Vitali's theorem in vector lattices

Dedicated to Professor Yoshie Katsurada on her 60th birthday

By Shozo KOSHI

§ 1. Introduction

Let Ω be a measure space with finite measure μ. Then, Vitali's theorem announces: *let f_n $(n=1, 2, \cdots)$ be a sequence of summable functions on Ω with equi-absolutely continuous integrals and f_n converges to f in measure. Then,*

$$\lim_{n\to\infty}\int_\Omega f_n\, d\mu = \int_\Omega f\, d\mu .$$

In this note, we shall generalize the Vitali's theorem to vector lattices.

§ 2. Convergences in vector lattices.

Let R be a σ-complete vector lattice i.e. $\bigcap_{n=1}^{\infty} a_n$ exists for every positive elements $0 \leqq a_n \in R$ $(n=1, 2, \cdots)$. In the sequel, we assume that R is σ-complete. For $a_n \in R$ $(n=1, 2, \cdots)$, if $\bigcap_{m=1}^{\infty}(\bigcup_{n\geqq m} a_n)$ and $\bigcup_{m=1}^{\infty}(\bigcap_{n\geqq m} a_n)$ exist and equal to a, then we denote that

$$o-\lim_{n\to\infty} a_n = a .$$

In this case, we say that the sequence $\{a_n\}$ is *order-convergent* to a. It is easy to see that o-$\lim a_n = a$ iff there exist $b_n \downarrow 0$ (i.e. $b_1 \geqq b_2 \geqq \cdots$ with $\bigcap_{n=1}^{\infty} b_n = 0$) such that

$$\left| a_n - a \right| \leqq b_n$$

where $|x| = x \cup (-x) = x^+ + x^-$ for $x \in R$.

We shall define star-order-convergence as follows: a sequence a_n $(n=1, 2, \cdots)$ is said to be *star-order-convergent* to a if for every subsequence of $\{a_n\}$, there exists its subsequence which is order-convergent to a. We shall denote

$$s-o-\lim a_n = a$$

if a_n $(n=1, 2, \cdots)$ is star-order-convergent to a.

For a subset M of R, we denote $M^{\perp} = \{a : |a| \cap |b| = 0 \text{ for all } b \in M\}$.

If we can decompose $a \in R$ into as follows:

$$(\mathrm{A}) \qquad a = a_1 + a_2$$

with $a_1 \in M$ and $a_2 \in M^{\perp}$, then M is called normal.

If M is normal, then $M = M^{\perp\perp}$ and the decomposition (A) of a is uniquely determined. Namely,

$$a = a_1 + a_2, \quad a = a_1' + a_2', \quad a_1, a_1' \in M, \quad a_2, a_2' \in M^{\perp}$$

imply $\quad a_1 = a_1' \quad$ and $\quad a_2 = a_2'$.

We see that the operator $[M]a = a_1$ is linear and lattice-homomorphic. $[M]$ is called a projection operator.

The normal subsets of R (or equivalently projection operators) constitutes a Boolean lattice by the usual order.

Let R be σ-complete. For $0 \leqq p \in R$, the subset $\{p\}^{\perp\perp}$ is normal and

$$\left[\{p\}^{\perp\perp}\right] a = \left(\text{denoted by } [p]\,a\right) = \bigcup_{n=1}^{\infty} (np \cap a) \quad \text{for} \quad a \geqq 0\,.$$

In general, $[p]a = [p]a^{+} - [p]a^{-}$.

$\{[N]\,;\ [p] \geqq [N]\}$ is σ-complete as a Boolean lattice and for every $[N]$ with $[p] \geqq [N]$ there exists $q \in R$ with $[N] = [q]$.

For arbitrary $p \in R$, $[p] = [|p|]$.

Let $[p_n]$ be a sequence of projection operators. $[P_n]$ is order-convergent to 0 if $\bigcap_{m=1}^{\infty}(\bigcup_{n \geqq m} P_n) = 0$, i.e. there exists $[Q_n] \geqq [P_n]$ with $[Q_1] \geqq [Q_2] \geqq \cdots$ and $\bigcap_{n=1}^{\infty}[Q_n] = 0$.

We shall write $[P_n] \downarrow 0$ if $[P_1] \geqq [P_2] \geqq \cdots$ and $\bigcap_{n=1}^{\infty} [P_n] = 0$.

We shall denote $[P_n] \downarrow\downarrow 0$ if for every subsequence of $[P_n]$ there exists its subsequence order-convergent to 0.

Now, we shall consider a special convergence in a σ-complete vector lattice.

We shall denote

$$\circledast - \lim_{n \to \infty} a_n = a$$

if there exists $[P_m] \downarrow\downarrow 0$ such that $(\mathrm{I} - [P_m])\,a_n$ is star-order-convergent to $(\mathrm{I} - [P_m])\,a$ $(m = 1, 2, \cdots)$. In the case of L_1-space (the totality of summable functions) $f_n \to f$ (in measure) implies $f_n \to f$ (in above sense).

It is easy to see that if $\circledast - \lim_{n \to \infty} a_n = a$, then there exists $[p_m] \downarrow\downarrow 0$ such that $p_m \in R\,(m = 1, 2, \cdots)$ and $(\mathrm{I} - [p_m])\,a_n$ is star-order-convergent to $(\mathrm{I} - [p_m])\,a$ for all $m = 1, 2, \cdots$.

We see easily $[p_n] \downarrow 0$ and $[q_n] \downarrow 0$ imply $[p_n] \cup [q_n] \downarrow 0$. Hence, we see that if $\circledast-\lim_{n\to\infty} a_n=a$ and $\circledast-\lim_{n\to\infty} b_n=b$, then

$$\circledast-\lim_{n\to\infty}(a_n+b_n)=\circledast-\lim_{n\to\infty} a_n+\circledast-\lim_{n\to\infty} b_n\,,$$
$$\circledast-\lim_{n\to\infty}(a_n\cup b)=a\cup b\,,\quad \circledast-\lim(a_n\cap b)=a\cap b\,,$$
$$\circledast-\lim_{n\to\infty}|a_n-a|=0\,.$$

§ 3. Equi-continuous subsets

Let $\bar{a}$ be a linear functional on R. $\bar{a}$ is said to be *o-continuous linear functional* if $o-\lim a_n=a$ implies $\bar{a}(a_n)\to\bar{a}(a)$.

It is easy to see that if $\bar{a}$ is an o-continuous linear functional on R, then

$$\sup_{0\leqq b\leqq|a|}|\bar{a}(b)|<+\infty \quad \text{for all} \quad a\in R\,.$$

The totality of o-continuous linear functionals is denoted by $\bar{R}$ and is called an order-conjugate space of R. Since R is reduced to 0 in some occasion, we assume that R is not trivial in the sense that for every $a\neq 0$, there exists $\bar{a}\in\bar{R}$ with $\bar{a}(a)\neq 0$.

$\bar{a}\geqq\bar{b}$ means that $\bar{a}(a)\geqq\bar{b}(a)$ for all $a\geqq 0$.

By this order, $\bar{R}$ is a complete vector lattice. In the sequel, we assume that R is not trivial. A subset Γ of R is *equi-continuous* if for $\bar{a}_n\downarrow 0$, $\bar{a}_n\in\bar{R}$ and $\varepsilon>0$, there exists a natural number n_0 with

$$\bar{a}_{n_0}(|a|)\leqq\varepsilon \quad \text{for all} \quad a\in\Gamma\,.$$

By definition, if Γ is an equi-continuous subset of R, then $N[\Gamma]=\{b\,;\,|b|\leqq|a|$ for some $a\in\Gamma\}$ is also an equi-continuous subset. It is known that Γ is equi-continuous iff Γ is relative compact by the weak topology induced by $\bar{R}$.

For $[P_n]\downarrow 0$ and $\bar{a}\in R$, $\bar{a}[P_n](a)=\bar{a}([P_n]a)$ is also o-continuous linear functional for all n and $\bar{a}[P_n]\downarrow 0$.

§ 4. Vitali's theorem

Vitali's theorem for summable functions can be formulated as follows in the case of vector lattices.

Theorem 1. *Let $a_n\in R$ $(n=1,2,\cdots)$ be an equi-continuous sequences and $\circledast-\lim_{n\to\infty} a_n=a$. Then, a_n is weakly convergent to a (i.e. $\bar{a}(a_n)\to\bar{a}(a)$ for all $\bar{a}\in\bar{R}$).*

Proof. *Let* $[p_n]\downarrow\downarrow 0$ and $0\leqq\bar{a}\in\bar{R}$. We shall prove that we can find a natural number n_0 such that

$$\bar{a}([p_n]|a_m|) \leqslant \varepsilon \quad \text{for} \quad n \geqq n_0 \quad \text{and} \quad m=1, 2, \cdots .$$

If not, we find $\varepsilon > 0$ and $m_\nu \leqq m_{\nu+1} \leqq \cdots$ and $n_\nu \leqq n_{\nu+1} \leqq \cdots$ such that

$$\bar{a}([p_{n_\nu}]|a_{m_\nu}|) \geqq \varepsilon \qquad \nu = 1, 2, \cdots .$$

By assumption, there exists a subsequence $[q_\nu]$ of $[p_{n_\nu}]$ $(\nu=1, 2, \cdots)$ order-convergent to 0 such that

$$\bar{a}([q_\nu]|a_{m_\nu}|) \geqq \varepsilon .$$

This contradicts to the equi-continuity of $\Gamma = \{a_n\}$.

We shall prove that $s-0-\lim a_n = a$ implies $\bar{a}(a_n) \rightarrow \bar{a}(a)$.

If not, there exists n_ν $(\nu=1, 2, \cdots)$ such that

$$|\bar{a}(a_{n_\nu} - a)| \geqq \varepsilon \quad \text{for some} \quad \varepsilon > 0 .$$

By assumption, we find a subsequence of $\{a_{n_\nu}\}$ order-convergent to a. This is a contradiction, since $\bar{a}$ is continuous by order-convergence.

Let $\circledast - \lim_{n\rightarrow\infty} a_n = a$. There exists $[a] \geqq [p_m] \downarrow\downarrow 0$ such that

$$s-o-\lim_{n\rightarrow\infty} (I - [p_m])\, a_n = (I - [p_m])\, a .$$

Hence, choosing m such that $|\bar{a}([p_m](a_n - a)| < \varepsilon$, we have

$$|\bar{a}(a_n) - \bar{a}(a)| \leqq |\bar{a}([p_m](a_n - a))| + |\bar{a}((I - [p_m])(a_n - a))| \leqq 2\varepsilon$$

for sufficient large n.

This proves Theorem 1.

COROLLARY. *Let* $\circledast - \lim_{n\rightarrow\infty} a_n = a$. $\{a_n\}$ *is weakly convergent to* a *if and only if* $\{a_n\}$ *is equi-continuous.*

§ 5. $|w|$-convergence.

By definition,

$|w| - \lim_{n\rightarrow\infty} a_n = a$ iff $\lim_{n\rightarrow\infty} \bar{a}(|a_n - a|) = 0$ for all $\bar{a} \in \bar{R}$. $|w| - \lim_{n\rightarrow\infty} a_n = a$ implies that a_n is weakly convergent to a. But, in general the converse is not true.

THEOREM 2. *Under the assumption of Theorem* 1, *we have* $|w| - \lim_{n\rightarrow\infty} a_n = a$.

PROOF. If $\{a_n\}$ is equi-continuous, then $\{x_n\}$ is equi-continuous where $x_n = |a_n - a|$. We see easily

$$\circledast - \lim_{n\rightarrow\infty} a_n = a \quad \text{iff} \quad \circledast - \lim_{n\rightarrow\infty} x_n = \circledast - \lim_{n\rightarrow\infty} |a_n - a| = 0 .$$

By Theorem 1, $\{x_n\}$ is weakly convergent to 0. But this means that

$$|w|-\lim_{n\to\infty} a_n = a\,.$$

REMARK. If R is a space of summable functions defined on a finite measure space, under the assumption of Theorem 1, we have $\lim_{n\to\infty} \|a_n - a\| = 0$ by Theorem 2.

Department of Mathematics
Hakkaido University

References

[1] H. NAKANO: Modulared semi-ordered linear spaces, Maruzen, Tokyo, 1950.
[2] I. NATANSON: Theory of functions of a real variable, (English translation) Ungar, New York, 1955.

(Received August 20, 1971)