory*, which appears to be particularly easy using the machinery developed here.

Since most of the work is concerned with the three-dimensional *Lorentz* group $SO^+(2, 1)$, we note that it has plenty of physical applications and should not be considered a mere toy model for $3 + 1$ dimensional *special relativity*. This has become particularly apparent since the breakthrough in *graphenes* [7].

However, the $3 + 1$ dimensional case seems like a natural extension and can be treated with similar techniques. Another investigation of physical relevance would be considering the case of coordinate frame, attached to the moving object. Actually, most of the results, obtained here, maintain their validity with a slight modification, but we reserve the pleasure of describing this correspondence for future work.

References

- [1] Ahlfors L., *Möbius Transformations in Several Dimensions*, School of Mathematics, University of Minnesota 1981.
- [2] Aravind P., *The Wigner Angle as an Anholonomy in Rapidity Space*, Am. J. Phys. **65** (1997) 634-636.
- [3] Arnold V., *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer, New York 1983.
- [4] Ben-Manahem A., *Wigner's Rotation Revisited*, Am. J. Phys. **53** (1985) 62-66.
- [5] Brezov D., Mladenova C. and Mladenov I., *Vector Decompositions of Rotations*, J. Geom. Symmetry Phys. **28** (2012) 67-103.
- [6] Bruno P., *The Berry Phase in Magnetism and the Anomalous Hall Effect*, Handbook of Magnetism and Advanced Magnetic Materials vol. 1, Wiley, Chichester (2007), pp 540-558.
- [7] Castro N., Guinea F., Peres N., Novoselov K. and Geim A., *Electronic Properties of Graphenes*, Rev. Mod. Phys. **81** (2009) 109-162.

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- [8] Chub V., *On the Possibility of Application of One System of Hypercomplex Numbers in Inertial Navigation*, Mech. Solids **37** (2002) 1-17.
- [9] Davenport P., *Rotations About Nonorthogonal Axes*, AIAA Journal **11** (1973) 853-857.
- [10] Fedorov F., *The Lorentz Group* (in Russian), Science, Moskow 1979.
- [11] Gibbs J., *Elements of Vector Analysis*, Tuttle, Morehouse & Taylor, New Haven 1881.
- [12] Gilmore R., *Relations Among Low-dimensional Simple Lie Groups*, J. Geom. Symmetry Phys. **28** (2012) 1-45.
- [13] Helgason S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, San Diego 1978.
- [14] Hurt N., *Geometric Quantization in Action*, Reidel, Dordrecht 1983.
- [15] Kandilarov B. and Detcheva V., *Transfer-Matrix Approach and Basic Ideas of the Zone Theory* (in Bulgarian), Bulgarian Academy of Sciences, Sofia 1988.
- [16] Kocik J., *Clifford Algebras and Euclid's Parameterization of Pythagorean Triples*, Adv. Appl. Clifford Algebras **17** (2007) 71-93.
- [17] Kuipers J., *Quaternions and Rotation Sequences*, Geometry, Integrability and Quantization **1** (2000) 127-143.
- [18] Lee, H. *On Plane Factorizations of Pseudo-Euclidean Rotations*, Quart. J. Math. **15** (1944) 7-10.
- [19] Mladenov I., *Geometry of the Saxon-Hutner Theorem*, Phys. Lett. A **230** (1997) 245-252.
- [20] Mladenov I., *On the Saxon-Hutner Theorem*, C. R. Bulg. Acad. Sci. **38** (1985) 993-996.
- [21] Mladenova C. and Mladenov I., *Vector Decomposition of Finite Rotations*, Rep. Math. Phys. **68** (2011) 107-117.
- [22] Mladenov I. and Vassileva J., *A Note on Hybridization*, Commun. Math. Chem. **11** (1981) 69-73.
- [23] Mladenova C., *Group Theory in the Problems of Modeling and Control of Multi-Body Systems*, J. Geom. Symmetry Phys. **8** (2006) 17-121.
- [24] Mladenova C., *An Approach to Description of a Rigid Body Motion*, C. R. Acad. Sci. Bulg. **38** (1985) 1657-1660.
- [25] Mukunda N., Aravind P. and Simon R., *Wigner Rotations, Bargmann Invariants and Geometric Phases*, J. Phys. A: Math. & Gen. **36** (2003) 2347-2370
- [26] Palais B., Palais R. and Rodi S., *A Disorienting Look at Euler's Theorem on the Axis of a Rotation*, Am. Math. Monthly **116** (2009) 892-909.

- [27] Piovan G. and Bullo F., *On Coordinate-Free Rotation Decomposition Euler Angles About Arbitrary Axes*, IEEE Trans. Robotics **28** (2012) 728-733.
- [28] Rhodes A. and Semon M., *Relativistic Velocity Space, Wigner Rotation and Thomas Precession*, Am. J. Phys. **72** (2004) 943-961
- [29] Rodrigues O., *Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace, et de la variation des coordonnées provenant de ces déplacements considérés indépendamment des causes qui peuvent les produire*, J. Math. Pures Appl. **5** (1840) 380-440.
- [30] Ungar A., *Analytic Hyperbolic Geometry: Mathematical Foundations and Applications*, World Scientific, Singapore (2005)
- [31] Ungar A., *The Proper-Time Lorentz Group Demystified*, J. Geom. Symmetry Phys. **4** (2005) 69-95.
- [32] Ward J., *Quaternions and Cayley Numbers: Algebra and Applications*, Kluwer, Dordrecht 1997.
- [33] Wittenburg J., *Dynamics of Multibody Systems*, Springer, Berlin 2008.
- [34] Wohlhart K., *Decomposition of a Finite Rotation into Three Consecutive Rotations About Given Axes*, In: Proc. VI-th Int Conf. on Theory of Machines and Mechanisms, Czechoslovakia, Liberec 1992, pp 325-332.

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